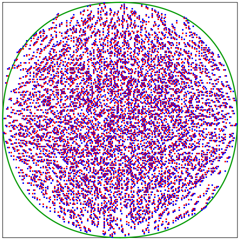
The Tangent Method: where do we stand?

Andrea Sportiello
work in collaboration with F. Colomo



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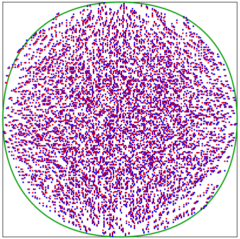
The topic of this talk is the determination of the "Arctic Curve" of 2-dimensional tiling models which are Yang–Baxter integrable, with special attention to the famous case of "Alternating Sign Matrices", that is, the 6-Vertex Model with domain-wall boundary conditions at the $\Delta = \frac{1}{2}$ combinatorial point.



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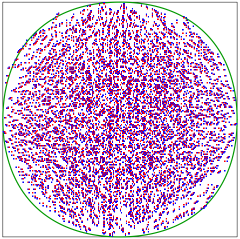

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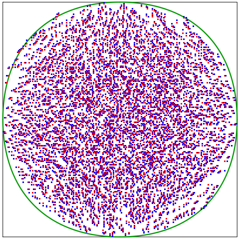

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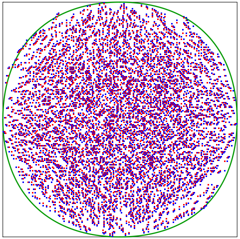

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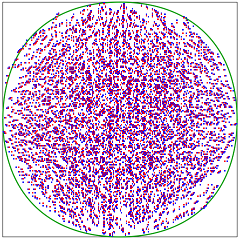

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hmm...

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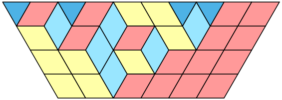

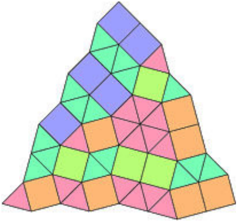

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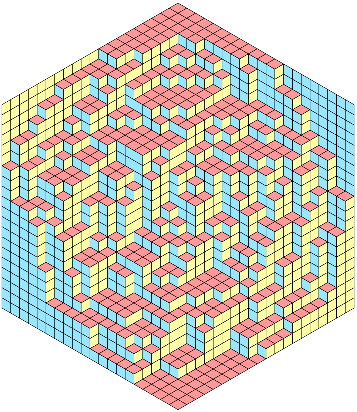
I will only mention here (and nowhere again)
that Gelfand–Tsetlin patterns, seen as lozenge tilings
of a trapezoid, are the simplest instance of the method,
and that Paul's interpretation of Littlewood–Richardson coefficients
(i.e. Knutson–Tao puzzles) as integrable square-triangle tilings
is also (in principle... to be done...) ameaneable to the method.

The dawn of Arctic Curves

The first examples of Arctic Curves have been on models of free fermions, realised as dimer models on bipartite graphs.
A famous case is lozenge tilings of a regular hexagon (the MacMahon problem of "boxed plane partitions")

This led to the famous arctic circle phenomenon



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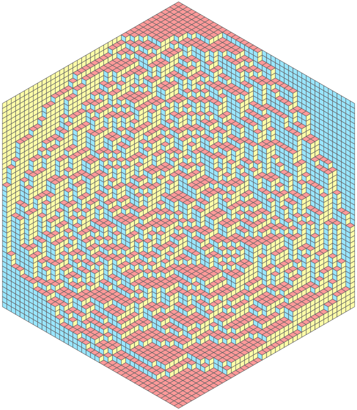

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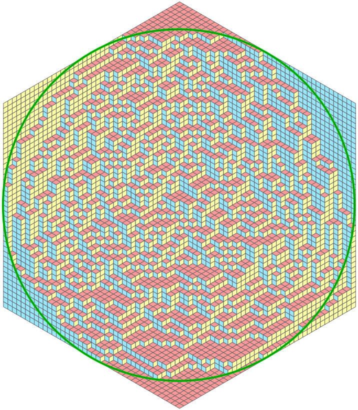
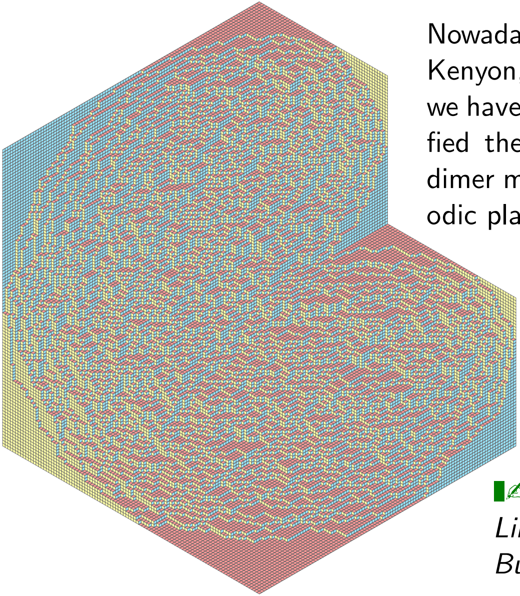


The ultimate theory for Arctic Curves in dimer models



Nowadays, thanks to the work of Kenyon, Okounkov and Sheffield, we have a (beautiful) general unified theory for Arctic Curves of dimer models on portions of periodic planar bipartite graphs.

An example: the cardioid for the hexagonal domain with a frozen corner

picture taken from:

R. Kenyon, A. Okounkov, *Limit shapes and the complex Burgers equation*, 2005

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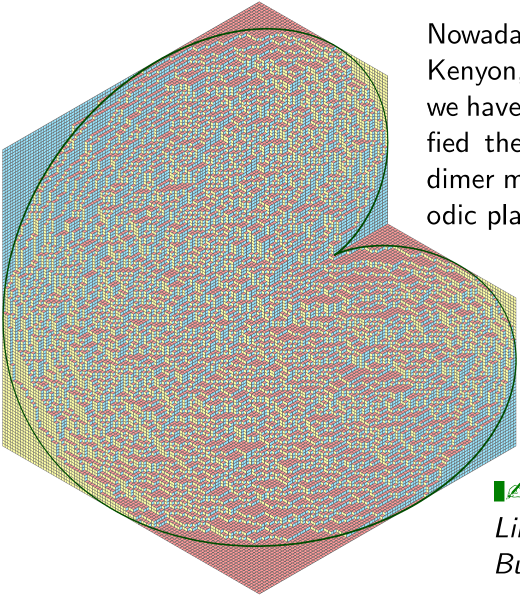

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Phase separation phenomena beyond free fermions

When we started this project, years ago,
my dream was to have an equivalent of the results
in Kenyon–Okounkov theory for integrable systems
out of the free-fermionic manifold.

This is a complicated goal for several reasons
that will be clear later on...

Let us just say, by now, that we want *some* result, in the fashion of the Arctic Circle phenomenon, for a non-free-fermionic integrable model that shows phase separation phenomena.

6VM: the ‘best’ model for phase separation phenomena

Although the Tangent Method applies to many different (integrable) models, we will mainly talk here of the 6-Vertex Model (6VM), on portions of $\mathbb{Z}^2$ with DWBC. Configs in this model are also bijectively related to Fully-Packed Loops (FPL) and Alternating-Sign Matrices (ASM).

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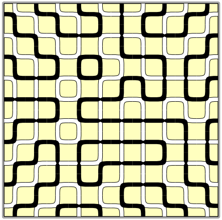
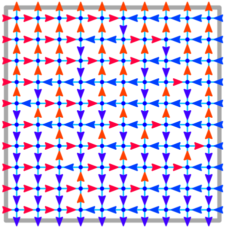


FPL



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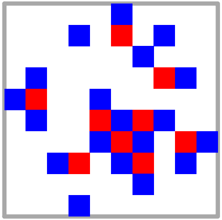

ASM

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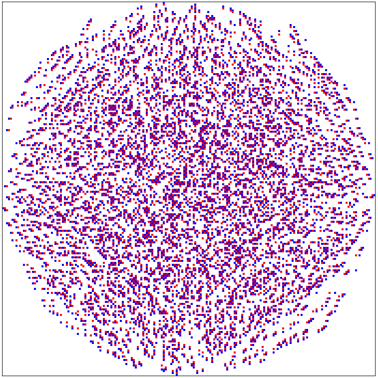
Also in large ASM's you see the emergence of an Arctic Curve



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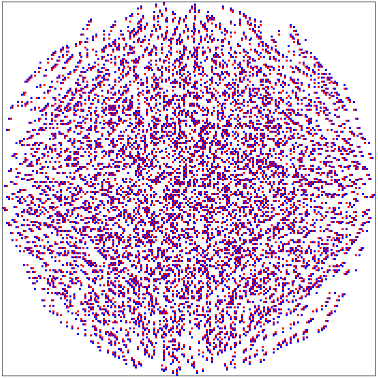
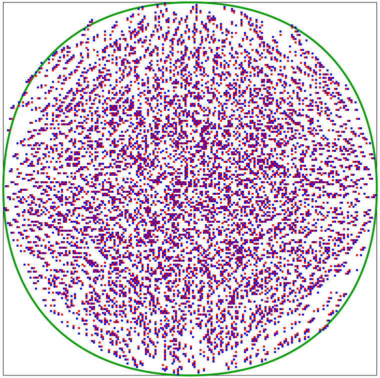
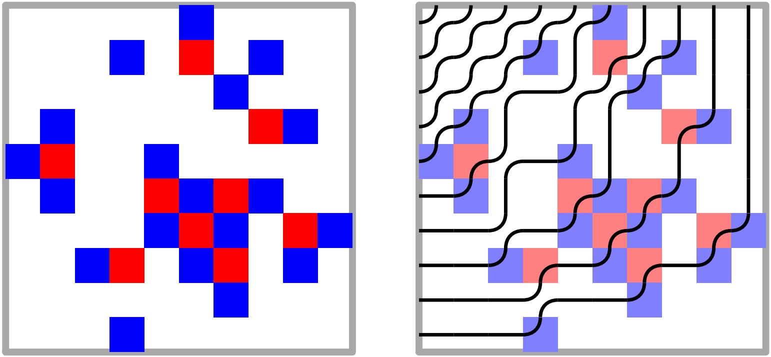




The path representation

Another local bijection is in terms of (interacting) non-intersecting paths.
Note that we have four ways of drawing paths,
each of which leaves empty one different “frozen corner”



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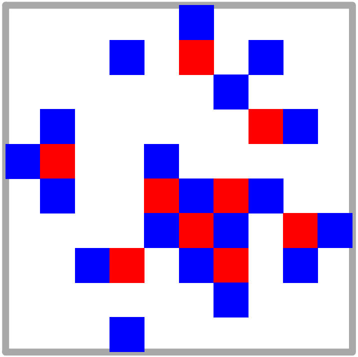

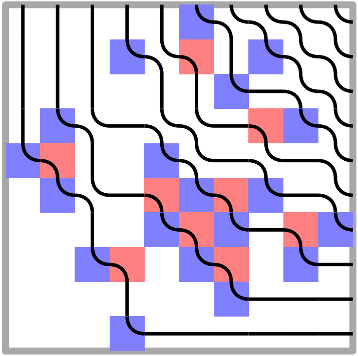

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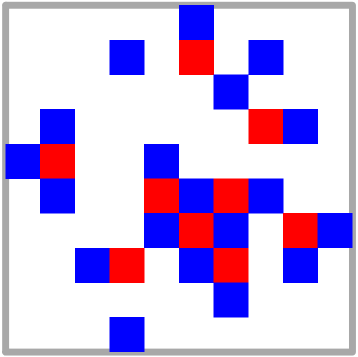

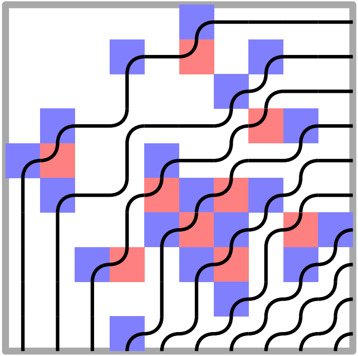

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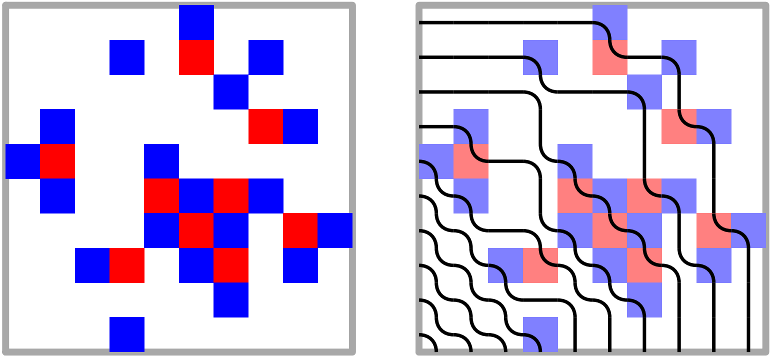


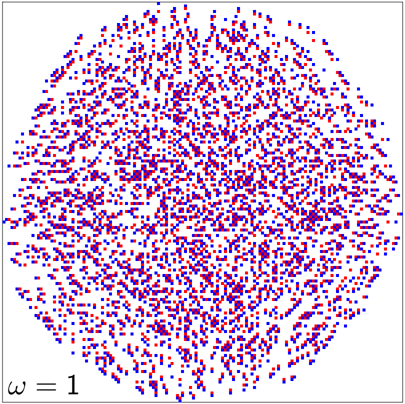
ω-enumeration of Alternating Sign Matrices

For $\omega \in \mathbb{R}^+$, let us weight ASMs with a measure $\omega^{\#\blacksquare}$

ω-enumerations of ASM form a YB-integrable line, with a fermionic point at $\omega = 2$ (domino tilings of the Aztec Diamond)

Numerical simulations suggest that the arctic curve varies smoothly with ω, at least in some range...

...but what was know theoretically when we started all this?



from this point on, ASM pictures are produced with C code based on a version kindly provided by Ben Wieland

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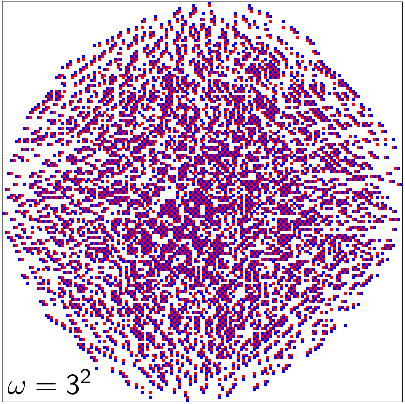

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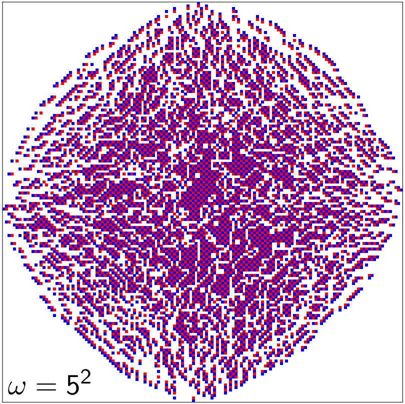

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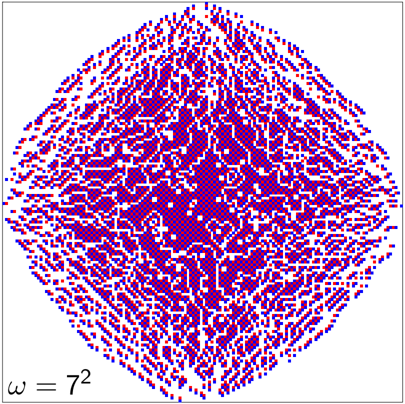

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Old facts from integrability

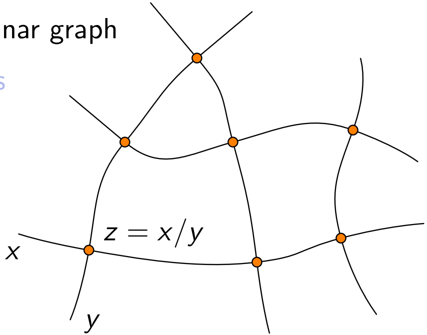
The ω-enumeration of ASMs is a 1-parameter deformation of the uniform measure within the much deeper and more general deformation inspired by the integrability of the underlying 6VM with DWBC...

The 6-Vertex Model in general

Let us define precisely our 6-Vertex Model:

- you have a degree-4 outer-planar graph
- variables are edge-orientations
- weights are on the vertices

depend on the four arrows,
through spectral parameters
attached to the lines,
and a global parameter q



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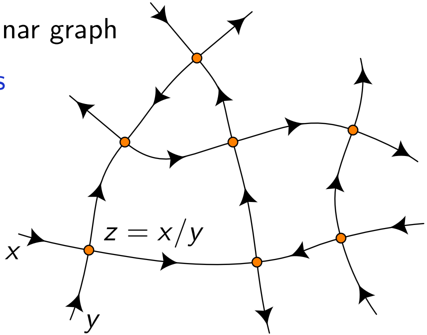
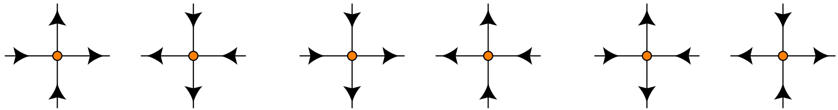


$a = zq - (zq)^{-1}$ $\qquad$ $b = z - z^{-1}$ $\qquad$ $c = q - q^{-1}$



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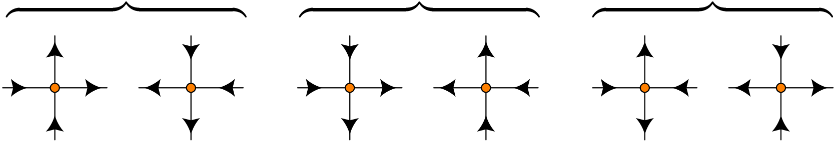
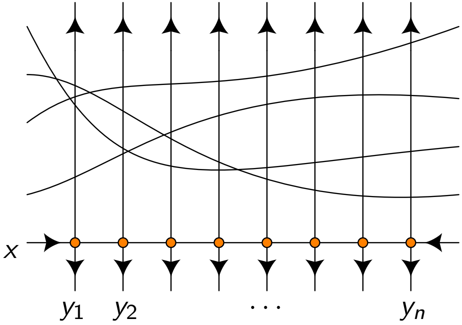
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}\left(q + \frac{1}{q}\right)$$

$$a = zq - (zq)^{-1} \qquad b = z - z^{-1} \qquad c = q - q^{-1}$$



The Izergin determinant

Now, suppose that, near to one boundary, your graph (and boundary conditions) look like this:



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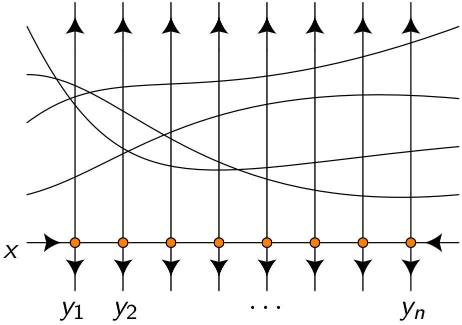

By YB, the partition function $Z(x, y_1, \dots, y_n, \cdots)$ is symmetric in $y_1, \dots, y_n$

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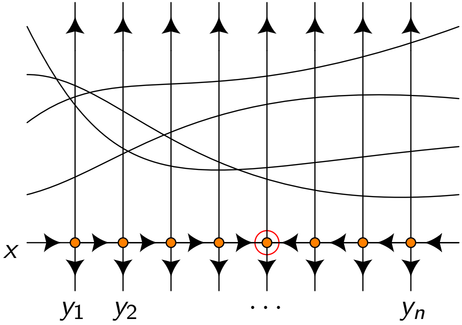

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The vertices in the bottom row form a sequence $aa \cdots acbb \cdots b$ Call refinement position the column index of this c-vertex. Also, $Z = x^{-n} P(x^2)$, where P ia a degree-$(n-1)$ polynomial

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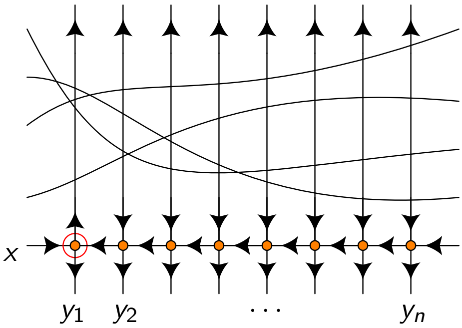

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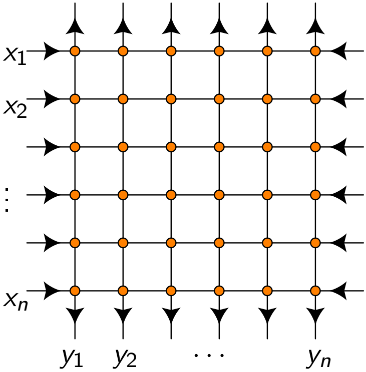
Also, $Z = x^{-n} P(x^2)$, where P ia a degree-$(n-1)$ polynomial

If $x = q^{-1} y_1$, the bottom-left vertex must be a "c",
if $x = y_n$, the bottom-right vertex must be a "c".
In both cases, the whole row can be filled in a unique way.

The Izergin determinant

Reasonings of this sort led Izergin to determine that

$$Z_n(x_1,\dots,x_n;y_1,\dots,y_n) = \frac{\prod_{i,j} a(x_i/y_j)b(x_i/y_j)}{\prod_{i<j} b(x_i/x_j)b(y_i/y_j)} \times \det\left(\frac{c}{a(x_i/y_j)b(x_i/y_j)}\right)_{1\le i,j\le n}$$



$Z_n(z,\dots,z;1,\dots,1)$ counts configs with homogeneous weights a, b and c. Due to DWBC, up to trivial factors, it only depends on $|a|/c$ and $|b|/c$. For $z = iq^{-\frac{1}{2}}$ we recover ω-enumerated ASMs, with $\omega = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$.

A.G. Izergin, *Partition function of the six-vertex model in a finite volume*, 1987

2-refined enumerations *(this is not old, but it could have been...)*

The homogeneous limit of the Izergin determinant is a bit tricky, as it requires multi-dimensional l'Hôpital.
A better idea is to write a recursion for $Z_n(xz, z \dots, z; y, 1, \dots, 1)$, by mean of the Desnanot–Jacobi formula

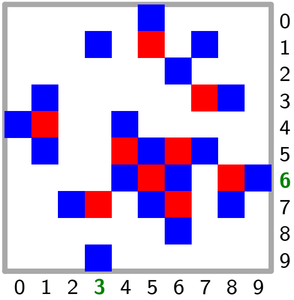
$$\det M \, \det M_{1n,1n} = \det M_{1,1} \, \det M_{n,n} - \det M_{1,n} \, \det M_{n,1}$$

As a bonus, we have a formula for the 2-refined enumeration polynomials $A_n^{\mathrm{rc}}(u, v)$.
In the parametrisation $a^2 = 1$, $b^2 = t$, $c^2 = w$ this gives

$$\begin{aligned}
(1-u)(1-v) \qquad & A_{n-1}\big(A_{n+1}^{\mathrm{rc}}(u,v) - (uvt)^n A_n\big) \\
- uvw \qquad & A_n \, A_n^{\mathrm{rc}}(u,v) \\
-\big((1-u)(1-v) - uvw\big) \qquad & A_n^{\mathrm{r}}(u) \, A_n^{\mathrm{c}}(v) = 0
\end{aligned}$$

$$A_n^{\mathrm{rc}}(u,v) := \sum_{A \in \mathcal{A}_n} t^{\frac{1}{2}\#b} w^{\frac{1}{2}(\#c-n)} u^{i(A)} v^{j(A)}$$

$$\begin{aligned} (1-u)(1-v) \quad & A_{n-1}\left(A_{n+1}^{\mathrm{rc}}(u,v) - (uvt)^n A_n\right) \\ -\,uvw \quad & A_n\, A_n^{\mathrm{rc}}(u,v) \\ -\big((1-u)(1-v) - uvw\big) \quad & A_n^{\mathrm{r}}(u)\, A_n^{\mathrm{c}}(v) = 0 \end{aligned}$$



This configuration contributes $u^3 v^6 w^9$ to $A_{10}^{\mathrm{rc}}(u,v)\big|_{t=1}$

Some special facts at the combinatorial point

When $q = e^{\frac{2}{3}\pi i}$ (i.e. $\omega = 1$), a number of further miracles occur:

- Some enumerations are 'round': $A_n = \prod_{j=0}^{n-1} \frac{j!(3j+1)!}{(2j)!(2j+1)!}$

  $A^{\mathrm{r}}_{n+1}[r] := [u^r]A^{\mathrm{r}}_{n+1}(u) = A_{n+1} \frac{(2n)!}{n!(3n+1)!} \frac{(n+r)!(n+\bar{r})!}{r!\,\bar{r}!}$
  $(\bar{r} = n - r)$
- Up to prefactors, Z_n is symmetric in the $2n$ variables $\{qx_1^2, \ldots, qx_n^2, y_1^2, \ldots, y_n^2\}$, and in fact a 'double-staircase' Schur function $s_{(n-1,n-1,n-2,n-2,\ldots,1,1)}(\cdots)$

  ✍ S. Okada, *Enumeration of symmetry classes of alternating sign matrices and characters of classical groups*, 2006
- As a result, the row-column 2-refined enums $A^{\mathrm{rc}}_n(u, v)$ can be related to their simpler row-row analogue $A^{\mathrm{rr}}_n(u, v)$

  ✍ Yu. Stroganov, *A new way to deal with Izergin-Korepin determinant at root of unity*, 2002

The Colomo–Pronko formula

Define the Emptiness Formation Probability, $EFP(n;r,s)$: the probability that there are no ±1 elements in the $s\times r$ top-left rectangle of the $n\times n$ domain.

$$h_{n,s}(z_1,\dots,z_s) := \frac{1}{\Delta(z)} \det\left(z_j^{k-1}(z_j-1)^{s-k} A^{\mathrm{r}}_{n-k+1}(z_j)\right)_{j,k}$$

$$EFP(n;r,s) = \oint_0 \frac{\mathrm{d}z_1}{2\pi i}\cdots\oint_0 \frac{\mathrm{d}z_s}{2\pi i} \prod_j \frac{((t^2-2t\Delta)z_j+1)^{s-j}}{z_j^{n-r}(z_j-1)^{s-j+1}}$$

$$\times \prod_{j<k} \frac{z_j-z_k}{t^2 z_j z_k - 2t\Delta z_j + 1}\; h_{n,s}(z_1,\dots,z_s)$$

For (r,s) crossing the Arctic Curve, $EFP(n;r,s)$ must show a 0-1 transition. However the asymptotic analysis is complicated and non-rigorous.

✍ F. Colomo and A.G. Pronko, 2008–'09

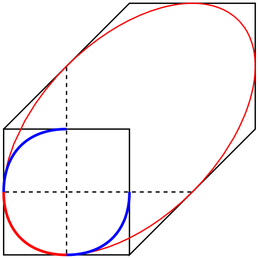
✍ F. Colomo, A.G. Pronko and P. Zinn-Justin, 2010

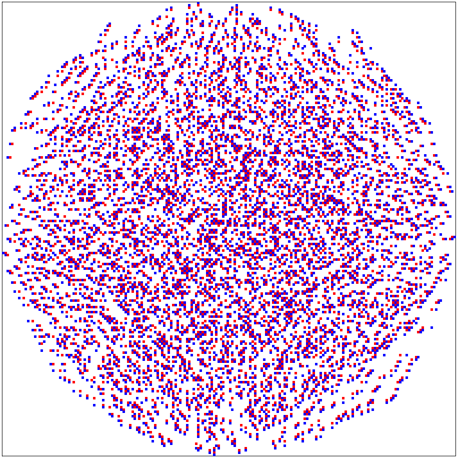
The Colomo–Pronko formula: $\omega = 1$

Picture and formula for $\omega = 1$:

The South-West arc satisfies
$x(1-x) + y(1-y) + xy = 1/4$
$x, y \in [0, 1/2]$

(just a "$+xy$" modification w.r.t. a circle)





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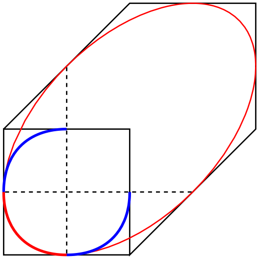
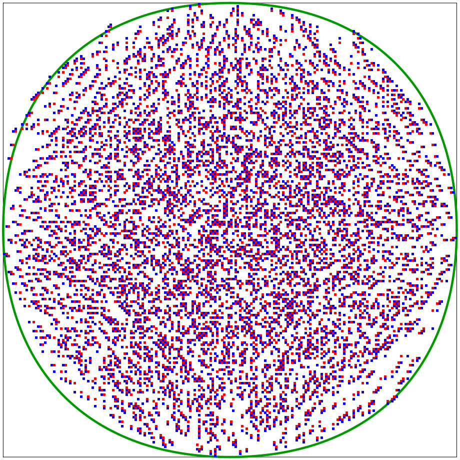




The Colomo–Pronko formula: generic ω

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$, in parametric form $x = x(z)$, $y = y(z)$ on the interval $z \in [1, +\infty)$, is the solution of the system of equations

$$F(z; x, y) = 0 \,; \qquad \frac{\partial}{\partial z} F(z; x, y) = 0 \,.$$

The function $F(z; x, y)$, that depends on x and y linearly, is

$$F(z; x, y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln A_n^{\mathrm{r}}(z) \,.$$

$\mathcal{C}(x, y)$ is algebraic only at discrete special values of ω (including $1, 2, 3$), namely when q is a root of unity

A few comments

In a few words, something very complicated already for the square.

Also, it's not clear that nice EFP formulas may exist in other domains.

Furthermore, differently from the curves in the Kenyon–Okounkov theory, already for $\omega = 1$, the curve is not $\mathcal{C}_\infty$ at the points of contact with the boundary of the domain, and for generic ω it is not even piecewise algebraic

How can we hope for an analog of Kenyon–Okounkov results for the 6-Vertex Model?

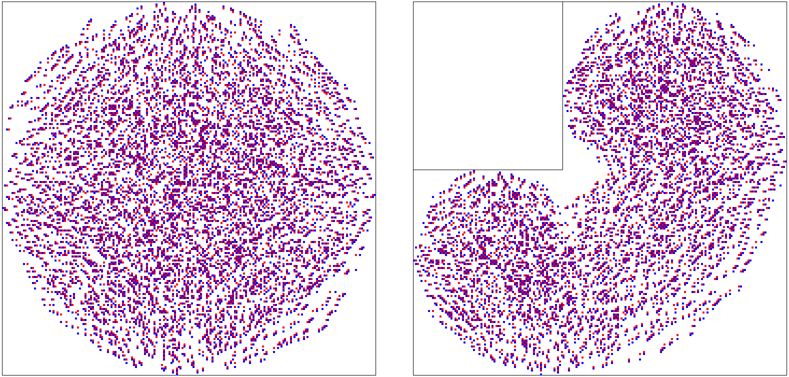
Can we hope to prove the Colomo–Pronko formula for the square?

Can we hope to determine new arctic curves, in domains more complicated than a square?

Emptiness Formation: typical configurations

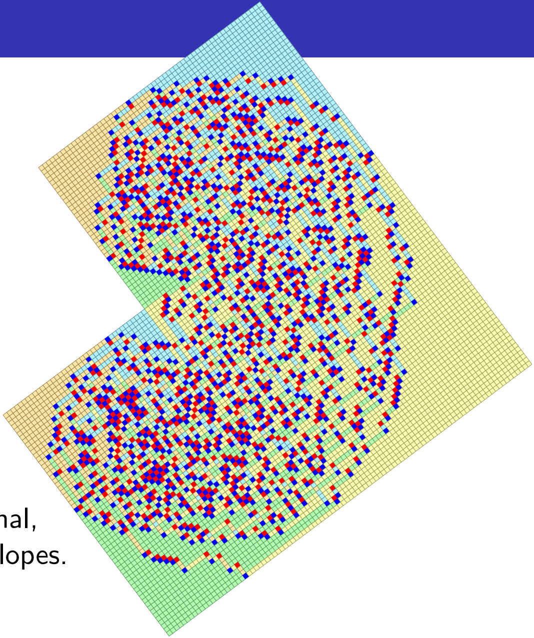
It is instructive to observe a typical configuration in the ensemble pertinent to $EFP(n; r, s)$.
For (r, s) inside the arctic curve, we see the emergence of a new (2-cusp) cardioid-like arctic curve (just like in Kenyon–Okounkov)

here $n = 200$, $(r, s) = (80, 90)$



Height function in ASMs

The height representation
is crucial in KO theory.
It also exists in the 6VM
(configs are the same as
in domino tilings, that's just
the weight that is different)
so this is good news...
As in Kenyon-Okounkov,
(and as in "soap bubbles"),
the surface minimizes a functional,
and DWBC correspond to ± 1 slopes.



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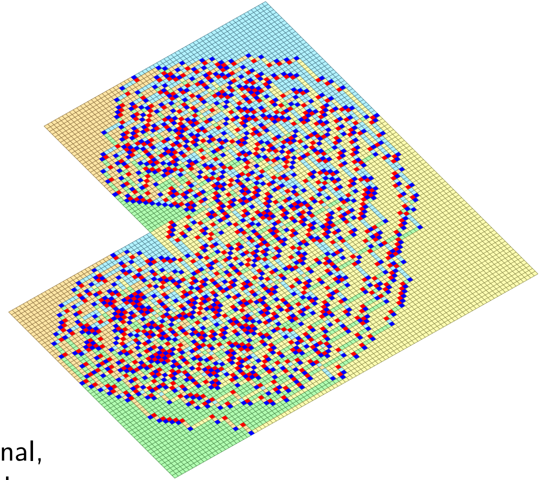

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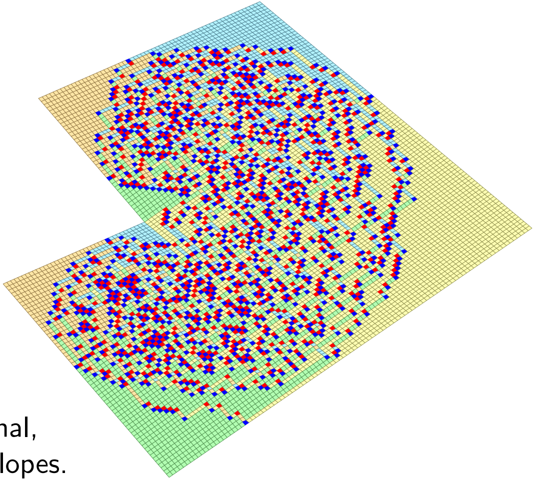

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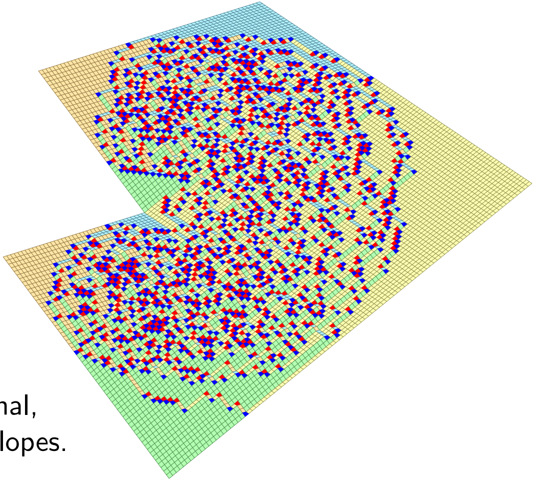

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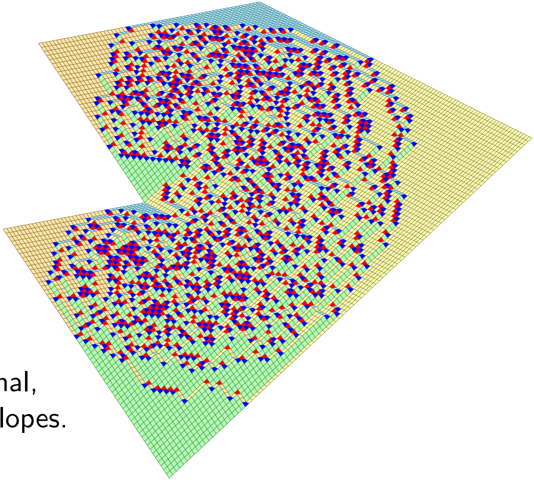

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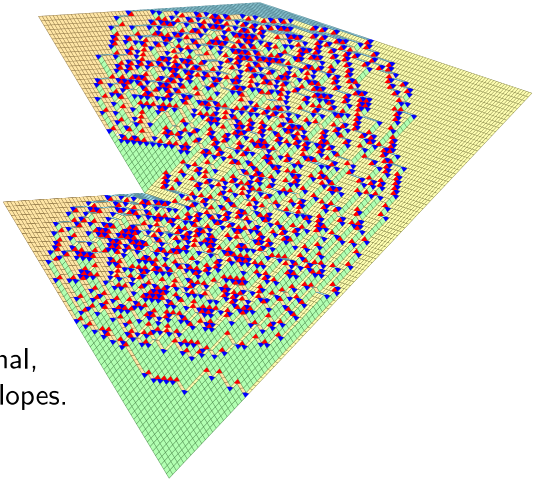

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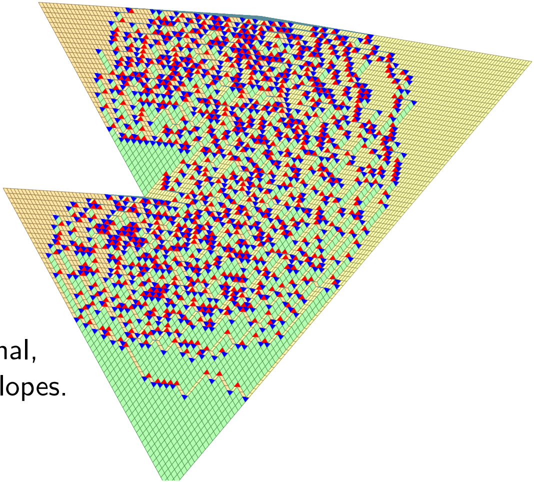

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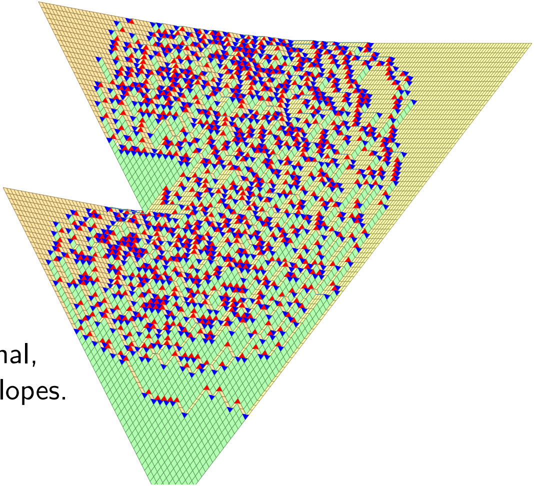

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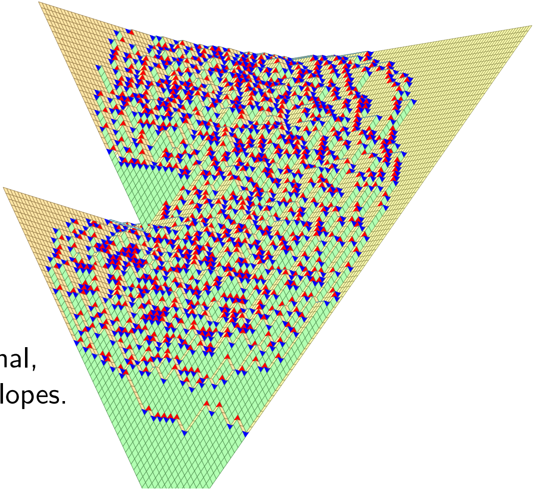


A reminder on the basic theory of Plane Curves

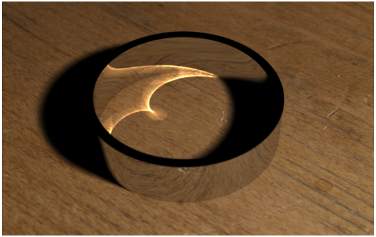
The *envelope* $\mathcal{E}$ of a one-parameter family of curves $\{\mathcal{C}_z\}_{z\in I}$ is the (minimal) curve that is tangent to every curve of the family.

If the equation of the family $\{\mathcal{C}_z\}$ is given in Cartesian coordinates by $U(z;x,y)=0$, the non-singular points (x,y) of the envelope $\mathcal{E}$ are the solutions of the system of equations

$$U(z;x,y)=0\,; \qquad \frac{\mathrm{d}}{\mathrm{d}z}U(z;x,y)=0\,.$$

We call *geometric caustic* the envelope of a family of straight lines. In this case U is linear in x and y:

$$U(z;x,y)=x\,A(z)+y\,B(z)+C(z)$$



A new hope

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x,y)$, in parametric form $x = x(z)$, $y = y(z)$ on the interval $z \in [1, +\infty)$, is the solution of the system of equations

$$F(z;x,y) = 0\,; \qquad \frac{\partial}{\partial z} F(z;x,y) = 0\,.$$

The function $F(z;x,y)$, that depends on x and y linearly, is

$$F(z;x,y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln A_n^{\mathrm{r}}(z)\,.$$

$\mathcal{C}(x,y)$ is algebraic only at discrete special values of ω (including $0, 1, 2, 3$).

A new hope

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$ is the geometric caustic of the family of lines, for z in the interval $[1, +\infty)$

$$F(z; x, y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n \to \infty} \frac{1}{n} \frac{\partial}{\partial z} \ln A_n^{\mathrm{r}}(z) \,.$$

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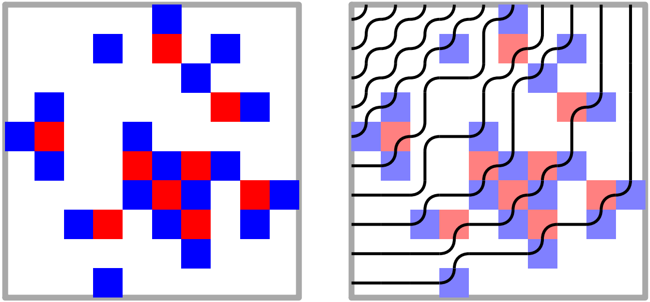
But this has not been derived geometrically!
Maybe, if we could interpret the meaning of the lines $\{F(z; x, y)\}_z$, we would have a new approach to the Arctic Curve!

This will lead to the Tangent Method

A reminder on interacting NILP

Recall that an ASM can be seen (in 4 different ways) as a configuration of interacting non-intersecting lattice paths (NILP), which are in fact non-interacting when $\omega = 2$.
Path weights: $\sqrt{\omega}$ for each corner, ω^{-1} for each contact.



The refinement position is the point at which the most external path leaves the boundary

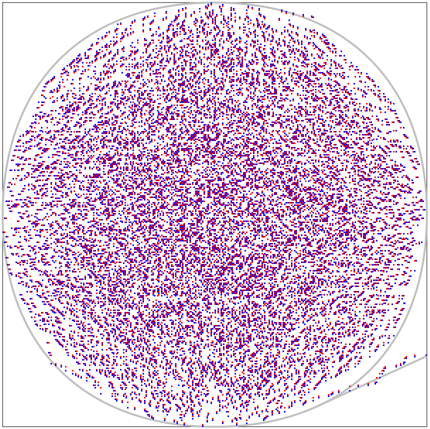
The structure of a typical refined ASM

Some thinking suggests how a typical large *ASM* of size n refined at r should look like.

It must be like a typical ASM, plus a straight line connecting $(0, r)$ to the Arctic Curve, and tangent to the Arctic Curve

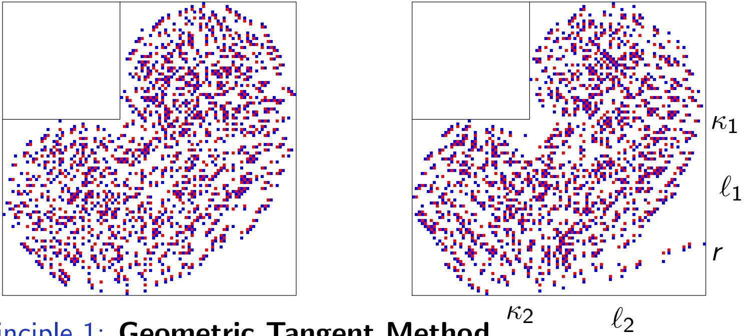
Indeed, this is what you see in simulations...

$n = 300, r = 250$



Let us turn this into the main idea of the first flavour of the Tangent Method

The Geometric Tangent Method

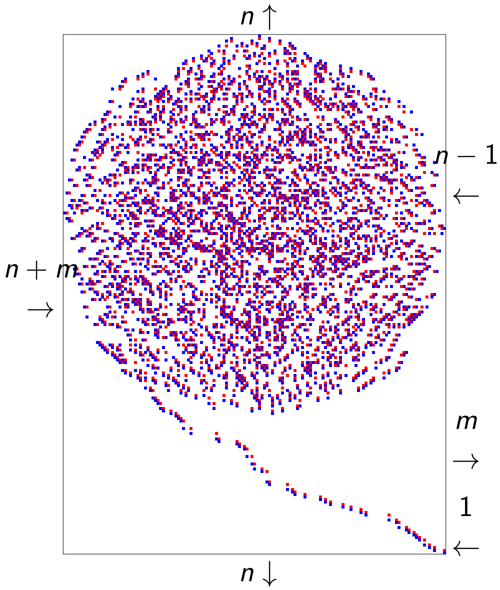


Principle 1: **Geometric Tangent Method**

Call Λ the domain shape, and $\mathcal{C}$ the corresponding Arctic Curve.

In the large n limit, a typical config on Λ, conditioned to have refinement position r along ℓ_1, shows the Arctic Curve $\mathcal{C}$ of the unrefined ensemble, plus a straight path from r to the tangency point on $\mathcal{C}$.

The Geometric Tangent Method in a picture



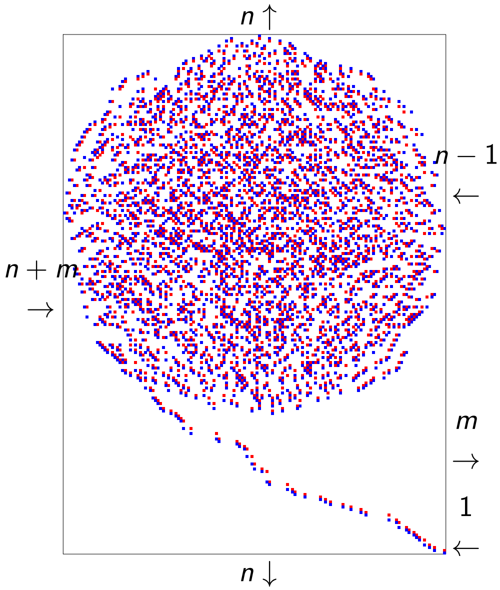
Let us see the consequences of this principle

In this geometry, there is no reason for the isolated line to change direction at row n. Then, if, according to our principle,

- the arctic curve exists
- it does not depend on m
- the segment reaches the curve tangentially

then the tangent method provides a caustic parametrisation of the curve

The Geometric Tangent Method in a picture



The trick is that

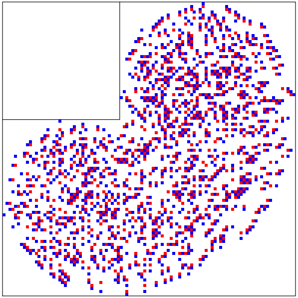
$$Z_{n,m} = \sum_r A_{n,r} B_{r,m}$$

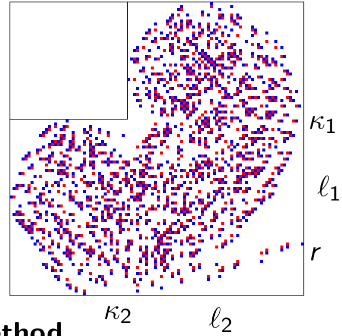
where $B_{r,s} = \sum_{\gamma:(0,0)\to(r,-s)} \sqrt{\omega}^{\#\text{corners}}$

For large n, the sum in r is dominated by a saddle point r^*. The resulting straight line goes through $(r^*, 0)$ with slope $-m/r^*$ and is tangent to $\mathcal{C}$.

Varying $m/n \in \mathbb{R}^+$, the caustic of these lines makes one arc of the curve $\mathcal{C}$.

The Entropic Tangent Method



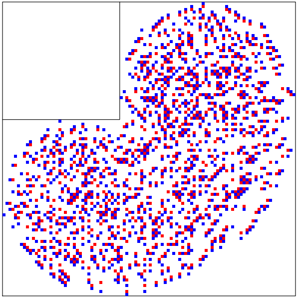


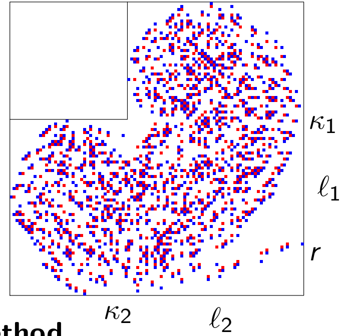
Principle 2: **Entropic Tangent Method**

Call Λ the domain shape, and $\mathcal{C}$ the corresponding Arctic Curve.

Call Λ' the domain Λ minus one row and one column, along the sides containing κ_1 and κ_2

The Entropic Tangent Method



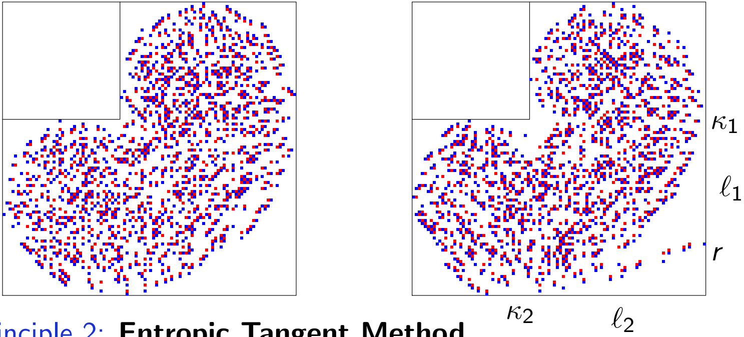


Principle 2: **Entropic Tangent Method**

Call $A(\Lambda)$ the number of configs in Λ, and $A^{(1)}(\Lambda, r)$, $A^{(2)}(\Lambda, s)$ the refined ASM enumerations along ℓ_1 and ℓ_2

Say $X(n) \sim Y(n)$ if $\lim_{n\to\infty} \frac{1}{n} \ln \frac{Y(n)}{X(n)} = \mathcal{O}(\ln n)$.

The Entropic Tangent Method



Principle 2: **Entropic Tangent Method**

Then

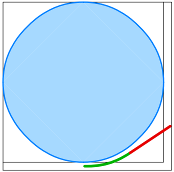
$$F_{\Lambda}^{(1,2)}(r,s) := \frac{A^{(1)}(\Lambda,r)\,A^{(2)}(\Lambda,s)}{A(\Lambda)\,A(\Lambda')\,B_{r,s}} \sim 1$$

If and only if the segment $\big((0,r),(s,0)\big)$ is tangent to $\mathcal{C}$ (otherwise $F(r,s) \sim \exp(-n\,\theta(1)))$

The Entropic Tangent Method in a picture

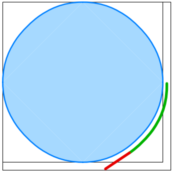
$A^{(1)}(\Lambda, r)A^{(2)}(\Lambda, s) \sim A(\Lambda)A(\Lambda')B_{r,s}$ iff $\big((0,r),(s,0)\big)$ is tangent to $\mathcal{C}$.

Free energy (logarithm) of...



$A^{(1)}(\Lambda, r)$

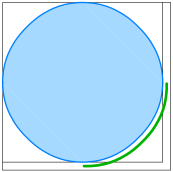
$+$



$A^{(2)}(\Lambda, s)$

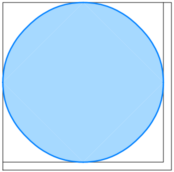
$=$

Free energy (logarithm) of...



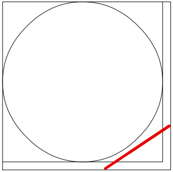
$A(\Lambda)$

$+$



$A(\Lambda')$

$+$

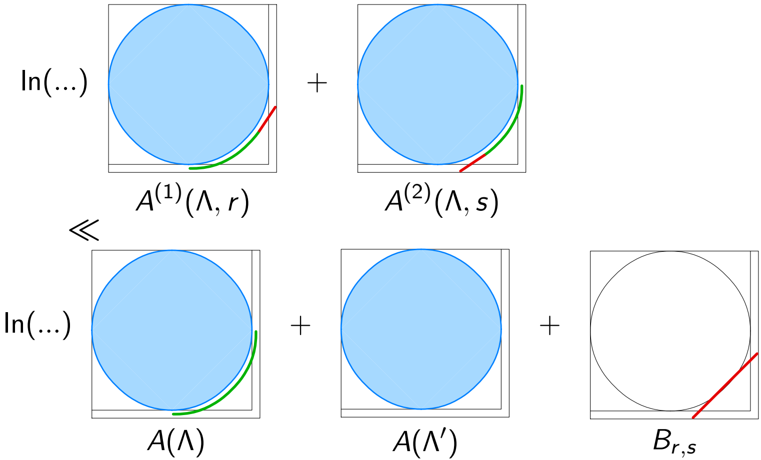


$B_{r,s}$

The Entropic Tangent Method's $F(r,s) \ll 1$

$A^{(1)}(\Lambda, r)A^{(2)}(\Lambda, s) \ll A(\Lambda)A(\Lambda')B_{r,s}$ if $\big((0,r),(s,0)\big)$ is not tangent to $\mathcal{C}$.

1) segment inside the arctic curve



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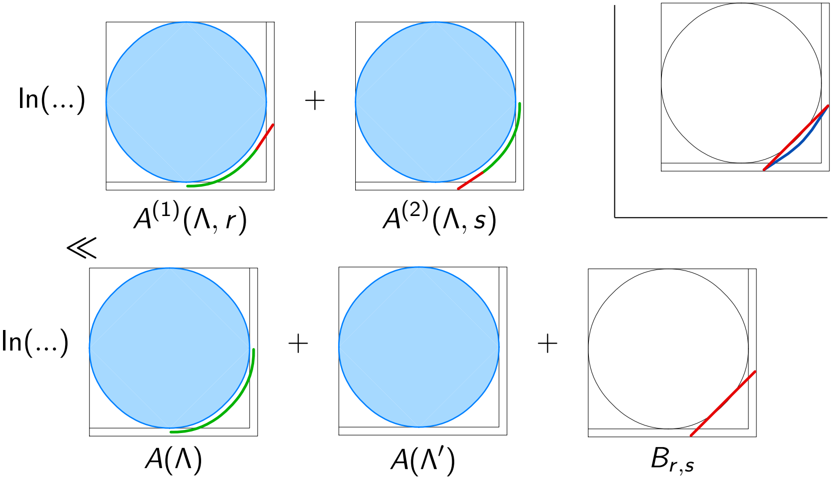

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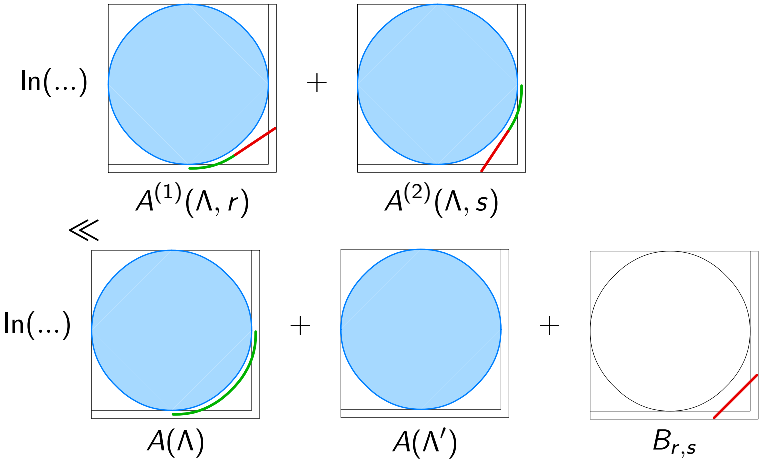
$A^{(1)}(\Lambda, r)A^{(2)}(\Lambda, s) \ll A(\Lambda)A(\Lambda')B_{r,s}$ if $\big((0,r),(s,0)\big)$ is not tangent to $\mathcal{C}$.

2) segment outside the arctic curve



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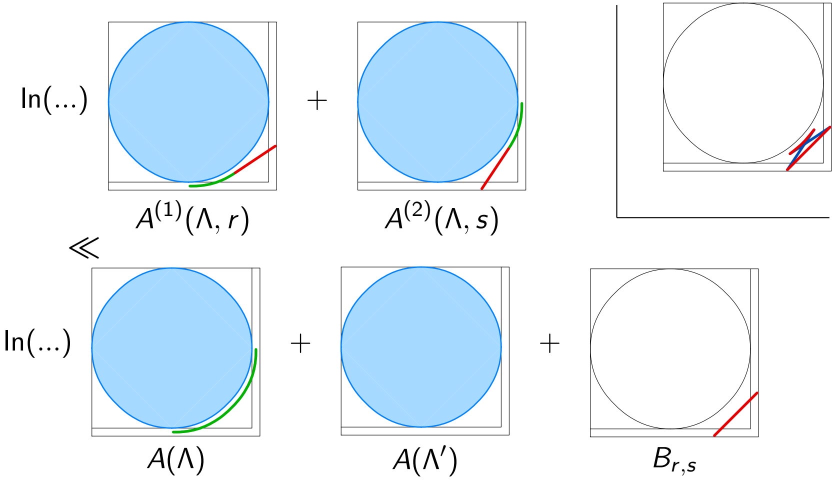
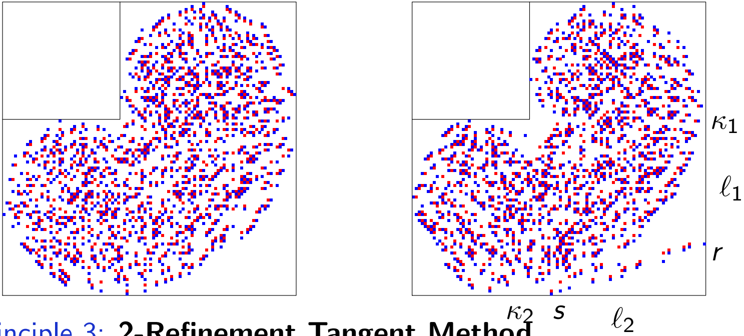


The 2-Refinement Tangent Method

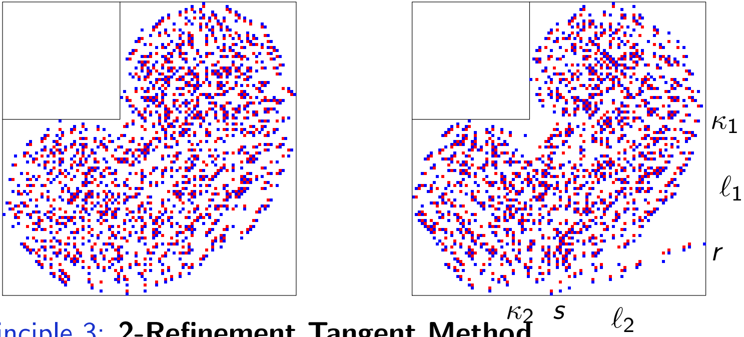


Principle 3: **2-Refinement Tangent Method**

Call Λ the domain shape, and $\mathcal{C}$ the corresponding Arctic Curve.

Call Λ' the domain Λ minus one row and one column, along the sides containing κ_1 and κ_2

The 2-Refinement Tangent Method



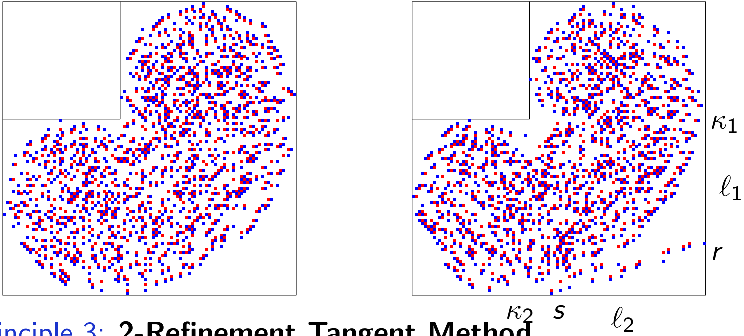
Principle 3: **2-Refinement Tangent Method**

Call $A(\Lambda)$ the number of configs in Λ, and $A^{(1,2)}(\Lambda, r, s)$, the refined ASM enumerations along (ℓ_1, ℓ_2)

Define $E_\Lambda^{(1,2)}(r, s)$ by $\quad A^{(1,2)}(\Lambda, r, s) = A(\Lambda')\; B_{r,s}\; E_\Lambda^{(1,2)}(r, s)$

Note that $0 \leq E_\Lambda^{(1,2)}(r, s) \leq 1$ when $\omega \geq 1$

The 2-Refinement Tangent Method



Principle 3: **2-Refinement Tangent Method**

Then $E_\Lambda^{(1,2)}(r,s)$ should have a sharp ($\sim \sqrt{n}$) 0-1 transition, when the segment is inside/outside the arctic curve.

If you can estimate $E_\Lambda^{(1,2)}(r,s)$, and see that it has such a transition, on a convex curve, then you have a **proof** that this curve is one arc of the Arctic Curve of the model.

Three flavours: advantages and disadvantages

So, we have seen that the Tangent Method exists in three flavours: geometric (G-TM), entropic (E-TM) and 2-refinement (2R-TM)

G-TM: Simplest one for calculations, easiest to visualise, links directly to Colomo–Pronko final formula. Seemed hard to make rigorous, see however the recent work of Amol Aggarwal.

E-TM: Looks crazy at first, but it seems to work pretty well. No hope for making its assumptions rigorous without extra tricks.

2R-TM: This method is fully rigorous, at least when the contact-term interaction of the paths is repulsive. But it requires to calculate $A^{\mathrm{rc}}(u, v)$, which is harder than $A^{\mathrm{r}}(u)$. Possibly, in some models/geometries we will have $A^{\mathrm{r}}(u)$, and never get to $A^{\mathrm{rc}}(u, v)$.

Three flavours of a unique method?

Recall that we have seen that YB implies the symmetry of Z over spectral parameters in the same bundle. In some lucky cases this leads to determinantal formulas like Izergin's one.

And determinants satisfy the Desnanot–Jacobi identity, which leads to a recursion for A^{rc} involving A^{r} and A^{c}

A nice surprise is that, as experience suggests in a few cases, DJ "blends our flavours", and allows to enjoy the rigour of 2R-TM together with the simplicity of G-TM and E-TM...

This works for the 6VM in general, but let us first illustrate this at $\omega = 1$, where it gets simpler...

A helping hand from Stroganov

At $\omega = 1$ we just have $B_{r,s} = \binom{r+s}{r}$, and the 2R-TM definition of the $0 \leq E_n(r,s) \leq 1$ quantity reads

$$A^{\mathrm{rc}}(n+1; r+1, s+1) = A(n) \binom{r+s}{r} E_n(r,s)$$

The knowledge of $A^{\mathrm{rc}}(n; r, s)$ (the "row-column" doubly-refined enumeration) is not so explicit as $A^{\mathrm{r}}(n; r)$, but is well under control (see e.g. Yu. Stroganov, *A new way to deal with Izergin-Korepin determinant at root of unity*)

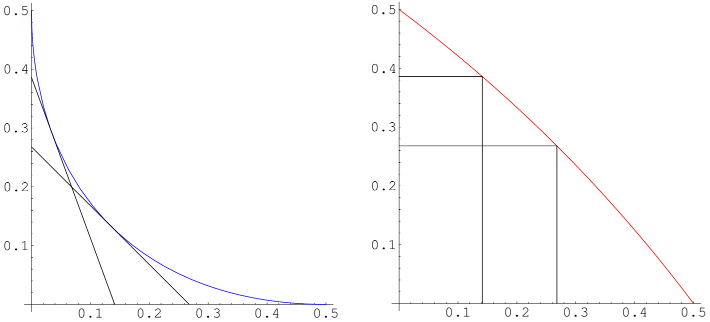
$$A^{\mathrm{rc}}(n; r, s+1) + A^{\mathrm{rc}}(n; r+1, s) - A^{\mathrm{rc}}(n; r+1, s+1) = A^{\mathrm{rr}}(n; r, s)$$

$$A^{\mathrm{rr}}(n; r, s) - A^{\mathrm{rr}}(n; r-1, s-1) = A(n-1)^{-1} \Big[A^{\mathrm{r}}(n-1, r-1) \big(A^{\mathrm{r}}(n, s) - A^{\mathrm{r}}(n, s-1) \big) + (r \leftrightarrow s) \Big]$$

To start: a simple transform

We want to find (the bottom-left corner of) the $\omega = 1$ arctic curve, which satisfies $x(1-x) + y(1-y) + xy = 1/4$

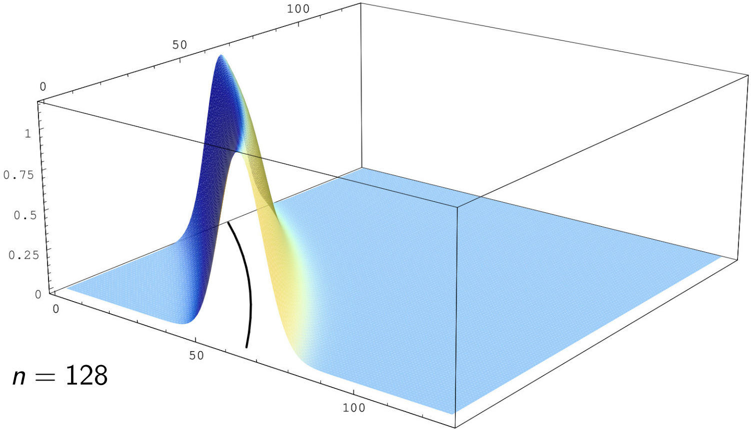
However, as our goal is to find it through the limit $n \to \infty$ of $E_n(\rho n, \sigma n)$, we shall equivalently represent it on the (ρ, σ) plane, where it gives $(\rho, \sigma)_\theta = \left(\frac{1-\sqrt{3}\tan\theta}{2}, \frac{1-\sqrt{3}\tan(\frac{\pi}{6}-\theta)}{2}\right)$, for $\theta \in [0, \frac{\pi}{6}]$



Let's have a look at $E_n(r, s)$

Let's have a look at $E_n(r, s)$, that shall converge to a step function

It is nicer to look at $-\sqrt{n}\partial_{(1,1)} E_n(r, s)$, that shall converge to a delta-function on our curve.



$n = 128$

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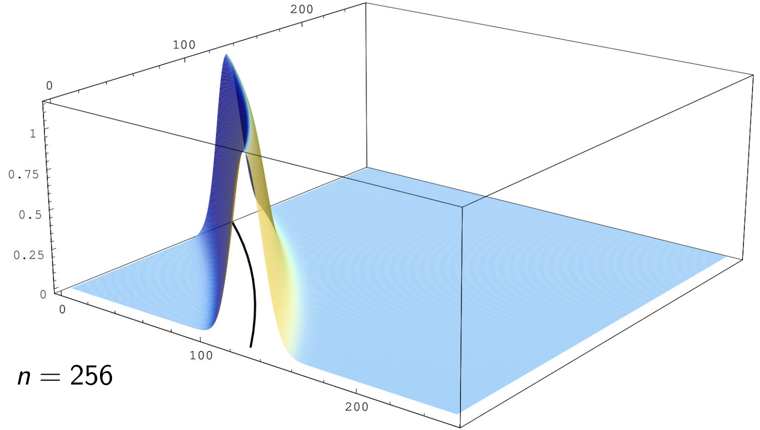

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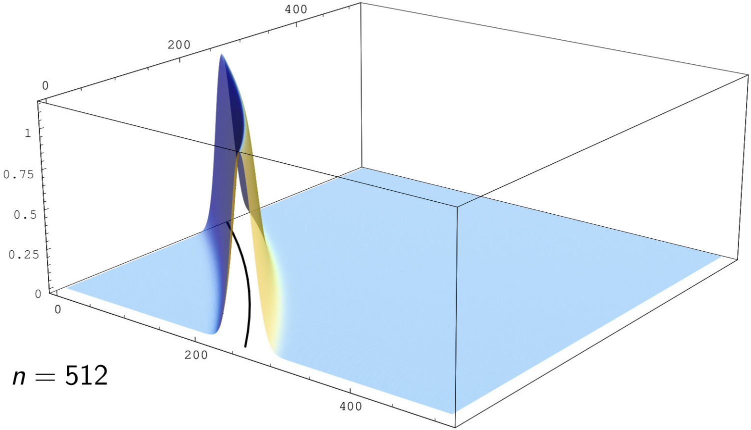

$n = 512$

A nice accident

In fact, although $A^{\mathrm{rc}}(n; r, s)$, the row-column 2-ref enumeration is well under control, $A^{\mathrm{rr}}(n; r, s)$, the row-row 2-ref enumeration is a bit easier

By a lucky accident, we have

$$A^{\mathrm{rc}}(n; r, s+1) + A^{\mathrm{rc}}(n; r+1, s) - A^{\mathrm{rc}}(n; r+1, s+1) = A^{\mathrm{rr}}(n; r, s)$$

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In fact, although $A^{\rm rc}(n;r,s)$, the row-column 2-ref enumeration is well under control, $A^{\rm rr}(n;r,s)$, the row-row 2-ref enumeration is a bit easier

By a lucky accident, we have

$$\frac{A^{\rm rc}(n;r,s+1)+A^{\rm rc}(n;r+1,s)-A^{\rm rc}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}} = \frac{A^{\rm rr}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

A nice accident

In fact, although $A^{\mathrm{rc}}(n;r,s)$, the row-column 2-ref enumeration is well under control, $A^{\mathrm{rr}}(n;r,s)$, the row-row 2-ref enumeration is a bit easier

By a lucky accident, we have

$$\frac{r}{r+s}\frac{A^{\mathrm{rc}}(n;r,s+1)}{A(n-1)\binom{r+s-1}{r-1}}+\frac{s}{r+s}\frac{A^{\mathrm{rc}}(n;r+1,s)}{A(n-1)\binom{r+s-1}{r}}-\frac{A^{\mathrm{rc}}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}}=\frac{A^{\mathrm{rr}}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

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By a lucky accident, we have

$$\tfrac{r}{r+s} E_n(r-1, s) + \tfrac{s}{r+s} E_n(r, s-1) - E_n(r, s) = \frac{A^{\mathrm{rr}}(n; r, s)}{A(n-1)\binom{r+s}{r}}$$

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By a lucky accident, we have

$$-\frac{r\partial_r^- + s\partial_s^-}{r+s} E_{n-1}^{\mathrm{RW}}(r, s) = \frac{A^{\mathrm{rr}}(n; r, s)}{A(n-1)\binom{r+s}{r}}$$

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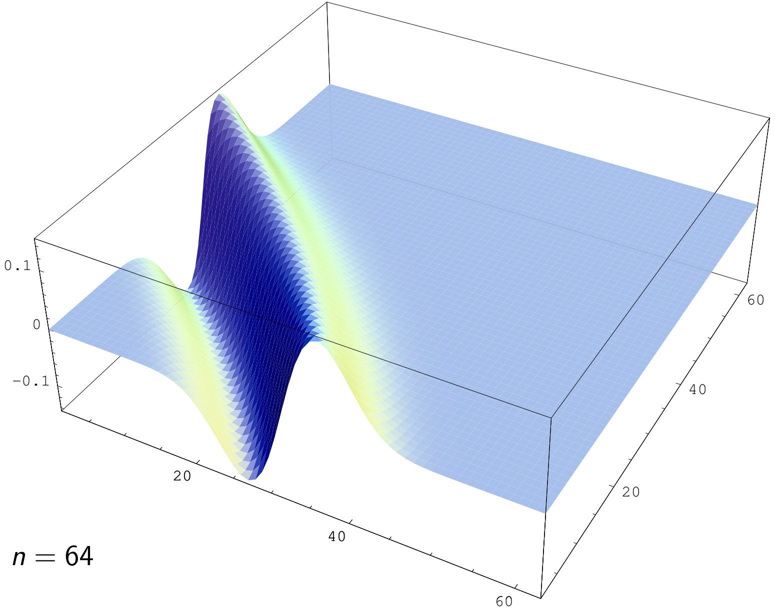
Thus we are done if we prove that $\frac{A^{\mathrm{rr}}(n;r,s)}{A(n-1)\binom{r+s}{r}} \neq \exp(-\Theta(n))$ only on the transform $\hat{\mathcal{C}}$ of the arctic curve.

Also its gradient along the $(1,1)$ direction shall concentrate on $\hat{\mathcal{C}}$, and also change sign on this curve.

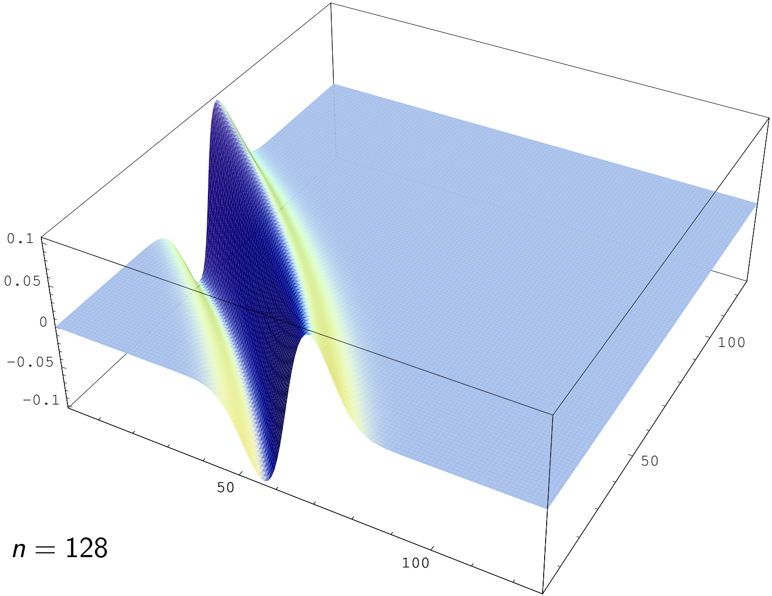
This can be done easily, as, by Stroganov formula, A^{rr} is just a Plücker-like combination of A and A^{r}, and both quantities are ‘round’.

A look at $\partial_{(1,1)}[A^{\mathrm{rr}}(n;r,s)/(A(n-1)\binom{r+s}{r})]$



$n = 64$

A look at $\partial_{(1,1)}[A^{\mathrm{rr}}(n; r, s)/(A(n-1)\binom{r+s}{r})]$



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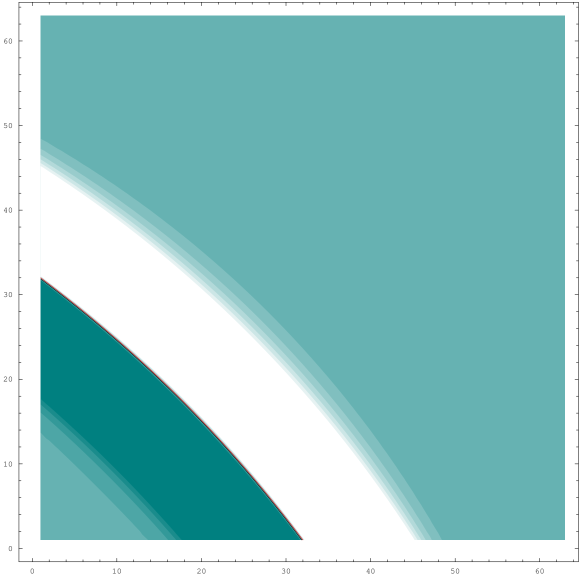
A look at $\partial_{(1,1)}[A^{\mathrm{rr}}(n; r, s)/(A(n-1)\binom{r+s}{r})]$



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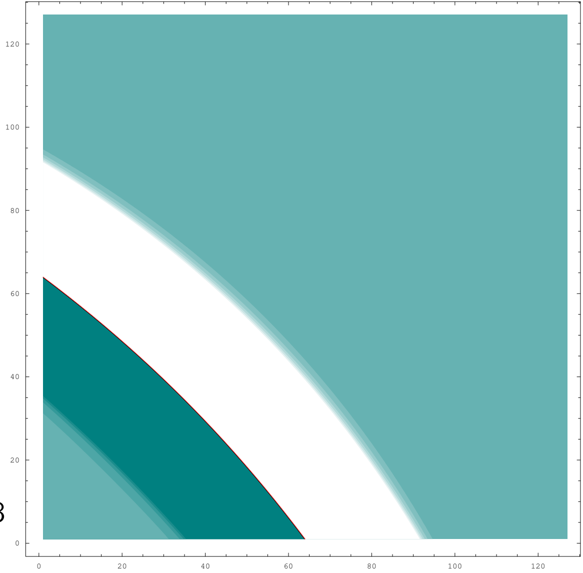


$n = 128$

Was this too easy? Let us try generic ω then!

So, where is this claimed blend of the E-TM and 2R-TM?

Actually, due to the lucky 'round' expressions of the combinatorial point, and the Stroganov relation among $A_n^{\mathrm{rc}}(u,v)$ and $A_n^{\mathrm{rr}}(u,v)$, in this very case we can solve the problem with 2R-TM, and no mention at all of the other flavours.

However, all the ingredients are there, and the case of generic ω will illustrate this more clearly...

Recall: for $a=b=1$ and $c=\sqrt{w}$,

$$\begin{aligned}(1-u)(1-v) \qquad & A_{n-1}\big(A_{n+1}^{\mathrm{rc}}(u,v)-(uv)^n A_n\big)\\ -\,uvw \qquad & A_n\,A_n^{\mathrm{rc}}(u,v)\\ -\big((1-u)(1-v)-uvw\big) \quad & A_n^{\mathrm{r}}(u)\,A_n^{\mathrm{c}}(v)=0\end{aligned}$$

Also note:

$$\sum_{r,s} B_{r,s} u^r v^s = \big((1-u)(1-v)-uvw\big)^{-1}$$

Was this too easy? Let us try generic ω then!

This can be slightly rephrased, by first noting that

$$1 = \frac{B_{r-1,s}}{B_{r,s}} + \frac{B_{r,s-1}}{B_{r,s}} + (w-1)\frac{B_{r-1,s-1}}{B_{r,s}} =: \pi_{1,0}(r,s) + \pi_{0,1}(r,s) + \pi_{1,1}(r,s)$$

and then looking at the $[u^r v^s]$ coeff of the DJ relation, divided by $A_n A_{n-1} B_{r,s}$. Calling T_x the "shift operators" $T_x f(\cdots, x, \cdots) = f(\cdots, x-1, \cdots)$, this reads, in terms of the quatities

$F(n,r,s) := \dfrac{A_n^{\mathrm{r}}[r]\, A_n^{\mathrm{c}}[s]}{A_n\, A_{n-1}\, B_{r,s}}$ of the E-TM and

$E(n,r,s) = \dfrac{A_n^{\mathrm{rc}}[r,s]}{A_{n-1}\, B_{r,s}}$ of the 2R-TM, in the form

$$P_1(T_r, T_s, T_n)E(n,r,s) = P_2(T_r, T_s, T_n)F(n,r,s)$$

where P_1, P_2 are certain polynomials, involving the $\pi_{ij}(r,s)$'s.

The structure of the DJ relation

$$P_1(T_r, T_s, T_n)E(n, r, s) = P_2(T_r, T_s, T_n)F(n, r, s)$$

where P_1, P_2 are certain polynomials, involving the $\pi_{ij}(r, s)$'s. P_1 and P_2 are crucially different: $P_1(1, 1, 1) = 0$, so it acts as a "discrete gradient" in a suitable direction of $(r, s, n) \in \mathbb{N}^3$, while $P_2(1, 1, 1) \neq 0$. From this we get the "moral" equation

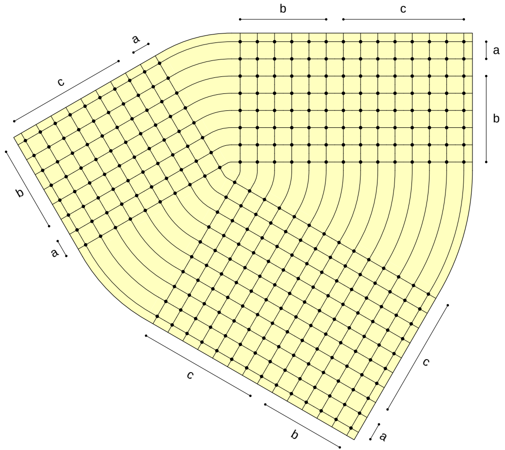
$$\nabla E(n, r, s) \simeq F(n, r, s)$$

which agrees with the fact that E is (asymptotically) a step function below $\hat{\mathcal{C}}$, while F is (asymptotically) a delta function on $\hat{\mathcal{C}}$.

Luckily enough, the inversion of our ∇ operator is sufficiently under control, so that the hypotheses of the E-TM imply the consequences of the 2R-TM

One Arctic Curve on a ‘new’ geometry

The severe bottleneck for obtaining arctic curves in new geometries is the absence of exact formulas for the refined enumerations...
...but we have a nice candidate, our favourite triangoloid domain!



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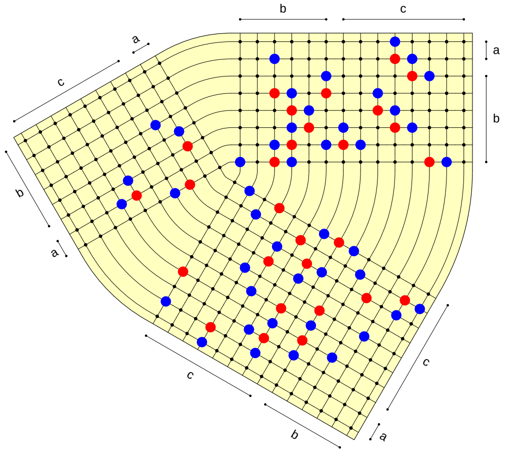

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This domain arises from the work of L. Cantini and myself on the classification of domains for which the Razumov–Stroganov correspondence holds (together with a 1-refined version of it, first conjectured by Di Francesco)

As a corollary, at $\omega = 1$ the enumeration of all configurations factorises into $\sum_\pi \Psi_\pi = A_n \cdot \Psi_{\pi_{\min}}$. And $\Psi_{\pi_{\min}}$ is equal the number of lozenge tilings of a hexagon, $M_{a,b,c}$.

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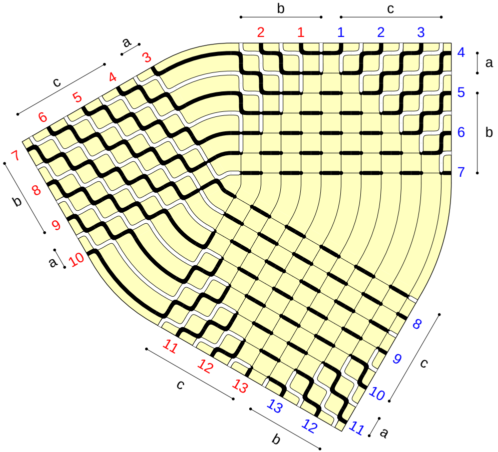

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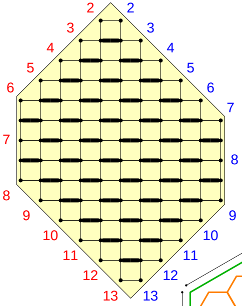
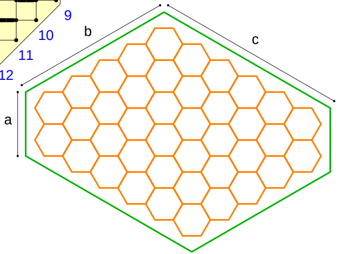


Thus $A_{a,b,c} = A_{a+b+c} M_{a,b,c}$
But in fact more is true: call $n = a + b + c$,
$A_{a,b,c}(r) = \sum_{r'} A(n, r - r') M_{a,b,c}(r')$

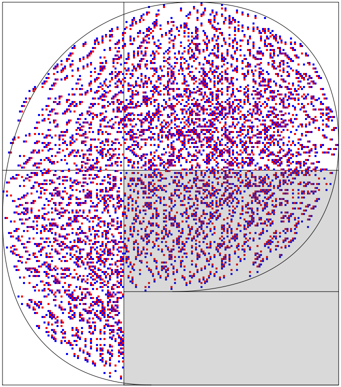


The arctic curve for the triangoloid

Very easy to find the position of tangence points κ_i.
Then, finding the arc between two of these points is harder but feasible (through the entropic method)... finally you get a parametric expression (here $a = 1 - b - c$, $p \in [0, 1]$, $q = 1 - p$)

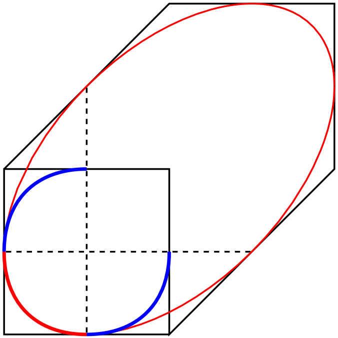
$$x(b, c, p) = \frac{3 - c}{2} - \frac{2 - p}{2\sqrt{1 - pq}} - \frac{(1 - c)(1 - (pb + qc)) - 2pbc}{2\sqrt{(pb - qc)^2 - 2(pb + qc) + 1}} ;$$

$$y(b, c, p) = x(c, b, 1 - p) .$$



Analytic continuation

The surprises are not over...
Just like the arc of the Colomo–Pronko Arctic Curve can be completed to a certain ellipse...



$$x(1-x)+y(1-y)+xy=1/4$$

Analytic continuation

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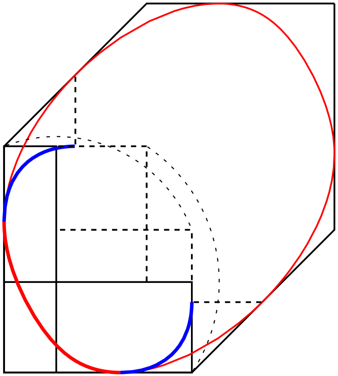
...we can try to continue analytically our curve. We get a closed curve composed of 6 arcs, for the intervals $p \in (-\infty, 0], [0, 1], [1, +\infty)$, and a $\pm$-choice for square roots.

This curve is framed into a hexagonal box, with side-slopes $0, 1, \infty$ and nice rational tangence points.



The shear phenomenon

Fact:

Consider a given arc of the triangoloid arctic curve $\mathcal{C}$ (the one "near vertex A")

The two other arcs of $\mathcal{C}$ (the ones "near vertices B and C") do coincide with the 45-degree shear of the neighbouring arcs in the boxed analytic continuation of the first arc.

This fact is of course true also in Colomo–Pronko ellipse, but here it sounds much more striking: we have two free parameters (b/a and c/a), and the single arcs do not have a polynomial Cartesian representation

It is believable that this points towards the universality of the shear phenomenon, for any tangent point of the arctic curve $\mathcal{C}$ on its boxing domain Λ, for $\omega = 1$ ASM.

The shear phenomenon

$$x(b,c,p) = \frac{3-c}{2} - \frac{2-p}{2\sqrt{1-pq}} - \frac{(1-c)(1-(pb+qc)) - 2pbc}{2\sqrt{(pb-qc)^2 - 2(pb+qc)+1}}\,;$$

$$y(b,c,p) = x(c,b,1-p)\,.$$