Gelfand–Zeitlin modules under Whittaker transform

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This talk is based on work in progess with A. Shapiro.

I'll start by talking about some problems in the representation theory of *non-compact* quantum groups: their split real forms.

Let $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$, and set

$$q = e^{\pi i \hbar}, \qquad \text{so} \quad |q| = 1.$$

If $U_q(\mathfrak{g})$ is a quantum group with Chevalley generators E_i, F_i, K_i, its split real form is defined by the $*$-involution

$$*E_i = E_i, \quad *F_i = F_i, \quad *K_i = K_i.$$

Principal series for $U_q(\mathfrak{sl}_2)$

Ponsot and Teschner, Faddeev '99: The split real quantum group $U_q(\mathfrak{sl}_2)$ has a *principal series* of $*$-representations

$$\mathcal{P}_s \simeq L^2(\mathbb{R}), \; s \in \mathbb{R}_{\geq 0},$$

with the following properties:

- the Chevalley generators E, F, K of the quantum group act on $\mathcal{P}_s$ by positive essentially self-adjoint operators;
- $\mathcal{P}_s$ is a bimodule for the quantum group $U_q(\mathfrak{sl}_2)$ and its **modular dual** $U_{q^\vee}(\mathfrak{sl}_2)$, where $q, q^\vee$ are related by the modular S-transformation:

$$q = e^{\pi i \hbar}, \qquad q^\vee = e^{\pi i / \hbar}.$$

Tensor product decomposition

Most importantly, **Ponsot and Teschner** showed that the class of principal series of $U_q(\mathfrak{sl}_2)$ is closed under taking tensor products by constructing an isomorphism of $U_q(\mathfrak{sl}_2)$–modules

$$\mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2} = \int_{\mathbb{R}_{\geq 0}}^{\oplus} \mathcal{P}_s d\mu(s).$$

The measure on the Weyl chamber $\mathbb{R}_{\geq 0}$ is

$$d\mu(s) = 4 \sinh\left(2\pi\hbar s\right) \sinh\left(2\pi\hbar^{-1} s\right) ds.$$

Higher rank principal series

Frenkel and Ip, '11 : For $\mathfrak{g}$ of any finite Dyknin type, the split real quantum group $U_q(\mathfrak{g}, \mathbb{R})$ has a family of principal series representations $\mathcal{P}_\lambda$ labelled by points of a Weyl chamber $\lambda \in \mathcal{C}^+$.

As in the rank 1 case:

- the Chevalley generators of the quantum group act on $\mathcal{P}_\lambda$ by positive essentially self-adjoint operators;
- $\mathcal{P}_\lambda$ is a bimodule for the quantum group $U_q(\mathfrak{g})$ and its Langlands dual $U_{q^\vee}({}^L\mathfrak{g})$, where $q, q^\vee$ are related by the modular S-transformation:

$$q^\vee = e^{\pi i/\hbar}.$$

Tensor decomposition, branching rules in higher rank?

Some natural representation theoretic questions to ask in higher rank:

Problem 1 (posed by Frenkel and Ip, '11) : Show the principal principal series representations $\mathcal{P}_\lambda$ of $U_q(\mathfrak{g})$ are closed under tensor product, and decompose the tensor product into irreducibles.

Problem 2: Decompose the $U_q(\mathfrak{sl}_{n+1})$ principal series representation $\mathcal{P}_\lambda$ into irreducibles as a $U_q(\mathfrak{sl}_n)$–module.

Two geometric realizations of $U_q(\mathfrak{sl}_n)$

With A. Shapiro, we solved Problem 2, and Problem 1 for $\mathfrak{g} = \mathfrak{sl}_n$, by relating each to a different geometric realization of $U_q(\mathfrak{g})$.

For the decomposition of tensor products, we used

Theorem (S.-Shapiro '16)

There is an algebra embedding

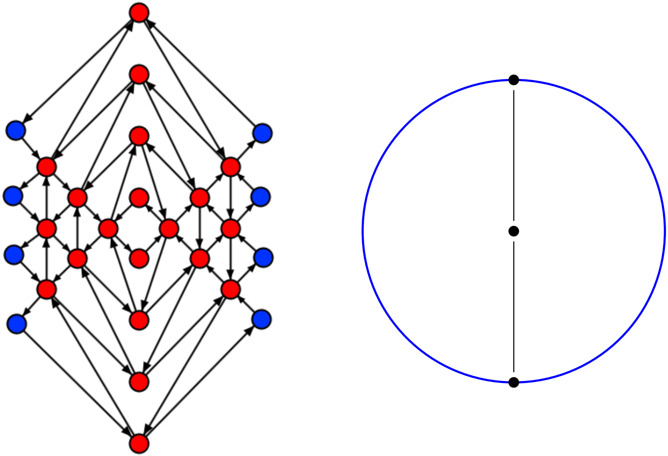
$$U_q(\mathfrak{sl}_n) \to \mathcal{X}^q_{G,S}$$

of $U_q(\mathfrak{sl}_n)$ into the quantized algebra of functions $\mathcal{X}^q_{PGL_n,S}$ on the moduli space of **framed** PGL_n**-local systems** *on the* **punctured disk** S *with* **2 marked points** *on its boundary.*

Cluster realization of $U_q(\mathfrak{sl}_n)$

Thanks to the fundamental results of Fock and Goncharov, this means the quantum group is embedded into a **quantum cluster algebra**: a combinatorially defined algebra encoded by a **quiver**.

Cluster realization of $U_q(\mathfrak{sl}_n)$



Another realization of $U_q(\mathfrak{sl}_n)$

In order to solve the **branching** Problem 2, we need to diagonalize the action of the center $Z(U_q(\mathfrak{sl}_n)) \subset U_q(\mathfrak{sl}_{n+1})$ on the principal series representation $\mathcal{P}_\lambda$ of $U_q(\mathfrak{sl}_{n+1})$.

It turns out this problem is related to another geometric realization of $U_q(\mathfrak{sl}_n)$, due to **Braverman, Finkelberg and Nakajima.**

Another realization of $U_q(\mathfrak{sl}_n)$

Given the data of a reductive group G and a representation V, **BFN '16** construct a space $\mathcal{R}_{G,V}$ equipped with an action of $\mathbb{C}^* \ltimes G[[z]]$, which fits into a **convolution diagram.**

The convolution equips

$$\mathcal{A}_{G,V} := K_{\mathbb{C}^* \ltimes G[[z]]}(\mathcal{R}_{G,V})$$

with the structure of an associative algebra, called the K**-theoretic quantized Coulomb branch algebra** associated to (G, V).

Important special case: Suppose Γ is a quiver with vertices Γ_0, edges Γ_1, and that we label each vertex $v \in \Gamma_0$ with an natural number n_v.

If $\Gamma_0' \subset \Gamma_0$ is a subset of nodes whose complement consists only of leaves, then we have a reductive group

$$G_\Gamma = \prod_{v \in \Gamma_0'} GL(n_v),$$

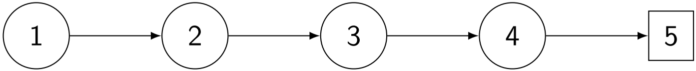
and a representation

$$V_\Gamma = \bigoplus_{e \in \Gamma_1} \mathrm{Hom}(\mathbb{C}^{n_{s(e)}}, \mathbb{C}^{n_{t(e)}})$$

of G_Γ.

Example

E.g.



$$\mathbb{C} \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^3 \longrightarrow \mathbb{C}^4 \longrightarrow \mathbb{C}^5$$

with $\Gamma_0 = \{1, 2, 3, 4\}$, so that

$$G_\Gamma = GL(1) \times GL(2) \times GL(3) \times GL(4),$$

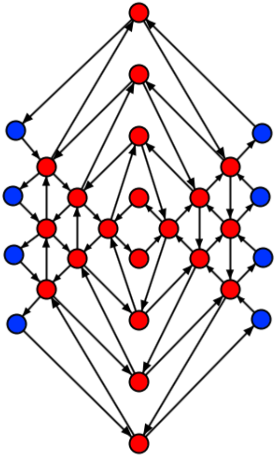
and

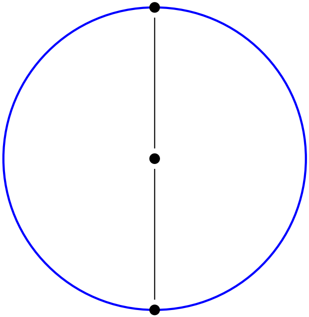
$$V_\Gamma = \mathrm{Mat}_{1\times 2} \oplus \mathrm{Mat}_{2\times 3} \oplus \mathrm{Mat}_{3\times 4} \oplus \mathrm{Mat}_{4\times 5}.$$

In this case,

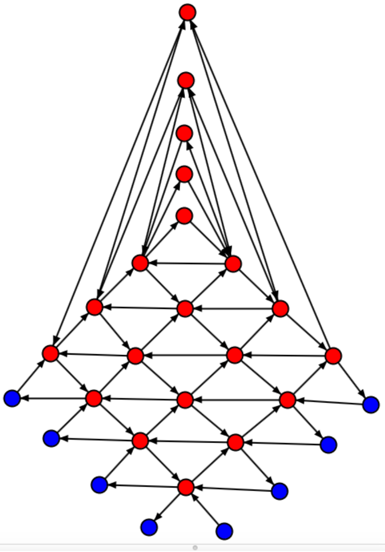
$$\mathcal{A}_{G,V} \simeq U_q(\mathfrak{sl}_5).$$

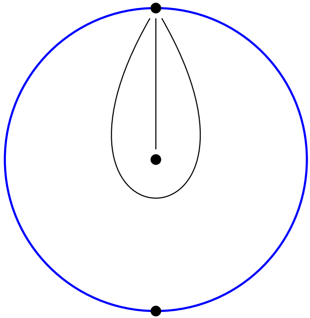
From one realization to another via mutations



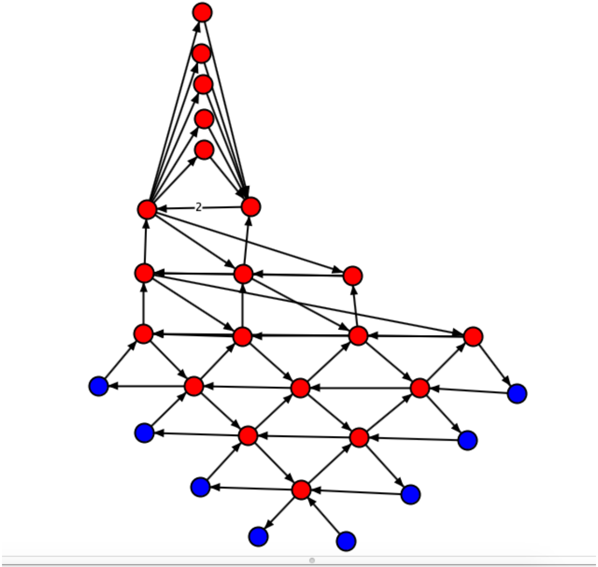


From one realization to the other via mutations





From one realization to the other via mutations



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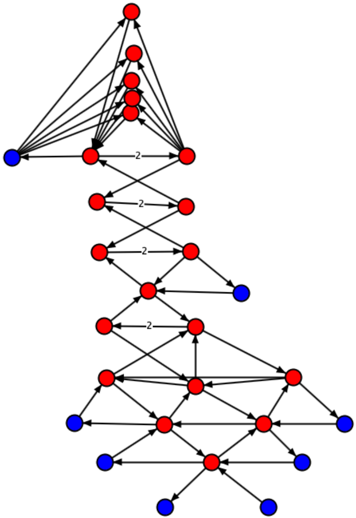

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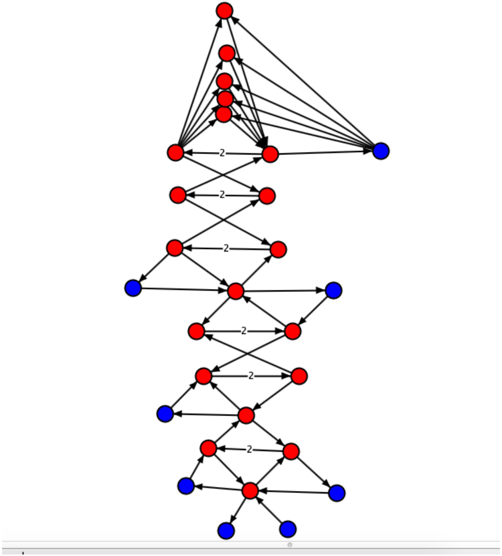


From Sasha's talk, we recognize several copies of quivers for **relativistic Toda spaces**:

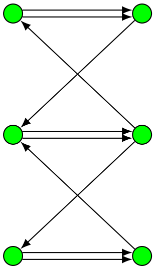
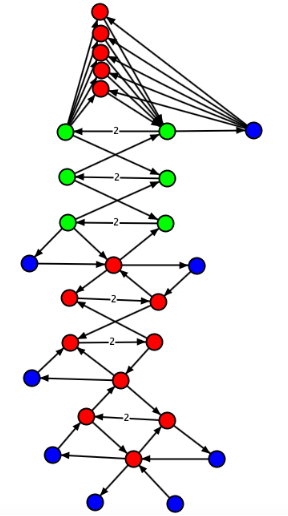
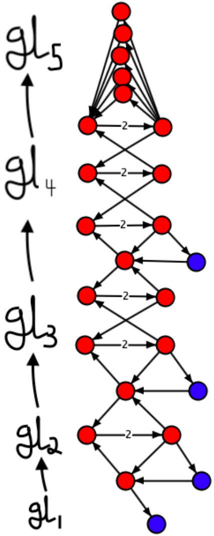


Figure: SL_4-Toda subquiver



So, the quiver for $U_q(\mathfrak{sl}_n)$ from Teichmüller theory is mutation equivalent to one obtained by gluing quivers for $\mathfrak{gl}_1, \mathfrak{gl}_2, \ldots, \mathfrak{gl}_{n-1}$ *relativistic Toda spaces*: **e.g.** $U_q(\mathfrak{sl}_5)$



Back to problem 2:
The chain of embeddings

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n,$$

where $\mathfrak{gl}_k$ is embedded as matrices zero outside of the top-left $k \times k$ square, induces embeddings

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \cdots \subset U_q(\mathfrak{gl}_n).$$

Idea: (Gelfand, Zeitlin) If

$$Z_k = \text{ center of } U_q(\mathfrak{gl}_k),$$

then $[Z_j, Z_k] = 0$ for all $1 \leq j, k \leq n$.

The subalgebra

$$\mathrm{GZ}_n \subset U_q(\mathfrak{gl}_n)$$

generated by $Z_1, \ldots, Z_n$ is a free commutative subalgebra in

$$1 + 2 + \ldots n = \frac{n(n+1)}{2} \text{ variables,}$$

called the **Gelfand-Zeitlin subalgebra.**

The quantum cluster algebra has a natural functional representation on the Laurent ring in $n(n-1)/2$ variables

$$\mathbb{C}[X_{ij}^{\pm 1}]_{1 \le j \le i \le n-1}.$$

The Gelfand-Zeitlin Hamiltonians are given by the collection of all $\mathfrak{gl}_1, \mathfrak{gl}_2, \ldots, \mathfrak{gl}_{n-1}$ open relativistic Toda Hamiltonians

$$H_k^{\mathfrak{gl}_i}, \qquad 1 \le k \le i \le n-1.$$

Toward the branching rule

For $|q| = 1$, the $\mathfrak{gl}_n$ quantum Coxeter-Toda operators

$$H_1, \ldots, H_n$$

act by **positive operators** on the Hilbert space $L^2(\mathbb{R}^n)$.

They have a set of eigendistributions

$$\Psi_{\lambda_1,\ldots,\lambda_n}(x_1, \ldots, x_n)$$

called $\hbar$-**Whittaker functions**. The joint spectrum is parameterized by

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}_{\mathbb{R}}/W.$$

The Whittaker transform

The $\mathfrak{gl}(n)$ **Coxeter-Toda Whittaker transform** is the integral transform

$$\mathcal{W}[f(x)] = \hat{f}(\lambda) = \int f(x)\overline{\Psi_\lambda(x)}dx.$$

Baby example: when $n = 1$,

$$\Psi_\lambda(x) = e^{2\pi i \lambda x},$$

so the Whittaker transform

$$\mathcal{W}^{\mathfrak{gl}_1}[f(x)] = \int f(x)e^{-2\pi i \lambda x}dx$$

is just the Fourier transform. In particular, it is a **unitary automorphism** of $L^2(\mathbb{R})$.

Unitarity of the Whittaker transform

For $\mathfrak{gl}(n)$, consider the Hilbert space

$$L^2(\mathbb{R}^n, m(\lambda)d\lambda),$$

where

$$m(\lambda) = \prod_{i<j} \sinh(\hbar(\lambda_j - \lambda_k)) \sinh(\hbar^{-1}(\lambda_j - \lambda_k))$$

is the modular Sklyanin measure.

Let us also write

$$e_k(\Lambda), \quad 1 \leq k \leq n$$

for the k-th **elementary symmetric function** in variables

$$\Lambda_1 = e^{2\pi\hbar\lambda_1}, \dots, \Lambda_n = e^{2\pi\hbar\lambda_n}$$

Unitarity of the Whittaker transform

Theorem (S.–Shapiro '17)

The Coxeter-Toda Whittaker transform

$$\mathcal{W} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, m(\lambda)d\lambda)$$

is a **unitary equivalence** *with inverse transform*

$$\mathcal{W}^*[\hat{f}] = \int \hat{f}(\lambda)\Psi_\lambda(x)m(\lambda)d\lambda,$$

which diagonalizes the Toda operators

$$\mathcal{W} \circ H_k = e_k(\Lambda) \circ \mathcal{W},$$

The branching rule

So we can apply the 'forward' $\mathfrak{sl}_n$–Whittaker transform to diagonalize the center of $Z(\mathfrak{sl}_n) \subset \mathfrak{sl}_{n+1}$, to get the branching rule

$$\mathcal{P}_\lambda^{\mathfrak{sl}_{n+1}} \simeq \int_{\nu \in \mathcal{C}^+} \mathcal{P}_\nu^{\mathfrak{sl}_n} m_{\mathfrak{sl}_n}(\nu) d\nu$$

Similarly, diagonalizing the full Gelfand–Zeitlin subalgebra gives an isomorphism of the principal series representations $\mathcal{P}_\lambda$ with a corresponding **Gelfand-Zeitlin representation** of $U_q(\mathfrak{sl}_n)$ by rational q-difference operators introduced by Gerasimov, Kharchev, Lebedev, Oblezin.

Now, when (G, V) comes from a quiver Γ the BFN Coulomb branch algebra $\mathcal{A}_{G,V}$ also has **Gelfand-Zeitlin representations** in which it acts by rational q-difference operators.

So we can try to turn the previous logic on its head: associate to (G, V) a quiver built from Coxeter-Toda quivers, and use the **inverse** Whittaker transform to try to embed $\mathcal{A}_{G,V}$ into the corresponding quantum cluster algebra.

Gelfand-Zeitlin representations of $\mathcal{A}_{G,V}$

Equivariant localization with respect to the maximal torus $T \subset G_\Gamma$ gives an embedding of $\mathcal{A}_{G,V}$ into an algebra of rational q-difference operators.

For each round node $i \in \Gamma_0$ with dimension n_i, we have a variable group $\Lambda_{i,\bullet}$, with n_i variables

$$\Lambda_{i,j}, \quad 1 \leq j \leq n_i.$$

(which we will think of as being on the 'spectral' side of the Whittaker transform for $\mathfrak{gl}_{n_i}$.)

Gelfand-Zeitlin representations of $\mathcal{A}_{G,V}$

We consider the ring of difference operators

$$\mathcal{D}_q^{rat}(\Gamma;\Lambda) = \bigotimes_{i\in\Gamma_0'} \mathcal{D}_q^{rat}(n_i),$$

where

$$\mathcal{D}_q(n_i) = \frac{\mathbb{C}\langle D_{ij}, \Lambda_{ij}\rangle_{j=1}^{n_i}}{\langle D_{ij}\Lambda_{rs} = q^{\delta_{ir}\delta_{js}}\Lambda_{rs}D_{ij}\rangle},$$

and the localization is taken with respect to a set of root hyperplanes.

Minuscule monopole operators

Weekes '19 proved that $\mathcal{A}_{G,V}$ is generated by certain explicit K-classes called **minuscule monopole operators,** and there is one (pair of) such generators for each node of the quiver determining (G, V).

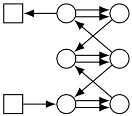
The cluster algebra $\mathcal{C}_{G,V}$

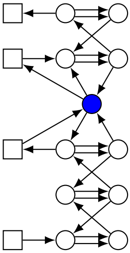
We now form a cluster algebra quiver $\mathcal{Q}(G,V)$ by taking for each round node $i \in \Gamma_0$ a $\mathfrak{gl}_{n_i}$ Coxeter-Toda quiver, and gluing them together by the following procedure.

How to build the quiver $\mathcal{Q}_{G,V}$

If Γ is the quiver of the gauge theory, associate to it a cluster algebra quiver $\mathcal{Q}_{G,V}$ as follows.

- To each round node $i \in \Gamma$ with dimension label n_i, we associate a **rigged** SL_{n_i} Coxeter–Toda quiver $\mathcal{Q}_i$ with two extra frozen variables;
- For each directed edge $e\colon i \to j$ in Γ_1, we add a new node (shown in blue) and use it to **glue** the top of $\mathcal{Q}_i$ to the bottom of $\mathcal{Q}_j$ as shown.





The cluster algebra $\mathcal{C}_{G,V}$

The cluster algebra $\mathcal{C}_{G,V}$ corresponding to quiver $\mathcal{Q}(G,V)$ is embedded as a subalgebra of q–difference operators on the Laurent ring

$$\mathbb{C}[X_{i,j}^{\pm 1}]_{1 \leq j \leq n_i}.$$

Embedding to the cluster algebra

So at this point we have a diagram

$$\begin{array}{ccc} \mathcal{D}_q^{rat}(\Gamma, \Lambda_\bullet) & \xrightarrow{\mathcal{W}^*} & \mathcal{D}_q^{pol}(\Gamma, X_\bullet) \\ \uparrow & & \uparrow \\ \mathcal{A}_{G,V} & \overset{?}{\dashrightarrow} & \mathcal{C}_{G,V} \end{array}$$

and we need to show that all minuscule monopole operators are contained in the cluster algebra $\mathcal{C}_{G,V} \subset \mathcal{D}_q^{pol}(\Gamma, X_\bullet)$.

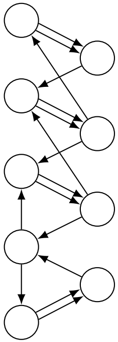
An easy special case

Suppose that the round node $i \in \Gamma_0$ has no in-pointing arrows: It's easy to see in this case that the corresponding monopole operator is a **cluster monomial**, so is by definition contained in the cluster algebra $\mathcal{C}_{G,V}$.

Bifundamental Baxter operator

Now imagine we have two gauge theory quivers Γ, Γ' which differ only by changing the orientation of a single edge between distinct vertices i, j with dimension labels (n, m).

e.g. for $(n, m) = (4, 2)$ the corresponding piece $\mathcal{Q}_{n,m}$ of the cluster algebra quiver looks like



Bifundamental Baxter operator

The cluster algebra quiver $\mathcal{Q}_{n,m}$ has a nontrivial mapping class group element $\beta_{n,m}$ given by a sequence of mutations, which we call the *bifundamental Baxter operator*.

This automorphism is a *cluster DT-transformation* in the sense of Keller for the quiver $\mathcal{Q}_{n,m}$.

Bi-fundamental Baxter operator

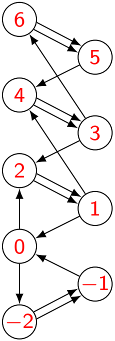
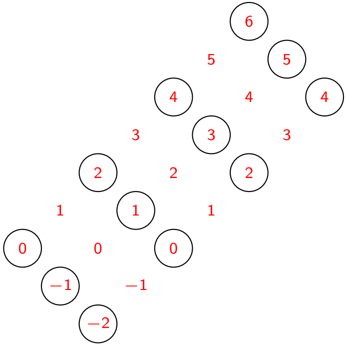


Figure: $\mathcal{Q}_{4,2}$



Mutate column-by-column, reading left to right. In each column mutate at circled vertices first bottom to top, and then in the rest top to bottom.

Bifundamental Baxter operator

For us, the crucial property of the Baxter operator is that it acts diagonally in the Whittaker basis:

$$\beta_{n,m} \cdot \left(\Psi_{\lambda}^{(n)}(\mathbf{x}) \boxtimes \Psi_{\mu}^{(m)}(\mathbf{y}) \right) = \prod_{j,k} \varphi(\lambda_j + \mu_k) \cdot \Psi_{\lambda}^{(n)} \boxtimes \Psi_{\mu}^{(m)},$$

where $\varphi(z)$ is **Faddeev's quantum dilogarithm.**

If $\{M_i, M_j\}, \{M'_i, M'_j\}$ are the monopole operators for nodes i, j of Γ, Γ' respectively, this relation implies

$$\beta M_i \beta^{-1} = M'_i, \quad \beta M_j \beta^{-1} = M'_j.$$

Bifundamental Baxter operator

So applying Baxter operators to reverse all incoming edges to a given vertex, we can mutate the corresponding monopole operator to a cluster monomial!

$$\implies \mathcal{A}_{Q,V} \subset \mathcal{C}_{Q,V}.$$

Cluster realizations of K-theoretic Coulomb branches for quiver theories

So, we've proved

Theorem (A. Shapiro-S.)

For each quiver without loops, its K-theoretic Coulomb branch algebra can be embedded into a quantum upper cluster algebra, such that each minuscule monopole operator is a cluster monomial.

In particular, this class of quivers realizes the generalized affine Grassmannian slices $\mathcal{W}^{\lambda}_{\mu}(\mathfrak{g})$ for $\mathfrak{g}$ of type ADE.

We **conjecture** (with M. Shapiro) that this embedding is an isomorphism.

Merci de votre attention!