Multiplicities from volumes

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Polyhedral models of multiplicities

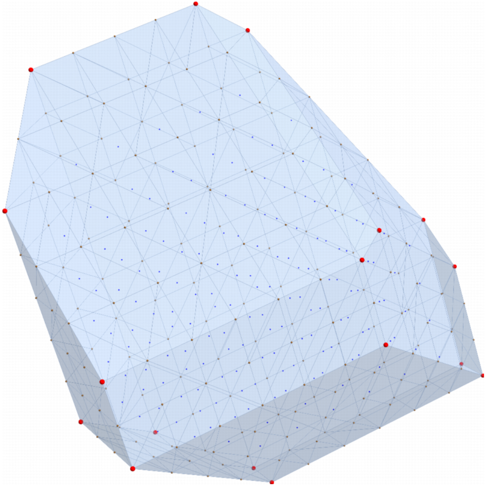
Tensor product of irreps of a compact semisimple Lie algebra $\mathfrak{g}$:

$$V_\lambda \otimes V_\mu = \bigoplus_\nu C_{\lambda\mu}^\nu V_\nu.$$

The multiplicity $C_{\lambda\mu}^\nu$ equals the number of integer points in a polytope $H_{\lambda\mu}^\nu \subset \mathbb{R}^N$. See e.g. Berenstein–Zelevinsky '88, Knutson–Tao '98.

Actually computing the multiplicities takes more work!

Polyhedral models of multiplicities



The $\mathfrak{su}(4)$ hive polytope for $\lambda = (21, 13, 5)$, $\mu = (7, 10, 12)$, $\nu = (20, 11, 9)$.
Figure: Coquereaux–Zuber '18.

What this talk is about

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Unsurprising, anticlimactic answer: Nope.

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More serious question: Given $\mathrm{Vol}(H_{\lambda\mu}^{\nu})$ for **all** (λ, μ, ν), can you compute **all** $C_{\lambda\mu}^{\nu}$?

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Answer: Yes! In fact there are several ways to do it.

Back to polyhedral models

We think of the weights λ, μ, ν as lying in the dominant chamber $\mathcal{C}_+$ of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.

Then the polytope $H_{\lambda\mu}^{\nu}$ is cut out by a system of linear inequalities depending on $x \in \mathbb{R}^N$ *and* on (λ, μ, ν):

$$H_{\lambda\mu}^{\nu} = \{ \ x \in \mathbb{R}^N \mid \ell(\lambda, \mu, \nu, x) \geq 0 \ \forall \ \ell \in L \ \},$$

where $L \subset (\mathfrak{t}^3 \times \mathbb{R}^N)^*$. So we can talk about $H_{\alpha\beta}^{\gamma}$ for $\alpha, \beta, \gamma \in \mathfrak{t}$.

The volume function

There is a special function $\mathcal{J} : \mathfrak{t}^3 \to \mathbb{R}$ associated to $\mathfrak{g}$, which computes $\mathrm{Vol}(H_{\alpha\beta}^{\gamma})$. First, some notation...

The discriminant of $\mathfrak{g}$:

$$\Delta_{\mathfrak{g}}(x) = \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle,$$

and the Harish-Chandra orbital integral:

$$\mathcal{H}(x, y) := \int_G e^{\langle \mathrm{Ad}_g y, x \rangle} dg, \qquad x, y \in \mathfrak{t} \otimes \mathbb{C},$$

where G is a connected group with Lie algebra $\mathfrak{g}$, and dg is the normalized Haar measure.

The volume function

For $\alpha, \beta, \gamma \in \mathfrak{t}$, define:

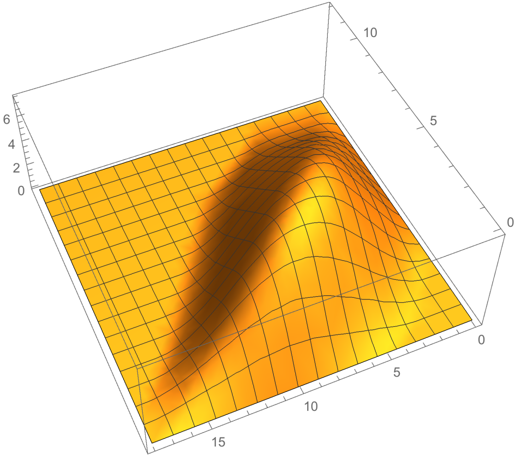
$$\mathcal{J}(\alpha, \beta; \gamma) :=$$
$$\frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{(2\pi)^r \, |W| \, \Delta_{\mathfrak{g}}(\rho)^3} \int_{\mathfrak{t}} \Delta_{\mathfrak{g}}(x)^2 \mathcal{H}(ix, \alpha)\mathcal{H}(ix, \beta)\mathcal{H}(ix, -\gamma)\, dx.$$

Then for α, β, γ dominant, $\mathcal{J}(\alpha, \beta; \gamma) = \mathrm{Vol}(H_{\alpha\beta}^{\gamma})$.

(See Coquereaux–M.–Zuber '19 for details.)

The volume function

We usually fix α, β and consider $\mathcal{J}$ as a W-skew-invariant function of $\gamma \in \mathfrak{t}$.



$\mathcal{J}(\alpha, \beta; \gamma)$ for $\mathfrak{so}(5)$, with $\alpha = (4, 7)$, $\beta = (5, 3)$. Coordinates are in the fundamental weight basis.

The volume function and random matrices

Let $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$ be the coadjoint orbits of $\alpha, \beta \in \mathcal{C}_+$.

Choose $A \in \mathcal{O}_\alpha$, $B \in \mathcal{O}_\beta$ uniformly at random. Let $p(\gamma|\alpha, \beta)$ be the probability density of $\gamma \in \mathcal{C}_+$ such that $A + B \in \mathcal{O}_\gamma$.

E.g.: Probability density of eigenvalues of sum of two uniform random Hermitian matrices with prescribed eigenvalues.

Then:

$$\mathcal{J}(\alpha, \beta; \gamma) = \frac{\Delta_\mathfrak{g}(\alpha)\Delta_\mathfrak{g}(\beta)}{\Delta_\mathfrak{g}(\gamma)\Delta_\mathfrak{g}(\rho)} \; p(\gamma|\alpha, \beta).$$

The volume function and symplectic geometry

The product of orbits $\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_{-\gamma}$ is also a symplectic G-manifold with moment map $(A, B, C) \mapsto A + B + C$.

For generic (α, β, γ) such that 0 is a regular value of the moment map,

$$\mathcal{J}(\alpha, \beta; \gamma) = (2\pi)^{|\Phi^+|} \Delta_{\mathfrak{g}}(\rho) \, \mathrm{Vol}[\, (\mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_{-\gamma}) \,//\, G \,],$$

where Vol is the Liouville volume.

Initial motivation: $\mathcal{J}$-LR relations

Write $\lambda' = \lambda + \rho$, etc. Let Q be the root lattice.

Theorem (Coquereaux–Zuber '18, C.–M.–Z. '19 + Etingof–Rains '18)

Suppose $\lambda + \mu - \nu \in Q$. Then

$$\mathcal{J}(\lambda', \mu'; \nu') = \sum_{\kappa \in K} \sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\ \mathcal{C}_+}} r_\kappa C^\tau_{\lambda\mu} C^\nu_{\tau\kappa}$$

where $K = Q \cap \mathrm{Conv}(W\rho)$ and r_κ are some computable coefficients.

This formula recovers the asymptotic relation between $\mathcal{J}$ and $C^\nu_{\lambda\mu}$ for "large representations," but is more precise. **Can we "invert" it?**

The box spline

Define a measure $B_c[\Phi^+]$ on $\mathfrak{t}$ by

$$\int_{\mathfrak{t}} f \; dB_c[\Phi^+] = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} f\Big(\sum_{\alpha \in \Phi^+} t_\alpha \alpha \Big) \prod_{\alpha \in \Phi^+} dt_\alpha, \quad f \in C^0(\mathfrak{t}).$$

This is the *centered box spline* associated to the positive roots. It has a piecewise polynomial density $b(x)$.

Four ways to think about the box spline

First way: As a convolution of uniform measures on line segments.

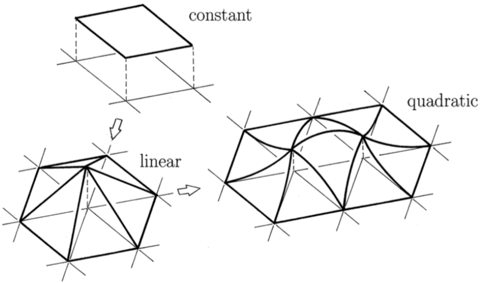


Figure: Boehm–Prautzsch '02, "Box Splines" (a good intro).

Second way: The density $b(x)$ computes the volume of the fibers of a projection of a polytope.

Four ways to think about the box spline

Third way: As the Duistermaat–Heckman measure for the action of the maximal torus on $\mathcal{O}_\rho$.

Fourth way: Define

$$j_{\mathfrak{g}}^{1/2}(x) = \prod_{\alpha \in \Phi^+} \frac{e^{i\langle \alpha, x\rangle/2} - e^{-i\langle \alpha, x\rangle/2}}{i\langle \alpha, x\rangle}$$

as in the Kirillov character formula. Then $b = \mathscr{F}^{-1}[j_{\mathfrak{g}}^{1/2}]$.

In brief, there are many ways to compute $b(x)$.

A convolution formula

An index-theoretic identity of De Concini–Procesi–Vergne '13 implies:

$$\mathcal{J}(\lambda', \mu'; \gamma) = b(\gamma) * \left(\sum_{\substack{\nu \in (\lambda+\mu)+Q \\ \cap\ \mathcal{C}_+}} C_{\lambda\mu}^{\nu} \sum_{w \in W} \epsilon(w) \delta_{w(\nu')} \right).$$

In other words, we can think of our question as a **deconvolution problem**.

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$$\mathcal{J}(\alpha, \beta; \gamma) := \frac{\Delta_{\mathfrak{g}}(\alpha)\Delta_{\mathfrak{g}}(\beta)\Delta_{\mathfrak{g}}(\gamma)}{(2\pi)^r \, |W| \, \Delta_{\mathfrak{g}}(\rho)^3} \int_{\mathfrak{t}} \Delta_{\mathfrak{g}}(x)^2 \mathcal{H}(ix, \alpha) \mathcal{H}(ix, \beta) \mathcal{H}(ix, -\gamma) \, dx.$$

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Dahmen–Micchelli, Vergne, etc. have studied box spline deconvolution in a general setting, but we'll do something simpler.

Restricting to the lattice

Idea: Consider only $\gamma = \nu'$ for $\nu \in \lambda + \mu + Q$. Then the convolution formula gives an equality of measures, or of functions on the weight lattice:

$$\sum_{\nu \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \nu')\, \delta_{\nu'}$$

$$= \left(\sum_{\tau \in Q} b(\tau)\, \delta_\tau \right) * \sum_{\substack{\tau \in \lambda+\mu+Q \\ \cap\ \mathcal{C}_+}} C_{\lambda\mu}^{\tau} \sum_{w \in W} \epsilon(w)\, \delta_{w(\tau')}.$$

We have reduced a hard deconvolution problem (measures on $\mathfrak{t}$) to an easier deconvolution problem (finitely supported functions on a lattice).

A first deconvolution formula

Moving to the discrete setting eliminates technical obstacles to “naive” deconvolution by Fourier analysis. We can also compute algebraically.

Theorem (M. ’19)

Part 1:

Part 2:

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$$C_{\lambda\mu}^{\nu} = \frac{1}{(2\pi)^r |Q^\vee|} \int_{\mathfrak{t}/2\pi Q^\vee} \frac{\sum_{\tau \in \lambda+\mu+Q} \mathcal{J}(\lambda', \mu'; \tau') e^{i\langle \tau - \nu, x\rangle}}{\sum_{\tau \in Q} b(\tau) \cos(\langle \tau, x \rangle)} dx.$$

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Part 2: *Moreover, one can compute $C_{\lambda\mu}^{\nu}$ algebraically from finitely many values of $\mathcal{J}(\lambda', \mu'; \gamma)$ via an explicit algorithm.*

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Part 2: *Moreover, one can compute* $C_{\lambda\mu}^{\nu}$ *algebraically from finitely many values of* $\mathcal{J}(\lambda', \mu'; \gamma)$ *via an explicit algorithm.*

For $\mathfrak{su}(n)$, we can do better.

Shielded triples

Take $\mathfrak{g} = \mathfrak{su}(n)$ and let $d := |\Phi^+| - r = \frac{1}{2}(n-1)(n-2)$.

Definition

We will say that a triple (λ, μ, ν) of dominant weights of $\mathfrak{su}(n)$ is *shielded* if $\lambda + \mu - \nu \in Q$ and if the points $\nu' + \lfloor d/2 \rfloor w(\rho)$, $w \in W$ are dominant and all lie in the interior of a single polynomial domain of $\mathcal{J}(\lambda', \mu'; \gamma)$.

Shielded triples are "typical"

The non-analyticities of $\mathcal{J}$ are contained in a finite hyperplane arrangement in $\mathfrak{t}^3$ (see e.g. C.–M.–Z. '19).

Any triple (λ, μ, ν) with $\lambda + \mu - \nu \in Q$ such that (λ', μ', ν') lies further than a distance $\lfloor d/2 \rfloor |\rho|$ from each of these hyperplanes is shielded.

In particular, as λ and μ both grow large, the ratio

$$\frac{\#\{\, \nu \mid C_{\lambda\mu}^{\nu} \neq 0,\ (\lambda, \mu, \nu) \text{ shielded} \,\}}{\#\{\, \nu \mid C_{\lambda\mu}^{\nu} \neq 0 \,\}}$$

goes to 1.

The box spline Laplacian

For $\tau \in Q$, let Δ_τ and ∇_τ denote respectively the forwards and backwards finite difference operators in the direction of τ:

$$\begin{aligned} \Delta_\tau f(x) &= f(x+\tau) - f(x), \\ \nabla_\tau f(x) &= f(x) - f(x-\tau), \qquad f : \mathfrak{t} \to \mathbb{C}. \end{aligned}$$

Define the *box spline Laplacian* $\mathcal{D}$ by

$$\mathcal{D} := \sum_{\tau \in Q} b(\tau) \nabla_\tau \Delta_\tau .$$

An explicit algebraic formula for $\mathfrak{su}(n)$

Theorem (M. '19)

For (λ, μ, ν) a shielded triple of dominant weights of $\mathfrak{su}(n)$,

$$C_{\lambda\mu}^{\nu} = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(-\frac{1}{2}\mathcal{D} \right)^k \mathcal{J}(\lambda', \mu'; \nu').$$

(Here $\mathcal{D}$ acts in the third argument of $\mathcal{J}$.)

Sketch of the proof

1. Define $\psi(\nu) := C^{\nu}_{\lambda\mu}$. Show that

$$\mathcal{J}(\lambda', \mu'; \nu') = \Big(1 + \frac{1}{2}\mathcal{D}\Big)\psi(\nu).$$

2. Introduce a space of degree d polynomials $D(\Phi^+)$, on which $\big(1 + \frac{1}{2}\mathcal{D}\big)$ is invertible by the Neumann series, which truncates:

$$\Big(1 + \frac{1}{2}\mathcal{D}\Big)^{-1} p = \sum_{k=0}^{\lfloor d/2 \rfloor} \Big(-\frac{1}{2}\mathcal{D}\Big)^{k} p, \qquad p \in D(\Phi^+).$$

3. Show that for (λ, μ, ν) shielded, ψ is locally equal to some $p \in D(\Phi^+)$ on a sufficiently large neighborhood of ν.

Formulae for low n

For $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, $\mathcal{D} = 0$. In these cases it is known (see C.–Z. '18) that *whenever* $\lambda + \mu - \nu \in Q$, $C_{\lambda\mu}^{\nu} = \mathcal{J}(\lambda', \mu'; \nu')$.

For (λ, μ, ν) a shielded triple of $\mathfrak{su}(4)$,

$$C_{\lambda\mu}^{\nu} = \left(1 - \frac{1}{24} \sum_{\alpha \in \Phi^+} \nabla_\alpha \Delta_\alpha\right) \mathcal{J}(\lambda', \mu', \nu').$$

For (λ, μ, ν) a shielded triple of $\mathfrak{su}(5)$, $C_{\lambda\mu}^{\nu} =$

$$\sum_{k=0}^{3} \left[-\frac{1}{30} \sum_{\alpha \in \Phi^+} \left(\nabla_\alpha \Delta_\alpha + \frac{1}{12} \sum_{\substack{\beta \in \Phi^+ \\ \langle \beta, \alpha \rangle = 0}} \left(\nabla_{\alpha+\beta} \Delta_{\alpha+\beta} + \nabla_{\alpha-\beta} \Delta_{\alpha-\beta} \right) \right) \right]^k \mathcal{J}(\lambda', \mu', \nu').$$

In conclusion...

- We can always compute $C^{\nu}_{\lambda\mu}$ from finitely many values of $\mathcal{J}(\lambda', \mu'; \gamma)$.
- We obtain more or less explicit expressions depending on $\mathfrak{g}$ and on (λ, μ, ν). The nicest formulae are for shielded triples of $\mathfrak{su}(n)$.
- Many questions remain: Exact algebraic formulae for unshielded triples or for $\mathfrak{g} \neq \mathfrak{su}(n)$? Combinatorial identities for $b(x)$? Full semiclassical expansion for $C^{\nu}_{\lambda\mu}$ from $\mathcal{J}$? Applications to other multiplicity problems? Etc...
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Thanks!