Opers for higher states of quantum $\widehat{\mathfrak{g}}-$KdV systems

Davide Masoero

Universidade de Lisboa

Hyeres ICR, 4/9/2019

Talk based on

- D. M., Andrea Raimondo, D.Valeri: *Bethe Ansatz and the Spectral Theory of affine Lie algebra–valued connections I. The simply–laced case.*
  CMP, 2016.
- D. M., Andrea Raimondo, D.Valeri: *Bethe Ansatz and the Spectral Theory of affine Lie algebra–valued connections II. The non simply–laced case.* CMP, 2017.
- D.M., Andrea Raimondo: *Opers for Higher States of Quantum KdV models*. arXiv, 2018
- D.M., Andrea Raimondo: *Opers for Higher States of the Quantum Boussinesq model*. arXiv, 2019

The ODE/IM correspondence

A remarkable relation between:

- *ODE* *Linear ordinary differential equations (more precisely, opers)*
- *IM* *Quantum integrable models*

In this talk:

- $\mathfrak{g}$ simply laced simple Lie algebra over $\mathbb{C}$, of rank n
- Untwisted affine Kac–Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g}[\lambda, \lambda^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad d = \lambda \partial_\lambda$$

- Langlands self-dual: ${}^L\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}$

Quantum $\widehat{\mathfrak{g}}$−Drinfeld Sokolov

- Quantization of second Poisson bracket of $\widehat{\mathfrak{g}}$−Drinfeld Sokolov (W−algebras)
  - $\mathfrak{g} = \mathfrak{sl}_2$: Bazhanov, Lukyanov, Zamolodchikov ['96]
  - $\mathfrak{g} = \mathfrak{sl}_3$: Bazhanov, Hibberd, Khoroshkin ['01]
  - Feigin, Frenkel ['94] (Hamiltonians of $\mathfrak{g}$ Toda field theory)

Highest weight representation $\mathcal{V}_{p,c}$ of W−algebras.

- $p \in \mathfrak{h} \subset \mathfrak{g}$ 'vacuum parameter'
- $-\infty < c < h - 1$ Moduli space of theories $\mathbb{C}^n \times \mathbb{R}$
- λ spectral parameter

Integrable structure: $\mathbf{Q}(\lambda)$-operators: $\mathbf{Q}(\lambda) : \mathcal{V}_{p,c} \to \mathcal{V}_{p,c}$

- Operator valued functions, expected to be entire in λ
- They encode the quantum integrals of motion
- Number of $\mathbf{Q}(\lambda)$−operators $= |\mathcal{W}|$, $\mathcal{W}$: Weyl group of $\mathfrak{g}$.
- They satisfy some algebraic identities

Quantum $\widehat{\mathfrak{g}}$−Drinfeld Sokolov

- $\mathbf{Q}(\lambda) : \mathcal{V}_{p,c} \to \mathcal{V}_{p,c}$ $\qquad \mathbf{Q}(\lambda)v = Q(\lambda)v$
- *The eigenvalue $Q(\lambda)$ is an entire function of λ.*

Functional relations among the eigenvalues $Q(\lambda)$
('quantum Wronskian', $Q\widetilde{Q}$−system)

$$\Downarrow$$

$\mathfrak{g}$-Bethe Ansatz equations
(e.g. Reshetikhin-Wiegmann [87,...], P. Zinn-Justin [98])

$$\prod_{j=1}^{n} e^{-2i\pi\beta_j C_{\ell j}} \frac{Q^{(j)}\left(e^{i\pi C_{lj}}\lambda^*\right)}{Q^{(j)}\left(e^{-i\pi C_{lj}}\lambda^*\right)} = -1\,,\ \forall\lambda^* \text{ s.t. } Q^{(l)}(\lambda^*) = 0$$

ODE/IM correspondence ideology

To any state of quantum $\widehat{\mathfrak{g}}-$KdV there corresponds a (unique) ${}^L\widehat{\mathfrak{g}}$ oper $\mathcal{L}(z,\lambda)$, such that the generalized monodromy data of $\mathcal{L}$ coincide with the $Q(\lambda)-$functions of the given state.

Examples

- $\mathfrak{sl}_2$, ground state / Schrödinger operator
  (Dorey Tateo ['98], Bazhanov, Lukyanov, Zamolodchikov ['98])
- $\mathfrak{sl}_2$, higher states / Schrödinger operator with 'monster potential'
  (Bazhanov, Lukyanov, Zamolodchikov ['03])
- $\mathfrak{sl}_{n+1}$, ground state / $(n+1)$th-order linear equation
  (Dorey, Dunning, Masoero, Suzuki, Tateo ['07])

Algebraic breakthrough

- Feigin-Frenkel ['07]: use ${}^L\widehat{\mathfrak{g}}-$opers on the ODE side.

Meromorphic opers on $\mathbb{P}^1$

Lie algebra data *

- $\mathfrak{g}$ simple Lie algebra of rank n, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- Borel subalgebra $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$.
- Chevalley generators $\{f_i, h_i, e_i, i = 1, \dots, n\}$
- $f = \sum_i f_i$ principal nilpotent element

Opers (Drinfeld–Sokolov ('84), Beilinson–Drinfeld ('93))

- Defined on an arbitrary Riemann surface Σ
- For $\Sigma = \mathbb{P}^1$ they can be written as

$$\{\partial_z + f + b(z) \,|\, b \in \mathfrak{b}_+(K_{\mathbb{P}^1})\} / \mathcal{N}(K_{\mathbb{P}^1})$$

- Gauge group: $\mathcal{N}(K_{\mathbb{P}^1}) = \{\exp y(z),\, y \in \mathfrak{n}_+(K_{\mathbb{P}^1})\}$

Canonical forms: Examples

Example ($\mathfrak{g} = A_1 = \mathfrak{sl}_2(\mathbb{C})$)

$$\partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix} \overset{\exp y(z)}{\longleftrightarrow} \partial_z + \begin{pmatrix} 0 & u(z) \\ 1 & 0 \end{pmatrix} \longrightarrow \partial_z^2 - u(z)$$

Example ($\mathfrak{g} = A_2 = \mathfrak{sl}_3(\mathbb{C})$)

$$\partial_z + \begin{pmatrix} * & * & * \\ 1 & * & * \\ 0 & 1 & * \end{pmatrix} \overset{\exp y(z)}{\longleftrightarrow} \partial_z + \begin{pmatrix} 0 & u^1 & u^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \partial_z^3 - u^1 \partial_z + u^2$$

Quantum $\widehat{\mathfrak{g}}-$KdV opers - Ground state

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f - \rho^\vee + r}{z} + (1 + \lambda z^{-\hat{k}}) e_\theta \qquad (*)$$

- $\begin{cases} \textit{Dual Weyl vector } \rho^\vee \in \mathfrak{h}. \qquad [\rho^\vee, e_i] = e_i, \quad i = 1, \dots, n \\ \textit{Highest root } \theta \in \mathfrak{h}^*. \ \ \textit{Corresponding coroot } \theta^\vee \in \mathfrak{h} \end{cases}$
- $\begin{cases} (r, \hat{k}) \in \mathfrak{h} \times (0,1) \\ \lambda \textit{ loop algebra variable} \end{cases}$ *moduli of opers* $\mathbb{C}^n \times \mathbb{R}$

$$\mathcal{L}(z,\lambda) \sim \partial_z + f + f_0 + \frac{r + (k-h)d}{z} + z^{-h} e_\theta$$

Quantum $\widehat{\mathfrak{g}}$−KdV opers - Higher states

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f - \rho^\vee + r}{z} + (1 + \lambda z^{-\hat{k}})e_\theta + \sum_{j=1}^{N} \frac{-\theta^\vee + X(j)}{z - w_j} \qquad (*)$$

Additional singularities

- Additional singularities do not change the singularity at $0, \infty$ and have trivial monodromy monodromy.
- Generically, the residue of a regular singularity with trivial monodromy is conjugated to $-\theta^\vee$, see Feigin-Frenkel ['07] (for $\mathfrak{sl}_2$ Duistermaat - Grunbaum ['86]).
- $N =$ number of additional poles $=$ level of the corresponding state.

The $\mathfrak{sl}_2$ case, 'Monster potential' (BLZ, ['03])

For $\mathfrak{g} = \mathfrak{sl}_2$, we get

$$L = -\partial_z^2 + \frac{r(r+1)}{z^2} + \frac{1}{z} + \lambda z^k + \sum_{j=1}^{N} \left(\frac{2}{(z-w_j)^2} + \frac{k}{z(z-w_j)} \right)$$

where the w_ℓ satisfy the following system of algebraic equations

$$\Delta - (k+1) w_\ell = \sum_{\substack{j=1 \dots N \\ j \neq \ell}} \frac{w_\ell \left((k+2)^2 w_\ell^2 - k(2k+5) w_j w_\ell + k(k+1) w_j^2 \right)}{(w_\ell - w_j)^3}$$

for $\ell = 1, \dots, N$, with $\Delta = \frac{1}{4}k^3 + k(k+1) - (k+2)r(r+1)$.

For generic r, k, the above system should have $p(n)N!$ solutions.

The $\mathfrak{sl}_3$ case, M.-Raimondo ['19]

$$L = \partial_z^3 - \left(\sum_{j=1}^{N} \left(\frac{3}{(z-w_j)^2} + \frac{k}{z(z-w_j)} \right) + \frac{\bar{r}^1}{z^2} \right) \partial_z$$

$$+ \sum_{j=1}^{N} \left(\frac{3}{(z-w_j)^3} + \frac{a_j}{z(z-w_j)^2} + \frac{2(k+3)a_j - k^2}{3z^2(z-w_j)} \right) + \frac{\bar{r}^2}{z^3} + \frac{1+\lambda z^{-k}}{z^2},$$

where

$$a_\ell^2 - k a_\ell + k^2 + 3k - 3\bar{r}^1 = \sum_{\substack{j=1,\dots,N \\ j \neq \ell}} \left(\frac{9 w_\ell^2}{(w_\ell - w_j)^2} + \frac{3k w_\ell}{w_\ell - w_j} \right)$$

$$A a_\ell + B - 9(k+2) w_\ell = \sum_{\substack{j=1 \\ j \neq \ell}}^{N} \left(\frac{18(k - a_\ell - a_j) w_\ell^3}{(w_\ell - w_j)^3} + \frac{(12k + 9k^2 - (63+6k)a_j - 9k a_\ell) w_\ell^2}{(w_\ell - w_j)^2} \right.$$

$$\left. + \frac{(9k + 16k^2 + 6(k^2+10k+6)a_j - 5k a_\ell) w_\ell}{w_\ell - w_j} \right).$$

Three statements

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f - \rho^\vee + r}{z} + (1 + \lambda z^{-\hat{k}})e_\theta + \sum_{j=1}^{N} \frac{-\theta^\vee + X(j)}{z - w_j} \qquad (*)$$

1. *The generalized monodromy data of* $(*)$ *satisfy the Bethe Ansatz equations of the quantum* $\hat{\mathfrak{g}}-$*KdV model*
2. *The most general opers satisfying the above conditions are* $(*)$
3. *The trivial monodromy conditions are a system of* $N\left(1+\frac{nh}{2}\right)$ *algebraic equations in* $N\left(1+\frac{nh}{2}\right)$ *unknowns (reduced to* $2N$ *equations in* $2N$ *unknowns for* A_n,D_n,$E_{6,7}$*).*

Conjecture

The expected number of opers of level N *is* $N!P_n(N)$, *the* $n-$*coloured partitions of* N. *(True for* $N=0,1$*)*

From Quantum $\mathfrak{g}-$KdV opers to the Bethe Ansatz

- $V^{(i)}$, $i = 1, \dots, n$ *fundamental representations of* $\mathfrak{g}$.
- $\widehat{\mathbb{C}} =$ *universal cover of* $\mathbb{C} \setminus \{0\}$
- *A distinguished space of (twisted) solutions:* $V^{(i)}(\lambda) \cong V^i \otimes \mathcal{O}_\lambda$

$$V^{(i)}(\lambda) = \{\psi : \widehat{\mathbb{C}} \times \mathbb{C} \to V^{(i)} \,|\, \mathcal{L}_{\frac{s(i)}{2}}(z,\lambda)\psi(z,\lambda) = 0, \psi \text{ entire in } \lambda\}$$

$\lambda-$dependent and $\mathcal{O}_\lambda-$linear monodromy

$$(M\psi)(z,\lambda) = e^{2\pi i \rho^\vee} \psi(e^{2\pi i} z, e^{2\pi i \hat{k}} \lambda)$$

- Discrete symmetry:

$$\psi(z,\lambda) \in V^{(i)}(\lambda) \quad \Longleftrightarrow \quad (M\psi)(z,\lambda) \in V^{(i)}(\lambda)$$

Singularities of Quantum $\mathfrak{g}-$KdV opers

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f - \rho^\vee + r}{z} + (1 + \lambda z^{-\hat{k}})e_\theta + \sum_{j=1}^{N} \frac{-\theta^\vee + X(j)}{z - w_j}$$

Regular singularity at $z = 0$

- Let $P_{\omega_i} \subset \mathfrak{h}^*$ set of weights of $V^{(i)}$, $i = 1, \dots, n$
- For every $\{\chi_\omega\}_{\omega \in P_{\omega_i}}$ eigenvector of $f - \rho^\vee + r$ with eigenvalue $\omega(r - \rho^\vee)$

$$\chi_\omega(z,\lambda) = z^{\omega(r-\rho^\vee)}(\chi_\omega + \sum_{(\ell,m)\in\mathbb{N}^2\setminus\{(0,0)\}} c_{\ell,m} z^\ell (\lambda z^{-\hat{k}})^m) \in V^{(i)}(\lambda)$$

- For generic $(r, \hat{k})$ generic and $f - \rho^\vee + r$ semisimple $\Longrightarrow \mathcal{O}_\lambda-$basis of eigenvectors of the monodromy M

Singularities of Quantum $\mathfrak{g}$−KdV opers

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f - \rho^\vee + r}{z} + (1 + \lambda z^{-\hat{k}})e_\theta + \sum_{j=1}^{N} \frac{-\theta^\vee + X(j)}{z - w_j}$$

Irregular singularity at $z = \infty$:

- $\mathcal{L}(z,\lambda) \sim \partial_z + z^{\frac{1}{h}-1}\Lambda^{(i)} + o(z^{-1})$, $\quad \Lambda^{(i)} = f + (-1)^{s(i)}e_\theta$
- $\Lambda^{(i)}\psi^{(i)} = \mu_i\psi^{(i)}$, $\quad \mu_i > 0$ maximal eigenvalue. $\quad \psi^{(i)} \in V^{(i)}$
- $\mu_1, \dots, \mu_n$ are the masses of affine Toda field theory.
- $\exists$ subdominant solution for $z \to +\infty$, $i = 1, \dots, n$:

$$\Psi^{(i)}(z,\lambda) \in V^{(i)}(\lambda), \qquad \Psi^{(i)} = e^{-\mu_i h z^{\frac{1}{h}}} R(z) \times (\psi^{(i)} + o(1))$$

Central connection problems

$$\mathcal{L}(z,\lambda) = \partial_z + \frac{f-\rho^\vee + r}{z} + (1+\lambda z^{-\hat{k}})e_\theta + \sum_{j=1}^{N} \frac{-\theta^\vee + X(j)}{z-w_j}$$

- *For $i \in 1,\dots,n$, there exist solutions*

$$\{\Psi^{(i)}(z,\lambda)\} \cup \{\chi_\omega(z,\lambda)\}_{\omega\in P_{\omega_i}} \subset V^{(i)}(\lambda)$$

- *Generalized monodromy data*

$$\Psi^{(i)}(z,\lambda) = \sum_{\omega\in P_{\omega_i}} Q_\omega(\lambda)\chi_\omega(z,\lambda), \qquad i=1,\dots,n$$

- *The $Q_\omega(\lambda)$ solve the Bethe Ansatz equations.*

How to get to the $Q\widetilde{Q}-$system?

The $\Psi-$system

- C_{ij} *Cartan matrix of* $\mathfrak{g}$, $B_{ij} = 2\delta_{ij} - C_{ij}$ *incidence matrix*
- *Homomorphism of representations,* $i = 1, \ldots, n$*:*

$$m_i : \bigwedge^2 V^{(i)} \to \bigotimes_{j=1}^{n} (V^{(j)})^{\otimes B_{ij}}$$

- *The* $\Psi-$*system,* $i = 1, \ldots, n$*:*

$$m_i(\Psi^{(i)}_{-\frac{1}{2}}(z,\lambda) \wedge \Psi^{(i)}_{\frac{1}{2}}(z,\lambda)) = \otimes_{j=1}^{n} \Psi^{(j)}(z,\lambda)^{\otimes B_{ij}}$$

- *Algebraic relations among subdominant solutions in different representations*
- $\mathfrak{g} = \mathfrak{sl}_2 \quad \Longrightarrow \quad Wr[\Psi_{-1/2}, \Psi_{1/2}] = 1$

The $Q\widetilde{Q}-$system

- *Recall*

$$\Psi^{(i)}(z,\lambda) = \sum_{\omega \in P_{\omega_i}} Q^{(i)}(\lambda)\chi_\omega(z,\lambda)$$

- *For $\sigma \in \mathcal{W}$, set*

$$Q^{(i)}_\sigma = Q^{(i)}_{\sigma(\omega_i)}(\lambda) \qquad \widetilde{Q}^{(i)}_\sigma = Q^{(i)}_{\sigma(\omega_i - \alpha_i)}(\lambda)$$

- *The $Q\widetilde{Q}-$system*

$$\prod_{j \in I} \left(Q^{(j)}_\sigma(\lambda) \right)^{B_{\ell j}} = e^{i\pi\theta_\ell} Q^{(\ell)}_\sigma(e^{-\pi i \hat{k}}\lambda)\widetilde{Q}^{(\ell)}_\sigma(e^{\pi i \hat{k}}\lambda) - e^{-i\pi\theta_\ell} Q^{(\ell)}_\sigma(e^{\pi i \hat{k}}\lambda)\widetilde{Q}^{(\ell)}_\sigma(e^{-\pi i \hat{k}}\lambda)\,,$$

where $\theta_\ell = \sigma(\alpha_\ell)(r - \rho^\vee)$.

$Q\widetilde{Q}-$system and affine quantum groups

- The $Q\widetilde{Q}-$system:

$$\prod_{j\in I}\left(Q_\sigma^{(j)}(\lambda)\right)^{B_{\ell j}} = e^{i\pi\theta_\ell}Q_\sigma^{(\ell)}(e^{-\pi i\hat{k}}\lambda)\widetilde{Q}_\sigma^{(\ell)}(e^{\pi i\hat{k}}\lambda) - e^{-i\pi\theta_\ell}Q_\sigma^{(\ell)}(e^{\pi i\hat{k}}\lambda)\widetilde{Q}_\sigma^{(\ell)}(e^{-\pi i\hat{k}}\lambda)\,, \qquad (**)$$

Theorem (Frenkel and Hernandez ['16])

The $Q\widetilde{Q}$-system $(**)$ is a universal system of relations in the (commutative) Grothendieck ring $K_0(\mathcal{O})$ of the category $\mathcal{O}$ (introduced by Hernandez and Jimbo) of representations of the Borel subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.