Deautonomization of cluster integrable systems

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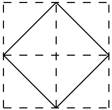
General scheme [BGM] (simplest example)

Non-autonomous q-difference system

$$G(qZ)G(q^{-1}Z) = \left(\frac{G(Z)+Z}{G(Z)+1}\right)^2$$

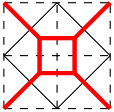
Cluster discrete flow

Newton polygon Δ



Topological string partition function

$=$ general solution (τ-function) of the equation



General scheme

Take Newton polygon Δ → **Compute topological string partition function for $\Delta \to$ get solution**

↓

Draw Thurston diagram → Write word in a double coextended Weyl group

↓

Draw bipartite graph

↓

Draw quiver

↓

Find quiver mapping class group

↓

Find Abelian subgroup of MCG and take corresponding flows. Each flow = equation.

Write word in a double coextended Weyl group

↓

Write element in $\widehat{PGL}^{\sharp}(N)$ [Fock-Goncharov], [Fock-Marshakov]

↓

Write q-isomonodromic Lax matrix

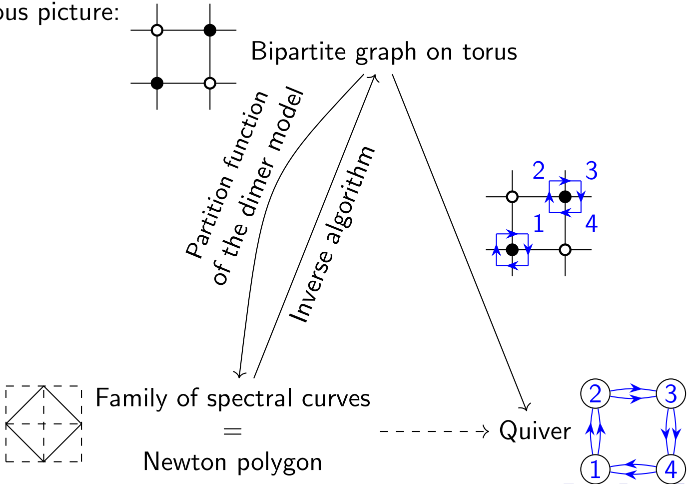
↓

Consider refactorization dynamics = q-isomonodromic dynamics

Goncharov-Kenyon map (simplest example)

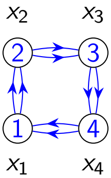
In the q=1 limit q-difference system turns into discrete symmetries of GK (cluster) integrable system. Switching on $q \neq 1$ is called *deautonomization*.

Autonomous picture:



Mapping class group $G_{\mathcal{Q}}$

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. This is purely combinatorial problem. Example:



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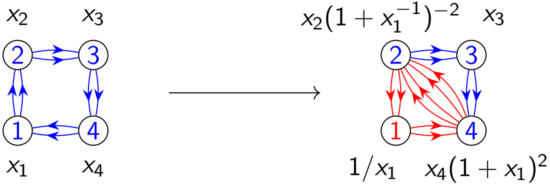

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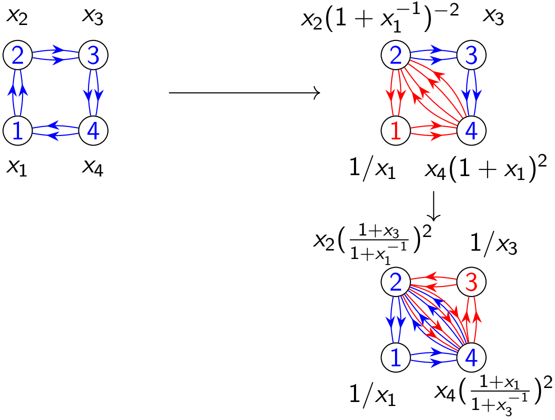

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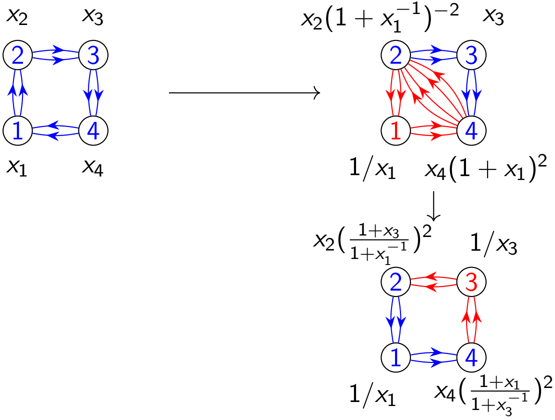

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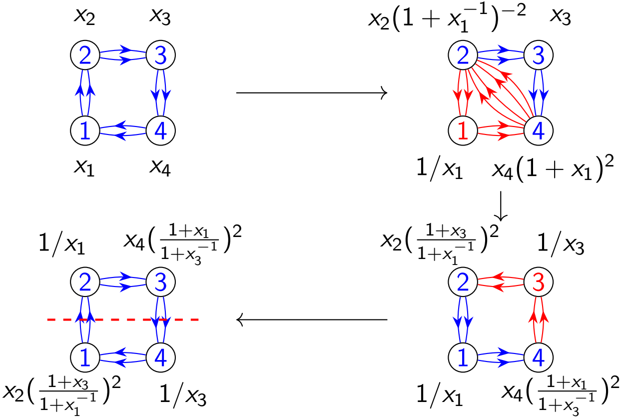


Discrete flow

Take a map coming from quiver automorphism.
Forget about $x_1x_2x_3x_4 = 1 \Longrightarrow$ no Hamiltonians. $x_1x_2x_3x_4 = q$

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 (\frac{1 + x_3}{1 + x_1^{-1}})^2, x_1^{-1}, x_4 (\frac{1 + x_1}{1 + x_3^{-1}})^2, x_3^{-1} \right)$$

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 (\frac{x_1 + z}{x_1 + 1})^2, x_1^{-1}, qz, q \right)$$

Casimir z becomes "time", so introduce $x_i = x_i(z)$, $T : x_i(z) \mapsto x_i(qz)$.

$$x_1(qz)x_1(q^{-1}z) = \left(\frac{x_1(z) + z}{x_1(z) + 1} \right)^2$$

This is q-Painlevé III$_3$ equation, or $P(A_7^{(1)'})$.

Only for $q = 1$ flow T preserves $H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + Z\sqrt{\frac{x_2}{x_1}}$.

Quantization (one of the main advantages)

- In addition to non-autonomous parameter q one may add quantum deformation p:

$$\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$$

- There are quantum mutations

$$\mu_j : \quad \hat{x}_j \mapsto \hat{x}_j^{-1}, \quad \hat{x}_i^{1/|\epsilon_{ij}|} \mapsto \hat{x}_i^{1/|\epsilon_{ij}|} \left(1 + p\hat{x}_j^{\operatorname{sgn} \epsilon_{ij}}\right)^{\operatorname{sgn} \epsilon_{ij}}, \; i \neq j$$

- All groups G_Q are the same.
- And so there are quantum deformations of all systems. For example, quantum q-Painlevé III_3:

$$\begin{cases} \hat{x}_1(q^{-1}z)^{1/2} \, \hat{x}_1(qz)^{1/2} = \dfrac{\hat{x}_1(z) + pz}{\hat{x}_1(z) + p}, \\ \hat{x}_1(z)\hat{x}_1(q^{-1}z) = p^4 \hat{x}_1(q^{-1}z)\hat{x}_1(z). \end{cases}$$

Different approaches to quantization were also considered long before by K. Hasegawa, G. Kuroki, H. Nagoya, Y. Yamada.

Generic solution [BShch]/[BGM] (quantum q-GIL formula)

$$\hat{x}_1(z) = p z^{1/2} \hat{\mathcal{T}}_1(z)^2 \hat{\mathcal{T}}_3(z)^{-2}$$

$$\hat{\mathcal{T}}_1(z) = \hat{a} \sum_{n\in\mathbb{Z}} \hat{s}^n Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$$

$$\hat{\mathcal{T}}_3(z) = i\hat{a} \sum_{n\in\frac{1}{2}+\mathbb{Z}} \hat{s}^n Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$$

Where

$$q_2 = q^{1/2}, \quad q_1 = q_2^{-1} p^2, \quad \hat{u}\hat{s} = p^4 \hat{s}\hat{u}$$

and also

$$q_2^2 \hat{a} = p^{-2} \hat{a} q_2^2 = \hat{a} q_1^{-1} q_2, \quad q_1 q_2^{-1} \hat{a} = p^2 \hat{a} q_1 q_2^{-1} = \hat{a} q_1^2$$

So here we have *operator* Fourier transformation.

$Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$ is a topological string partition function (with prefactor), or 5D Nekrasov function for $SU(2)$ pure gauge theory.

Proofs for the classical case: M. Bershtein, A. Shchechkin (also conjectured this); M. Jimbo, H. Nagoya, H. Sakai ($P(A_3^{(1)})$) + Y. Matsuhira, H. Nagoya (limit)

Bilinear relations for $q-$PIII eqaution

There is a blow-up relation conjectured by Bershtein and Shchechkin:

$$\sum_{2n\in\mathbb{Z}}\left(u^{2n}(q_1q_2)^{4n^2}z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1^2z)\,\mathrm{F}^{(2)}(uq_2^{4n}|q_2^2z)\right)=$$

$$=(1-q_1q_2z)\sum_{2n\in\mathbb{Z}}\left(z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|z)\,\mathrm{F}^{(2)}(uq_2^{4n}|z)\right)$$

where

$$\mathrm{F}^{(1)}(u|z)=\mathrm{F}(u;q_1^2,q_1^{-1}q_2|z),\quad \mathrm{F}^{(2)}(u|z)=\mathrm{F}(u;q_1q_2^{-1},q_2^2|z)$$

and $\mathrm{F}(u;q_1^2,q_1^{-1}q_2|z)$ is appropriately normalized q-deformed Virasoro conformal block = Nekrasov partition function.

$$Z^{2,0}(u;q_1,q_2|z)=\exp\left(\frac{-\log z\,(\log u)^2}{4\log q_1\log q_2}\right)\mathrm{F}(u;q_1,q_2|z)$$

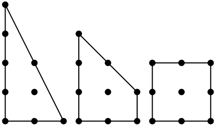
For $q_1=q_2^{-1}$ shifts are symmetric, and one has classical bilinear relations for the usual commutative Fourier transformations of $Z^{2,0}$:
$\tau_1(qz)\tau_1(q^{-1}z)=\tau_1(z)^2+z^{1/2}\tau_3(z)^2$

Numerology of cluster integrable systems

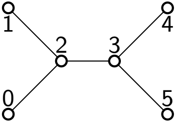
- (# of variables) $=$ 2*Area(Δ).
- Dimension of the phase space $=$ 2*(# of internal points).
- Number of Casimirs (without q) $=$ (# of boundary points) - 3.
- (# of discrete flows) $=$ number of Casimirs (without q)

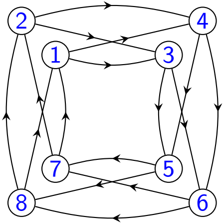
Simplest cases: 1) one discrete flow, 2) one Hamiltonian.

Example of the MCG: $q-$PVI equation

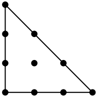


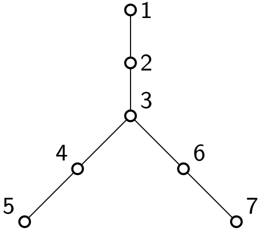
$s_0 = (1,2),$
$s_1 = (5,6),$
$s_2 = (1,5) \circ \mu_5 \circ \mu_1,$
$s_3 = (3,7) \circ \mu_3 \circ \mu_7,$
$s_4 = (3,4),$
$s_5 = (7,8),$
$\pi = (1,7,5,3)(2,8,6,4),$
$\sigma = (1,7)(2,8)(3,5)(4,6) \circ \varsigma,$
here ς — inversion of all arrows

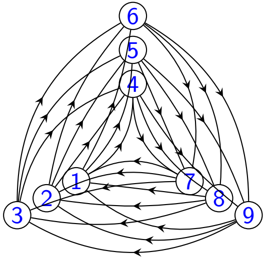




Example of the MCG: $E_6^{(1)}$ equation

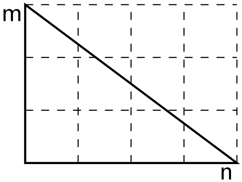






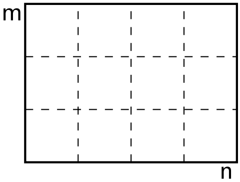
$$
\begin{aligned}
s_1 &= (2,3),\\
s_2 &= (1,2),\\
s_4 &= (4,5),\\
s_5 &= (5,6),\\
s_6 &= (7,8),\\
s_0 &= (8,9),\\
s_3 &= (4,7)\circ\mu_1\circ\mu_4\circ\mu_7\circ\mu_1,\\
\pi &= (1,4,7)(2,5,8)(3,6,9),\\
\sigma &= (1,7)(2,8)(3,9)\circ\varsigma
\end{aligned}
$$

Some other examples

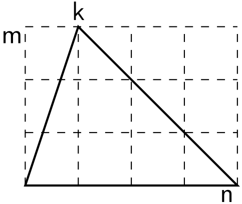


K. Kajiwara, M. Noumi, Y. Yamada: action of the $W(A^{(1)}_{n-1}) \times W(A^{(1)}_{m-1})$ Weyl group

(?) $m = 2$, $n = 2k$ case: N. Okubo, T. Suzuki (?)



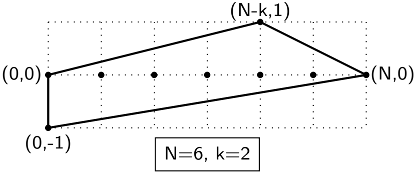
M. Semenyakin, A. Marshakov:
$W(A^{(1)}_{n-1} \times A^{(1)}_{m-1})^2 \rtimes \mathbb{Z}$

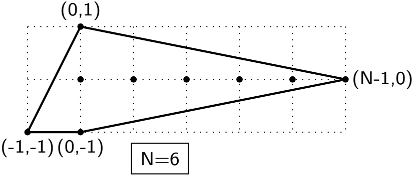


R. Inoue, T. Ishibashi, T. Lam, H. Oya, P. Pylyavskyy: quiver $Q_m(A^{(1)}_{n-1})_k$

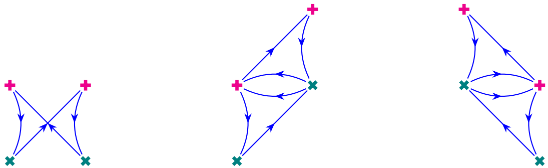
4 boundary points, hyperelliptic curves (Toda family)

Classification: $Y^{N,k}$ polygons with $0 \leq k \leq N$ (left picture) and $L^{1,2N-1,2}$ polygons (right picture):



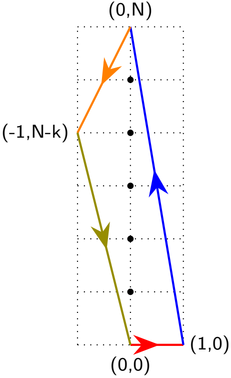


Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1, respectively. $N = N_1 + N_0 + N_{-1}$, $k = N_1 - N_{-1}$.

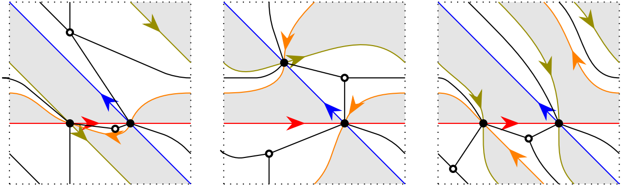


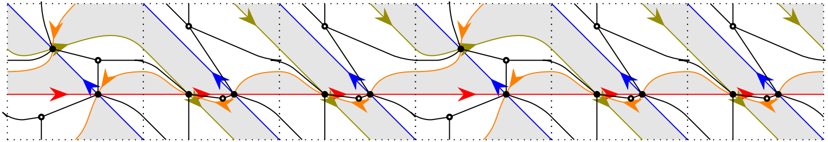
Similar non-cyclic quivers appeared in Di Francesco's paper.
Graphs also computed by S. Franco, A. Hanany, K. Kennaway, D. Vegh, B. Wecht

Building blocks for Thurston diagrams and dimer lattices



N=6, k=2





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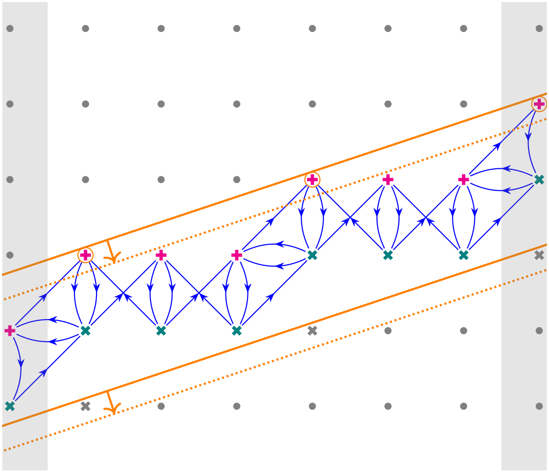
Action of the automorphism group



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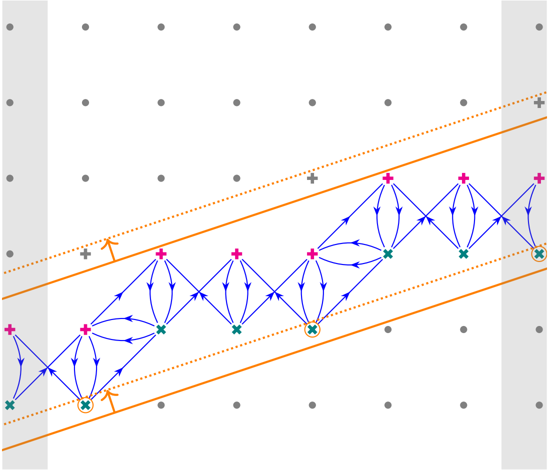


N=6, k=2

Equations

Mutable "+"-variables are labelled by the points of integer lattice: $x_{(n,m)}$. They satisfy periodicity condition and Y-system in order to be compatible with mutations:

$$\frac{x_{(n,m+1)}x_{(n,m-1)}}{x_{(n,m)}^2} = \frac{(1+x_{(n+1,m)})(1+x_{(n-1,m)})}{(1+x_{(n,m)})^2}, \quad x_{(n,m)} = x_{(n+N,m+k)}$$

One can move from Y-system to T-system (from X-clusters to A-clusters):

$$x_{(n,m)} = z_0^{1/N} q^{(kn-Nm+N)/N^2} \frac{\tau_{(n-1,m-1)}\tau_{(n+1,m-1)}}{\tau_{(n,m-1)}^2}, \quad \tau_{(n,m)} = \tau_{(n+N,m+k)}$$

$$\tau_{(n,m+1)}\tau_{(n,m-1)} = \tau_{(n,m)}^2 + z_0^{1/N} q^{(kn-Nm)/N^2} \tau_{(n+1,m)}\tau_{(n-1,m)}$$

And after some change of labeling:

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

Solution

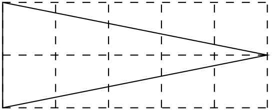
$$\tau_j(qz)\,\tau_j(q^{-1}z) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

Fourier transformation of partition function of 5D $SU(N)$ pure gauge theory with Chern-Simons term at level k:

$$\tau_j(z) = \sum_{\vec{\Lambda}\in Q_{N-1}+\omega_j} \prod_i (s_i^{\Lambda_i}) \cdot Z^{N,k}(\{u_i q^{\Lambda_i}\}; q, q^{-1}|z), \qquad j \in \mathbb{Z}/N\mathbb{Z}.$$

where Q_{N-1} is $SL(N)$ root lattice, and ω_i are $SL(N)$ fundamental weights ($\omega_0 = 0$).

K. Takasaki constructed solution for special u's: $u_i \approx q^{\frac{N+1-2i}{2N}}$, $k = N$



Nekrasov functions

$$Z^{N,k}(\vec{u};q_1,q_2|z) = Z^{N,k}_{\text{cl}}(\vec{u};q_1,q_2|z)\cdot Z^{N}_{1-\text{loop}}(\vec{u};q_1,q_2)\cdot Z^{N,k}_{\text{inst}}(\vec{u};q_1,q_2|z)\,,$$

where

$$Z^{N,k}_{\text{cl}}(\vec{u};q_1,q_2|z) = \exp\left(\log z\frac{\sum(\log u_i)^2}{-2\log q_1\log q_2}+k\frac{\sum(\log u_i)^3}{-6\log q_1\log q_2}\right),$$

$$Z^{N}_{1-\text{loop}}(\vec{u};q_1,q_2) = \prod_{1\le i\ne j\le N}(u_i/u_j;q_1,q_2)_\infty\,,$$

$$Z^{N,k}_{\text{inst}}(\vec{u};q_1,q_2|z) = \sum_{\vec{\lambda}}\frac{z^{|\vec{\lambda}|}\prod_{i=1}^N(\mathsf{T}_{\lambda^{(i)}}(u_i;q_1,q_2))^k}{\prod_{i,j=1}^N\mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)}\,,$$

$$\vec{\lambda}=(\lambda^{(1)},\ldots,\lambda^{(N)}),\;\;|\vec{\lambda}|=\sum|\lambda^{(i)}|,\;\;|\lambda|=\sum\lambda_j,$$

$$\mathsf{N}_{\lambda,\mu}(u,q_1,q_2)=\prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\cdot\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})\,,$$

$$\mathsf{T}_\lambda(u;q_1,q_2)=u^{-|\lambda|}q_1^{|\lambda'|-\frac12(\|\lambda'\|)}q_2^{\frac12(|\lambda|-\|\lambda\|)}=\prod_{(i,j)\in\lambda}u^{-1}q_1^{1-i}q_2^{1-j},$$

$$\|\lambda\|=\sum\lambda_j^2\,.$$

Differential limit (5D $\to$ 4D)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$

We take $q = \exp R$, $z = R^{2N}\mathsf{z}$ and send $R \to 0$:

$$(\partial_{\log \mathsf{z}})^2 \log\tau_j = \mathsf{z}^{1/N}\frac{\tau_{j+1}\tau_{j-1}}{\tau_j^2}, \quad j \in \mathbb{Z}/N\mathbb{Z}$$

So we see no dependence on k. In the different variables

$$\phi_j = \log\tau_j - \log\tau_{j-1}, \quad r = 2N\mathsf{z}^{\frac{1}{2N}}$$

We have

$$\frac{d^2\phi_n}{dr^2} + \frac{1}{r}\frac{d\phi_n}{dr} = e^{\phi_{n+1}-\phi_n} - e^{\phi_n-\phi_{n-1}}$$

This is radial Toda equation, for $N = 2$ — radial sinh-Gordon equation (PIII_3).

Thurston diagrams and words in the double coextended Weyl group

Picture from Fock-Marshakov paper:

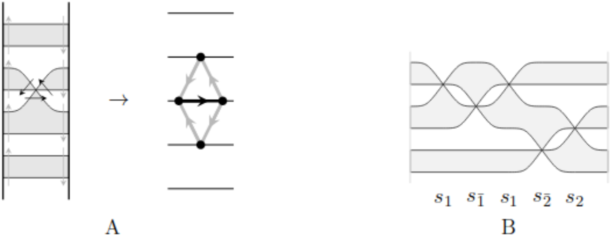


Figure 1: Thurston diagrams: (A) for an elementary generator s_i of the Weyl group; (B) for the word $s_1 s_{\bar{1}} s_1 s_{\bar{2}} s_2$.

s_i permutes strands $2i-1$ and $2i+1$
$s_{\bar{i}}$ permutes strands $2i$ and $2i+2$
Λ rotates strands cyclically, $\Lambda^N = 1$

Together we have $(\mathbb{Z}/N\mathbb{Z}) \ltimes (S_N \times S_N)$

Coextended loop group

Coextended loop group (Kac-Moody with zero level) $\widehat{PGL}^{\sharp}(N)$:

$$\widehat{\boldsymbol{L}} = L(\lambda) q^{\lambda \frac{\partial}{\partial \lambda}}$$

Generators for $\widehat{PGL}^{\sharp}(2)$:

$$\boldsymbol{H}_i(x) = H_i(x) x^{\lambda \frac{\partial}{\partial \lambda}}, \quad H_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1(x) = x^{-1/2} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

$$E_0(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$F_0(\lambda) = E_{\bar{0}}(\lambda) = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}, \quad F_1 = E_{\bar{1}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \Lambda(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}.$$

Formulas from Fock-Marshakov paper:

B Relations among the generators of a simply laced Lie group

1. $H_i(x)H_j(y) = H_j(y)H_i(x)$,
2. $H_i(x)H_i(y) = H_i(xy)$,
3. $E_iH_i(-1)E_i = H_i(-1)$
4. $H_i(x)E_j = E_jH_i(x)$ for $i \neq |j|$,
5. $E_iE_j = E_jE_i$ if $C_{ij} = 0$,
6. $E_iH_i(x)E_i = H_i(1+x)E_iH_i(1+x^{-1})^{-1}$,
7. $E_iH_i(x)E_jE_i = H_i(1+x)H_j(1+x^{-1})^{-1}E_jH_j(x)^{-1}E_iE_jH_i(1+x^{-1})^{-1}H_j(1+x)$ for $C_{ij} = -1$ and $i, j > 0$,
8. $E_{\bar{i}}H_i(x)E_{\bar{j}}E_{\bar{i}} = H_i(1+x^{-1})^{-1}H_j(1+x)E_{\bar{j}}H_j(x)^{-1}E_{\bar{i}}E_{\bar{i}}H_i(1+x)H_j(1+x^{-1})^{-1}$ for $C_{ij} = -1$ and $i, j > 0$,
9. $E_{\bar{i}}H_i(x)E_i = \prod_{j \neq i} H_j(1+x)^{-C_{ij}}H_i(1+x^{-1})^{-1}E_iH_i(x^{-1})E_{\bar{i}}H_i(1+x^{-1})^{-1}$ for $i > 0$.

From Weyl group to $\widehat{PGL}^{\sharp}(N)$

Make a replacement $s_i \mapsto E_i$, $s_{\bar{i}} \mapsto E_{\bar{i}}$, $\Lambda \mapsto \Lambda(\lambda)$, then add appropriate number of $\boldsymbol{H}_i(x_k)$ in between. In the simplest case: $s_i s_{\bar{i}} \mapsto E_i \boldsymbol{H}_i(x_k) E_{\bar{i}} \boldsymbol{H}_i(x_{k+1})$

Example: $q-$Painlevé III: $u = s_0 s_1 s_{\bar{0}} s_{\bar{1}} \in S_2 \times S_2$

$$\widehat{\boldsymbol{L}} = \boldsymbol{H}_0(x_1 x_2)^{-1/2} \cdot \boldsymbol{H}_0(x_1) E_0(\lambda) \boldsymbol{H}_1(x_3) E_1 \boldsymbol{H}_0(x_2) F_0(\lambda) \boldsymbol{H}_1(x_4) F_1 \cdot \boldsymbol{H}_0(x_1 x_2)^{1/2}$$

Decompose Lax operator according to the decomposition u: $u = u_+ u_-$, where u_+ contains all s_i, and u_- contains all $s_{\bar{i}}$:

$$\widehat{\boldsymbol{L}} = L_+(\lambda, \boldsymbol{x}) L_-(\lambda, \boldsymbol{x}) \cdot (x_1 x_2 x_3 x_4)^{\lambda \frac{\partial}{\partial \lambda}}$$

where

$$L_+(\lambda, \boldsymbol{x}) = E_0 \left(\lambda \cdot x_1 (x_1 x_2)^{-1/2} \right) H_1(x_3) E_1 H_1(1 + x_3^{-1}),$$

$$L_-(\lambda, \boldsymbol{x}) = H_1\left(\frac{x_3}{1 + x_3}\right) H_0(x_2) F_0 \left(\lambda \cdot x_1 x_2 x_3 (x_1 x_2)^{-1/2} \right) H_1(x_4) F_1$$

q-isomonodromic systems [work in progress]

q-difference linear system:

$$L_+(\lambda, \boldsymbol{x}) L_-(\lambda, \boldsymbol{x}) \psi(q\lambda) = \psi(\lambda)$$

q-isomonodromic transformation:

$$\psi(\lambda) = L_+(\lambda, \boldsymbol{x}) \psi'(\lambda)$$

Resulting system:

$$L_-(\lambda, \boldsymbol{x}) L_+(q\lambda, \boldsymbol{x}) \psi'(q\lambda) = \psi'(\lambda)$$

Using the relation (analog of $s_i s_{\bar{j}} = s_{\bar{j}} s_i$)

$$F_i H_i(x) E_i = \prod_{j \neq i} H_j (1+x)^{-C_{ij}} H_i (1+x^{-1})^{-1} E_i H_i(x^{-1}) F_i H_i (1+x^{-1})^{-1}$$

refactorize Lax matrix:

$$L'(\lambda, \boldsymbol{x}) = L_-(\lambda, \boldsymbol{x}) L_+(q\lambda, \boldsymbol{x}) = L_+(\lambda, \boldsymbol{x}') L_-(\lambda, \boldsymbol{x}') = L(\lambda, \boldsymbol{x}')$$

where

$$x_1' = x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, \quad x_2 = x_1^{-1}, \quad z' = qz, \quad q' = q.$$

Thank you for your attention!