TRIANGULAR ICE COMBINATORICS

(PDF + E. Guitter IPHT Saclay)

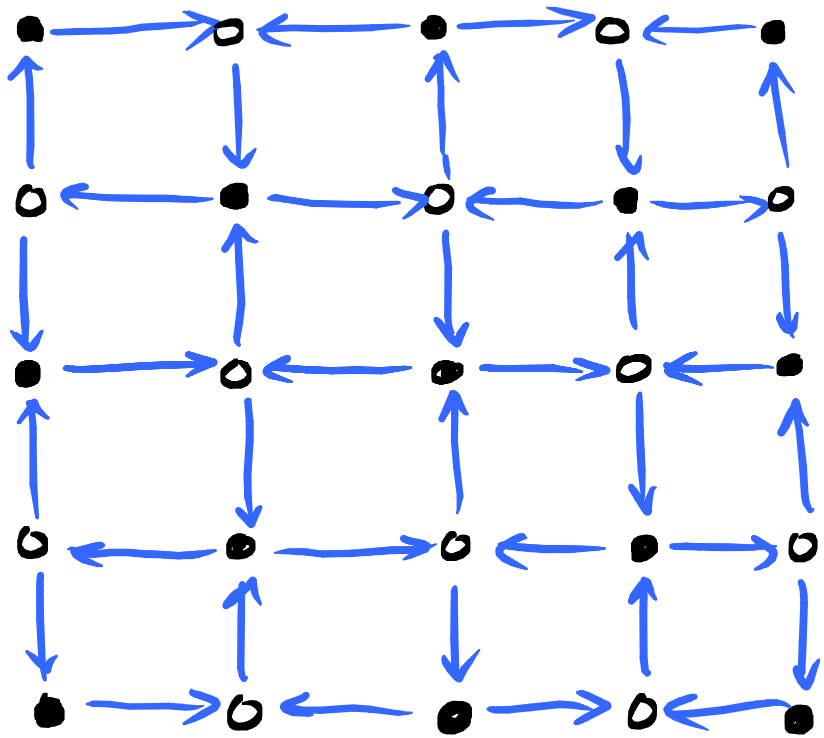
1. ASM, square ice, 6V model and integrability
2. Triangular ice, DWBC, and APM
3. Domino Tilings of the Holey Aztec Square
4. Proof of the APM – HASDT correspondence
5. Combinatorial Conjectures
6. Limit shape / Arctic Phenomenon
7. Conclusion

A Tale of 2 sequences $\begin{cases} 1, 3, 23, 433, 19705, 2151843, \ldots \\ 1, 3, 29, 901, 89893, 28793575, \ldots \end{cases}$

Giens 2019

1. Alternating Sign Matrices (ASM)

- Laurent phenomenon in T-system cluster algebra



● = time 0 $\quad X = 1 = M^{(0)}_{ij}$

○ = time 1 $\quad X = M^{(1)}_{ij}$

iterated mutations compute

λ-determinants of M

[Mills-Robbins-Rumsey82]

$$M^{(k+1)}_{ij} M^{(k-1)}_{ij} = M^{(k)}_{ij} M^{(k)}_{i+1,j+1} + \lambda M^{(k)}_{i,j+1} M^{(k)}_{i+1,j}$$

take $\lambda = 1$

Alternating Sign Matrices (ASM)

- Laurent phenomenon in T-system cluster algebra

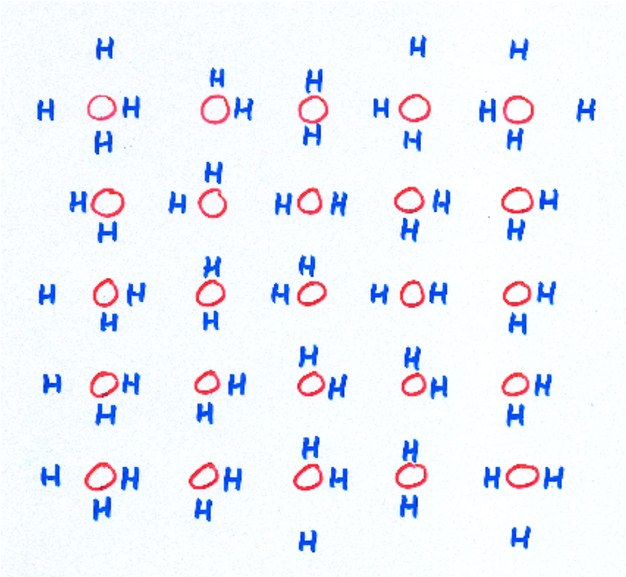
after many iterations:

$$M_{i,j}^{(k)} = \sum_{\substack{\text{ASM } k\times k \\ A=(a_{ij})}} 2^{\#(-1)} \prod_{i,j=1}^{n} M_{ij}^{a_{ij}} \in \mathbb{Z}_+[\{M_{ij}^{\pm 1}\}]$$

$a_{ij} \in \{0, \pm 1\}$ and alternation conditions along rows and columns

$(0..010..0\,{-1}\,0..01\,...\ 00010..0)$

ASM and Square Ice

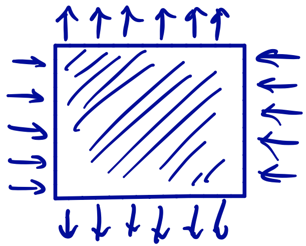


Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

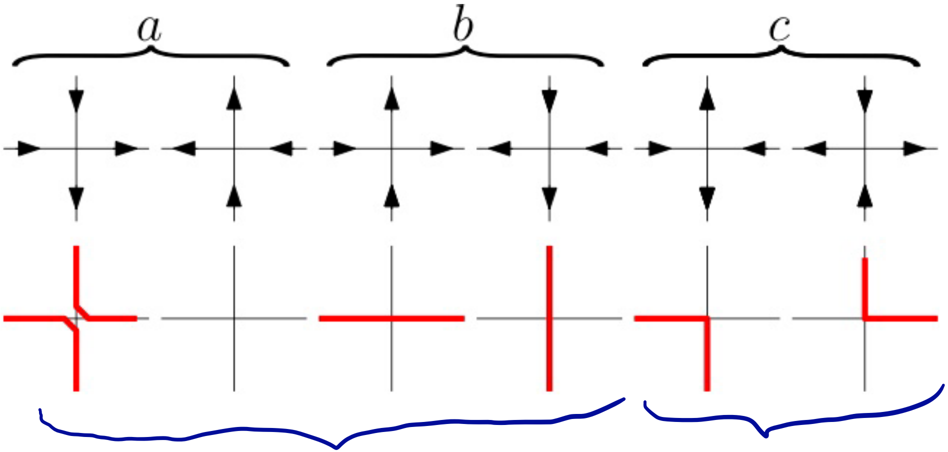
Ice Rule at each vertex

incoming arrows = # outgoing arrows $\Rightarrow$ 6V

+ Domain Wall Boundary Conditions



6V – ASM Correspondence



6V configs

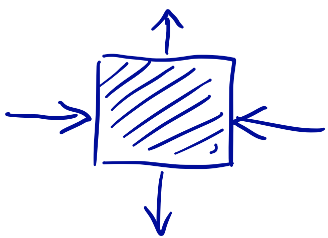
osculating paths

Transmittor vertices | Reflector vertices

0 0 0 0 +1 −1

ASM entries

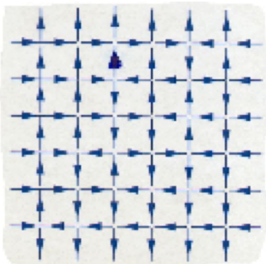
Alternance conditions



odd # of reflections!

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ASM



6V DWBC

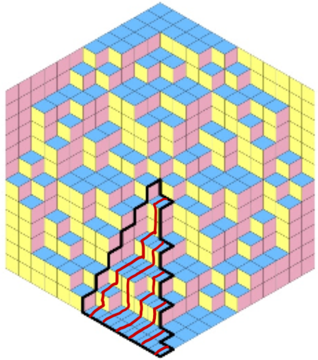
n

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

n

ASM

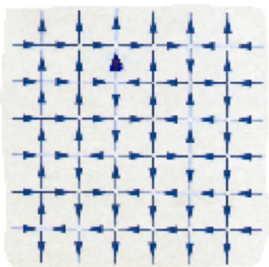
$$ASM_n = \frac{\prod_{i=0}^{n-1} (3i+1)!}{\prod_{i=0}^{n-1} (n+i)!}$$



2n

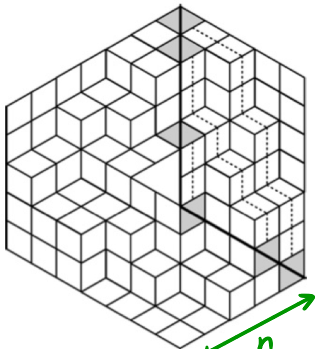
TSSC PP

n



n

6V DWBC



n+2

n

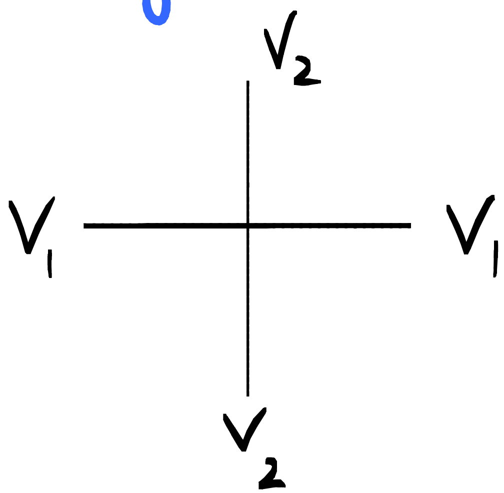
DPP

n

INTEGRABILITY

• Boltzmann weights

R operator



$$V_1 \otimes V_2 \xrightarrow{R} V_2 \otimes V_1$$

$\dim V_i = 2$

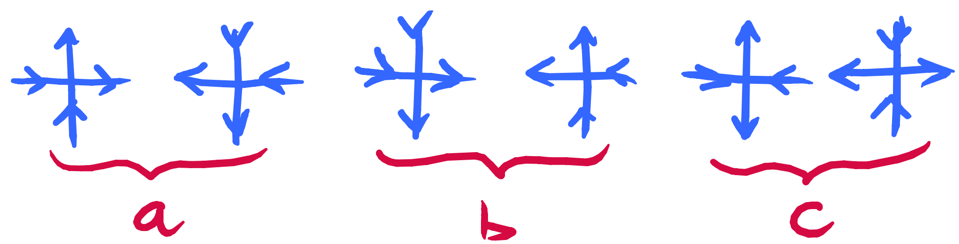
$V_1 = \langle \rightarrow, \leftarrow \rangle = \alpha$

$V_2 = \langle \uparrow, \downarrow \rangle = \beta$

matrix entries in $\alpha \otimes \beta \to \beta \otimes \alpha$

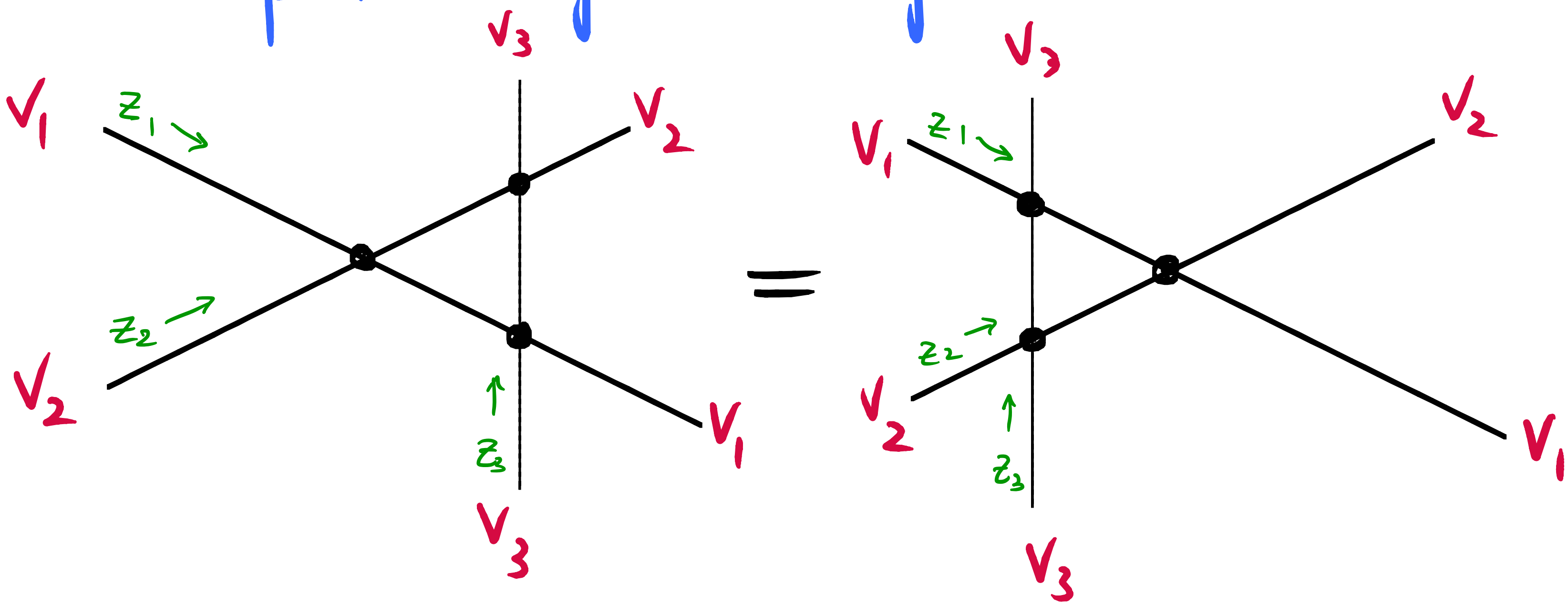
6 non-zero entries out of 16

(ice rule).



YANG-BAXTER RELATION

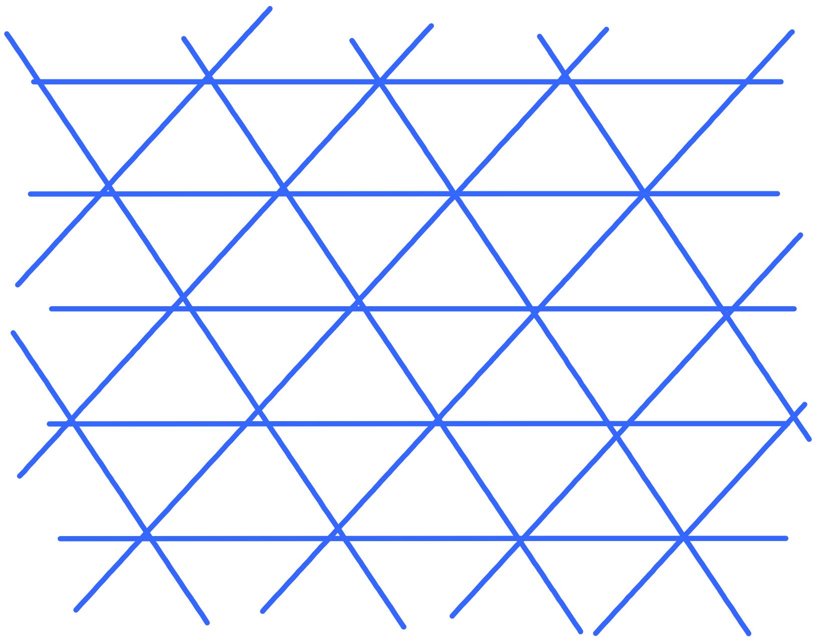
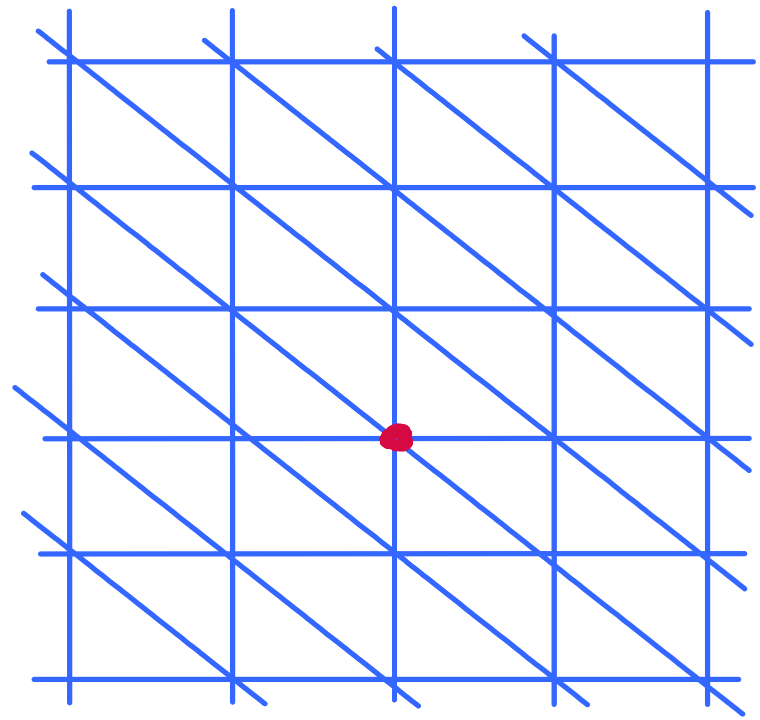
One can pick "integrable weights" such that:

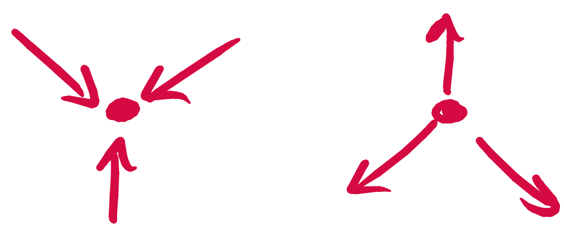


(a cubic identity for R operators from $V_1 \otimes V_2 \otimes V_3$ to $V_3 \otimes V_2 \otimes V_1$)

$$\boxed{a(z,w) = z - w \; ; \; b(z,w) = q^{-2} z - q^{2} w \; ; \; c(z,w) = (q^{2} - q^{-2})\sqrt{zw}}$$

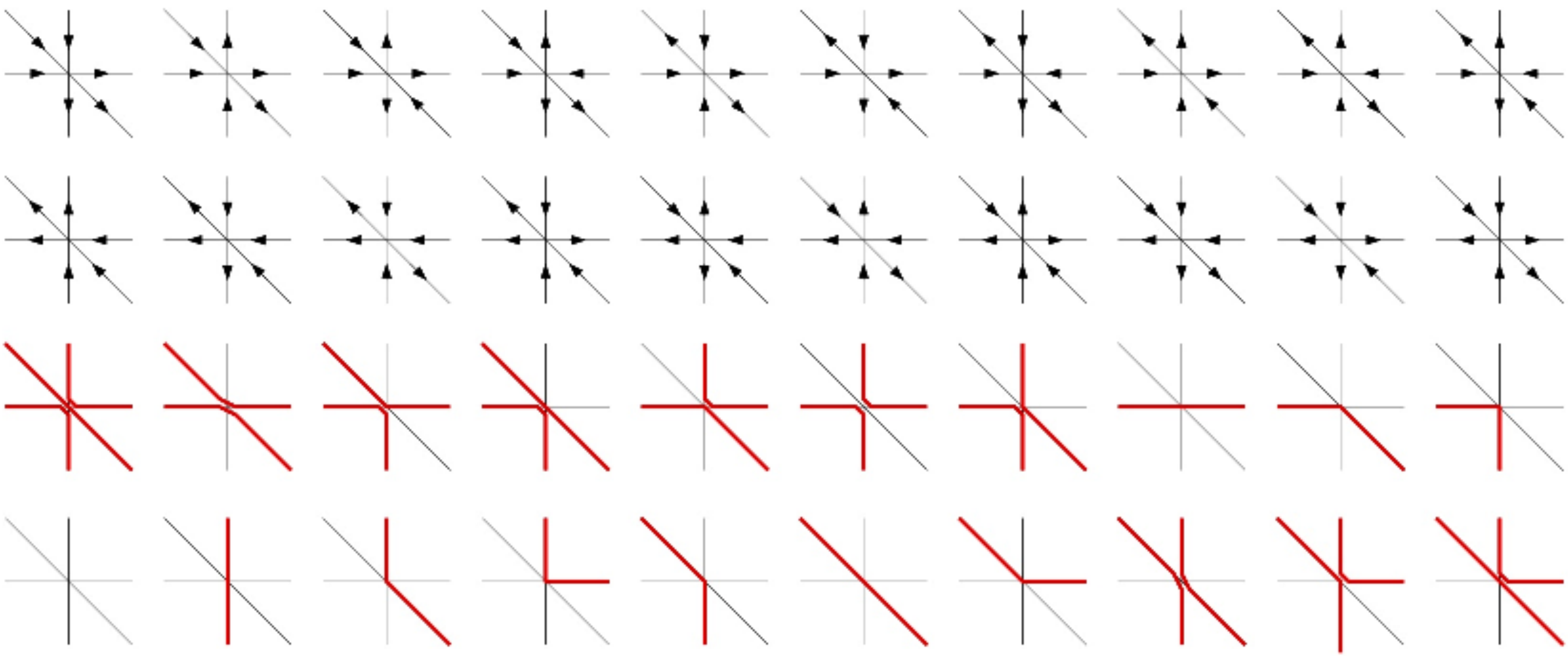
2. TRIANGULAR ICE (20V model)

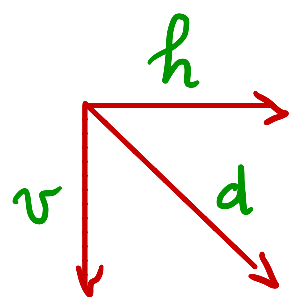
ice rule  at each vertex

[Kelland, Baxter]

TRIANGULAR ICE (20V model)



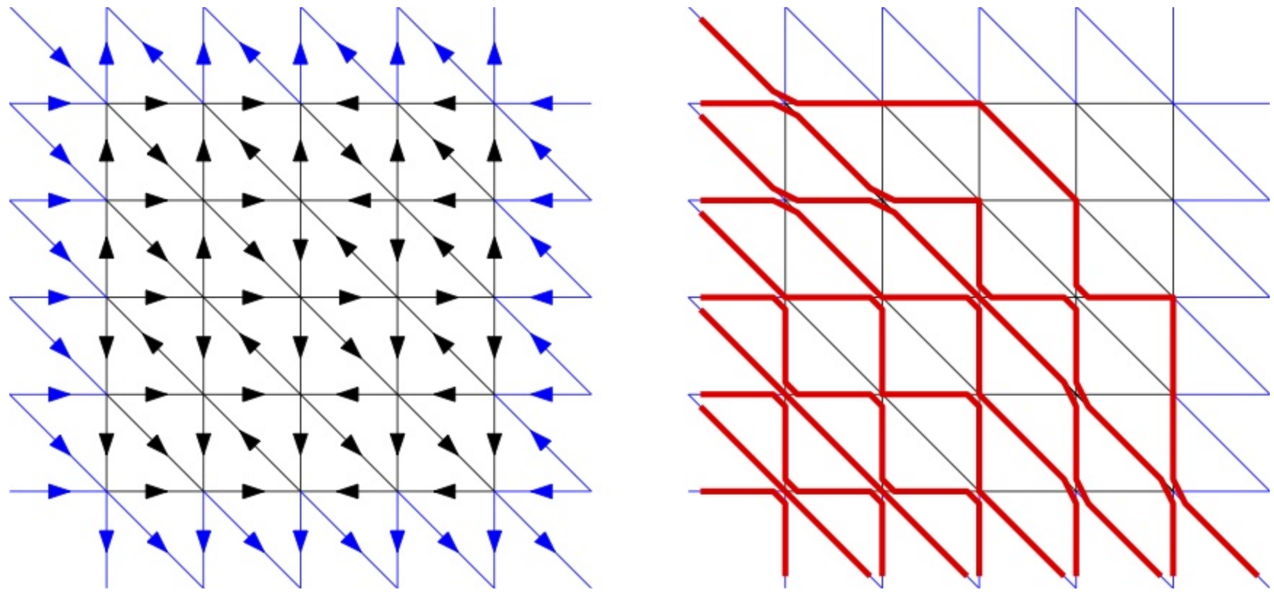
Osculating Schröder paths



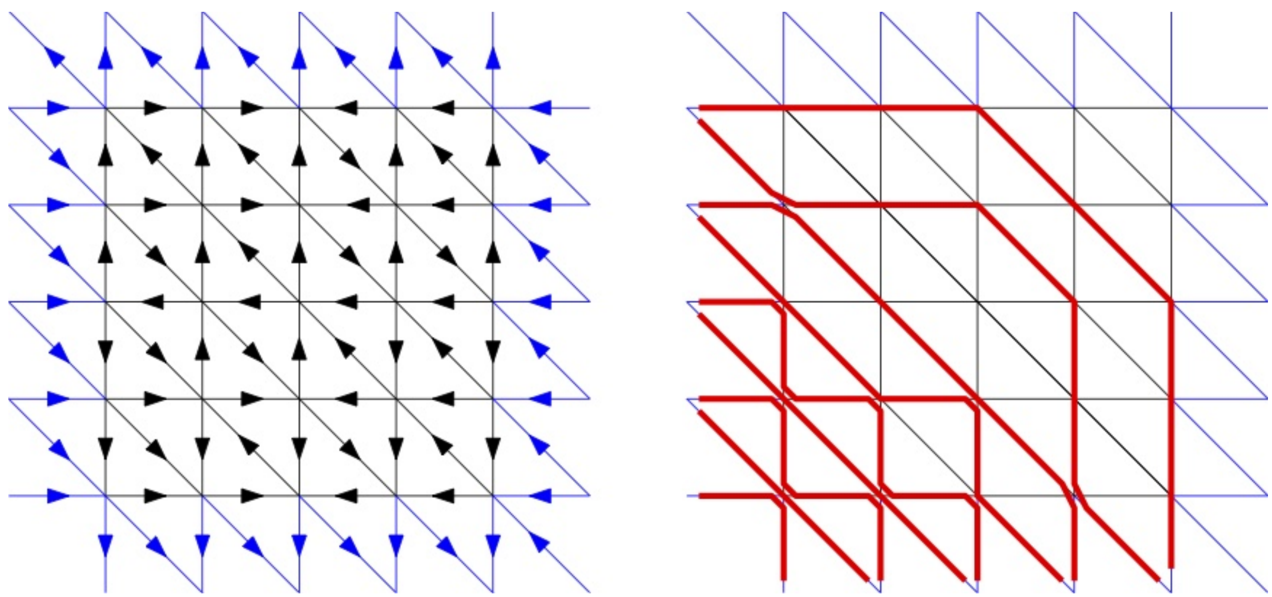
h, v, d steps

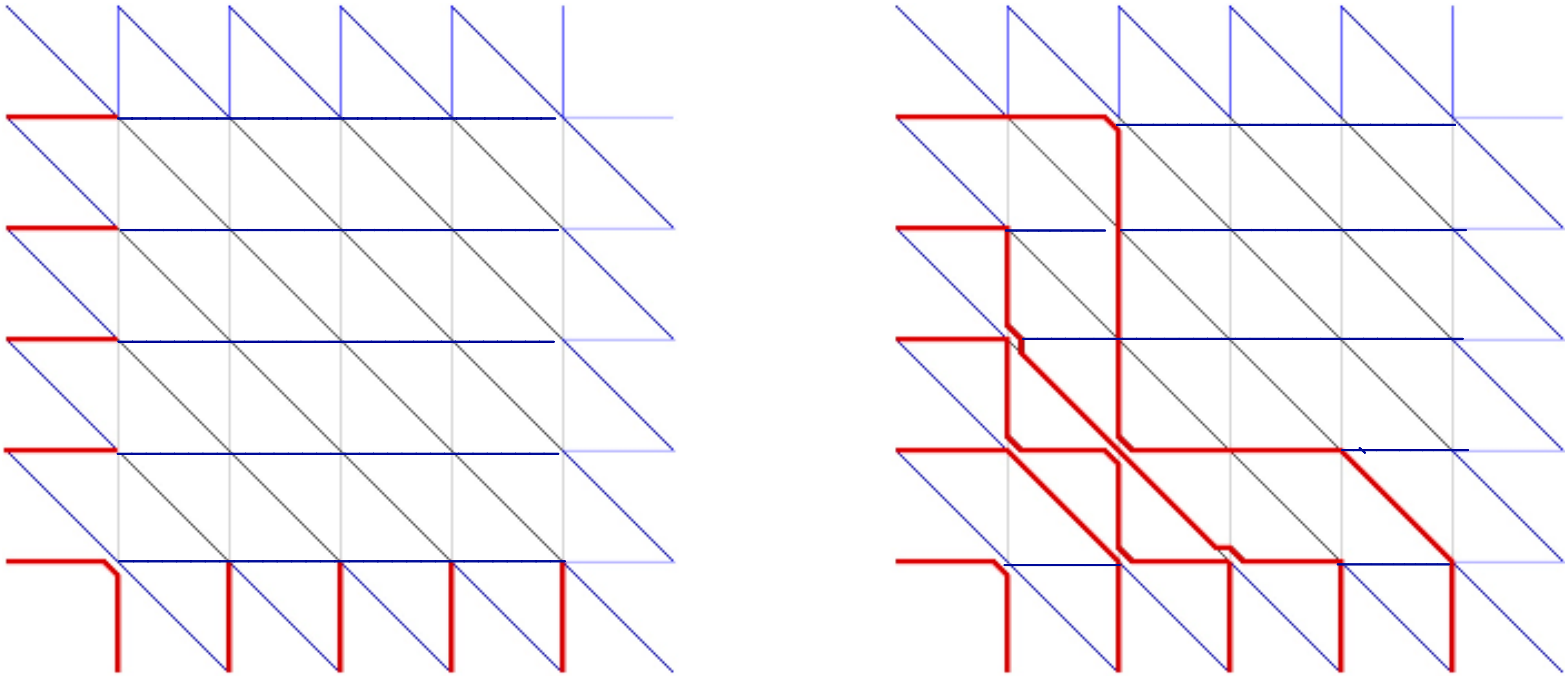
DOMAIN WALL BOUNDARY CONDITIONS

DWBC1



DWBC2





DWBC 3

Numbers of Configurations on an $n \times n$ grid:

DWBC 1,2

$$A_n = 1, 3, 23, 433, 19705, 2151843, \ldots$$

DWBC 3

$$B_n = 1, 3, 29, 901, 89893, 28793575, \ldots$$

[computed by transfer matrix]

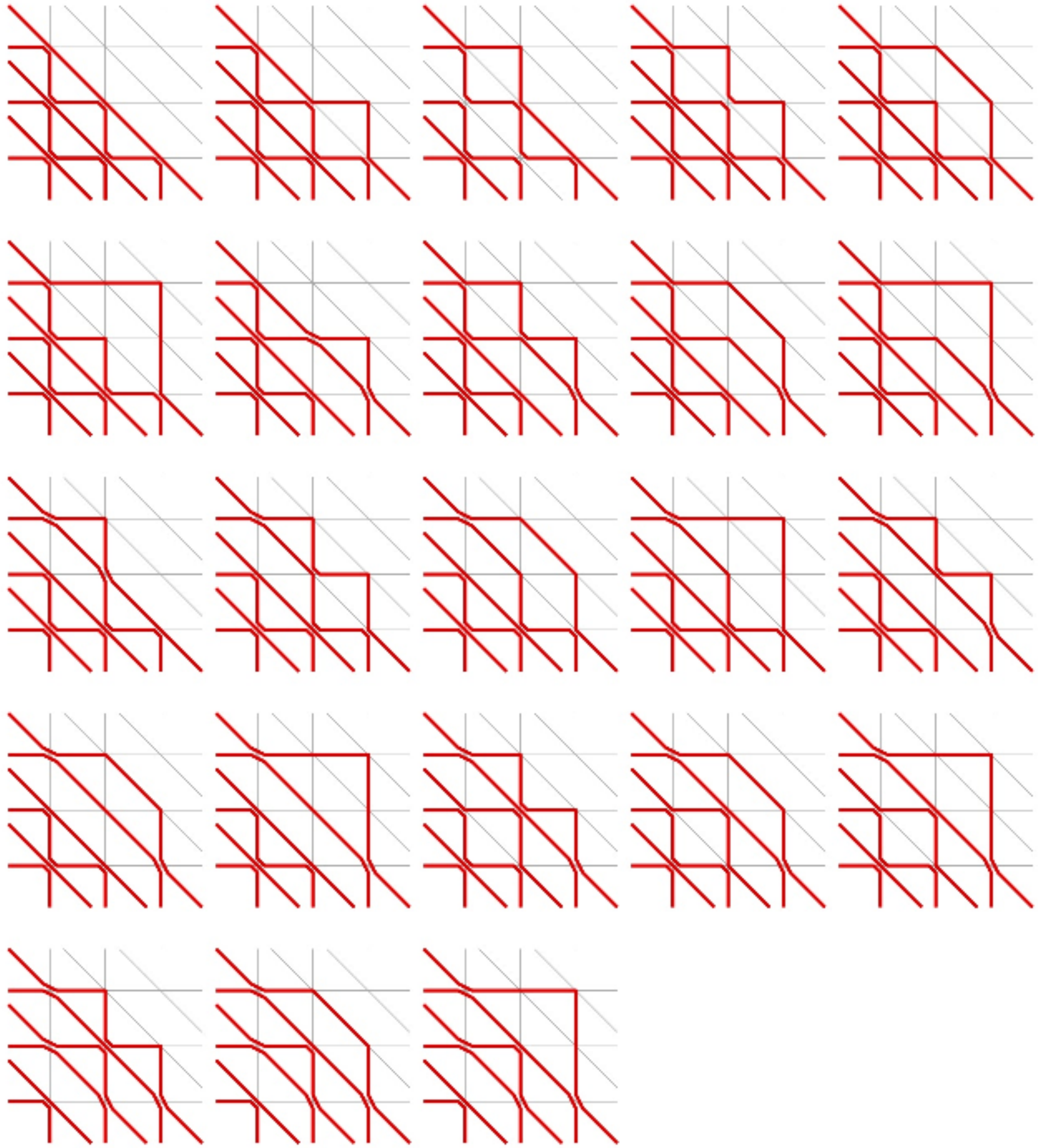
20 v

DWBC 1

configurations

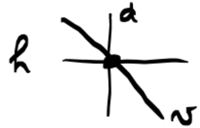
$n=3$

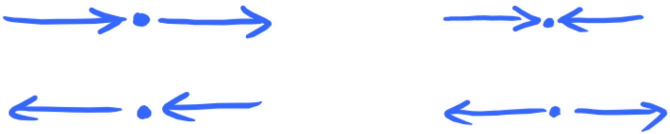
(23)



ALTERNATING PHASE MATRICES

Coding of the vertex configurations





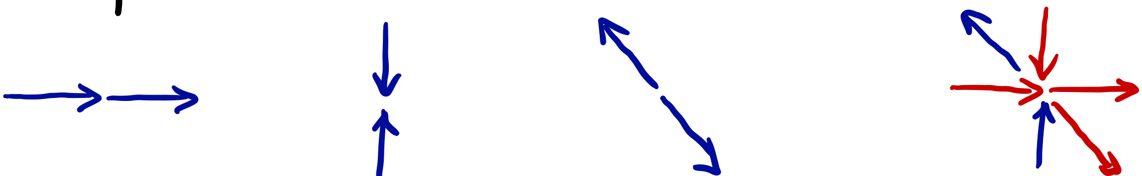
Transmittor — 0

Reflector — 1, -1

$n \times n$ matrix with entries $\{h, v, d\} \in \{0, \pm 1\}^3$ with $h + v + d = 0$

$\Leftrightarrow$ Sixth roots of unity & 0.

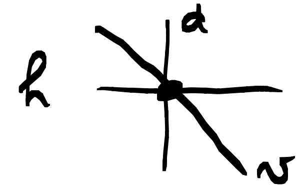
Example:



h: 0 v: 1 d: -1 $\longrightarrow$ $(0, 1, -1)$

ALTERNATING PHASE MATRICES

Coding of the vertex configurations



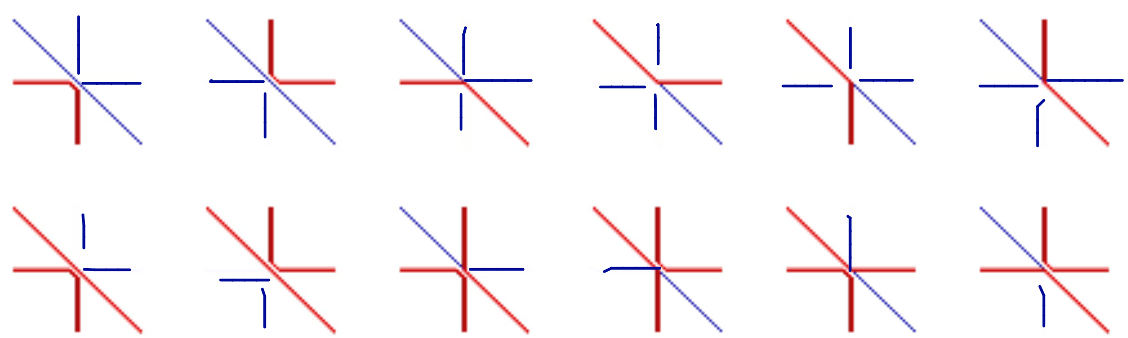
Transmittor | Reflector

0 | 1, -1

$n \times n$ matrix with entries $\{h, v, d\} \in \{0, \pm 1\}^3$ with $h + v + d = 0$

$\Leftrightarrow$ Sixth roots of unity & 0.

8 transmittor vertices + 12 reflector vertices

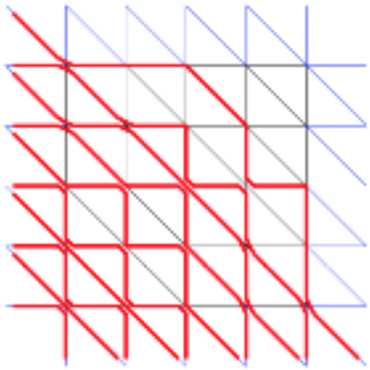


$$0 \quad\quad 1 \quad\quad -1 \quad\quad -\omega \quad\quad \omega \quad\quad -\omega^2 \quad\quad \omega^2$$

$$(0,0,0) \leftarrow (h,v,d) \rightarrow (1,-1,0) \quad (-1,1,0) \quad (1,0,-1) \quad (-1,0,1) \quad (0,-1,1) \quad (0,1,-1)$$

(h,v,d): $(1,-1,0)$ $(-1,1,0)$ $(1,0,-1)$ $(-1,0,1)$ $(0,-1,1)$ $(0,1,-1)$

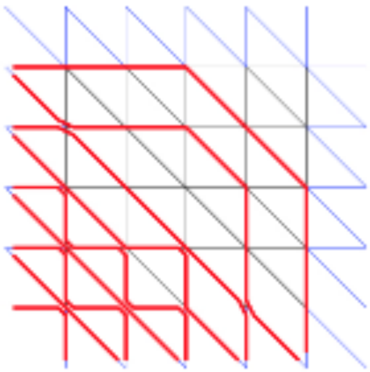
$1 \quad -1 \quad -\omega \quad \omega \quad -\omega^2 \quad \omega^2$



$$\mapsto \begin{pmatrix} 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 1 & -\omega^2 & 0 \\ -\omega^2 & -\omega^2 & 0 & 0 & 1 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 \end{pmatrix}$$

APM of type 1

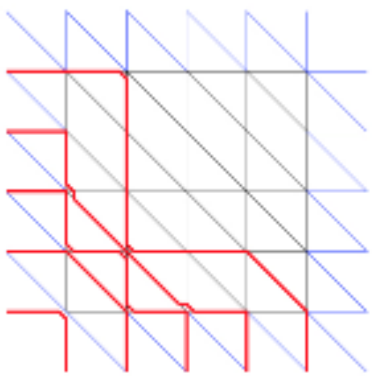
$\equiv$ 20V- DWBC1



$$\mapsto \begin{pmatrix} 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 1 & 0 & 0 & -\omega^2 & -\omega^2 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 \end{pmatrix}$$

APM of type 2

$\equiv$ 20V- DWBC2



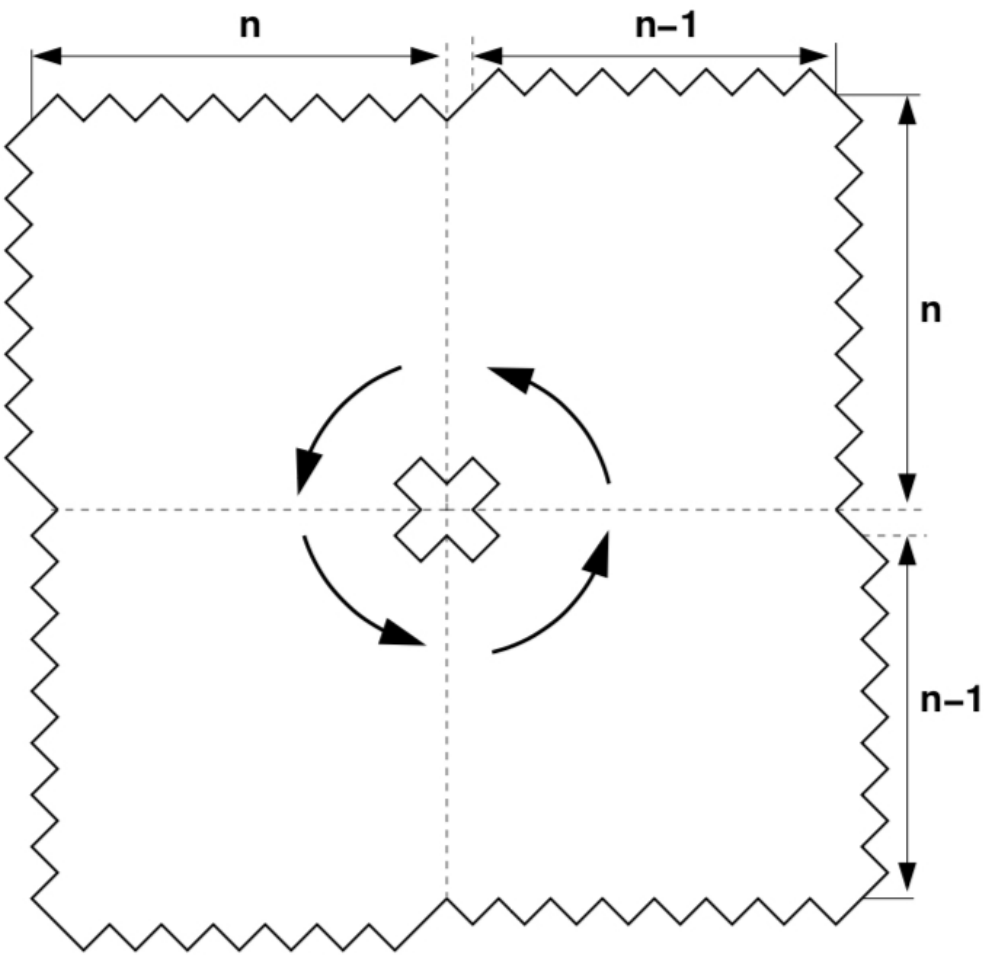
$$\mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & -\omega & 0 \\ 1 & \omega & -\omega^2 & 1 & -\omega^2 \end{pmatrix}$$

APM of type 3

$\equiv$ 20V- DWBC3

Thm All ASM are APM.

3. DOMINO TILINGS OF THE HOLEY SQUARE WITH QUARTER-TURN SYMMETRY



$\mathcal{S}_n$

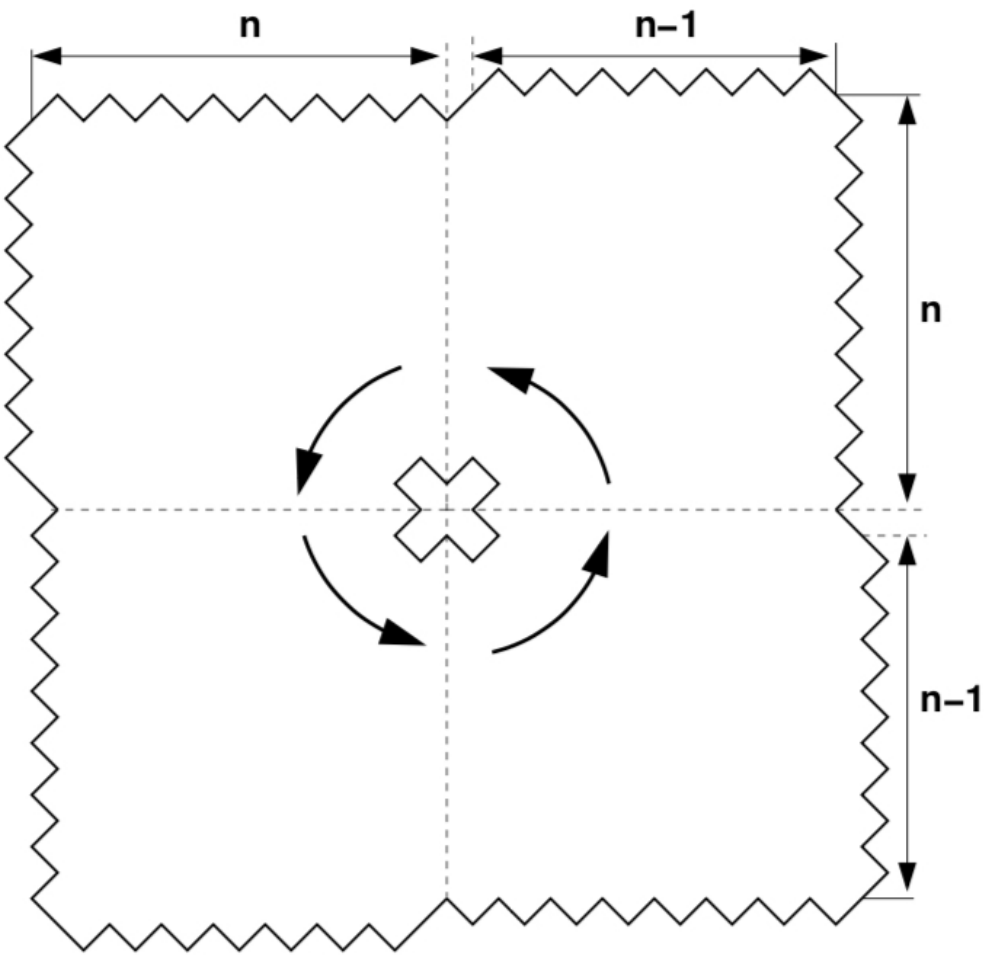
Domino Tilings: use

and 2×1 dominos

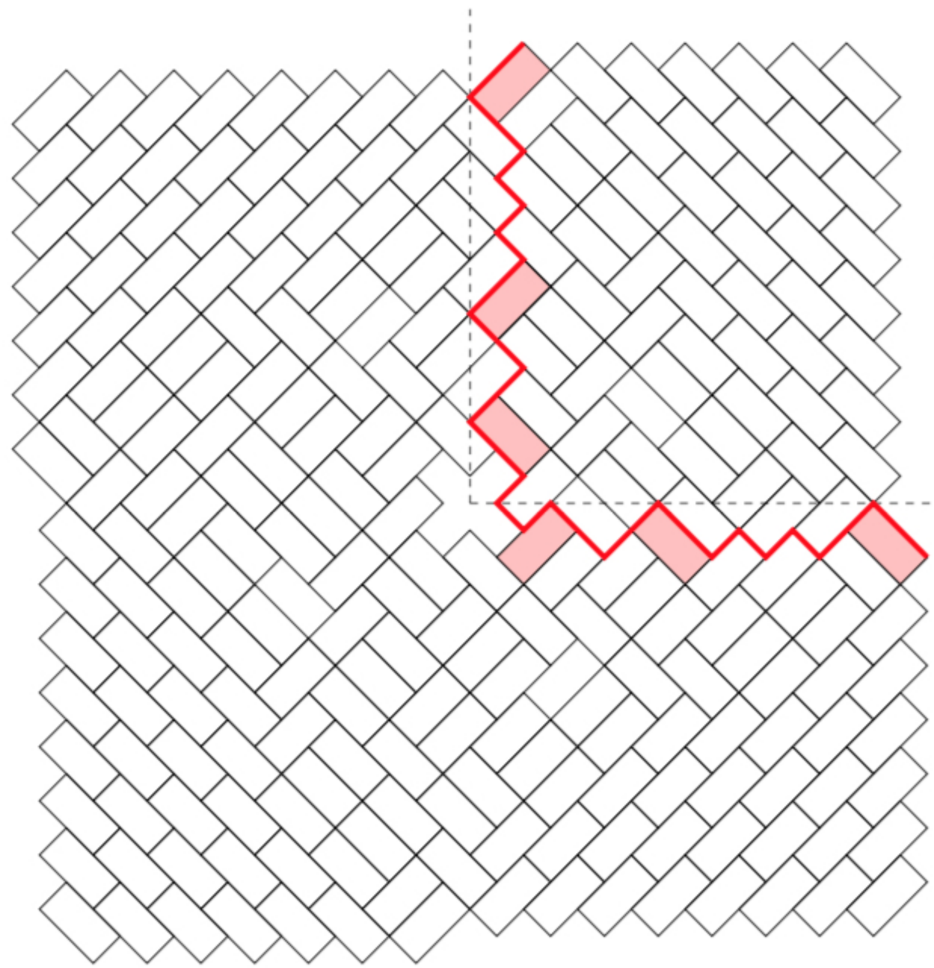
Rotational symmetry by $\frac{\pi}{2}$

NB: the hole makes it tileable!

DOMINO TILINGS OF THE HOLEY SQUARE WITH QUARTER-TURN SYMMETRY

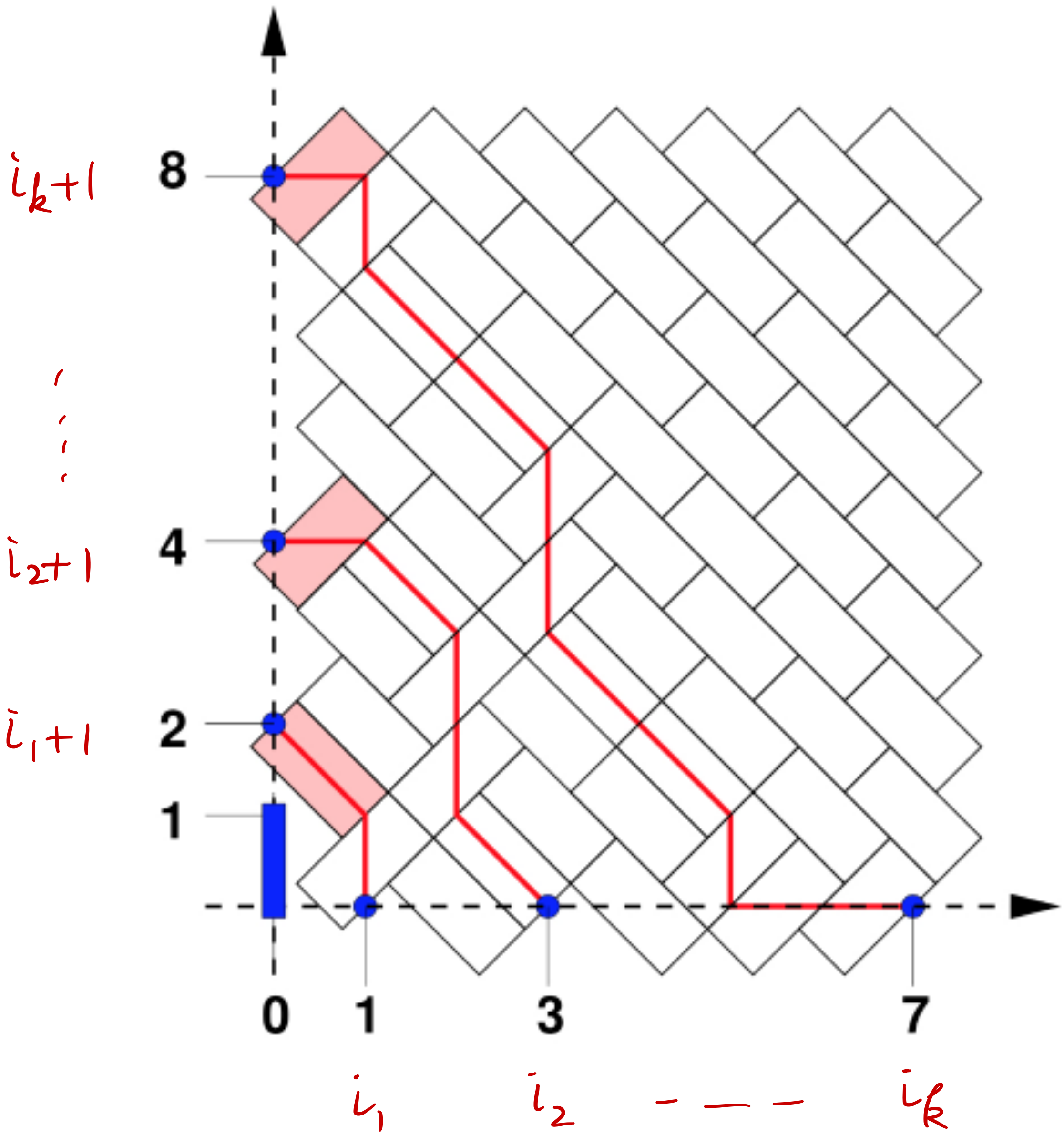


$\mathcal{S}_n$



sample tiling

Counting Configurations



- Non-intersecting Schröder paths w fixed ends
- first step cannot be |
- start and ends identified (cone).

Counting Configurations

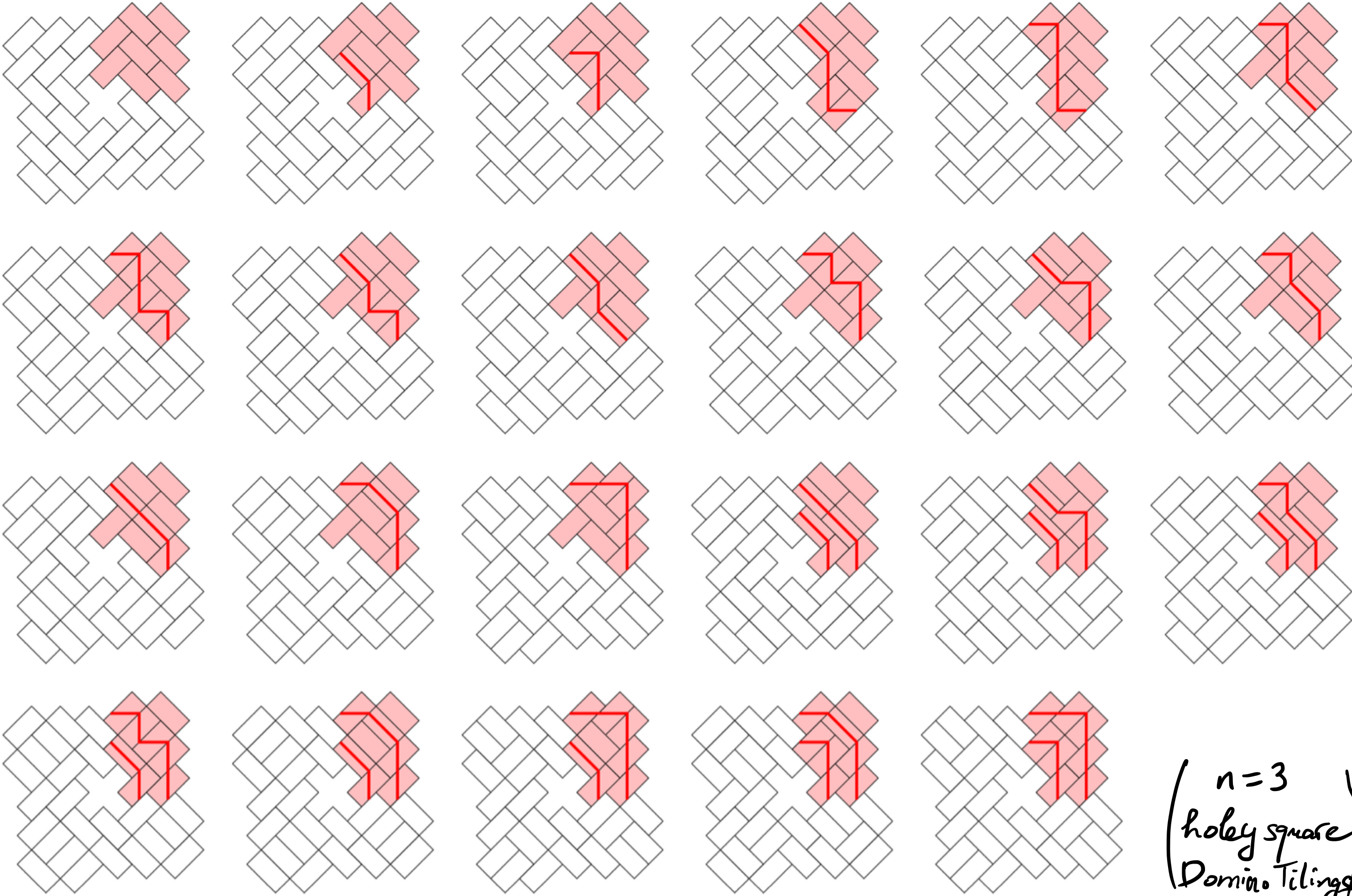
Thm [PDF-Guitter 19]

$$T_4(\mathcal{J}_n) = \det_{1\le i,j\le n}\left(\left\{ \frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)} \right\}_{z^i w^j} \right)$$

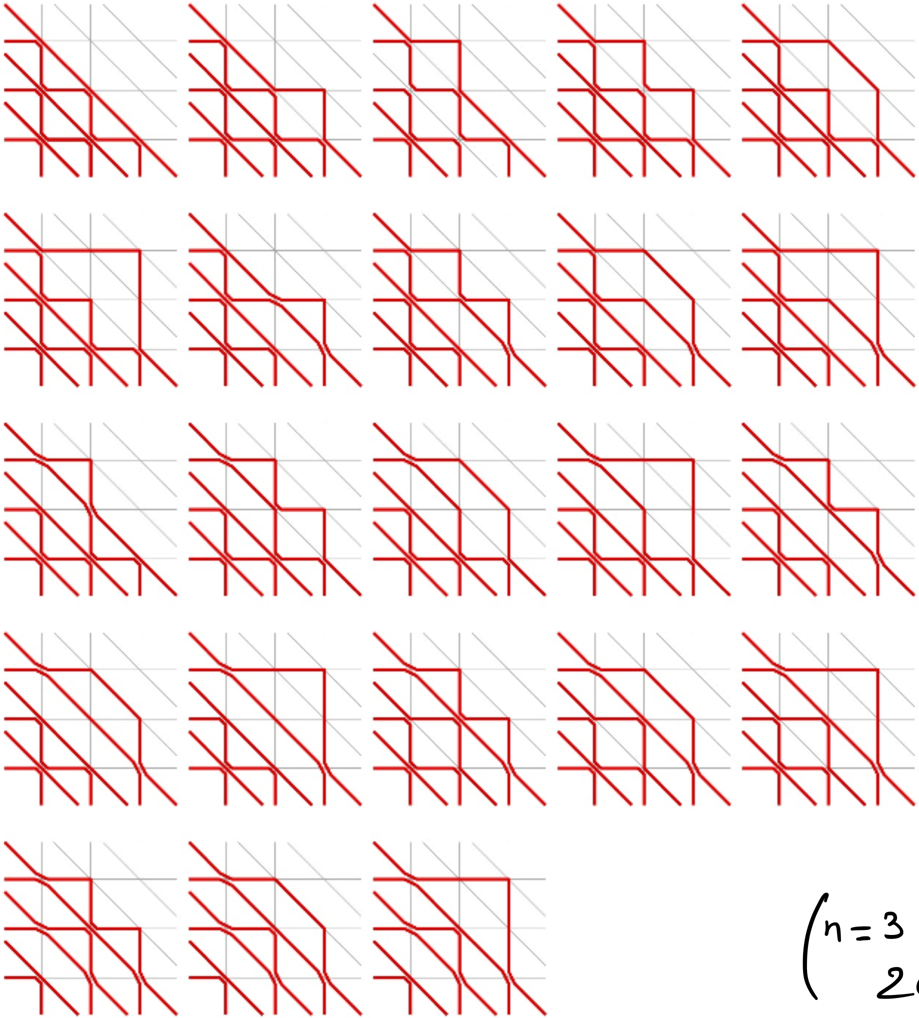
Proof: (Cauchy Binet) $\det(\mathrm{Id} + M) = \sum_{i_1<\dots<i_k} \underbrace{\left|M^{i_1\dots i_k}_{i_1\dots i_k}\right|}_{\text{(Gessel-Viennot)}}$

$$T_4(\mathcal{J}_n) = 1,\ 3,\ 23,\ 433,\ 19705,\ 2151843,\ \dots$$

Ex: $n=3$ $\quad \det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\2&2&2\\4&8&12\end{pmatrix}\right] = 23$ Domino Tiling configurations $\longrightarrow$



$\left(\begin{array}{c} n=3 \\ \text{holey square} \\ \text{Domino Tilings} \end{array}\right)$



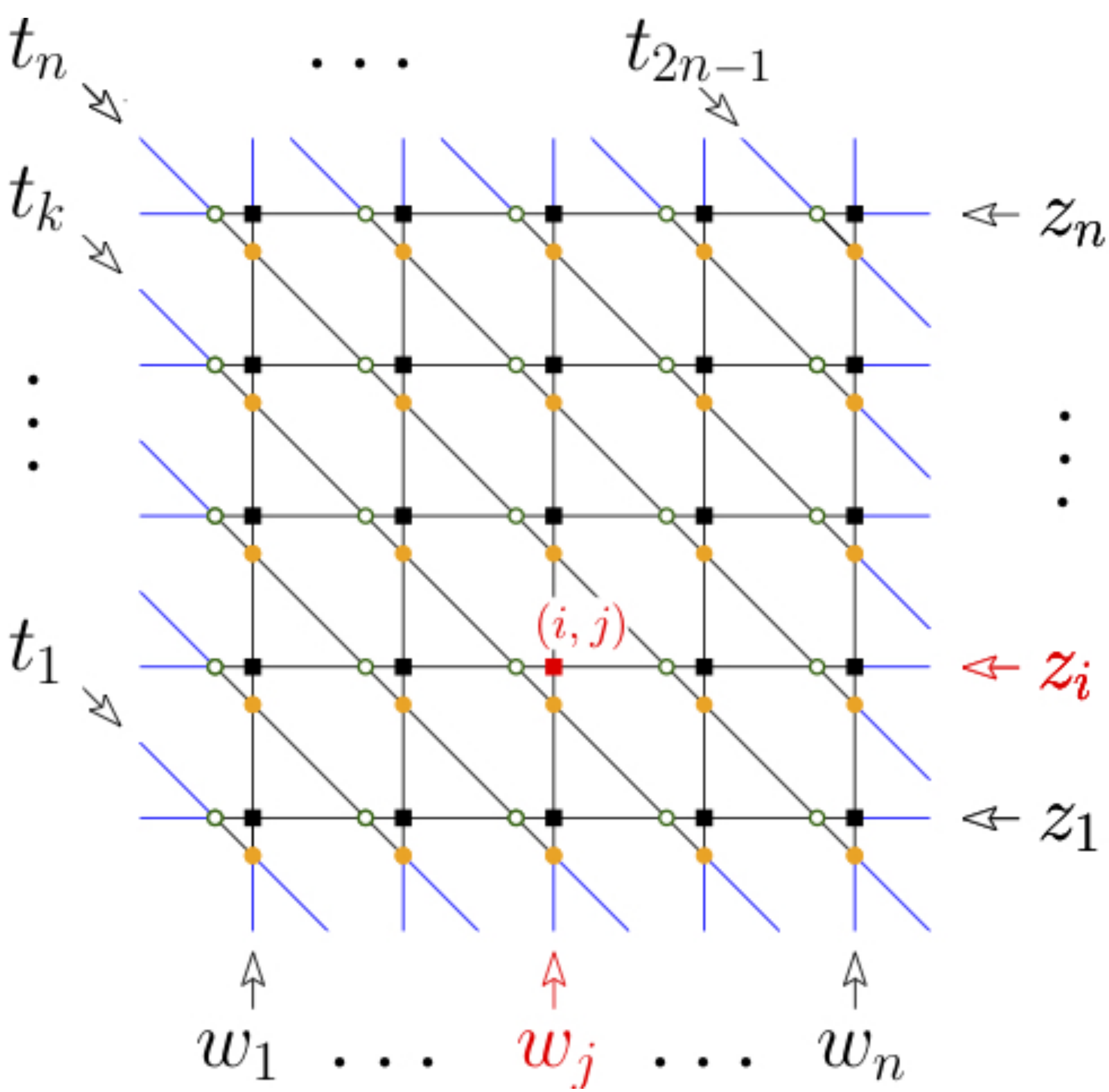
(n = 3 DWBC1
20V configurations)

4. PROOF OF THE CORRESPONDENCE WITH 20V - DWBC1,2

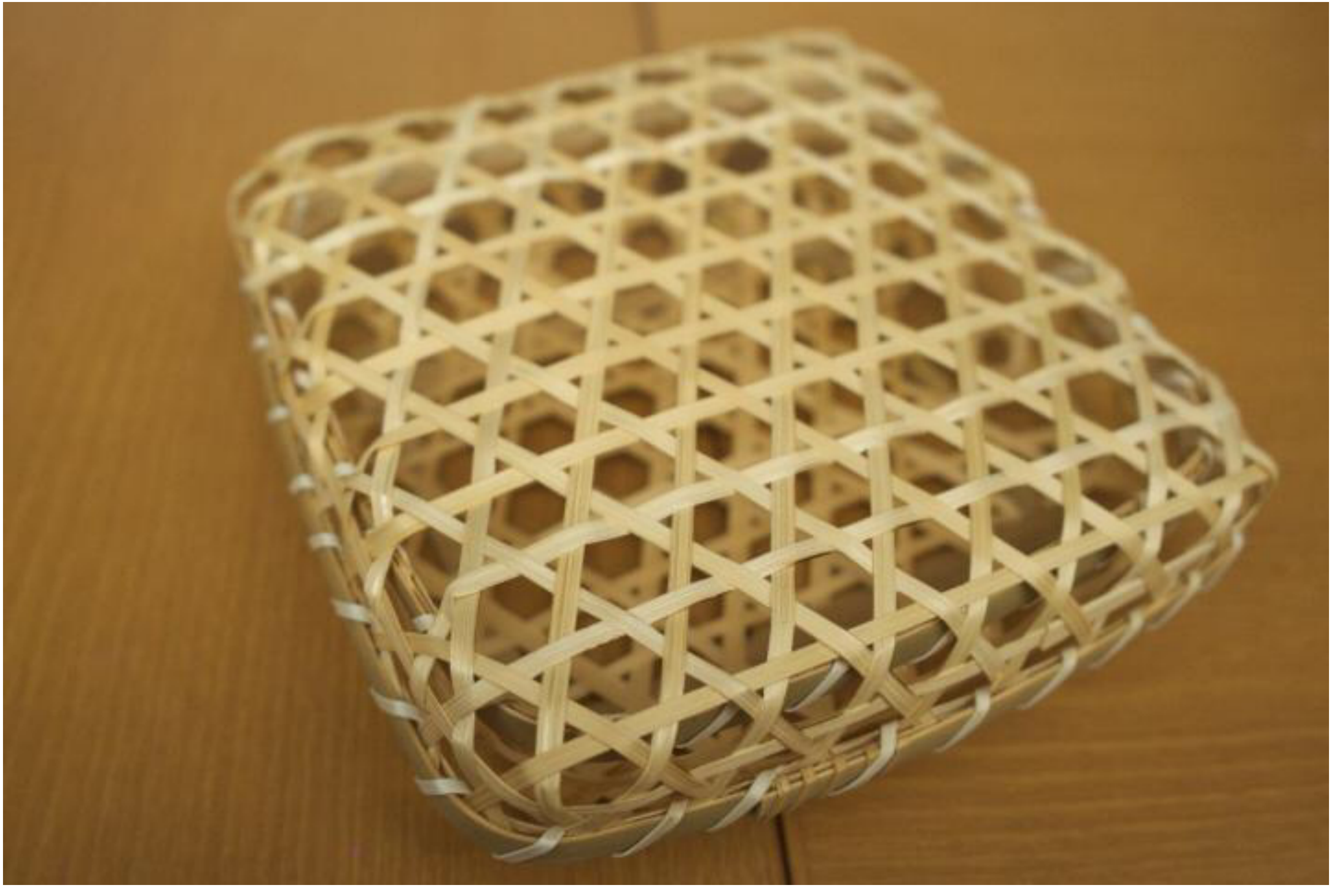
idea

- use integrable weights for the 20V
- deform the line arrangement into a 6V
- use 6V results (Izergin-Korepin det)
- refinement

ICE ON THE KAGOME LATTICE

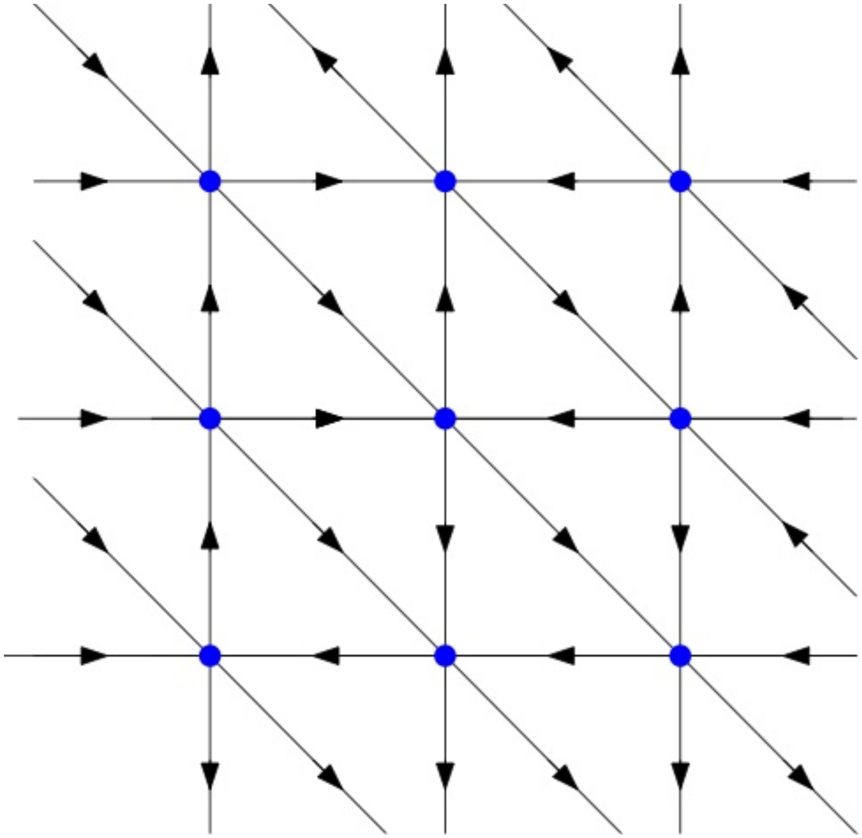


- z_i, w_j, t_k are complex (spectral) parameters
- The weights are functions of a pair of spectral parameters and obey the Yang-Baxter eqn

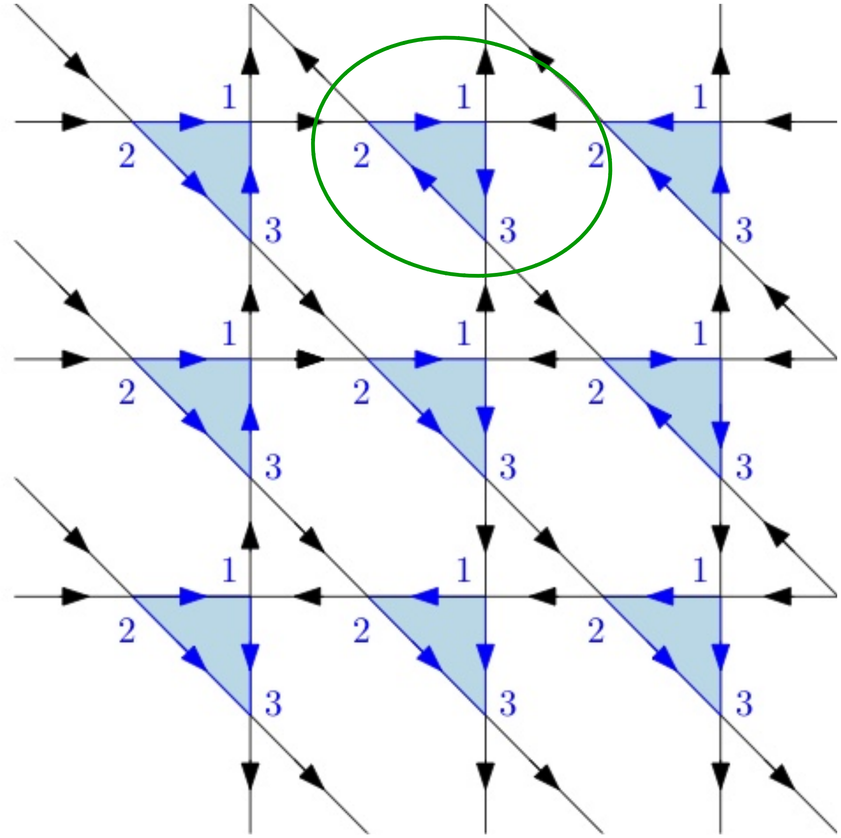


籠 (*kago*, “basket”) + 目 (*me*, “eye, hole”)

A lattice of KAGOME (Daikokuya, Kitashirakawa)

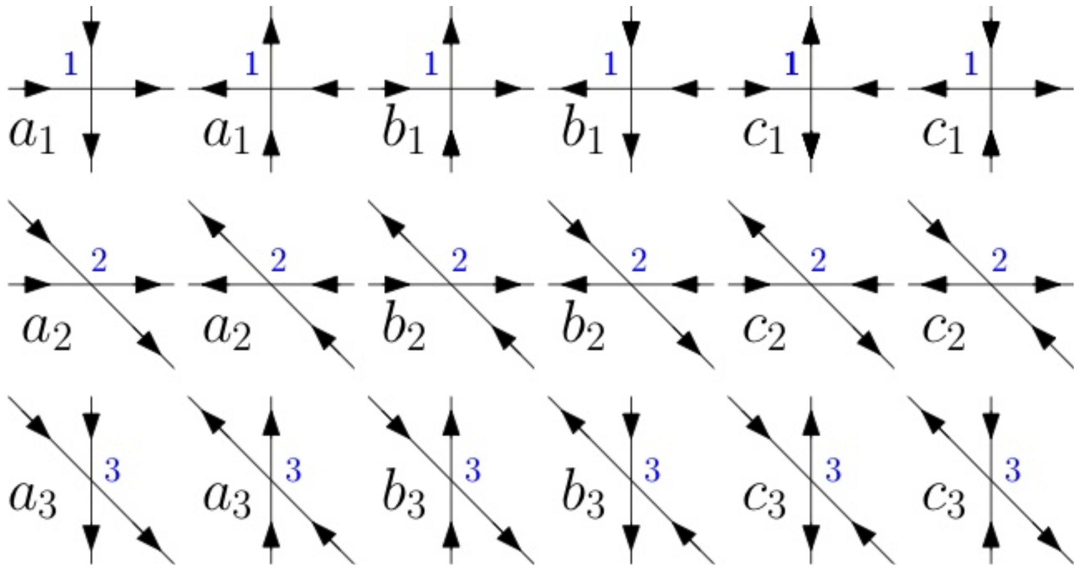


Triangular lattice
ice



Kagome Lattice
ice

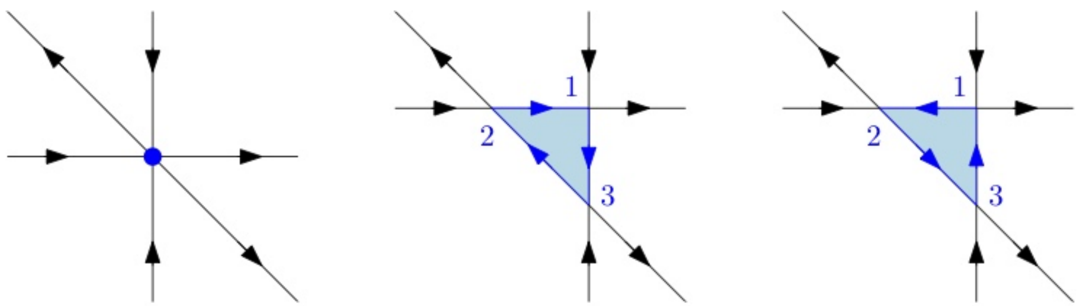
BOLTZMANN WEIGHTS



weights of the 6V models on the 3 sublattices

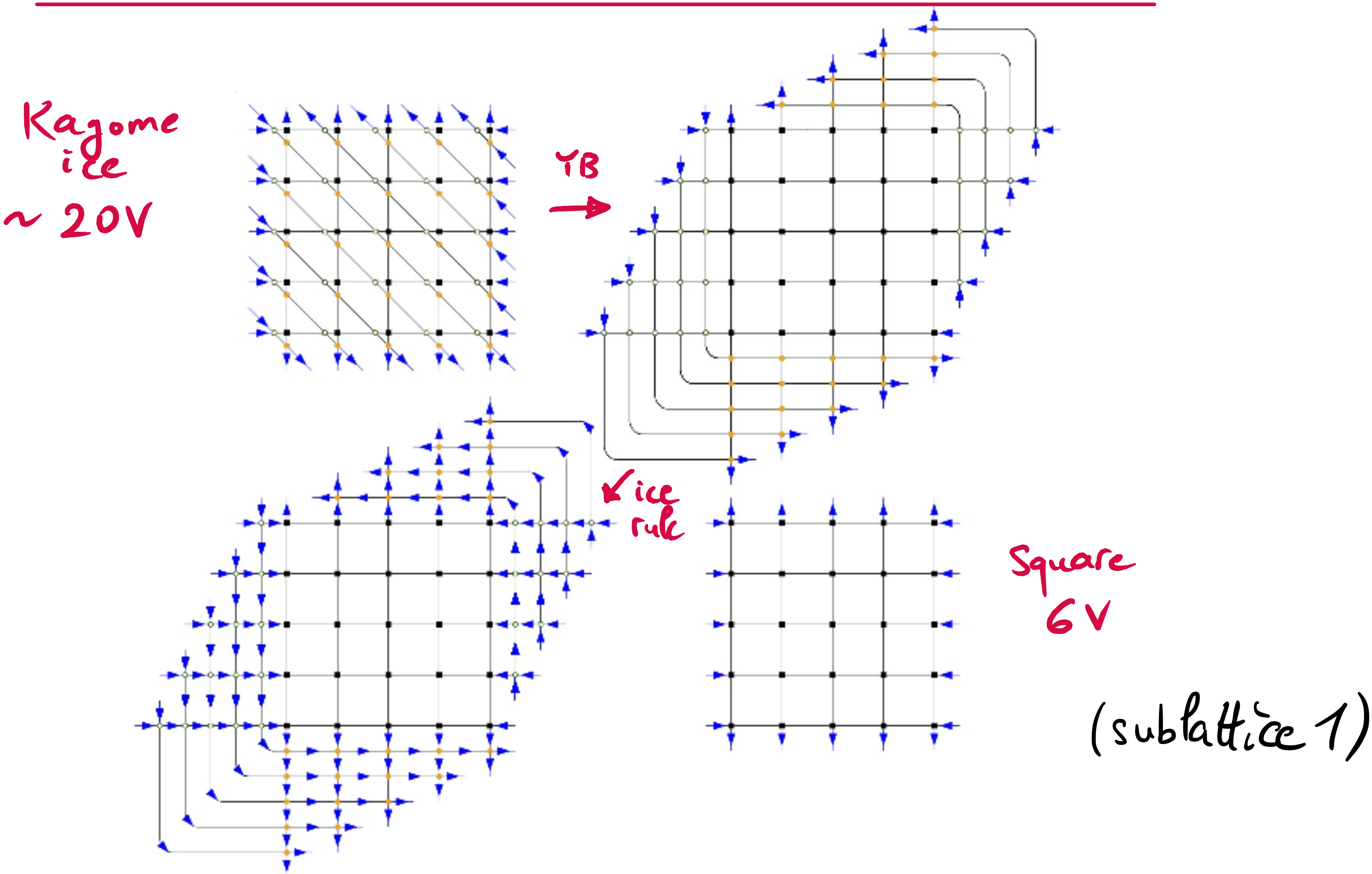
- 20V weights are given by sums over inner triangle configs

Example:

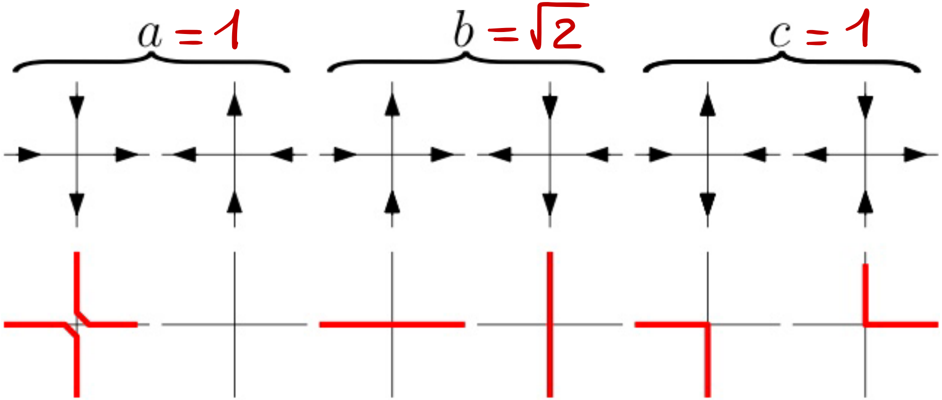


$$w = a_1 b_2 c_3 + c_1 c_2 b_3$$

TRANSFORMATION INTO A 6V MODEL

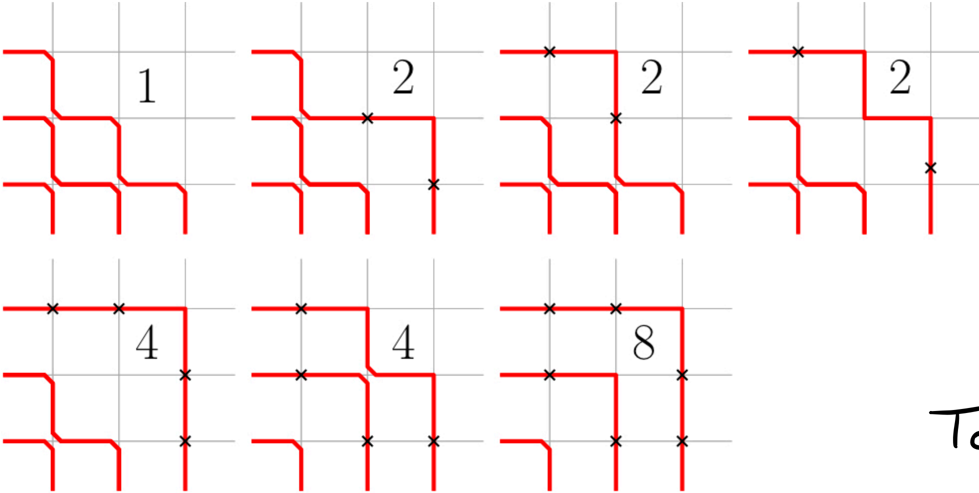


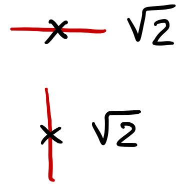
Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights = 1 is equal to that of the 6V model with weights $(a,b,c) = (1, \sqrt{2}, 1)$ and DWBC




20種の野菜と6種の果実
野菜汁60%+果汁40%=100%
KAGOME
野菜生活
100
ベリーサラダ
1食分の野菜
20
6

Example of size $n=3$ — 20V-DWBC1 v/s 6V aka ASM





(b weights)

Total = 23

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

7 ASM of size 3.

Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights = 1 is equal to that of the 6V model with weights $(a,b,c)=(1,\sqrt{2},1)$ and DWBC

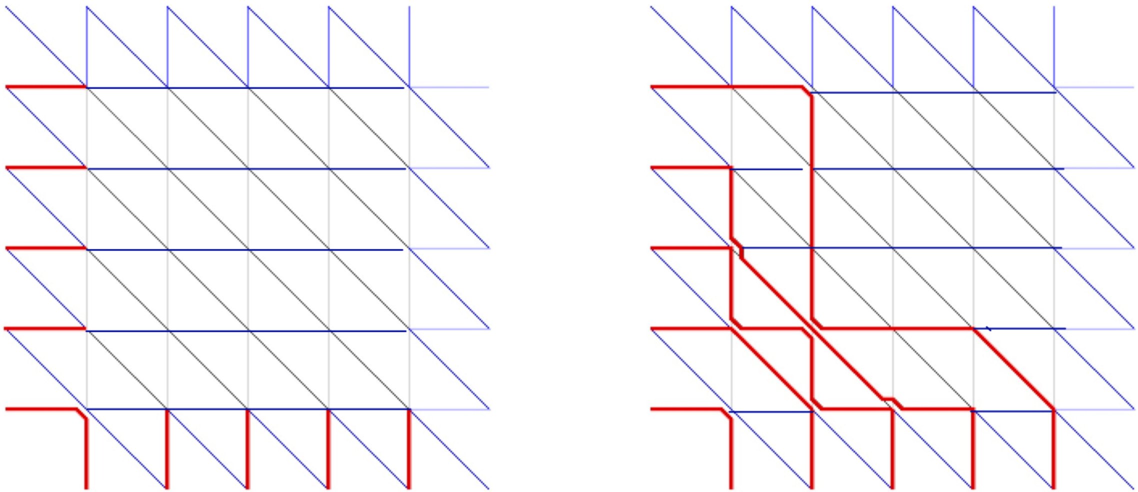
Then use classical result by Korepin-Izergin for the 6V-DWBC and spectral parameters $(z_1 \dots z_n, w_1 \dots w_n)$

$$Z_{6VDWBC}(z_1 \dots z_n, w_1 \dots w_n) = \frac{\prod_{i=1}^{n} c(z_i,w_i) \prod_{i,j=1}^{n} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{1\leq i<j\leq n} (z_i - z_j)(w_i - w_j)} \det_{1\leq i,j\leq n} \left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}$$

→ Limiting procedure → same det as holey square DT!

→ Refinements

5. The DWBC 3 Conjecture



DWBC 3

$b_n = 1, 3, 29, 901, \ldots$

5. The DWBC 3 Conjecture

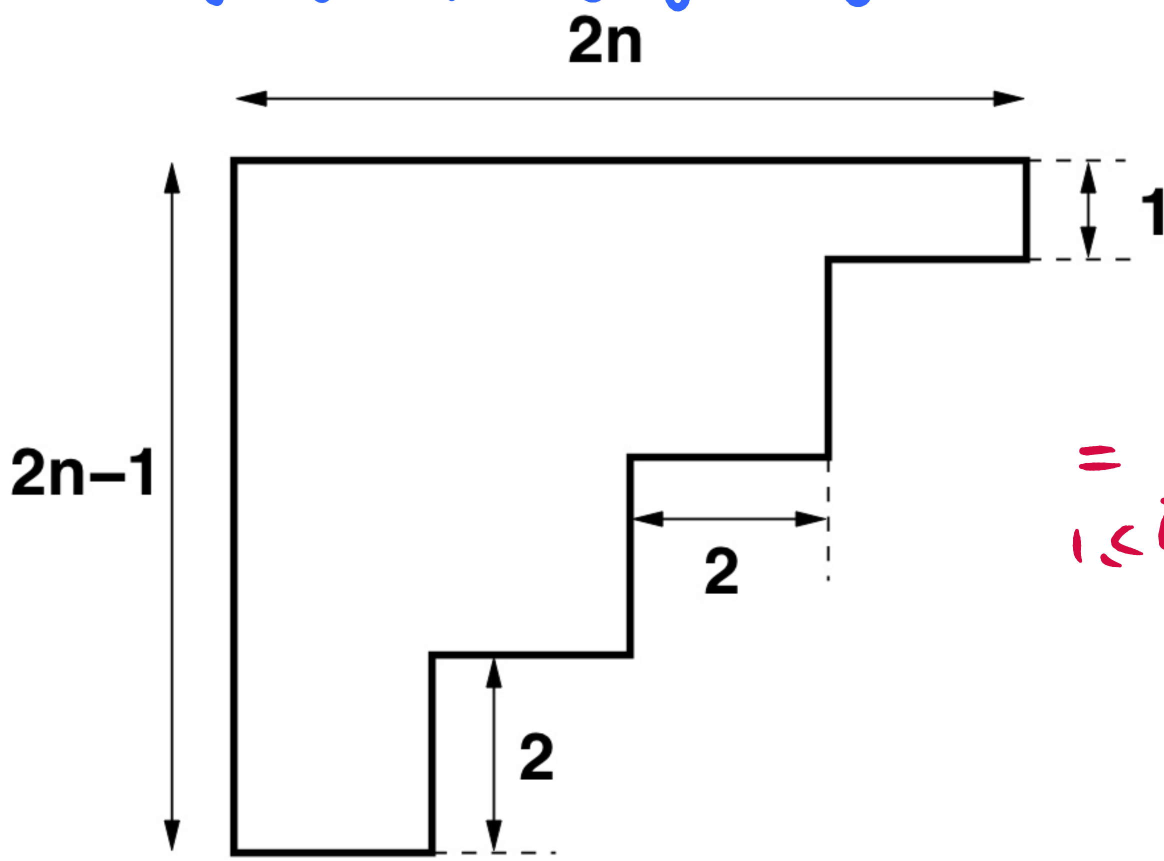
[OEIS for B_n] $\longrightarrow$ Domino Tilings of a $2n \times 2n$ square

$$= 2^n b_n^2 \qquad b_n = 1, 3, 29, 901, \ldots$$

[Temperley-Fisher 61]

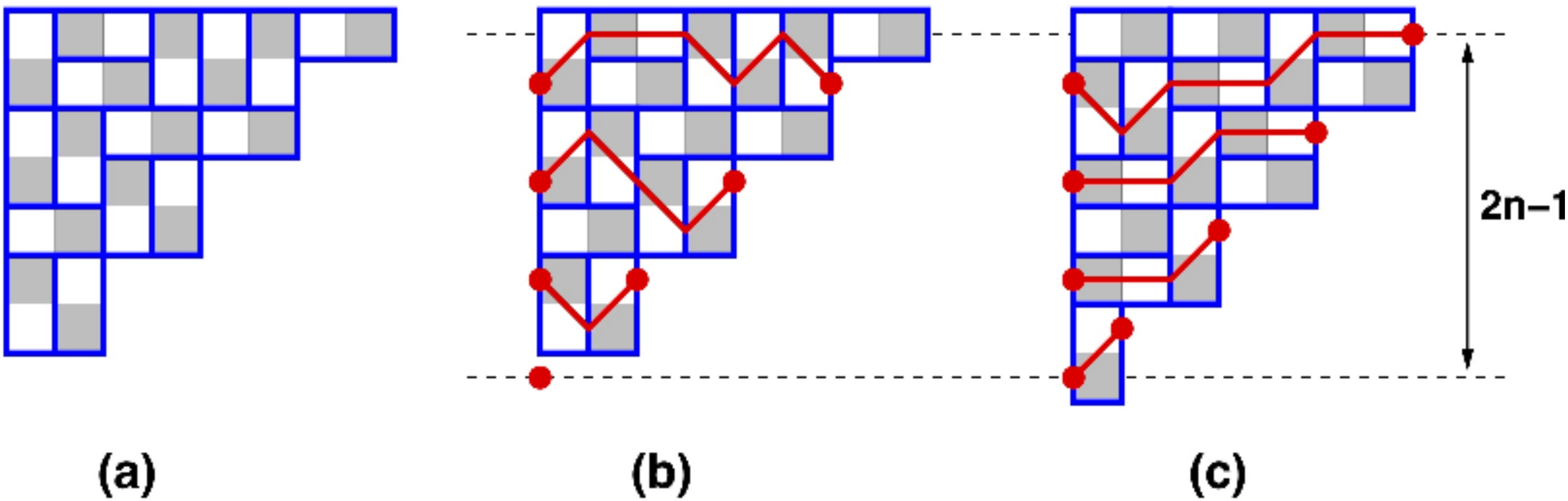
[Patcher 97] proof of integrality of b_n found a Domino Tiling interpretation

$b_n =$



$$= \prod_{1 \le i < j \le n} \left(4\cos^2 \frac{\pi i}{2n+1} + 4\cos^2 \frac{\pi j}{2n+1} \right)$$

Counting the domino tilings of Patcher's triangle

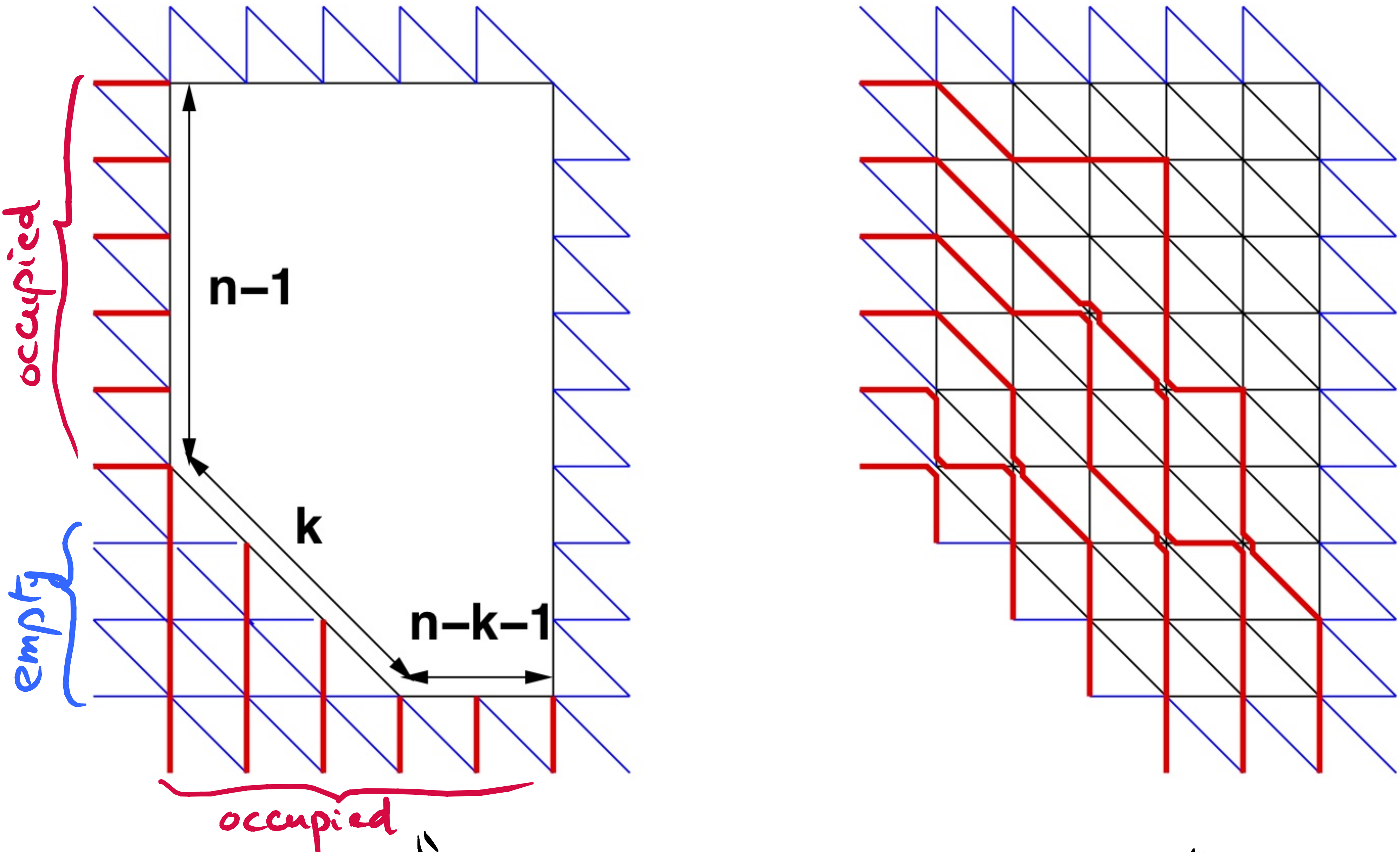


use Gessel-Viennot for Schröder paths under a roof

Result = 1, 3, 29, 901, 89893, 28793575, ...

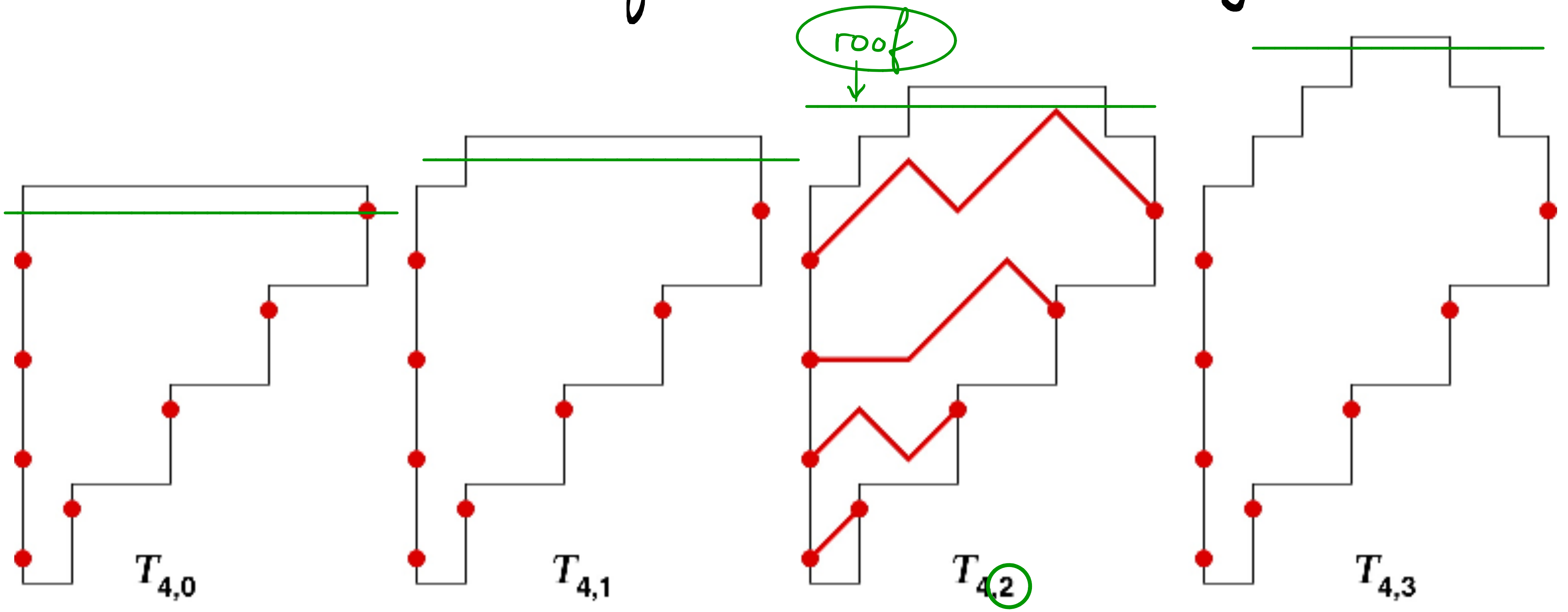
Conjecture [PDF- E.Quitter19] The configurations of the 20V-DWBC3 model on an $n \times n$ grid are counted by the Domino Tilings of Patcher's triangle

But we can do better....

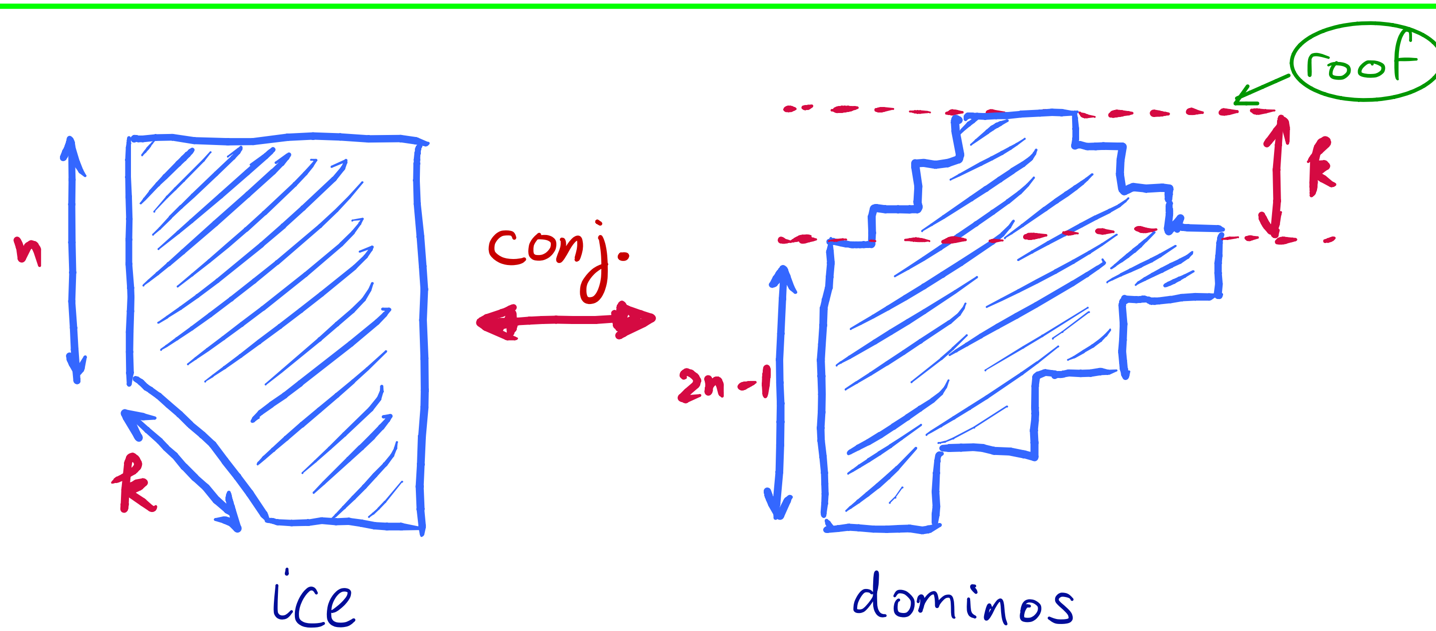


“Pentagon of triangular ice”

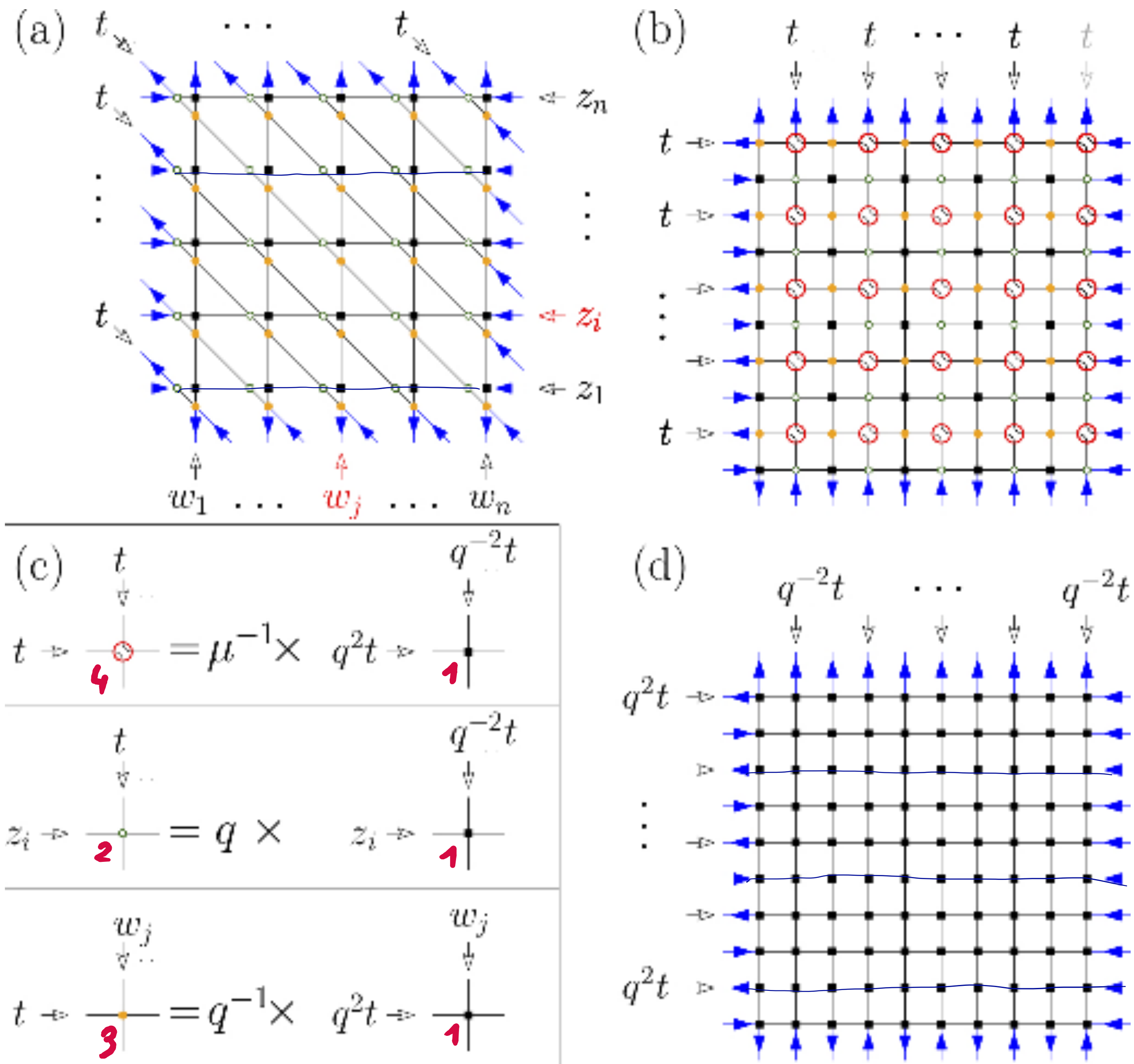
"raise the roof" above Pascal's triangle



Conjecture [PDF+E. Guitter 19] The number of configurations of triangular ice in a pentagon w/ DWBC3 is equal to that of domino tilings of Patcher's raised triangle



TRANSFORMATION TO STAGGERED 6V



(a) 20V in Kagome version

(b) move the diags to form kissing pts

(c) change the spectral parameters to produce 6V weight

(d) 6V model w staggered BC and weights:

$(\sqrt{2}, 1, 1) \to$ lattice 1

$(1, \sqrt{2}, 1) \to$ lattice 2

$(1, \sqrt{2}, 1) \to$ lattice 3

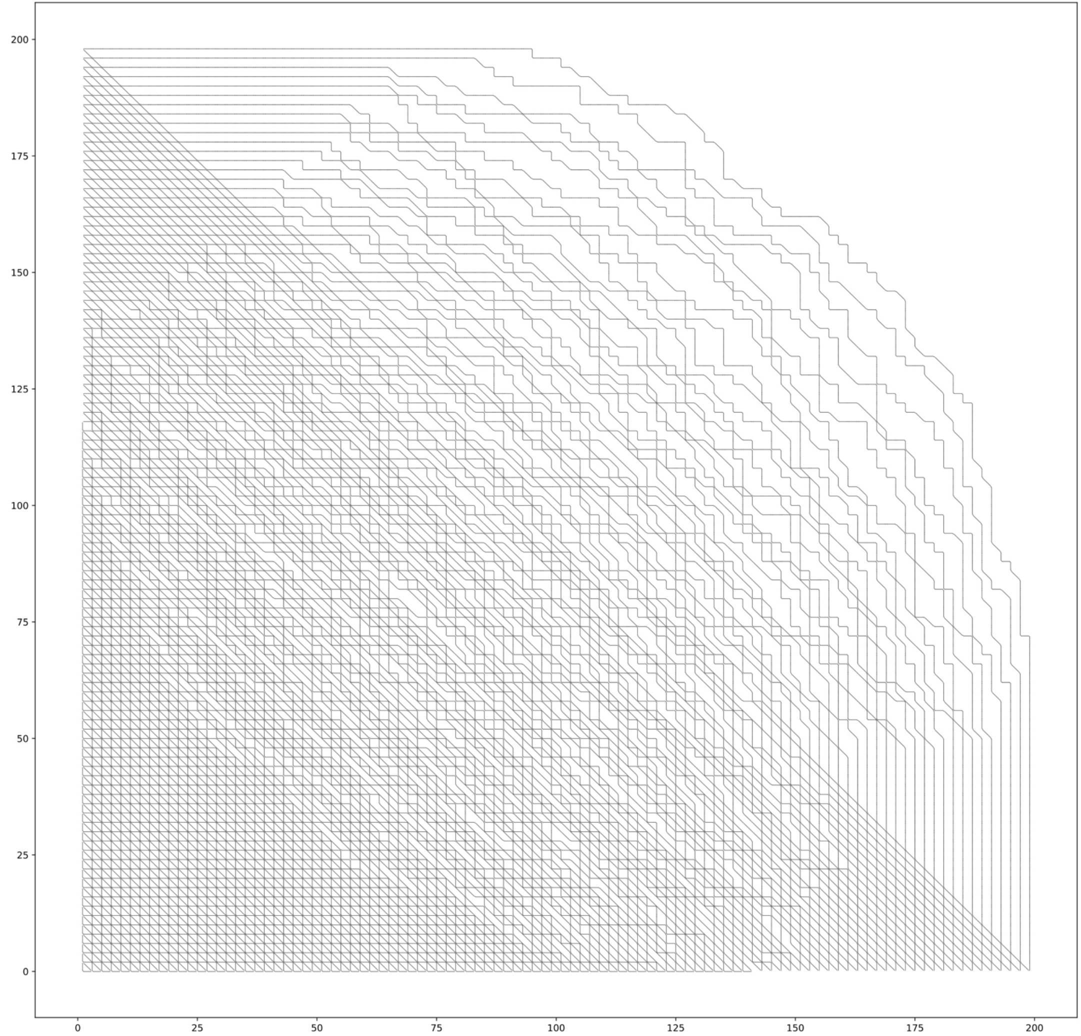
$(1, 0, 1) \to$ lattice 4

6. LIMIT SHAPE : THE ARCTIC PHENOMENON

- large size N ; typical configuration exhibits "frozen" domains / "liquid" domains

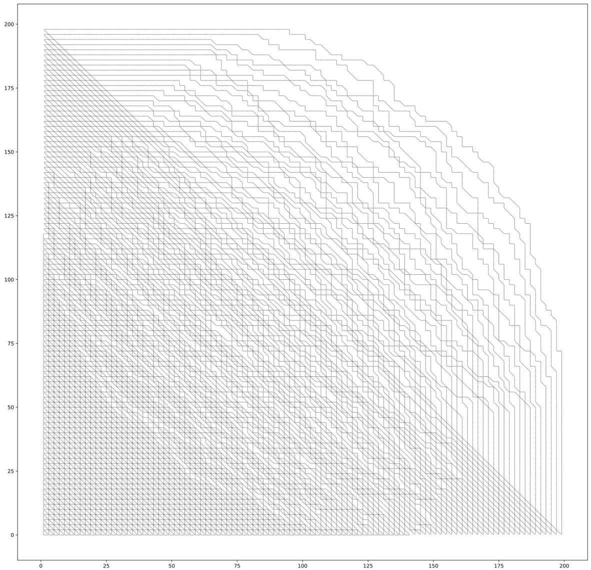
"frozen" domains → regularly ordered paths

"liquid" domains → disordered paths

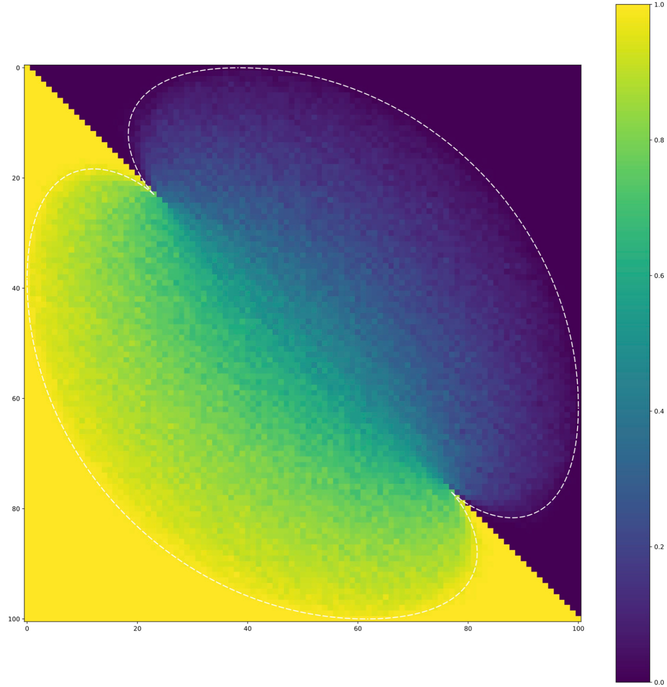


DWBC1
uniform
weights
N = 200

20V-DWBC1 uniform weights

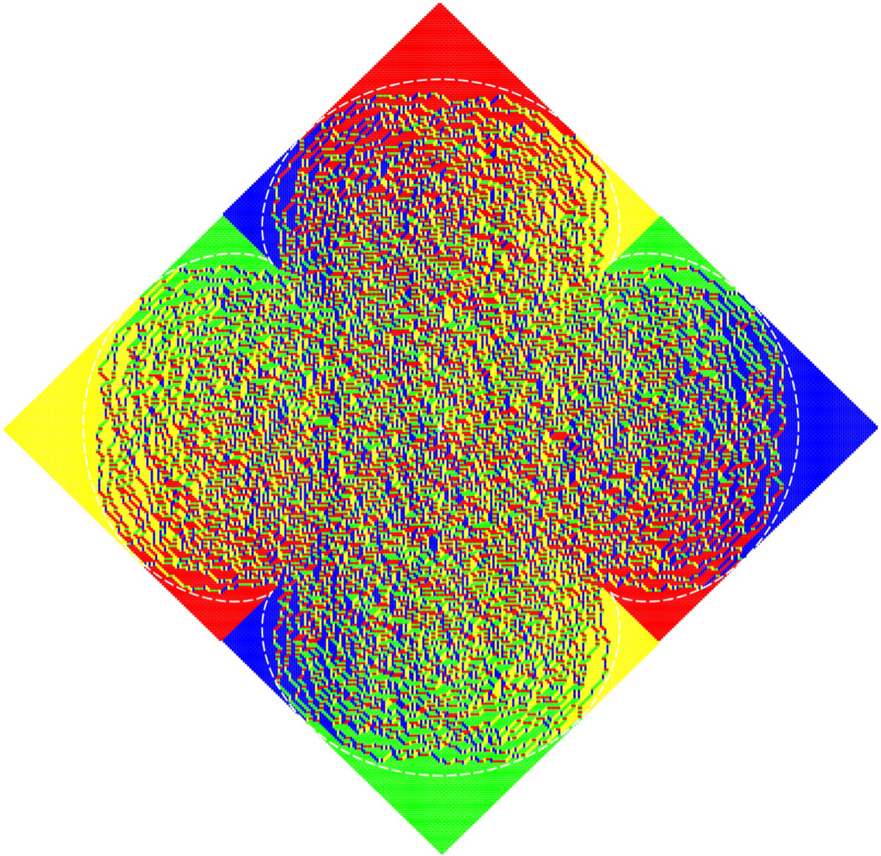


N=200

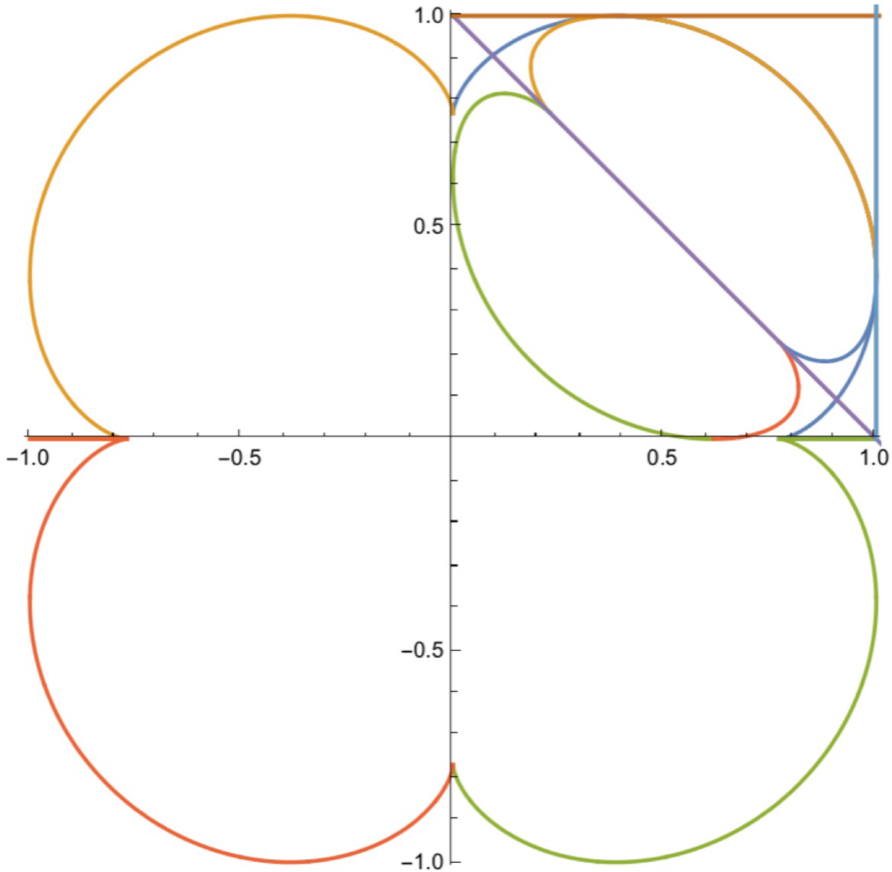


N=100

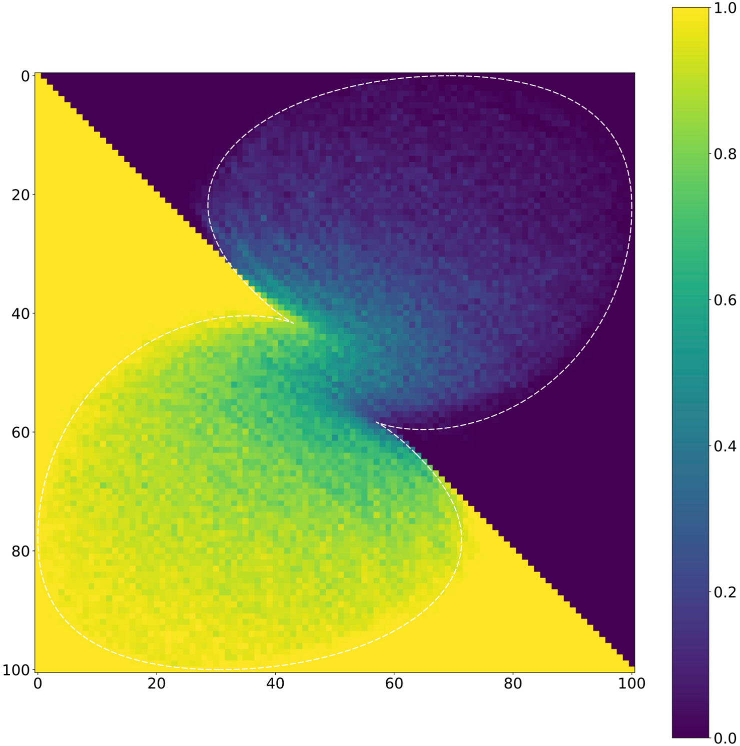
Holey Aztec square domino tilings (uniform weights)



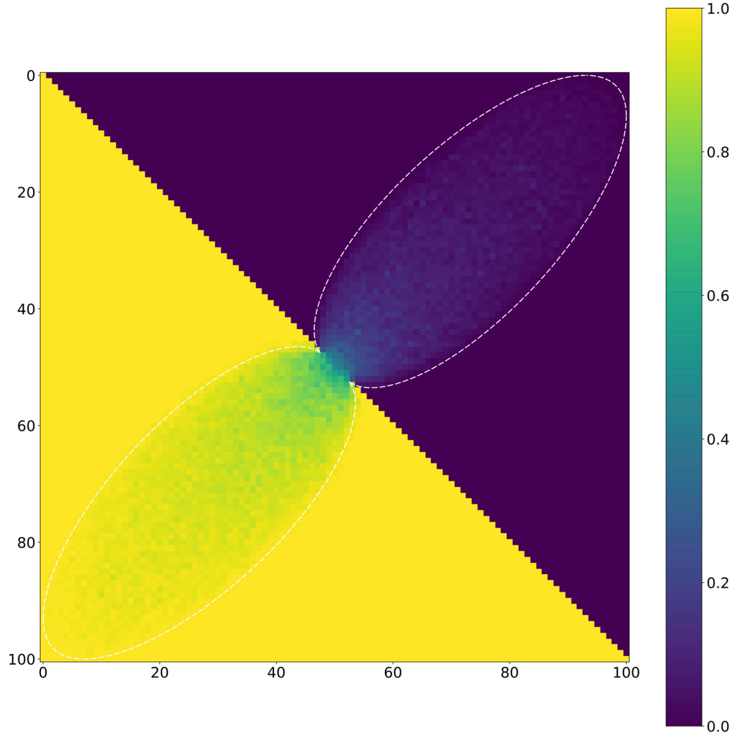
APM - holey Aztec Domino Tiling



20V-DWBC1 – Non-uniform integrable weights



N=100

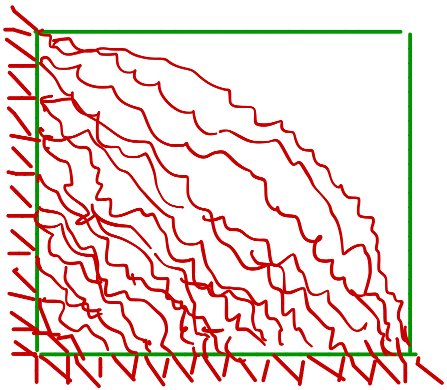


N=100

ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration

→ use "tangent method"

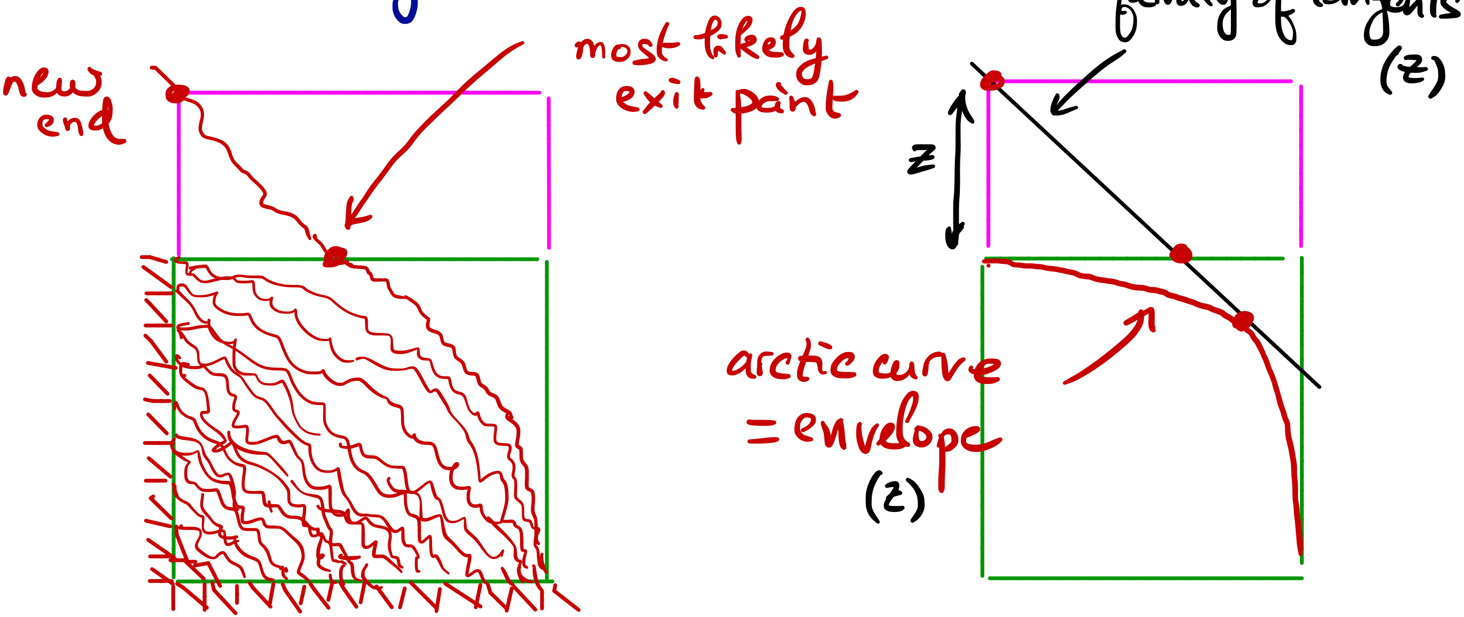


- modify last path entry point
- use this new path as probe for the limit shape

ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration

→ use "tangent method"



7. CONCLUSION

- Triangular ice does have interesting combinatorics!
- APM : they're new. Are they useful? Symmetries?
- Staggered 6V model : study it!
- Arctic Phenomenon DWBC1/2 have one!

→ use tangent method

→ use refinements and connection to 6V) Analytic predictions

[Debin, PDF, Guitter, in progress]

Ref. [P. Di Francesco and E. Guitter "20-Vertex model with Domain Wall boundaries and domino tilings", ArXiv 1905.12387 math.CO]