Strong positivity for quantum cluster algebras

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Edinburgh

Algèbres amassées

Combinatorial input

Let Q be a quiver (e.g. $s, t : Q_0 \to Q_1$) without loops or 2-cycles(determined by signed adjacency matrix $B_{ij} = \#\{i \to j\} - \#\{j \to i\}$), and $\{n+1, \ldots, m\} =\subset Q_0 = \{1, \ldots, m\}$ a subset of *frozen* vertices. Let $L = \mathbb{Z}^m = \mathbb{Z} \cdot Q_0$.

- Initial seed defined to be $(y_1 := y^{e_1}, \ldots, y_m = y^{e_m}) \subset \mathbb{Z}[L]$.
- For i unfrozen, mutate via

$$\mu_i(y_j) = \begin{cases} y_j & \text{if } i \neq j \\ y_i + y^{\sum_{i \to k} e_k - \sum_{k \to i} e_k + e_i} & \text{if } i = j \end{cases}$$

  and substitution rule $y_i' = y^{-e_i + \sum_{k \to i} e_k}$
- $\to$ usual mutation rule for Q.
- $\mathcal{A}_Q$ is the algebra genrated by all *cluster monomials* $\mu_{\mathbf{s}}(y^{\mathbf{v}})$ for s a sequence of the unfrozen vertices, and $\mathbf{v}_i \geqslant 0$ for $i \leqslant n$.

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Algèbres amassées, le palmarès

- Laurent phenomenon (Fomin and Zelevinsky): This definition makes sense. e.g. for $\mathbf{s}$ a sequence of unfrozen vertices $\mu_{\mathbf{s}}(y^{\mathbf{v}}) = \sum_{\mathbf{w}\in L} c_{\mathbf{v},\mathbf{w}} y^{\mathbf{w}}$ for constants $c_{\mathbf{v},\mathbf{w}} \in \mathbb{Z}$
- Linear independence (Cerullli–Irelli, Keller, Labardini, Plamondon): The cluster monomials are linearly independent
- Positivity (Lee, Schiffler): The constants $c_{\mathbf{v},\mathbf{w}} \in \mathbb{Z}$ are positive
- Strong positivity (Gross, Hacking, Keel, Kontsevich) $\mathcal{A}_Q \subset \mathcal{A}_Q^{can}$, where $\mathcal{A}_Q^{can}$ has a basis of *theta functions* $\{\vartheta_{\mathbf{p}}\}_{\mathbf{p}\in\Theta} \subset \mathbb{Z}[L]$ indexed by $\Theta \subset L$, containing all of the cluster monomials. The structure constants w.r.t. this basis are positive.

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Bond quantique

How to quantize? Set $\beta_i = \sum_{i\to k} e_k - \sum_{k\to i} e_k = Be_i$. Then

$$\mu_i(y^{\mathbf{v}}) = \sum_{r=0}^{\mathbf{v}_i} \binom{\mathbf{v}_i}{r} y^{\mathbf{v}+r\beta_i}$$

Set

- $(n)_t = (t^n - t^{-n})/(t^1 - t^{-1})$,
- $(n)!_t := (1)_t \cdots (n)_t$,
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q-commutativity

How about setting $\mu_i(y^{\mathbf{v}}) = \sum_{r=0}^{\mathbf{v}_i} \binom{\mathbf{v}_i}{r}_t y^{\mathbf{v}+r\beta_i}$? Not obviously a homomorphism Solution: find a skew-symmetric form on $\mathbb{Z}^{Q_0}$ such that $\Lambda(Be_i, Be_j) = B(e_i, e_j)$ for unfrozen i (Berenstein+Zelevinsky call this a *compatible* form) and decree $\mathbf{y}^{\mathbf{v}}\mathbf{y}^{\mathbf{v}'} = t^{\Lambda(\mathbf{v},\mathbf{v}')}\mathbf{y}^{\mathbf{v}+\mathbf{v}'}$

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- $(n)_t = (t^n - t^{-n})/(t^1 - t^{-1})$,
- $(n)!_t := (1)_t \cdots (n)_t$,
- $\binom{n}{m}_t = (n)!_t / \left((m)!_t (n-m)!_t\right)$

q-commutativity

How about setting $\mu_i(y^{\mathbf{v}}) = \sum_{r=0}^{\mathbf{v}_i} \binom{\mathbf{v}_i}{r}_t y^{\mathbf{v}+r\beta_i}$? Not obviously a homomorphism Solution: find a skew-symmetric form on $\mathbb{Z}^{Q_0}$ such that $\Lambda(Be_i, Be_j) = B(e_i, e_j)$ for unfrozen i (Berenstein+Zelevinsky call this a *compatible* form) and decree $\mathbf{y}^{\mathbf{v}}\mathbf{y}^{\mathbf{v}'} = t^{\Lambda(\mathbf{v},\mathbf{v}')}\mathbf{y}^{\mathbf{v}+\mathbf{v}'}$

Quantum version

How to quantize? Set $\beta_i = \sum_{i\to k} e_k - \sum_{k\to i} e_k = Be_i$. Then

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L'algèbre amassée quantique

The ambient ring

Given a lattice L with skew-symmetric form ω we define $\mathbb{Z}_{\omega,t}[L]$, the free $\mathbb{Z}[t^{\pm 1}]$-module with

- generators $\mathbf{y}^{\mathbf{v}}$ for $\mathbf{v} \in L$,
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The **quantum cluster algebra** $\mathcal{A}_\Lambda \subset \mathbb{Z}_{\Lambda,t}[L]$ is the algebra generated by all the quantum cluster monomials ($\mu_{\mathbf{s}}(\mathbf{y}^{\mathbf{v}})$ s.t. $\mathbf{v}_i \geq 0$ for $i \leq n$) as defined before. E.g. mutate via

$$\mu_i(\mathbf{y}_j) = \begin{cases} \mathbf{y}_j & \text{if } i \neq j \\ \mathbf{y}_i + \mathbf{y}^{\sum_{i\to k} e_k - \sum_{k\to i} e_k + e_i} & \text{if } i = j \end{cases}$$

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Palmarès quantique

- Quantum Laurent phenomenon (Berenstein+Zelevinsky): This definition makes sense e.g. for $\mathbf{s}$ a sequence of unfrozen vertices, $\mu_{\mathbf{s}}(\mathbf{y}^{\mathbf{v}}) = \sum_{\mathbf{w} \in L} c_{\mathbf{v},\mathbf{w}}(t)\mathbf{y}^{\mathbf{w}}$ with $c_{\mathbf{v},\mathbf{w}}(t) \in \mathbb{Z}[t^{\pm 1}]$ if $\mathbf{v}_i \geqslant 0$ for all unfrozen i (e.g. $i \leqslant n$).
- Quantum positivity (D): In the above expressions, all the $c_{\mathbf{v},\mathbf{w}}(t)$ have positive coefficients, and are moreover Lefschetz.

Aside: Lefschetz type polynomials

A sum of polynomials of the form $(n)_t$ is called Lefschetz. So called because the Poincaré polynomial of a smooth projective variety is of Lefschetz type thanks to the hard Lefschetz theorem.

Quantum cluster algebras: main results

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Plethysme

Plethystic exponentials

Let $f(t, z_1, \ldots, z_r) = \sum_{(n,\mathbf{v})\in\mathbb{Z}\times\mathbb{N}^r} f_{n,\mathbf{v}} \cdot (-t)^n z^{\mathbf{v}} \in \mathbb{Z}((t))[[z_1, \ldots, z_r]]$.
Then

$$\mathrm{Exp}(f(t, z_1, \ldots, z_r)) := \prod_{(n,\mathbf{v})\in\mathbb{Z}\times\mathbb{N}^r} (1 - (-t)^n z^{\mathbf{v}})^{-f_{n,\mathbf{v}}}.$$

$$\mathbb{E}(f(t, z_1, \ldots, z_r)) := \mathrm{Exp}(f(t, z_1, \ldots, z_r) t(1 - t^2)^{-1}).$$

Let V be a $\mathbb{Z} \times \mathbb{N}^r$-graded vector space with $\chi_t(V) \in \mathbb{Z}((t))[[z_1, \ldots, z_r]]_+$.
Then

- $\mathrm{Exp}(\chi_t(V)) = \chi_t(\mathrm{Sym}(V))$: "Plethystic exponential is decategorification of symmetric algebra"
- Let $V = \bigoplus_{n\geq 0} H(\mathrm{pt} / \mathrm{Gl}_n)[-n^2]$. Then $\chi_t(V) = \mathbb{E}(z) \in \mathbb{Z}((t))[[z]]$.
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Very different!: $\mathrm{Exp}(tz) = 1 + tz$, and $\mathrm{Exp}(z) = 1 + z + z^2 + \ldots$ These are the *DT invariants* of the zero/one loop quiver. ("fermionic" vs "bosonic")

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Very different!: $\mathrm{Exp}(tz) = 1 + tz$, and $\mathrm{Exp}(z) = 1 + z + z^2 + \ldots$ These are the *DT invariants* of the zero/one loop quiver. ("fermionic" vs "bosonic")

Plethysm

Plethystic exponentials

Let $f(t, z_1, \ldots, z_r) = \sum_{(n,\mathbf{v}) \in \mathbb{Z} \times \mathbb{N}^r} f_{n,\mathbf{v}} \cdot (-t)^n z^{\mathbf{v}} \in \mathbb{Z}((t))[[z_1, \ldots, z_r]]$. Then

$$\mathrm{Exp}(f(t, z_1, \ldots, z_r)) := \prod_{(n,\mathbf{v}) \in \mathbb{Z} \times \mathbb{N}^r} (1 - (-t)^n z^{\mathbf{v}})^{-f_{n,\mathbf{v}}}.$$

$$\mathbb{E}(f(t, z_1, \ldots, z_r)) := \mathrm{Exp}(f(t, z_1, \ldots, z_r) t (1 - t^2)^{-1}).$$

Let V be a $\mathbb{Z} \times \mathbb{N}^r$-graded vector space with $\chi_t(V) \in \mathbb{Z}((t))[[z_1, \ldots, z_r]]_+$. Then

- $\mathrm{Exp}(\chi_t(V)) = \chi_t(\mathrm{Sym}(V))$: "Plethystic exponential is decategorification of symmetric algebra"
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Quantum $\mathcal{X}$ space

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$$\mathbf{x}^\gamma \mapsto \mathbf{y}^{B\gamma}$$

defines a homomorphism $\iota\colon \mathbb{Z}_{B,t}[K] \to \mathbb{Z}_{\Lambda,t}[L]$

Easy calculation: if $i \neq j$ $\iota\,\mathbb{E}(\mathbf{x}_i)$ and $\mathbf{y}_i$ commute, otherwise

$$\begin{aligned}\mathrm{Ad}_{\iota\,\mathbb{E}(\mathbf{x}_i)^{-1}}(\mathbf{y}_i) =& \iota\,\mathbb{E}(\mathbf{x}_i)^{-1}\mathbf{y}_i\iota\,\mathbb{E}(\mathbf{x}_i)\\ =& \mathbf{y}_i\,\mathbb{E}((1-t^2)\mathbf{x}_i) = \mathbf{y}_i + \mathbf{y}^{e_i+Be_i}\end{aligned}$$

E.g. cluster mutation is effected by letting $\mathcal{X}$ coordinates act on $\mathcal{A}$ coordinates via conjugation.

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Positivité: chemin dur

The old proof

- From what we saw before, $\mathbb{E}(\mathbf{x}_i) = \chi_t(\mathsf{H}(\mathsf{Rep}_{\mathbb{N}e_i} Q, \mathbb{Q})_{\mathsf{vir}})$.
- For iterated mutation along **s**, Nagao showed there is a stack $\mathcal{T}_{\mathbf{s}}$ of $\mathrm{Jac}(Q, W)$-reps such that $\mathsf{Ad}_{\chi_{\mathrm{wt}}(\mathsf{H}(\mathcal{T}_{\mathbf{s}}))}$ recreates cluster mutation.
- **Cohomological wall crossing**(CWC) shows that this cohomology is pure, so we can take χ_t instead of strange χ_{wt} (e.g. categorify)
- Similarly CWC shows that expression $\mathsf{Ad}_{\chi_{K,\mathrm{wt}}(\mathsf{H}(\mathcal{T}_{\mathbf{s}}))}(\mathbf{y}^{\mathbf{v}})$ can be categorified → positivity (+ some Hodge theory) → Lefschetz type.

The take-away: Categorification is our friend, the rest of the proof maybe less so.

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Contre les murs

Write $L_+ = L \cap B(\mathbb{N}^n)_{\mathbb{R}}$. $\mathbf{v} \in L_+$ primitive, $\mathbf{p}$ a positive multiple of $\mathbf{v}$. Define an automorphism of $\mathbb{Z}[L]$

$$l_{\mathbf{p}} : y^{\mathbf{v}'} \mapsto y^{\mathbf{v}'}(1 + y^{\mathbf{p}})^{\Lambda(\mathbf{p},\mathbf{v}')}$$

define $G_{\mathbf{v}}^{\text{class}}$ to be group generated by all such automorphisms.

Definition

A (classical) **wall** $(\mathfrak{d}, f)$ is a $(n-1)$-dimensional rational polyhedral cone $\mathfrak{d}$ in $L_{\mathbb{R}}$ parallel to $\mathbf{v}^{\Lambda\perp}$ for some $\mathbf{v} \in L_+ \setminus \mathrm{Ker}(\Lambda)$, along with a $f \in G_{\mathbf{v}}^{\text{class}}$. The wall is called **incoming** if closed under adding $\mathbf{v}$.

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Scattering diagrams

Definition

A **scattering diagram** $\mathfrak{D}$ is a union of walls $(\mathfrak{d}_a, f_a)$ in $L_{\mathbb{R}}$ such that $(\forall N$ only finitely many functions of order N and below)

- **Joints**$(\mathfrak{D}) = \left(\bigcup_{a \neq a'} \mathfrak{d}_a \cap \mathfrak{d}_{a'}\right) \cup \left(\bigcup_a \delta \mathfrak{d}_a\right)$
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Droites brisés

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Let $\mathcal{Q} \in L_{\mathbb{R}} \backslash \mathfrak{D}$. A broken line with ends $(\mathbf{p}, \mathcal{Q})$ is a piecewise linear path $(-\infty, 0] \to L_{\mathbb{R}}$ avoiding joints, meeting walls $w_1, \ldots, w_r$ at $t_1, \ldots, t_r$ with each piecewise linear section labelled by a monomial $c_i y^{\mathbf{v}_i}$ such that

1. $\gamma(0) = \mathcal{Q}$
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Remarks

- As long as only functions of the form $f_{\mathbf{v}}$ (not their inverses) appear on the walls, all resulting monomials are positive.
- (GHKK) Cluster monomials and structure constants are given by sums of broken lines
- Easy to quantize: replace $f_{\mathbf{p}}$ with $\mathrm{Ad}_{\iota\,\mathbb{E}(x^{\mathbf{p}})}$

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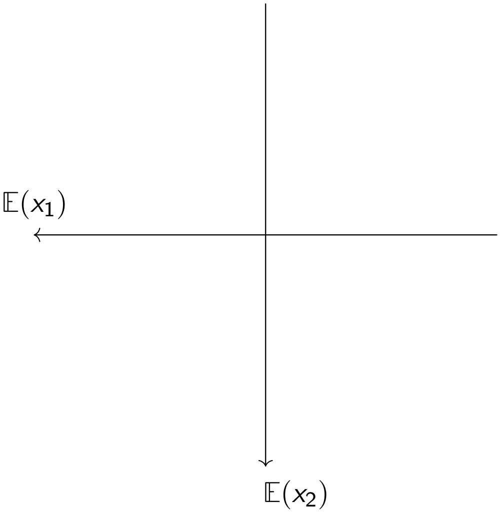
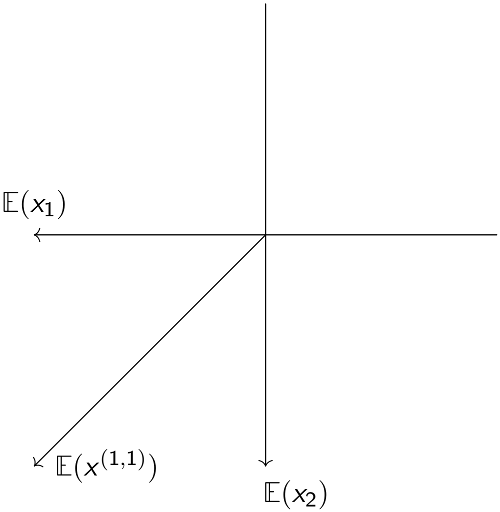
Exemple facile

$L = \mathbb{Z}^2$, $\Lambda(e_1, e_2) = 1$. Start with *inconsistent* scattering diagram



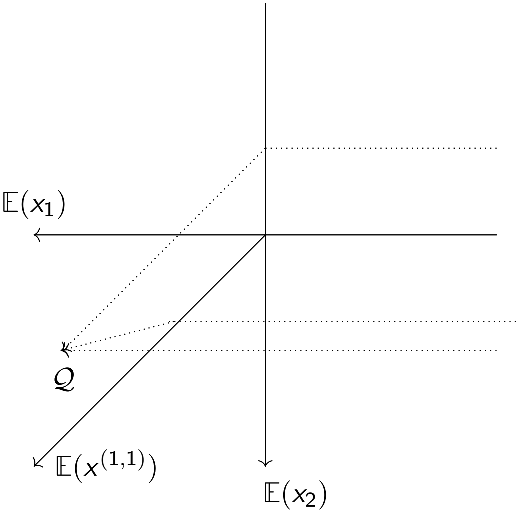
Easy example

$L = \mathbb{Z}^2$, $\Lambda(e_1, e_2) = 1$. Add walls to make it consistent



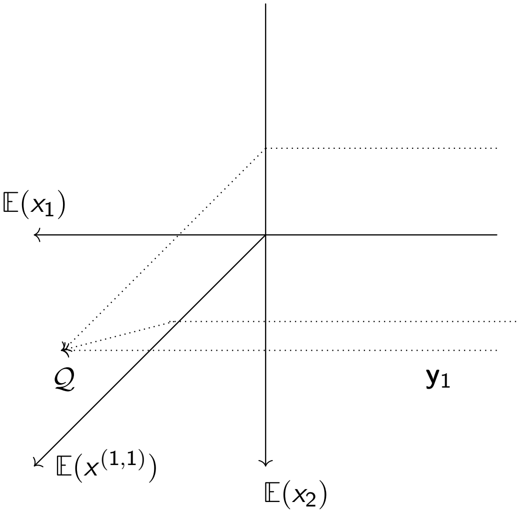
Easy example

$L = \mathbb{Z}^2$, $\Lambda(e_1, e_2) = 1$. Count broken lines with ends $(1,0)$, $\mathcal{Q}$



Easy example

$L = \mathbb{Z}^2$, $\Lambda(e_1, e_2) = 1$. Add up contributions from broken lines



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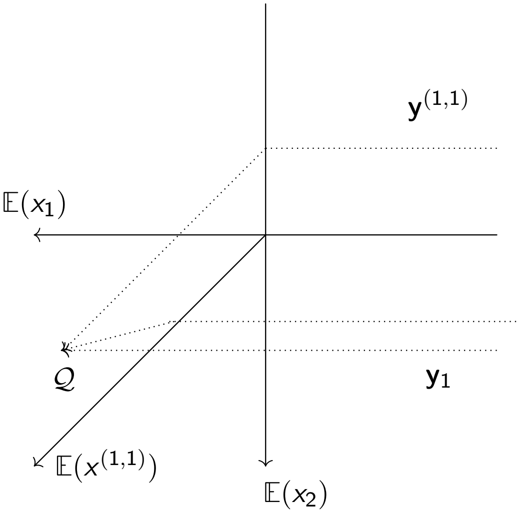

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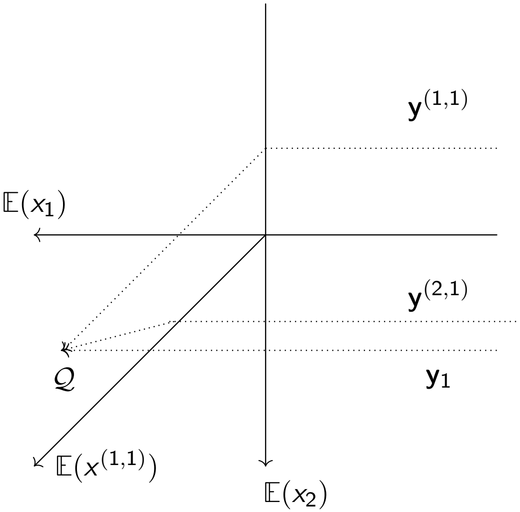

Positivitè de base theta

Via perturbations of scattering diagrams and recursive arguments, we can reduce the construction of consistent scattering diagrams to the two wall case with $\mathbb{E}(t^{\alpha_1}\mathsf{x}_1)$ and $\mathbb{E}(t^{\alpha_2}\mathsf{x}_2)$ on the walls with $\alpha_i \in \{0, t-1\}$, and $\Lambda(e_1, e_2) = n \in \mathbb{N}$. These examples get pretty hard to calculate...

- Let Q be the quiver with vertices $\{1, 2\}$, $1 + \alpha_i$ loops at each i, and n arrows from 1 to 2. In general, the problem of what goes on the walls comes down to factorizing

$$\mathbb{E}(t^{\alpha_2}\mathsf{x}_2)\,\mathbb{E}(t^{\alpha_1}\mathsf{x}_1) = \mathbb{E}(t^{\alpha_1}\mathsf{x}_1)\left(\prod_{\infty \xrightarrow{a/b} 0} \mathbb{E}(f(\mathsf{x}^{(a,b)}, t))\right)\mathbb{E}(t^{\alpha_2}\mathsf{x}_2)$$

  in $\mathbb{Z}_{B,t}[\mathbb{N}^2]$.
- The wall crossing formula plus CWC plus earlier caclulations for zero/one loop quiver tell us that

$$\mathbb{E}(f(\mathsf{x}^{(a,b)}, t)) = \chi\Big(\bigoplus_{n \geqslant 0} \mathrm{H}(\mathrm{Rep}^{\mathrm{sst}}_{(na,nb)}\, Q, \mathbb{Q})_{\mathrm{vir}}\Big)$$

Positivity for quantum theta functions

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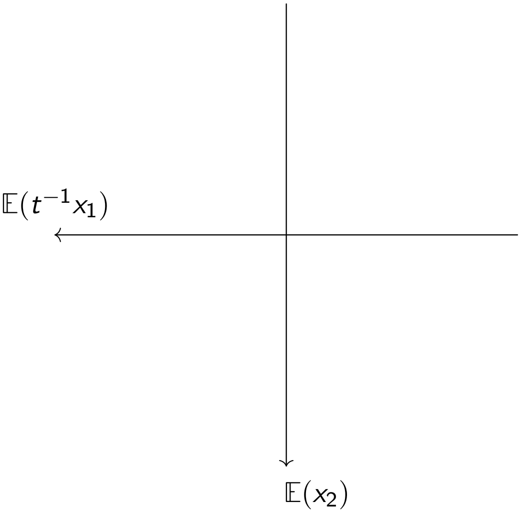
$\Lambda(e_1, e_2) = 1$; inconsistent



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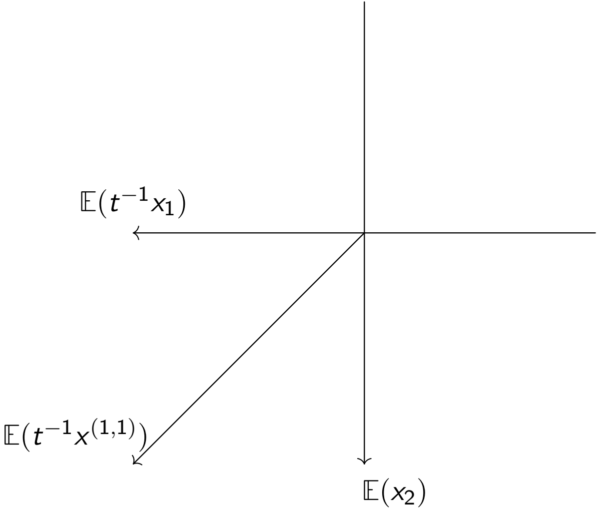

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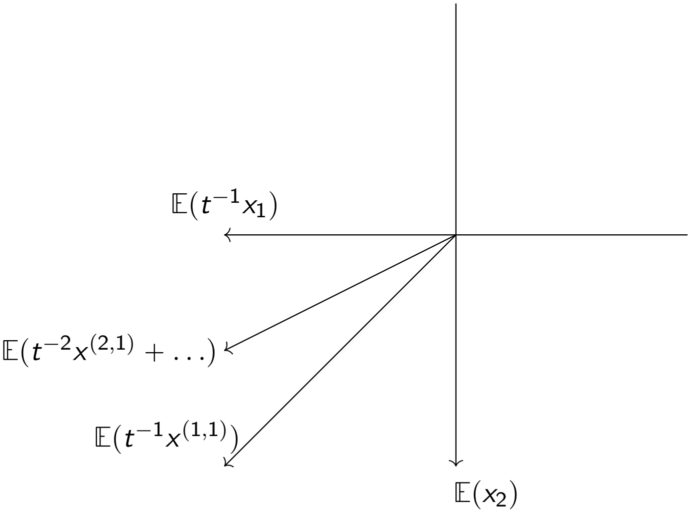
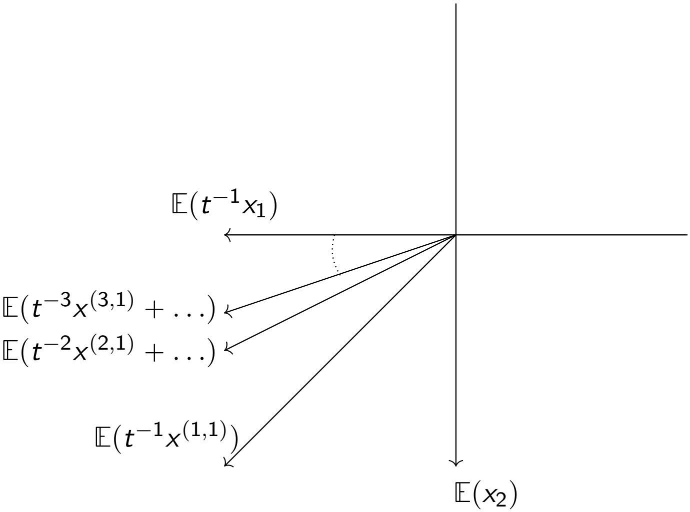


Positivity for quantum theta functions

$\Lambda(e_1, e_2) = 1$; after infinitely many steps... consistent but infinite



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- Integrality theorem (-,Meinhardt): $RHS = \mathbb{E}(\chi(\mathcal{BPS}_{a/b}))$ is manifestly positive.

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$$\mathbb{E}(f(\mathsf{x}^{(a,b)},t)) = \chi_{wt}\Big(\bigoplus_{n\geqslant 0} \mathrm{H}(\mathrm{Rep}^{\mathrm{sst}}_{(na,nb)}\,Q,\mathbb{Q})_{\mathrm{vir}}\Big)$$

- Integrality theorem (-,Meinhardt): $RHS = \mathbb{E}(\chi(\mathcal{BPS}_{a/b}))$ is manifestly positive.

Positivity for quantum theta functions

- Let Q be the quiver with vertices $\{1,2\}$, $1+\alpha_i$ loops at each i, and n arrows from 1 to 2. In general, the problem of what goes on the walls comes down to factorizing

$$\mathbb{E}(t^{\alpha_2}\mathsf{x}_2)\,\mathbb{E}(t^{\alpha_1}\mathsf{x}_1) = \mathbb{E}(t^{\alpha_1}\mathsf{x}_1)\left(\prod_{\infty \xrightarrow{a/b} 0} \mathbb{E}(f(\mathsf{x}^{(a,b)},t))\right)\mathbb{E}(t^{\alpha_2}\mathsf{x}_2)$$

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Énoncé principal (avec T. Mandel)

Theorem

There is a subset $\Theta \subset L$ and quantum theta functions $\{\vartheta_{\mathbf{p}}\}_{\mathbf{p}\in\Theta} \subset \mathbb{Z}_{\Lambda,t}[L]$ such that

- *Each $\vartheta_{\mathbf{p}}$ can be written*

$$\vartheta_{\mathbf{p}} = \mathbf{y}^{\mathbf{p}} + \sum_{\mathbf{v}\in L^+\setminus 0} c_{\mathbf{p},\mathbf{v}}(t) y^{\mathbf{p}+\mathbf{v}}$$

  with $c_{\mathbf{p},\mathbf{v}}(t) \in \mathbb{N}[t^{\pm 1}]$
- *$\mathbb{Z}[t^{\pm 1}]$-module generated by $\vartheta_{\mathbf{p}}$ is strongly positive algebra and*
- *contains all of the cluster monomials.*
- *This is a universally positive atomic basis.*

Conjecture

Each of the $c_{\mathbf{p},\mathbf{v}}(t)$ are Lefschetz

Main result (with T. Mandel)

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