Kac-Moody theory via nil-DAHA

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ABSTRACT. Classically, we express symmetric polynomials In terms of Schur ones, but this can be not optimal, especially if you need to expand theta functions and the Kac-Moody characters. And of course we need some canonical bases in all polynomials (not only symmetric)! Presumably level-1 Demazure characters (generally non-symmetric), and relatively new level-1 thick ("upper") ones are just fine. They are the key in nil-Daha theory serving the limits $t=0$ and at infinity; they provide the characters of local Weyl modules, as well as the so-called nonsymmetric global Weyl modules (E.Feigin, Kato, Macedonskiy). Furthermore, they generalize

classical q-Hermite polynomials and serve perfectly theory of Rogers-Ramanujan sums. We will connect the latter with a version of 2d TQFT with levels. Nil-DAHA is DAHA where $T_i(T_i+1)=0$; we will not use it in this particular talk.

We will begin with the refined Rogers-Ramanujan sums, where the matrix entries of the (operators of) multiplication by θ^ℓ in the basis $\{P_a\}$ are the key. These matrix elements are also some sums of the DAHA-Jones polynomials of chains of Hopf 2-links (Ch, Danilenko). When $t \to 0$, we arrive at Rogers-Ramanujan-type presentations for certain basic string functions of level ℓ (Ch, B.Feigin). Importantly, the level-rank duality for $\widehat{gl_n}$ can be seen directly from these sums. Then we will briefly discuss the nonsymmetric Rogers-Ramanujan sums and Demazure slices (Ch, Kato).

REFINED "LITTLEWOOD-RICHARDSON"

Let $R=\{\alpha\}\in\mathbb{R}^n$ be a simple root system, $(\cdot,\cdot)$ the corresponding inner product normalized by $(\alpha_{\text{sht}},\alpha_{\text{sht}})=2$, $\{\alpha_i\}$ simple roots, $W=\langle s_i=s_{\alpha_i}\rangle=\langle s_\alpha\rangle$ the Weyl group, $\rho_k=(1/2)\sum_{\alpha>0}k_\alpha\alpha$, $P=\oplus_i\mathbb{Z}\omega_i$ the weight lattice (for fundamental ω_i), $P_+=\oplus\mathbb{Z}_+\omega_i$, $Q=\sum_\alpha\mathbb{Z}\alpha, Q_+=\sum_{\alpha>0}\mathbb{Z}_+\alpha$. We set $\mathbb{C}[X_a]=\mathbb{C}[X^{\pm1}_{\omega_i}]$, where $X_{a+b}=X_aX_b$ for $a,b\in P$, $w(X_a)=X_{w(a)}$ for $w\in W$, $\mathbb{C}[X]^W=\{F\in\mathbb{C}[X_a],w(F)=F\}$, $\langle F\rangle$ the constant term of Laurent series F, $X^\iota_a=X_{\iota(a)}$, where $\iota(a)=-w_0(a)$ for the longest element $w_0\in W$.

Let $\theta_u(X)\ \stackrel{\text{def}}{=\!=}\ \sum_{a\in P}u(a)q^{(a,a)/2}X_a,\theta=\theta_{triv}$ for characters $u:P/Q\to\mathbb{C}^*$, playing the role of the classical theta- characteristics **(necessary in the level-rank duality for R of type A). Also: $\theta^{(\ell)}_{\mathbf{u}}=\theta_{u_1}\cdots\theta_{u_\ell}$ for $\mathbf{u}=\{u_1,\ldots,u_\ell\},\ell\geq0$. Given a system of orthogonal polynomials $\{P_a,a\in P_+\}$ linearly generating $\mathbb{C}[X]^W$, the problem is to calculate/interpret $\widetilde{P}_a\widetilde{P}_b=\sum_c\mathbb{C}^{c\mathbf{u}}_{ab}\widetilde{P}_c$ for $\widetilde{P}_a\stackrel{\text{def}}{=\!=}P_a\theta^{(\ell)}_{\mathbf{u}}$, $a,b\in P_+$.**

TOPOLOGICAL VERTEX ALGEBRAICALLY

Assuming $\langle P_a P_b^\iota \mu\rangle = \delta_{ab} C_a$ for a measure μ (a Laurent series), $\mathbb{C}_{ab}^{c\mathbf{u}} = \langle P_a P_b P_c^\iota \theta_{\mathbf{u}}^{(\ell)} \mu\rangle / C_c$. When $\ell = 0$, this is essentially the setting of $2d\ TQFT$ ($\simeq$ commutative finite-dimensional Frobenius algebras), though ∞-dimensional and with ι. The "associativity" of $\mathbb{C}_{ab}^{c}$ is in most theories related to that for proper bordisms (classically, pairs of pants); it is granted for any (Laurent series) θ, but only theories with sufficiently simple 2-point functions $\mathbb{C}_{0b}^{cu}$ are expected interesting (they are δ_{bc} for $\ell = 0$ as $P_0 = 1$). This is what DAHA and Macdonald polynomials provide (at least) for products of θ-functions.

Quite a few theorems/conjectures connect $\mathbb{C}_{ab}^{c}$ for proper orthogonal polynomials (mostly Macdonald-type ones) with open Gromov-Witten invariants counting holomorphic maps from bordered Riemann surfaces to "reasonable" CY 3-folds with boundary in 3 specific Lagrangian submanifolds (like $\mathbb{C}^3$, various conifolds, toric CY).

TWO-POINT FUNCTIONS

Let c_+ **be such that** $c_+\in W(c)\cap P_+$**. Given** $b\in P_+$**, let** $b\neq c_+\in b-Q_+$**,** $P_b-\sum_{a\in W(b)}X_a\in\oplus_c\mathbb{C}X_c$, $\langle\, P_b X_{c^\iota}\,\mu(X;q,t)\,\rangle=0$ **for such** c**, where** $\mu(X;q,t)\stackrel{\mathbf{def}}{=\!=}\prod_{\alpha\in R_+}\prod_{j=0}^{\infty}\frac{(1-X_\alpha q_\alpha^j)(1-X_\alpha^{-1}q_\alpha^{j+1})}{(1-X_\alpha t_\alpha q_\alpha^j)(1-X_\alpha^{-1}t_\alpha q_\alpha^{j+1})}$ **, considered a Laurent series of** X_b **(expanded in terms of positive powers of** q**),** $q_\alpha=q^{\nu_\alpha},\nu_\alpha=\frac{(\alpha,\alpha)}{2},t_\alpha=t_{\nu_\alpha}$**; the coefficients of** P_b **belong to the field** $\mathbb{Q}(q,t_\nu)$**. Setting** $t_\alpha=q_\alpha^{k_\alpha}$**,** $k_\alpha=k_{\nu_\alpha}$**,** $X_a(q^b)=q^{(a,b)}$**,** $P_b(q^{\rho_k})=q^{-(\rho_k,b)}\prod_{\alpha>0}\prod_{j=0}^{(\alpha^\vee,b)-1}\Big(\frac{1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1-q_\alpha^j X_\alpha(q^{\rho_k})}\Big)$**,** $\langle P_b P_c^\iota\mu\rangle=$ $\langle\mu\rangle\,\delta_{bc}\prod_{\alpha>0}\prod_{j=0}^{(\alpha^\vee,b)-1}\frac{(1-q_\alpha^{j+1}t_\alpha^{-1}X_\alpha(q^{\rho_k}))(1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1-q_\alpha^j X_\alpha(q^{\rho_k}))(1-q_\alpha^{j+1}X_\alpha(q^{\rho_k}))}$.

For any $b,c\in P_+$**,** $\mathbf{u}=(u_1,\ldots,u_\ell)$**, and** $\mathbb{C}_{ab}^{c\mathbf{u}}$ **for** $\theta_{\mathbf{u}}^{(\ell)}$ **above:**

$$\mathbb{C}_{0b}^{cu}\stackrel{\mathbf{def}}{=\!=}\frac{\langle P_b P_c^\iota\theta_u\,\mu\rangle}{\langle P_c P_c^\iota\mu\rangle}=\frac{q^{b^2/2+c^2/2+(b+c,\rho_k)}}{u(b-c)\langle P_c P_c^\iota\mu\rangle}P_b^\iota(q^{c+\rho_k})P_c(q^{\rho_k})\langle\theta\mu\rangle,$$

$$\mathbb{C}_{\mathbf{0}\boldsymbol{b}}^{\boldsymbol{c}\mathbf{u}}=\sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+}\mathbb{C}_{\mathbf{0}\boldsymbol{b}}^{\boldsymbol{c_1}u_1}\mathbb{C}_{\mathbf{0}\boldsymbol{c_1}}^{\boldsymbol{c_2}u_2}\mathbb{C}_{\mathbf{0}\boldsymbol{c_2}}^{\boldsymbol{c_3}u_3}\cdots\mathbb{C}_{\mathbf{0}\boldsymbol{c_{\ell-1}}}^{\boldsymbol{c}u_\ell}.\ \textbf{(R-R)}$$

DAHA VERTEX, HOPF LINKS

The series for $\mathbb{C}^{0\mathbf{u}}_{00}$ are refined Rogers-Ramanujan sums**; they become modular 0-weight functions as $t \to 0$ $[$*Ch,B.Feigin,2013*$]$. Similarly, for $\mathbf{b}=(b_i, 1\le i\le m)\subset P_+ \ni c$ and $P_{\mathbf{b}} \stackrel{\mathbf{def}}{=\!=} \prod_i P_{b_i}$, $\mathbb{C}^{cu}_{\mathbf{b}} =$**

$$\frac{\langle P_{\mathbf{b}} P_c^{\iota}\theta_u\mu\rangle}{\langle P_c P_c^{\iota}\mu\rangle} = \frac{\langle P_{\mathbf{b}} P_{c^{\iota}}\theta\mu\rangle}{u(\Sigma_i b_i - c)\,\langle P_c P_c^{\iota}\mu\rangle} = \frac{\dot{\tau}_-^{-1}(P_{\mathbf{b}}P_{c^{\iota}})(q^{\rho_k})\langle\theta\mu\rangle}{u(\sum_i b_i - c)\,\langle P_c P_c^{\iota}\mu\rangle},$$

where $\dot{\tau}_-(P_b) = q^{-(b,b)/2-(b,\rho_k)}P_b$ for $b\in P_+$ define the action of the DAHA automorphism τ_- in the polynomial representation; $\dot{\tau}_-^{-1}(P_{\mathbf{b}}P_{c^{\iota}})(q^{\rho_k})/P_c(q^{\rho_k})$ is the DAHA-Jones "polynomial" **(for the root system R) from $[$*Ch,Danilenko,2015*$]$ for Hopf $(m{+}1)$-link with the** pairwise **linking numbers -1 for colors b and $+1$ between b and c (i.e. the orientation of the c-component is reversed). As above, the case of any ℓ can be reduced to $\ell=1$, which gives Rogers-Ramanujan-type formulas for the key** matrix entries of the operators of multiplication by $\theta^{(\ell)}_{\mathbf{u}}$ in the basis $\{P_a\}$**, topologically related to chains of Hopf links.**

DAHA VIA ELLIPTIC CONFIGURATION SPACE

For $E = T^2$, we set $\mathcal{H} = \mathbb{C}\mathbf{B}_{ell}/\{T_i^2 + aT_i + b = 0\}$ for $\mathbf{B}_{ell} = \pi_1\big((E^N \setminus \{x_i = x_j\})/\mathbf{S}_N\big)$; $T_i (1 \le i < N)$ are the usual "half-turns". $\mathcal{H}$ can be generalized to any root systems, but then orbifold π_1 must be used. Here the action of the *projective* **$PSL_2(\mathbb{Z})$ ($= B_3$ due to Steinberg) in $\mathcal{H}$ is granted, which is far from obvious in other approaches: via $K_{T\times C^*}(\widehat{G/B})$ and Harmonic Analysis. DAHA is a universal flat deformation of the Heisenberg-Weyl algebra extended by W. Its Fock representation is the** *polynomial representation*. **The eigenfunctions of "Y-operators" are** *nonsymmetric Macdonald polynomials*. **The symmetric polynomials are obtained upon the t-symmetrization. The limit $t \to 0$ results in nil-DAHA and generalized Hermite polynomials.**

DAHA INVARIANTS OF HOPF CHAINS

Let us consider now the chains $\cup_j \mathbf{b_j}$ of Hopf links ($b_{i,j}$ are the colors of the corresponding unknots). Here $1 \le j \le p$ and $\mathbf{b_j} = \{b_{i,j}, 1 \le i \le m_j\}$ and the unknots $\{m_j, j\}$ and $\{1, j+1\}$ are identified ($1 \le j < p$). We assume that pairwise linking numbers are -1 within any given component $\mathbf{b_j}$ unless with $i = m_j$, when they are all $+1$; otherwise the linking numbers are zero. Any tree formed by such chains can be taken, but we will consider here only "paths". The DJ polynomials normalized at the "top" unknots $\{m_j, j\}$ are:

$$\frac{\langle P_{b_{1,1}} \cdots P_{b_{m_1-1,1}}\, P^{\iota}_{b_{m_1,1}}\ \theta\mu\rangle}{P_{b_{m_1,1}}(q^{\rho_k})\,\langle\theta\mu\rangle} \cdots \frac{\langle P_{b_{1,p}} \cdots P_{b_{m_p-1,p}}\, P^{\iota}_{b_{m_p,p}}\ \theta\mu\rangle}{P_{b_{m_p,p}}(q^{\rho_k})\,\langle\theta\mu\rangle}.$$

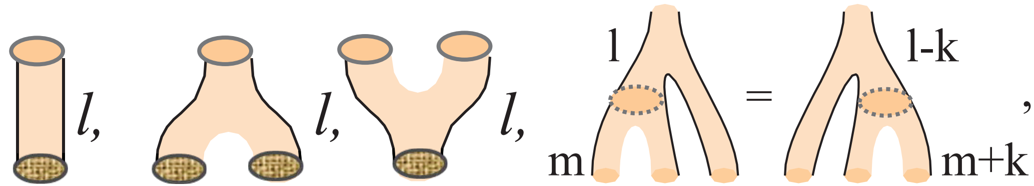
They are defined as products of DAHA invariants of Hopf links (and products for direct sums); the comatibility is due [Ch,Danilenko,2015]. The corresponding general LR-coefficients $\mathbb{C}_{\mathbf{b_1},\ldots,\mathbf{b_p}}$ have the same numerators, but $P_b(q^{\rho_k})\,\langle\theta\mu\rangle$ are replaced in $\mathbb{C}$ by $\langle P_b P_b^{\iota}\mu\rangle$ in the denominators. There is a connection to [Aganagic,Klemm,Marino,Vafa, 2005].

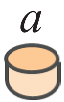
ASSOCIATIVITY VIA TQFT

Following $TQFT$ **(the unoriented one due to Turaev-Tuner with** ι**), the relations between** $\mathbb{C}^{cu}_{\mathbf{b}}$ **can be interpreted as follows. Let** $\mathscr{A}$ **be a commutative algebra with** 1 **and a symmetric non-degenerate form** $\langle f,g\rangle = \langle f g^{\iota}\mu_1\rangle$ **for** $\epsilon : \mathscr{A}\ni f \mapsto \langle f\mu_1\rangle$, $\mu_1^{\iota}=\mu_1$, $1^{\iota}=1, \epsilon(1)=1$**. Define** $\Delta : \mathscr{A}\to\mathscr{A}\widehat{\otimes}\mathscr{A}$ **via** $\langle\Delta(f), x\otimes y\rangle = \langle f, xy\rangle$**. In the basis of orthogonal polynomials/functions** $\{P_a\in\mathscr{A}\}$ **under** $P_0=1, \langle 1,1\rangle=1$**:** $\Delta(P_aV) = \sum_{b,c}\frac{\langle P_aV, P_bP_c\rangle P_b\otimes P_c}{\langle P_b,P_b\rangle\langle P_c,P_c\rangle}$ **for any** ι**-invariant function** V**. The invariant of** S^2 **is then** $\langle V\mu_1\rangle$**. Taking** $V=\theta^{(\ell)}_{\mathbf{u}}$**,** $P_a (a\in P_+)$ **etc., as above, it is** $\langle\theta^{(\ell)}_{\mathbf{u}}\mu\rangle/\langle\mu\rangle$**. The corresponding invariant for the torus** T^2 **is** $\sum_{b\in P_+}\frac{\langle\theta^{(\ell)}_{\mathbf{u}}, P_bP_b^{\iota}\rangle}{\langle P_b,P_b\rangle}$**. For** $A_1, \theta^{(\ell)}_{\mathbf{u}}=\theta$ **as** $t\to 0$**, it is proportional to** $1+\sum_{m\geq 1}\frac{1}{(1-q)\cdots(1-q^m)}$**, which diverges as** $|q|<1$**. One can use here some renormalization (and analytic continuation), roots of unity** q**, ... or proper** V**. There are no convergence problems though for** $\theta^{(\ell)}_{\mathbf{u}}$ $(\ell\geq 0)$ **if no "cycles" are allowed (the next page)!**

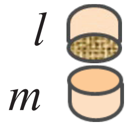
TQFT WITH LEVELS

Generators, relations and some amplitudes:

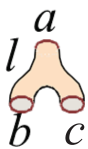


$\vartheta^l = \sum l$ 

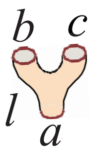
P_a, $\vartheta^l P_a =\ l$ 

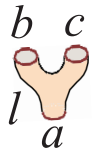
$1+\ldots$, 

$= \dfrac{<\vartheta^{l+m}\mu>}{<\mu>}$,

where $P_0 = 1$, 

$= \dfrac{<P_a P_b^{\iota} P_c^{\iota}\, \vartheta^l \mu>}{<P_a P_a^{\iota} \mu>} = \dfrac{<P_a^{\iota}\, P_b P_c \vartheta^l \mu>}{<P_a P_a^{\iota} \mu>}$. $\quad \Delta$:

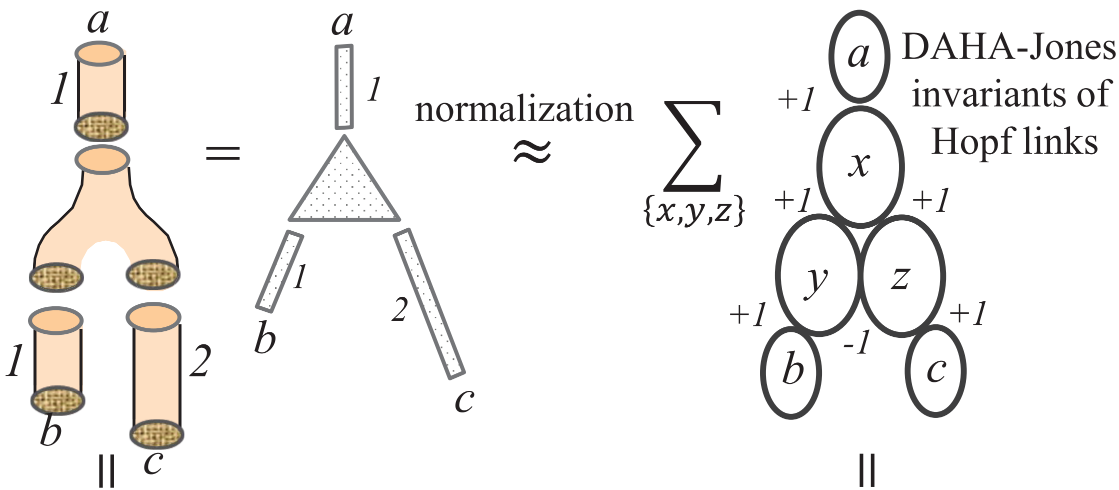
$P_a\, \vartheta^l \rightarrow \sum_{\{b,c\}}$ 

$P_b \otimes P_c$, 

$= \dfrac{<P_a\, P_b^{\iota} P_c^{\iota} \vartheta^l \mu><\mu>}{<P_b\, P_b^{\iota}\, \mu><P_c\, P_c^{\iota}\, \mu>}$.

CONNECTION TO HOPF LINKS

From pairs of pants to Hopf links:



$$\frac{\langle P_a^{\iota} P_b P_c \,\vartheta^4 \mu\rangle}{\langle P_a P_a^{\iota}\, \mu\rangle} \qquad \frac{\langle P_y^{\iota} P_b \vartheta\, \mu\rangle\langle P_z^{\iota}\, P_c \vartheta\, \mu\rangle\langle P_x^{\iota} P_y\, P_z \vartheta \mu\rangle\langle P_a^{\iota} P_x\, \vartheta\, \mu\rangle}{P_y(t^{\rho})P_z(t^{\rho})P_x(t^{\rho})P_a(t^{\rho})\ \ \langle\vartheta\mu\rangle^4}.$$

So renormalized DJ/super-polynomials result in 2d TQFT!

NIL-THEORY: THE LIMIT $t\to 0$

The usual Rogers-Ramanujan sums occur as $t\to 0$ ($t_\nu\to 0$, to be exact). The μ-function and P-polynomials are well-defined at $t=0$; we put then $\bar\mu$, $\bar P_b$, $\bar{\mathbb{C}}^*_*$. Also, $\lim_{t\to 0} q^{(b,\rho_k)}P_b(q^{c+\rho_k})=q^{(b,c)}$. **One has at $t=0$:**

$$\bar{\mathbb{C}}^{cu}_{0b}\stackrel{\mathbf{def}}{=\!=}\frac{\langle \bar P_b \bar P_c^\iota \theta_u\,\bar\mu\rangle}{\langle \bar P_c \bar P_c^\iota \bar\mu\rangle}=\frac{q^{(b-c)^2/2}}{u(b-c)\prod_{i=1}^n\prod_{j=1}^{(c,\alpha_i^\vee)}(1-q_i^j)},$$

$$\bar{\mathbb{C}}^{c\mathbf{u}}_{0b}=\sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+}\bar{\mathbb{C}}^{c_1u_1}_{0b}\,\bar{\mathbb{C}}^{c_2u_2}_{0c_1}\,\bar{\mathbb{C}}^{c_3u_3}_{0c_2}\cdots\bar{\mathbb{C}}^{cu_\ell}_{0,c_{\ell-1}}\qquad \textbf{(R-R)}$$

$$=\sum_{c_1,c_2,\ldots,c_{\ell-1}}\frac{q^{(c_0-c_1)^2/2+(c_1-c_2)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell}u_p(c_{p-1}-c_p)\prod_{i=1}^n\prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)},$$ **where**

$c_i\in P_+$, $q_i=q_{\alpha_i}$, $\alpha_i^\vee=2\alpha_i/(\alpha_i,\alpha_i)$, **and we set** $c_0=b$, $c_\ell=c\in P_+$.

Here q-Hermite polynomials $\bar P_b$ **coincide with dominant Demazure level-one characters (*Sanders, Ion*). Upon the division by their norms, they coincide with the characters of some natural quotients of the** upper **level-one Demazure modules and those of** global Weyl modules.

RELATION TO STRING FUNCTIONS

Let us discuss briefly the connections with string functions**. Here** $\widehat{\theta}_v(X) \stackrel{\mathbf{def}}{=\!=} \sum_{a\in v+Q} q^{\frac{(a,a)}{2}} X_a$ **for** $v \in P/Q$ **are more convenient. Then the corresponding** $\langle \bar{P}_b \bar{P}_c^{\iota} \widehat{\theta}_{\mathbf{v}} \bar{\mu}\rangle / \langle \bar{P}_c \bar{P}_c^{\iota} \bar{\mu}\rangle$ **for** $c_0 = b, c_\ell = c$ **are**

$$\widehat{\mathbb{C}}^{\mathbf{v}}_{b,c} = \sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \frac{q^{(c_0-c_1)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell}\prod_{i=1}^{n}\prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)},$$ **where** $\mathbf{v} = \{v_1,\ldots,v_\ell\} \subset P/Q$ **and the summation is over** $c_i - c_{i+1} \in v_i + Q$**. They are zero unless** $b - c + v_1 + \ldots + v_\ell \in Q$**. When** $b = 0$**, they are modular weight-zero functions for minuscule** c**, w.r.t. some congruence subgroups of** $SL(2,\mathbb{Z})$ **and up to** $q^{\bullet}$**. Let** $\eta = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1-q^i)$**.**

First, $q^{-\frac{1}{4}}\widehat{\mathbb{C}}^{111}_{0,1} = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{\prod_{j=1}^{m}(1-q^{2j})} \prod_{j=1}(1+q^j)^2$ **for** A_1 **and** $\ell = 3$**; the sum is the** Rogers-Ramanujan "G" **after** $q^2 \mapsto q$**. Upon** $\frac{q^{\bullet}}{\eta^2}\times$**,** $\widehat{\mathbb{C}}^{000}_{0,0}, \widehat{\mathbb{C}}^{110}_{0,0}, \widehat{\mathbb{C}}^{100}_{0,1}, \widehat{\mathbb{C}}^{111}_{0,1}$ **coincide with the basic string functions for** $\widehat{sl_3}$ **of level 2:** $C_0^{2\widehat{\omega}_0}, C_{\alpha_1+\alpha_2}^{2\widehat{\omega}_0}, C_{\omega_1}^{\widehat{\omega}_0+\widehat{\omega}_1}, C_{\omega_1+\alpha_2}^{\widehat{\omega}_0+\widehat{\omega}_1}$ [*Georgiev, 1995*].

LEVEL-RANK DUALITY

Here $\widehat{\lambda} = \lambda + \delta$ for $\lambda \in P_+$, $\omega_0 = 0$, and string functions for affine dominant Λ of level ℓ are the coefficients of the decomposition of the character of the integrable Kac-Moody module L_Λ in terms of the standard affine orbit sums ϑ_ν^ℓ; namely, $\chi(L_\Lambda) = \sum_\nu C_\nu^\Lambda \vartheta_\nu^\ell$.

The calculations are quite involved here (based on parafermions). Thus we arrived at the level-rank duality (*I.Frenkel and others*) for certain string functions. Surprisingly, this duality is simple to observe in terms of the sums $\widehat{\mathbb{C}}$. The quadratic q-powers here are given in terms of the (inverse) Cartan matrix for the root system $R \otimes A_{\ell-1}$. So for $R = A_{n-1}$, a straightforward analysis shows that they satisfy $n \leftrightarrow \ell$. At the level of sets v: the ℓ-sets of the element from $P/Q = \mathbb{Z}_n$ for A_{n-1} are naturally identified with n-sets of the elements from $P/Q = \mathbb{Z}_\ell$ for $A_{\ell-1}$. Note that counting *classes* of integrable modules, you have essentially $\binom{n+\ell-1}{n-1}/n = \binom{n+\ell-1}{\ell-1}/\ell$, but the duality for the corresponding string functions is generally much more subtle.

NONSYMMETRIC THEORY

Let us briefly discuss [*Ch, Kato*]. The focus on the identification of the (recent) nonsymmetric global Weyl modules [*E.Feigin, Kato, Macedonskiy, 2017*)] **with the** Demazure slices **of the** **upper Demazure filtration** **in the (basic) level-one module L. The upper Demazure modules are with respect to $\widehat{\mathfrak{b}}_-$ in contrast to the Borel subalgebra $\widehat{\mathfrak{b}}_+$, resulting in the usual level-one Demazure modules D_b, $b \in P$. The characters of the latter coincide with non-symmetric q-Hermite polynomials $\bar{E}_b = E_b(t \to 0)$ (*Sanderson, Ion*), where E_b are** *nonsymmetric Macdonald polynomials* **for $b \in P$. They are orthogonal for the same μ, but now form a basis in the whole $\mathbb{C}[X_b]$. The characters of Demazure slices are identified with $E_b^\dagger = E_b(t \to \infty)$, divided by their norms h_b^0, which can be defined as the limits $t \to 0$ of the norms of E_b. The** dag-polynomials **are significantly more subtle than $\bar{E}_b$, though $P_b^\dagger$ are closely related to $\bar{P}_b$ (for $b \in P_+$). Let us relate the decomposition of $L^{\otimes \ell}$ via the Demazure slices to R-R sums.**

DEMAZURE SLICES

The first part is entirely numerical (based on the DAHA theory). Let $\widehat{\theta} \stackrel{\mathbf{def}}{=\!=} \theta \frac{\langle\bar{\mu}\rangle}{\langle\theta\bar{\mu}\rangle}$, $\bar{\mu} = \mu(t \to 0)$ (actually, $\langle\theta\bar{\mu}\rangle = 1$); then $\widehat{\theta}$ can be identified with the graded character of the level-one (basic) integrable representation L of the twisted affinization $\widehat{\mathfrak{g}}$ of the simple Lie algebra $\mathfrak{g}$ corresponding to the root system R.

For $\ell \in \mathbb{N}, b \in P$ and $\mathbf{c} = \{c_i \in P,\ 1 \le i \le \ell\}$, $\bar{E}_{b^\iota}\,\widehat{\theta}^{\ell} = \sum_{\mathbf{c}} C_{\mathbf{c}}\, \frac{q^{\left(((b_+ - (c_1)_+)^2 + \ldots + ((c_{\ell-1})_+ - (c_\ell)_+)^2\right)/\,2}}{\prod_{i=1}^{\ell-1}\, h^0_{c_i}}\ \frac{E^{\dagger *}_{c_\ell}}{h^0_{c_\ell}}$, where $C_{\mathbf{c}}$ is some (non-trivial) power of q, $E^{\dagger *}_c$ is $E^{\dagger}_c$ where $X_a \to X_a^{-1}, q \to q^{-1}$.

Its Kac-Moody interpretation is essentially as follows. For a level one usual Demazure module D_b associated to $b \in P$ and its dual $D_b^{\vee}$, the module $D_b^{\vee} \otimes L^{\otimes \ell}$ admits a filtration by the Demazure slices (as constituents). Its multiplicities are provided by the formula above. Actually any integrable modules have such decompositions (*Chari,...*).