Painlevé equations from Nakajima-Yoshioka blowup relations

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Painlevé equation

- The Painlevé equations are second order differential equations without movable critical points except poles. They are equations of the isomonodromic deformation of linear differential equation.
- Parameterless Painlevé equations (other names: Painlevé III $D_8^{(1)}$ equation or Painlevé III_3 equation)

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$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{2w^2}{z^2} - \frac{2}{z}$$

- Can be rewritten as a system of Toda-like bilinear equations

$$\begin{cases} 1/2 D^2_{[\log z]}(\tau_0(z), \tau_0(z)) = z^{1/2}\tau_1(z)\tau_1(z), \\ 1/2 D^2_{[\log z]}(\tau_1(z), \tau_1(z)) = z^{1/2}\tau_0(z)\tau_0(z), \end{cases}$$

where $D^2_{[\log z]}$ denotes second Hirota operator with respect to $\log z$. The function $w(z)$ is equal to $-z^{1/2}\tau_0(z)^2/\tau_1(z)^2$.

Formulas for tau functions

Painlevé tau function

$$\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}_{c=1}(\sigma + n|z). \tag{1}$$

Due to AGT relation there are two ways to define $\mathcal{Z}$

- Algebraically, $\mathcal{Z}$ is a Virasoro conformal block.
  In Liouville parameterization $c = 1 + 6(b^{-1} + b)^2$, the condition $c = 1$ corresponds to $b = \sqrt{-1}$.
- Geometrically, $\mathcal{Z}$ is a generating function of equiaveriant volumes of ADHM moduli space of instantons.
  In physical language $\mathcal{Z}_{c=1}$ — 4d Nekrasov partition $\mathcal{Z}$ function $SU(2)$ with $\epsilon_1 = \epsilon, \epsilon_2 = -\epsilon$.

Incomplete list of people: [Gamayun, Iorgov, Lisovyy, Teschner, Shchechkin, Gavrylenko, Marshakov, Its, Bonelli, Grassi, Tanzini, Nagoya, Tykhyy, Maruyoshi, Sciarappa, Mironov, Morozov, Iwaki, Del Monte,. . .]

Another central charges

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There are several reasons to believe the existence of such analogue for central charges of (logarithmic extension of) minimal models $\mathcal{M}(1, n)$

$$c = 1 - 6\frac{(n-1)^2}{n}, n \in \mathbb{Z} \setminus \{0\}. \quad (2)$$

Equvalently $b^2 = \sqrt{-n}$, or $\epsilon_1 = -\epsilon$, $\epsilon_2 = n\epsilon$.

- Operator valued monodromies commute [Iorgov, Lisovyy, Teschner 2014].
- Bilinear relations on conformal blocks [M.B., Shchechkin 2014]
- Action of $SL(2, \mathbb{C})$ on the vertex algebra [Feigin 2017]

Today: $c = -2$ tau functions

$$\tau^{\pm}(\sigma, s|z) = \sum_{n \in \mathbb{Z}} s^{n/2} \mathcal{Z}_{c=-2}(\sigma + n|z). \quad (3)$$

Blowup relations

$$\tau(a, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(a + 2n\epsilon, \epsilon, -\epsilon|z), \tag{4}$$

$$\tau^{\pm}(a, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2} \mathcal{Z}(a + 2n\epsilon; \mp\epsilon, \pm 2\epsilon|z). \tag{5}$$

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- [Nakajima Yoshioka], [Göttshe, Nakajima, Yoshioka], [MB, Feigin, Litvinov],

$$\beta_D \mathcal{Z}(a,\epsilon_1,\epsilon_2|z)=\sum_{m\in\mathbb{Z}+j/2} \mathrm{D}\Big(\mathcal{Z}(a+m\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2|z),\mathcal{Z}(a+m\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2|z)\Big),$$

D is some differential operator, $j=0,1$, β_D is some function (may be zero).

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  D is some differential operator, $j=0,1$, β_D is some function (may be zero).
- Set $\epsilon_1=\epsilon$,$\epsilon_2=-\epsilon$, and take the sum of these relations with coefficients s^n

$$\beta_D \tau(z) = \mathrm{D}(\tau^+(z), \tau^-(z)). \tag{6}$$

  Excluding $\tau(z)$ one gets system of bilinear relations on $\tau^+(z)$, $\tau^-(z)$.
- This system can be used to prove the (Painlevé) bilinear relations on $\tau(z)$.

Plan of the talk

1. Introduction
2. The function $\mathcal{Z}$
3. Blowup relations
4. Painlevé equations
5. Discussion

The function $\mathcal{Z}$

There are two ways to define:

- Geometric, through ADHM moduli space of instantons.
- Algebraically, through Virasoro algebra (or more generally W-algebras).

Geometric definition: $\mathcal{M}(r,N)$

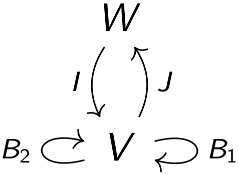
- Denote by $\mathcal{M}(r,N)$ the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$ of rank r, $c_1=0$, $c_2=N$.
- Description as a quver variety (ADHM description)

$$\mathcal{M}(r,N) \cong \left\{ \begin{pmatrix} B_1, \\ B_2, \\ I, \\ J \end{pmatrix} \middle| \begin{array}{ll} \text{(i)} & [B_1,B_2]+IJ=0 \\ \text{(ii)} & \begin{array}{l}\text{there are } N \text{ linear independent vec-} \\ \text{tors obtained by the action of algebra} \\ \text{genareted by } B_1 \text{ and } B_2 \text{ on } I_1, I_2, \dots, I_r\end{array} \end{array} \right\} \Big/ \mathrm{GL_N},$$

- B_j, I and J are $N\times N$, $N\times r$ and $r\times N$ matrices.
- $I_1,\dots,I_r$ denote the columns of the matrix I.
- The GL_N action is given by

$$g\cdot(B_1,B_2,I,J)=(gB_1g^{-1},gB_2g^{-1},gI,Jg^{-1}),$$

for $g\in\mathrm{GL}_N$.

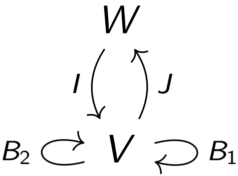


Geometric definition: $\mathcal{Z}$

- $\mathcal{M}(r,N)$ is smooth manifold of complex dimension $2rN$.
- There is a natural action of the $r+2$ dimensional torus T on the $\mathcal{M}(r,N)$: $(\mathbb{C}^*)^2$ acts on the base $\mathbb{CP}^2$ and $(\mathbb{C}^*)^r$ acts on the framing at the infinity.

$$B_1 \mapsto t_1 B_1; \quad B_2 \mapsto t_2 B_2; \quad I \mapsto It; \quad J \mapsto t_1 t_2 t^{-1} J,$$

Here $(t_1, t_2, t) \in \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^r$. Denote by $\epsilon_1, \epsilon_2, a_1, \ldots, a_r$ coordinates on Lie T.



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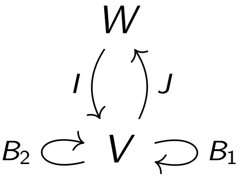


Definition

$$\mathcal{Z}(\epsilon_1, \epsilon_2, \vec{a}; q) = \sum_{N=0}^{\infty} q^N \int_{\mathcal{M}(r,N)} [1],$$

These equivariant integrals can be computed by localization method and equal to the sum of contributions of torus fixed points (which are labeled by r-tuple of Young diagrams $\lambda_1 \dots, \lambda_r$).

Algebraic definition: Virasoro algebra

- By Vir we denote the Virasoro Lie algebra with the generators $C, L_n, n \in \mathbb{Z}$ subject of relation:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}C, \quad [L_n, C] = 0$$

- Denote by $\mathrm{V}_{\Delta,c}$ the Verma module of the Virasoro algebra generated by the highest weight vector v:

$$L_n v = 0, \text{ for } n > 0 \qquad L_0 v = \Delta v, \ Cv = cv.$$

- It is convenient to parametrize Δ and c as

$$\Delta = \Delta(P, b) = \frac{(b^{-1}+b)^2}{4} - P^2, \qquad c = 1 + 6(b^{-1}+b)^2$$

Algebraic definition: function $\mathcal{Z}$

- The Whittaker vector $\mathrm{W}(z)=\sum_{N=0} w_N z^N$, defined by the equations:

$$L_0 w_N = (\Delta+N) w_N, \quad L_1 w_N = w_{N-1}, \quad L_k w_N = 0, \text{for } k>1.$$

These equations can be simply rewritten as

$$L_1 \mathrm{W}(z) = z\mathrm{W}(z), \quad L_k \mathrm{W}(z) = 0, \text{ for } k>1.$$

- One can use normalization of W such that $\langle w_0, w_0\rangle = 1$. Therefore

$$w_0 = v, \qquad w_1 = \frac{1}{2\Delta} L_{-1} v$$

$$w_2 = \frac{c+8\Delta}{4\Delta(c-10\Delta+2c\Delta+16\Delta^2)} L_{-1}^2 v - \frac{3}{c-10\Delta+2c\Delta+16\Delta^2} L_{-2} v$$

- The Whittaker vector corresponding to $\mathrm{V}_{\mathrm{P},b}$ will be denoted by $\mathrm{W}_{\mathrm{P},b}(z)$.
- The Whittaker limit of the 4 point conformal block defined by:

$$\mathcal{Z}(P,b;z) = \langle \mathrm{W}_{\mathrm{P},b}(1), \mathrm{W}_{\mathrm{P},b}(z)\rangle = \sum_{N=0}^{\infty} \langle w_{\mathrm{P},b,N}, w_{\mathrm{P},b,N}\rangle z^N \tag{7}$$

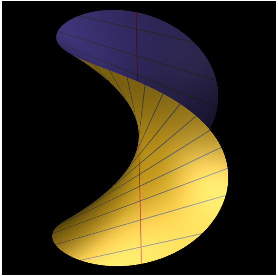
$$\mathcal{Z}(P,b;z) = 1 + \frac{2}{(b+b^{-1})^2 - 4P^2} z + \dots$$

Plan of the talk

Blow up equations

- Denote by $\widehat{\mathbb{CP}^2}$ blowup in origin.
- Denote by $\widehat{\mathcal{M}}(r,k,N)$ moduli space framed torsion free sheaves on $\widehat{\mathbb{CP}^2}$, r is a rank, k is a first Chern class, N is a second Chern class.

$$\widehat{\mathcal{Z}}(\epsilon_1,\epsilon_2,\vec{a};q)=\sum_{N=0}^{\infty}q^N\int_{\widehat{\mathcal{M}}(r,0,N)}[1],$$



- There is a map $\widehat{\pi}\colon \widehat{\mathcal{M}}(r,0,N)\to\mathcal{M}_0(r,N)$ [Nakajima, Yoshioka]

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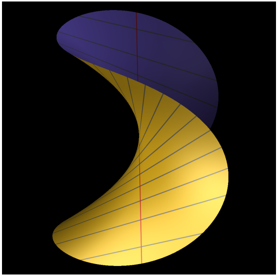

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$$\widehat{\mathcal{Z}}(\epsilon_1,\epsilon_2,\vec{a};q)=\mathcal{Z}(\epsilon_1,\epsilon_2,\vec{a};q)$$

- There are two torus invariant points on the $\widehat{\mathbb{C}^2}$.
- The torus fixed points on the $\widehat{\mathcal{M}}(r,0,N)$ are labelled by $\vec{\lambda}^1,\vec{\lambda}^2,k$

$$\widehat{\mathcal{Z}}(\epsilon_1,\epsilon_2,a;q)=\sum_{k\in\mathbb{Z}}\mathcal{Z}(\epsilon_1,\epsilon_2-\epsilon_1,a+k\epsilon_1;q)\cdot\mathcal{Z}(\epsilon_1-\epsilon_2,\epsilon_2,a+k\epsilon_2;q),$$

Blowup equations: representations

$$\mathcal{Z}(\epsilon_1,\epsilon_2,a;q)=\sum_{k\in\mathbb{Z}}\mathcal{Z}(\epsilon_1,\epsilon_2{-}\epsilon_1,a{+}k\epsilon_1;q)\cdot\mathcal{Z}(\epsilon_1{-}\epsilon_2,\epsilon_2,a{+}k\epsilon_2;q),$$

• In terms of representation theory on the left side we have Vir_b, where $b^2=\epsilon_1/\epsilon_2$. On the right side we have a sum of Vir_{b_1} and Vir_{b_2}, where $b_1=b/\sqrt{1-b^2}$, $b_2=\sqrt{b^2-1}$.

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Theorem (M.B., Feigin, Litvinov)

There is a isomorphism of chiral algebas the (extended) product of $Vir_{b_1} \otimes Vir_{b_2}$ and a product $Vir_b \otimes \mathcal{U}$

Here $\mathcal{U}$ is a special chiral algebra of central charge -5. As a vertex algebra U is isomorphic to a lattice algebra $V_{\sqrt{2}\mathbb{Z}}$ or $\widehat{\mathfrak{sl}(2)}_1$.

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• If $b^2=-2/3$ then $\mathcal{U}$ is isomorphic to (extended) product of minimal models 2/5 and 3/5. Therefore

$$\begin{aligned}\chi(\mathcal{L}_{0,1})&=q^{-1/4}\left(\chi^{2/5}_{(1,2)}\cdot\chi^{5/3}_{(2,1)}+\chi^{2/5}_{(1,4)}\cdot\chi^{5/3}_{(4,1)}\right),\\ \chi(\mathcal{L}_{1,1})&=q^{-1/4}\left(\chi^{2/5}_{(1,1)}\cdot\chi^{5/3}_{(1,1)}+\chi^{2/5}_{(1,3)}\cdot\chi^{5/3}_{(3,1)}\right).\end{aligned}\tag{8}$$

Blowup equations: combinatorics

$$\chi(\mathcal{L}_{0,1}) = q^{-1/4}\left(\chi^{2/5}_{(1,2)}\cdot\chi^{5/3}_{(2,1)} + \chi^{2/5}_{(1,4)}\cdot\chi^{5/3}_{(4,1)}\right),$$

- Due to Weyl-Kac formula

$$\chi(\mathcal{L}_{0,1}) = \sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{(q)_\infty} = 1 + 3q + 4q^2 + \cdots .$$

- Fermionic formulas for minimal models [Feigin Frenkel], [Feigin Foda Welsh]

$$\chi^{2/5}_{1,1} = q^{\Delta(P_{1,1},b_{2/5})}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}, \quad \chi^{2/5}_{1,2} = q^{\Delta(P_{1,2},b_{2/5})}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n},$$
$$\chi^{5/3}_{2,1} = q^{\Delta(P_{1,2},b_{3/5})}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_{2n}}, \quad \chi^{5/3}_{4,1} = q^{\Delta(P_{1,4},b_{3/5})}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q)_{2n+1}}.$$

here $(q)_n = \prod_{k=1}^{n}(1-q^k)$

Blowup equations: combinatorics

Definition

We call the function $f: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ a (l,k) configuration if

1. $f(m) + f(m+1) \leq k$
2. $f(2m+1) = k - l$, $f(2m) = l$, for $m << 0$
3. $f(m) = 0$, for $m >> 0$

The set of such configurations we denote by $\Sigma_{l,k}$. Extremal configuration:

$f_{2n}(m)$	$\cdots$	$k-l$	l	$k-l$	l	$k-l$	l	0	0	0	$\cdots$
m			$\cdots$	$2n-3$	$2n-2$	$2n-1$	$2n$	$2n+1$	$2n+2$	$\cdots$	

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m			$\cdots$	$2n-3$	$2n-2$	$2n-1$	$2n$	$2n+1$	$2n+2$	$\cdots$	

Define a weight

$$w_q(f)=-\sum_{m<0}(2m+1)(k-l-f(2m+1))-\sum_{m<0}2m(l-f(2m))+\sum_{m\geq 0}mf(m)$$

Theorem (Feigin Stoyanovsky)

$$\chi(\mathcal{L}_{l,k})=q^{\frac{l(l+2)}{4(k+2)}}\sum_{f\in\Sigma_{l,k}}q^{w_q(f)}.$$

Blowup equations: combinatorics

- $\Sigma_{l,k} = \sqcup \Sigma^r_{l,k}$, where $\Sigma^r_{l,k}$ consists of (l,k) configurations such that $f(0) = r$.
  $\Sigma^r_{l,k} = \Sigma^{+,k-r}_{k} \times \Sigma^{-,k-r}_{l,k}$, where
  $\Sigma^{+,k-r}_{k}$: functions $f\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $f(1) \leq k-r$ and (1), (3) hold;
  $\Sigma^{-,k-r}_{l,k}$: functions $f\colon -\mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $f(-1) \leq k-r$ and (1), (2) hold.

$$q^{-\frac{l(l+2)}{4(k+2)}} \cdot \chi(\mathcal{L}_{l,k}) = \sum_{f\in\Sigma_{l,k}} q^{w_q(f)} = \sum_{0\leq r\leq k} \left(\sum_{f\in\Sigma^{+,k-r}_{k}} q^{w_q(f)} \right) \cdot \left(\sum_{f\in\Sigma^{-,k-r}_{l,k}} q^{w_q(f)} \right)$$

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- [Feigin Frenkel], [Feigin Foda Welsh]

$$\chi_{1,1}^{2/5} = q^{\Delta(P_{1,1}, b_{2/5})} \sum_{f \in \Sigma_{0,1}^{+,0}} q^{w_q(f)}, \quad \chi_{1,2}^{2/5} = q^{\Delta(P_{1,2}, b_{2/5})} \sum_{f \in \Sigma_{0,1}^{+,1}} q^{w_q(f)},$$

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$$\chi_{1,1}^{2/5} = q^{\Delta(P_{1,1}, b_{2/5})} \sum_{f \in \Sigma_{0,1}^{+,0}} q^{w_q(f)}, \quad \chi_{1,2}^{2/5} = q^{\Delta(P_{1,2}, b_{2/5})} \sum_{f \in \Sigma_{0,1}^{+,1}} q^{w_q(f)},$$

$$\chi_{1,2}^{3/5} = q^{\Delta(P_{1,2}, b_{3/5})} \sum_{f \in \Sigma_{0,1}^{-,1}} q^{w_q(f)}, \quad \chi_{1,4}^{3/5} = q^{\Delta(P_{1,4}, b_{3/5})} \sum_{f \in \Sigma_{0,1}^{-,0}} q^{w_q(f)}.$$

$$\chi(\mathcal{L}_{0,1}) = q^{-1/4} \left(\chi_{(1,2)}^{2/5} \cdot \chi_{(2,1)}^{5/3} + \chi_{(1,4)}^{2/5} \cdot \chi_{(4,1)}^{5/3} \right),$$

Plan of the talk

1. Introduction
2. The function $\mathcal{Z}$
3. Blowup relations
4. Painlevé equations
5. Discussion

Blowup relations

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2|z)=\sum_{m\in\mathbb{Z}}\mathcal{Z}(a+m\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2|z)\mathcal{Z}(a+m\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2|z),$$

Blowup relations

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2|z)=\sum_{m\in\mathbb{Z}}\mathcal{Z}(a+m\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2|z)\mathcal{Z}(a+m\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2|z),$$

- Imposing condition $\epsilon_1+\epsilon_2=0$ we get in the CFT notations

$$\mathcal{Z}_{c=1}(\sigma|z)=\sum_{n\in\mathbb{Z}}\mathcal{Z}^{+}_{c=-2}\left(\sigma-n\,\middle|\,\frac{z}{4}\right)\mathcal{Z}^{-}_{c=-2}\left(\sigma+n\,\middle|\,\frac{z}{4}\right), \qquad (9)$$

We get $\qquad \tau(\sigma,s|z)=\tau^{+}(\sigma,s|z)\tau^{-}(\sigma,s|z),$

Recall that in CFT notation

$$\tau(\sigma,s|z)=\sum_{n\in\mathbb{Z}}s^n\mathcal{Z}_{c=1}(\sigma+n|z),\quad \tau^{\pm}(\sigma,s|z)=\sum_{n\in\mathbb{Z}}s^{n/2}\mathcal{Z}^{\pm}_{c=-2}(\sigma+n|z/4).$$

Blowup relations 2

We get $\qquad \tau(\sigma, s|z) = \uptau^{+}(\sigma, s|z)\uptau^{-}(\sigma, s|z),$

$$\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}_{c=1}(\sigma + n|z), \quad \uptau^{\pm}(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2} \mathcal{Z}^{\pm}_{c=-2}(\sigma + n|z/4).$$

Blowup relations 2

We get $\qquad \tau(\sigma, s|z) = \tau^+(\sigma, s|z)\tau^-(\sigma, s|z),$

$$\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}_{c=1}(\sigma+n|z), \quad \tau^{\pm}(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2} \mathcal{Z}^{\pm}_{c=-2}(\sigma+n|z/4).$$

Differential blowup relations

$$\sum_{n\in\mathbb{Z}} \mathcal{Z}(a+2\epsilon_1 n; \epsilon_1, \epsilon_2-\epsilon_1|ze^{-\frac{1}{2}\epsilon_1\alpha})\mathcal{Z}(a+2\epsilon_2 n; \epsilon_1-\epsilon_2, \epsilon_2|ze^{-\frac{1}{2}\epsilon_2\alpha})|_{\alpha^4} = \\ = \mathcal{Z}(a;\epsilon_1,\epsilon_2|z) + \frac{(2\alpha)^4}{4!}\left(\left(\frac{\epsilon_1+\epsilon_2}{4}\right)^4 - 2z^4\right)\mathcal{Z}(a;\epsilon_1,\epsilon_2|z) + O(\alpha^5). \tag{10}$$

We get

$$D^1_{[\log z]}(\tau^+,\tau^-) = z^{1/4}\tau_1, \qquad D^2_{[\log z]}(\tau^+,\tau^-) = 0, \\ D^3_{[\log z]}(\tau^+,\tau^-) = z^{1/4}\left(z\frac{d}{dz}\right)\tau_1, \qquad D^4_{[\log z]}(\tau^+,\tau^-) = -2z\tau. \tag{11}$$

Painlevé equations from Nakajima-Yoshioka blowup relations

$$\tau_0 = \tau^+\tau^-,\ D^1_{[\log z]}(\tau^+, \tau^-) = z^{1/4}\tau_1,\ D^2_{[\log z]}(\tau^+, \tau^-) = 0. \qquad (12)$$

Theorem (MB, Shchechkin)

Let $\tau^\pm$ satisfy equations (12). Then τ_0 and τ_1 satisfy Toda-like equation

$$D^2_{[\log z]}(\tau_0, \tau_0) = -2z^{1/2}\tau_1^2 \qquad (13)$$

Since we know from blowup relations that $\tau^\pm(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}\mathcal{Z}^\pm_{c=-2}(\sigma + n|z/4)$ satisfy (12) we proved that $\tau(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^n\mathcal{Z}_{c=1}(\sigma + n|z)$ satisfy Painlevé equation.

Plan of the talk

Blowup relations for $\mathbb{C}^2/\mathbb{Z}_2$

$$\tau(a,s|z)=\sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(a+2n\epsilon,\epsilon,-\epsilon|z), \tag{14}$$

Blowup relations for $\mathbb{C}^2/\mathbb{Z}_2$

$$\tau(a,s|z) = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(a+2n\epsilon,\epsilon,-\epsilon|z), \tag{14}$$

- [Bruzzo, Poghossian, Tanzini 09], [Bruzzo, Pedrini, Sala, Szabo 2013], [Ohkawa 2018], [Belavin, M.B., Feigin, Litvinov, Tarnopolsky 2011]

$$\tilde{\mathcal{Z}}(a,\epsilon_1,\epsilon_2|z) = \sum_n \mathrm{D}\Big(\mathcal{Z}(a+n\epsilon_1,2\epsilon_1,-\epsilon_1+\epsilon_2|z),\mathcal{Z}(a+n\epsilon_2,\epsilon_1-\epsilon_2,2\epsilon_2|z)\Big). \tag{15}$$

Here $\tilde{\mathcal{Z}}$ is Nekrasov partition function for $\mathbb{C}^2/\mathbb{Z}_2$.

- After specialization $\epsilon_1+\epsilon_2=0$ and exclusion $\tilde{\mathcal{Z}}$ we get bilinear relations on $\mathcal{Z}_{c=1}$, which lead to bilinear relations of $\tau(z)$

$$\tilde{\mathrm{D}}(\tau(z),\tau(z)) = 0. \tag{16}$$

These are (Paivlevé) bilinear equations, without additional τ^+,τ^-.

Painlevé and blowup after Nekrasov

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2|z)=\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+n\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2|z)\cdot\mathcal{Z}(a+n\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2|z),$$

Take the limit $\epsilon_1\to 0$. In this limit

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2|z)\sim\exp(\frac{1}{\epsilon_1}f(a,z)),$$

where f is a classical conformal block.
The limit of the blowup relations takes the form

$$\exp\left(\frac{\partial f}{\partial\epsilon_2}\right)=\sum_{n\in\mathbb{Z}}e^{n\frac{\partial f}{\partial a}}\mathcal{Z}_{c=1}(a+n,-\epsilon_2,\epsilon_2|z)\Big).$$

For the left side [Reshetikhin], [Teschner], [Litvinov, Lukyanov, Nekrasov, Zamolodchikov].

Thank you for the attention!

Explicit formulas

$$\mathcal{Z} = \mathcal{Z}_{cl}\mathcal{Z}_{1-loop}\mathcal{Z}_{inst}.$$

where

$$\mathcal{Z}_{cl}(a;\epsilon_1,\epsilon_2|\Lambda) = \Lambda^{-\frac{a^2}{\epsilon_1\epsilon_2}},$$

$$\mathcal{Z}_{1-loop}(a;\epsilon_1,\epsilon_2) = \exp(-\gamma_{\epsilon_1,\epsilon_2}(a;1) - \gamma_{\epsilon_1,\epsilon_2}(-a;1)),$$

$$\mathcal{Z}_{inst}(a;\epsilon_1,\epsilon_2|\Lambda) = \sum_{\lambda^{(1)},\lambda^{(2)}} \frac{(\Lambda^4)^{|\lambda^{(1)}|+|\lambda^{(2)}|}}{\prod_{i,j=1}^{2} \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(a_i - a_j;\epsilon_1,\epsilon_2)}, \quad |\lambda| = \sum \lambda_j,$$

$$\mathsf{N}_{\lambda,\mu}(a;\epsilon_1,\epsilon_2) = \prod_{s\in\lambda}(a - \epsilon_2(a_\mu(s)+1) + \epsilon_1 l_\lambda(s)) \prod_{s\in\mu}(a + \epsilon_2 a_\lambda(s) - \epsilon_1(l_\mu(s)+1))$$

$$\gamma_\epsilon(x;\Lambda) = \frac{d}{ds}|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^{+\infty} \frac{dt}{t} t^s \frac{e^{-tx}}{e^{\epsilon t} - 1}, \quad \mathrm{Re}\, x > 0.$$

where $\lambda^{(1)}, \lambda^{(2)}$ are partitions, $a_\lambda(s), l_\lambda(s)$ denote the lengths of arms and legs for the box s in the Young diagram corresponding to the partition λ.

$\mathcal{U}$

Definition

The conformal algebra $\mathcal{U}$ coincide with the $V_{\sqrt{2}\mathbb{Z}}$ as the operator algebra, but the stress–energy tensor is modified:

$$T_{\mathcal{U}} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{\sqrt{2}}(\partial^2\varphi) + \epsilon\left(2(\partial\varphi)^2 e^{\sqrt{2}\varphi} + \sqrt{2}(\partial^2\varphi)e^{\sqrt{2}\varphi}\right) = \\ = \frac{1}{2}\partial_z\varphi(z)^2 + \frac{1}{\sqrt{2}}\partial_z^2\varphi(z) + \epsilon\partial_z^2(e^{\sqrt{2}\varphi(z)}), \quad \varepsilon \neq 0 \quad (17)$$

- The conformal algebras $\mathcal{U}$ isomorphic for different values $\varepsilon \neq 0$. For the $\varepsilon = 0$ $T_{\mathcal{U}}(z)$ has the from discussed above form for $u = \frac{1}{\sqrt{2}}$ and central charge -5.
- The spaces $U_0 = \bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{k\sqrt{2}}$ and $U_1 = \bigoplus_{k\in\mathbb{Z}+1/2} \mathrm{F}_{k\sqrt{2}}$ become a representations of $\mathcal{U}$.

Calculation

$$\sum_{n_1,n_2\in\mathbb{Z}} s^{n_1}\mathcal{Z}^+_{c=-2}\left(\sigma+n_1-n_2\,\middle|\,\frac{z}{4}\right)\mathcal{Z}^-_{c=-2}\left(\sigma+n_1+n_2\,\middle|\,\frac{z}{4}\right)=$$

$$=\sum_{n_1,n_2\in\mathbb{Z}|n_1+n_2\in2\mathbb{Z}}+\sum_{n_1,n_2\in\mathbb{Z}|n_1+n_2\in2\mathbb{Z}+1}=\left\|n_\pm=\frac{1}{2}(n_1\pm n_2)\right\|=$$

$$=\sum_{n_+\in\mathbb{Z}} s^{n_+}\mathcal{Z}^+_{c=-2}\left(\sigma+2n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}} s^{n_-}\mathcal{Z}^-_{c=-2}\left(\sigma+2n_-\,\middle|\,\frac{z}{4}\right)+ \tag{18}$$

$$+\sum_{n_+\in\mathbb{Z}+1/2} s^{n_+}\mathcal{Z}^+_{c=-2}\left(\sigma+2n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}+1/2} s^{n_-}\mathcal{Z}^-_{c=-2}\left(\sigma+2n_-\,\middle|\,\frac{z}{4}\right)=$$

$$=\sum_{n_+\in\mathbb{Z}} s^{n_+/2}\mathcal{Z}^+_{c=-2}\left(\sigma+n_+\,\middle|\,\frac{z}{4}\right)\sum_{n_-\in\mathbb{Z}} s^{n_-/2}\mathcal{Z}^-_{c=-2}\left(\sigma+n_-\,\middle|\,\frac{z}{4}\right),$$

where the last equality follows from the

$$\mathcal{Z}^+(\sigma+n_++1/2)\mathcal{Z}^-(\sigma+n_-)+\mathcal{Z}^-(\sigma+n_++1/2)\mathcal{Z}^+(\sigma+n_-)=0,\quad n_+,n_-\in\mathbb{Z},$$

$$\tau(\sigma,s|z)=\tau^+(\sigma,s|z)\tau^-(\sigma,s|z), \tag{19}$$