Universal K-matrices for quantum symmetric pairs

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IRC 2019

If you like:

1. quantum enveloping algebras
2. R matrices
3. the quantum Yang Baxter equation
4. braided tensor categories

...then you should also like:

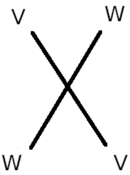
1. quantum symmetric pairs
2. K matrices
3. the reflection equation
4. braided module categories

[B, Kolb, *The bar involution for quantum symmetric pairs, Represent. Theory 19 (2015), 186-210*]

[B, Kolb, *Universal K-matrix for quantum symmetric pairs, Journal für die reine und angewandte Mathematik 747 (2019), 299–353*]

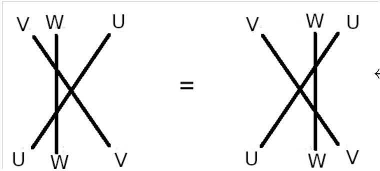
[Kolb, *Braided module categories via quantum symmetric pairs*]

- Particle on a line $\leftrightsquigarrow V$
- Two particles $\leftrightsquigarrow V \otimes W$
- Scattering:



$$\leftrightsquigarrow c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$$

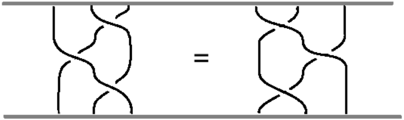
- Quantum Yang Baxter equation:



$$\leftrightsquigarrow (c_{W,U} \otimes 1)(1 \otimes c_{V,U})(c_{V,W} \otimes 1) =$$
$$= (1 \otimes c_{V,W})(c_{V,U} \otimes 1)(1 \otimes c_{W,U})$$

Braided tensor categories

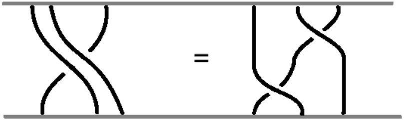
- tensor category, commutativity $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes W$
- QYBE = the action of the braid group of type on $V^{\otimes n}$



[Reshethikhin-Turaev]

- hexagon axiom (similar for $c_{V,W\otimes U}$):

$$c_{V\otimes W,U} = (c_{V,U} \otimes 1) \circ (1 \otimes c_{W,U})$$



Quasitriangular Hopf algebras

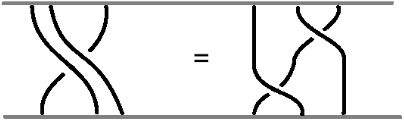
- Hopf algebra H, $\mathcal{V}$ (some nice) category of representations
- quasitraingular = exists $\mathcal{R} \in H \otimes H$, $\check{\mathcal{R}} = \mathrm{flip} \circ \mathcal{R}$

$$c_{V,W} = \check{\mathcal{R}}|_{V\otimes W} : V \otimes W \to W \otimes V$$

$$\check{\mathcal{R}}\Delta(a) = \Delta(a)\check{\mathcal{R}}$$

- The hexagon axiom becomes:

$$(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$



- QYBE

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

Quantum enveloping algebra

- $\mathfrak{g}$, $U_q\mathfrak{g}$, $\mathcal{O}_{int}$
- The construction of the R-matrix [Lusztig]:
  - Define the bar involution on $U_q\mathfrak{g}$:

$$E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_i \mapsto K_i^{-1}, \quad q \mapsto q^{-1}$$

  - Find the quasi R-matrix $\mathcal{R}_0 \in U_q\mathfrak{n}^- \otimes U_q\mathfrak{n}^+$ such that

$$\mathcal{R}_0\overline{\Delta(a)} = \Delta(\bar{a})\mathcal{R}_0$$

  - Set $\mathcal{R} = \mathcal{R}_0 \cdot q^{-H\otimes H}$,

$$\check{\mathcal{R}} = \mathcal{R}_0 \circ q^{-H\otimes H} \circ \mathrm{flip}$$

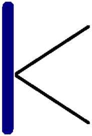
  - Prove

$$(\Delta \otimes 1)(\mathcal{R}) = \dots$$
$$(1 \otimes \Delta)(\mathcal{R}) = \dots$$

  - $\Rightarrow$ QYBE

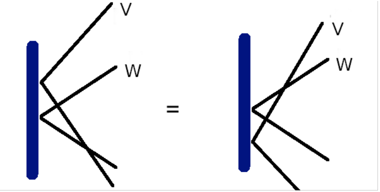
Reflection equation

- particle on a line + a wall:



$\leftrightsquigarrow t_V : V \to V$

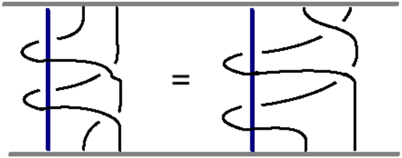
- Reflection Equation:



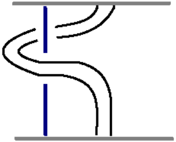
$$c_{W,V}\,(t_W \otimes 1)\,c_{V,W}\,(t_V \otimes 1) = (t_V \otimes 1)\,c_{W,V}\,(t_W \otimes 1)\,c_{V,W}$$

- braids with a fixed pole:

$$c_{W,V}\,(t_W\otimes 1)\,c_{V,W}\,(t_V\otimes 1)=(t_V\otimes 1)\,c_{W,V}\,(t_W\otimes 1)\,c_{V,W}$$



- Naturality condition in $\otimes$:

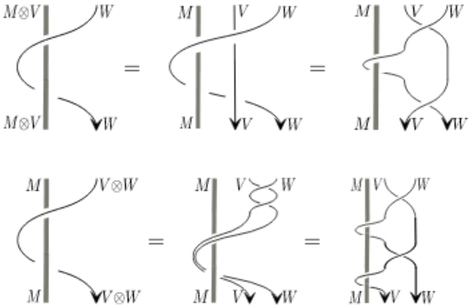


$$t_{V\otimes W}=(t_V\otimes 1)\,c_{W,V}\,(t_W\otimes 1)\,c_{V,W}$$

- Naturality condition $\Rightarrow$ RE

Braided module categories

- $\mathcal{V}$ braided category, $\mathcal{M}$ module category ($\boxtimes : \mathcal{M} \times \mathcal{V} \to \mathcal{M}$)
- $e_{M,V} : M \boxtimes V \to M \boxtimes V$
- $e_{M\boxtimes V,W} = (id_M \boxtimes c_{V,W})(e_{M,W} \boxtimes id_V)(id_M \boxtimes c_{W,V})$
- $e_{M,V\otimes W} = (id_M \boxtimes c_{W,V})(e_{M,W} \boxtimes id_V)(id_M \boxtimes c_{V,W})(e_{M,V} \boxtimes id_W)$



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- $e_{M, V \otimes W} = (id_M \boxtimes c_{W,V})(e_{M,W} \boxtimes id_V)(id_M \boxtimes c_{V,W})(e_{M,V} \boxtimes id_W)$
- Recover $t_V = e_{Triv,V}$
- Representation of the braid group of type B on $M \boxtimes V^{\otimes n}$

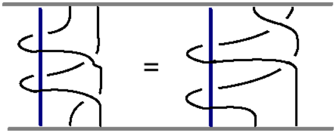
[Kolb, *Braided module categories via quantum symmetric pairs*]

[Brochier, *Cyclotomic associators and finite type invariants for tangles in the solid torus, Algebraic and Geometric Topology, 2013.*]

Quasitriangular comodule algebras

- H quasitriangular Hopf algebra, B algebra, $\Delta_B : B \to B \otimes H$
- $\mathcal{V} = Rep(H)$, $\mathcal{M} = Rep(B)$, $\boxtimes : \mathcal{M} \times \mathcal{V} \to \mathcal{M}$
- Want: element $\mathcal{K} \in B \otimes H$, $e_{M,V} = \mathcal{K}|_{M \boxtimes V}$
- Conditions:
  - $\mathcal{K}\Delta_B(b) = \Delta_B(b)\mathcal{K}$
  - $(\Delta_B \otimes id)(\mathcal{K}) = \mathcal{R}_{32}\mathcal{K}_{13}\mathcal{R}_{23}$
  - $(id \otimes \Delta)(\mathcal{K}) = \mathcal{R}_{32}\mathcal{K}_{13}\mathcal{R}_{23}\mathcal{K}_{12}$
- $K = (\varepsilon \otimes id)(\mathcal{K})$ will then satisfy the reflection equation:

$$(K \otimes 1)\check{\mathcal{R}}(K \otimes 1)\check{\mathcal{R}} = \check{\mathcal{R}}(K \otimes 1)\check{\mathcal{R}}(K \otimes 1)$$



Main point:

Theorem
Quantum symmetric pairs provide examples of this structure.

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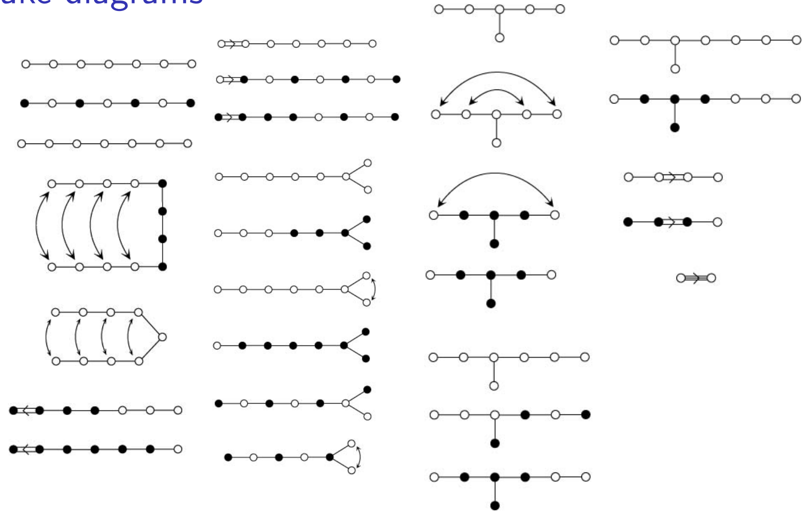
Classical symmetric pairs:

- $\mathfrak{g}$
- $\theta : \mathfrak{g} \to \mathfrak{g}$ an involution
- $\mathfrak{k} = \mathfrak{g}^{\theta}$ fixed points
- $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair

Classification:

- (X, τ) Satake diagrams
- $\theta = const \circ Ad(w_X) \circ \tau \circ \omega$
- [Araki]
- [Kac, Wang]

Satake diagrams



Quantum symmetric pairs:

- $(\mathfrak{g}, \mathfrak{k})$ a symmetric pair
- $(U_q\mathfrak{g}, U_q\mathfrak{k})$ not compatible deformations
- better deformation: $(U_q\mathfrak{g}, B_{\mathbf{c},\mathbf{s}})$:
  - subalgebra $B_{\mathbf{c},\mathbf{s}} \subseteq U_q\mathfrak{g}$
  - coideal $\Delta(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}} \otimes U_q\mathfrak{g}$
  - parameters $\mathbf{c}, \mathbf{s}$
  - at $q \to 1$, $B_{\mathbf{c},\mathbf{s}} \to U\mathfrak{k}$
- [G. Letzter, *Symmetric pairs for quantized enveloping algebras, 1999.]*

  [G. Letzter, *Coideal subalgebras and quantum symmetric pairs, 2002.]*

  [G. Letzter, *Quantum symmetric pairs and their zonal spherical functions, 2003.]*

  [S. Kolb, *Quantum symmetric Kac-Moody pairs, 2012.]*

Presentation

Theorem (Letzter; Kolb; B-Kolb, Hadewijch De Clercq)

$B_{\mathbf{c},\mathbf{s}}$ has a presentation with generators and relations which look a little like the relations of $U_q\mathfrak{k}$.
Generated over $(U_q\mathfrak{h}^\theta)\cdot(U_q\mathfrak{g}_X)$ with generators

$$B_i = F_i + c_i\theta_q(F_iK_i)K_i^{-1} + s_iK_i^{-1},$$

relations:

- $K_\beta B_i K_\beta^{-1} = q^{-(\beta,\alpha_i)}B_i$;
- $[E_i, B_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$;
- $\mathrm{Serre}(B_i, B_j) =$ *lower order terms in* B_k.

Strategy

$U_q\mathfrak{g}$	$B_{\mathbf{c},\mathbf{s}} \subseteq U_q\mathfrak{g}$ quantum symmetric pair
1. bar involution	1. bar involution
2. quasi R-matrix $\mathcal{R}_0$	2. quasi K-matrix $\mathfrak{X}$
3. universal R-matrix	3. universal K-matrices $K, \mathcal{K}$
4. $(1 \otimes \Delta)(\mathcal{R})$	4. $\Delta(K)$, $(\Delta \otimes id)(\mathcal{K}), (id \otimes \Delta)(\mathcal{K})$
5. prove $\mathcal{R}$ sats QYBE	5. prove K sats RE

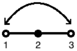
- Bar involution $U_q\mathfrak{g} \to U_q\mathfrak{g}$ does not preserve $B_{\mathbf{c},\mathbf{s}}$
- [H. Bao, W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs , 2013.]*

  [M. Ehrig, C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality, 2013.]*
- Want the internal bar involution $B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c},\mathbf{s}}$ such that:

$$\overline{q}^B = q^{-1} \qquad \overline{E_i}^B = E_i \qquad \overline{B_i}^B = B_i$$

$$\overline{K_\beta}^B = K_\beta^{-1} \qquad \overline{F_i}^B = F_i$$

- Relations must be bar invariant



$$C_{12}(\mathbf{c}) = 0$$

$$C_{13}(\mathbf{c}) = \frac{-1}{(q-q^{-1})^2}\left(q^{-1}(1-q^2)c_1\mathcal{Z}_1 + q(1-q^{-2})c_3\mathcal{Z}_3\right)$$

$$\Leftrightarrow \quad \overline{c_1\mathcal{Z}_1} = c_3\mathcal{Z}_3$$

$$\Leftrightarrow \overline{c_1} = q^{-2}c_3$$

Theorem (B-Kolb)

For every Satake diagram, and for a good choice of parameters $\mathbf{c}, \mathbf{s}$*, there exists a bar involution on* $B_{\mathbf{c},\mathbf{s}}$*,* $b \mapsto \overline{b}^{B}$*.*

- fix a good choice of c_i, s_i
- two bar involutions: $a \mapsto \overline{a}$ on $U_q\mathfrak{g}$ and $b \mapsto \overline{b}^B$ on $B_{\mathbf{c},\mathbf{s}}$
- $\overline{B_i} \neq \overline{B_i}^B$

Theorem (B-Kolb)

There exists a unique invertible $\mathfrak{X} \in \widehat{U_q\mathfrak{n}^+}$ *such that for all* $b \in B_{\mathbf{c},\mathbf{s}}$

$$\mathfrak{X} \cdot \overline{b} = \overline{b}^B \cdot \mathfrak{X}$$

$$\mathfrak{X} = \sum_{\mu} \mathfrak{X}_\mu, \quad \mathfrak{X}_\mu \in U^+_\mu, \quad \mathfrak{X}_0 = 1$$

Rewrite $\mathfrak{X} \cdot \overline{b} = \overline{b}^B \cdot \mathfrak{X}$ as

$$r_i(\mathfrak{X}_\mu) = \text{some expression in lower } \mathfrak{X}_\nu$$
$${}_ir(\mathfrak{X}_\mu) = \text{some expression in lower } \mathfrak{X}_\nu$$

Proposition

For given A_i, ${}_iA$, $i \in I$, the following are equivalent:

1. *The following system has a unique solution:*

$$r_i(X) = A_i$$
$${}_ir(X) = {}_iA.$$

2. *A_i, ${}_iA$ satisfy:*
   - i) $r_i({}_jA) = {}_jr(A_i)$
   - ii) *Some analogue of Serre relations.*

From now on:

- $\mathfrak{g}$ finite type
- w_0 longest element of the Weyl group of $\mathfrak{g}$, $w_0(\alpha_i) = \alpha_{\tau_0(i)}$
- w_X longest element of the Weyl group of $\mathfrak{g}_X$
- ξ a certain character of weight lattice
- τ the diagram automorphism from Satake data

Definition (B-Kolb, Kolb)

The universal K-matrix is

$$K = \mathfrak{X} \circ \xi \circ T_{w_0}^{-1} \circ T_{w_X}^{-1} \circ \tau\tau_0.$$

[H. Bao, W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs , 2013.]*

[T. tom Dieck, R. Häring-Oldenburg, *Quantum groups and cylinder braiding, 1998.*]

$$K = \mathfrak{X} \circ \xi \circ T_{w_0}^{-1} \circ T_{w_X}^{-1} \circ \tau\tau_0.$$

Theorem (B-Kolb)

Let V be a finite dimensional $U_q\mathfrak{g}$ module. Then

$$K : V \to V$$

is a $B_{\mathbf{c},\mathbf{s}}$-isomorphism.

Theorem (B-Kolb)

$$\Delta(K) = (K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}}$$

Theorem (B-Kolb)

K satisfies the reflection equation,

$$(K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}} = \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1)$$

Proof:

$$\Delta(K) = (K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}}$$

$$\Delta(K) = \check{\mathcal{R}} \cdot \Delta(K) \cdot \check{\mathcal{R}}^{-1}$$

$$(K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}} = \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}} \cdot (K \otimes 1) \cdot \check{\mathcal{R}} \cdot \check{\mathcal{R}}^{-1}$$

$\square$

Theorem (Kolb)

$\mathcal{K} = \check{\mathcal{R}}(K \otimes 1)\check{\mathcal{R}}$ lies in the suitable completion of $B_{\mathbf{c},\mathbf{s}} \otimes U_q\mathfrak{g}$ and satisfies

- $\mathcal{K}\Delta_B(b) = \Delta_B(b)\mathcal{K}$
- $(\Delta_B \otimes id)(\mathcal{K}) = \mathcal{R}_{32}\mathcal{K}_{13}\mathcal{R}_{23}$
- $(id \otimes \Delta)(\mathcal{K}) = \mathcal{R}_{32}\mathcal{K}_{13}\mathcal{R}_{23}\mathcal{K}_{12}$

Corollary

$B_{\mathbf{c},\mathbf{s}}$ is a quasitriangular comodule algebra for $U_q\mathfrak{g}$, with the universal R-matrix $\mathcal{R}$ and the universal K-matrix $\mathcal{K}$. The category $\mathcal{M}$ of finite dimensional $B_{\mathbf{c},\mathbf{s}}$ representations is a braided module category for the category $\mathcal{O}_{int}$ of finite dimensional $U_q\mathfrak{g}$ modules.

- [Dobson, Kolb, *Factorisation of quasi K-matrices for quantum symmetric pairs]*
- [Bao, Wang et al]
- [Regelskis, Vlaar]:
  - *Quasitriangular coideal subalgebras of* $U_q(\mathfrak{g})$ *in terms of generalized Satake diagrams*
  - *Reflection matrices, coideal subalgebras and generalized Satake diagrams of affine type*
  - *Solutions of the* $U_q(\hat{\mathfrak{sl}}_N)$ *reflection equations*
- [De Commer, Matassa, *Quantum flag manifolds, quantum symmetric spaces and their associated universal K-matrices]*
- [De Commer, Neshveyev, Tuset, Yamashita, *Ribbon braided module categories, quantum symmetric pairs and Knizhnik-Zamolodchikov equations]*
- [Appel, Vlaar, in progress]

THANK YOU!