# Demazure crystals for Kohnert polynomials

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# Outline

# Representatives of Schubert classes for Flag manifold

Consider the cohomology theory for the complete flag manifold of nested subspaces

$$\mathscr{F} \ : \ \mathscr{F}^{(0)} \subset \mathscr{F}^{(1)} \subset \cdots \subset \mathscr{F}^{(n)} \qquad \dim(\mathscr{F}^{(i)}) = i$$

Let $\mathcal{I}_n$ be the ideal generated by symmetric polynomials with no constant term.

**Theorem (Borel 1953)**

*The cohomology ring $H^*(\mathscr{F})$ is $\mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n$ and has a linear basis of cosets $[X_w]$ for each $w \in \mathcal{S}_n$, and so we may compute $[X_u \cap X_v] = [X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w]$.*

Divided difference operators $\partial_i$ act by

$$\partial_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

For any reduced expression $w = s_{i_k} \cdots s_{i_1}$

$$\partial_w = \partial_{i_k} \cdots \partial_{i_1}$$

**Definition (Lascoux–Schützenberger 1982)**

The Schubert polynomial $\mathfrak{S}_w$ is

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} \left( x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \right)$$

where $w_0 = n \cdots 21$ is the long element.

Schubert polynomials are a $\mathbb{Z}$-basis for the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$.

**Example (Schubert polynomials)**

- $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$
- $\mathfrak{S}_{v(\lambda,n)} = s_\lambda(x_1, \ldots, x_n)$

**Theorem (Lascoux–Schützenberger 1982)**

*$\mathfrak{S}_w$ represents the Schubert class $[X_w]$, and*

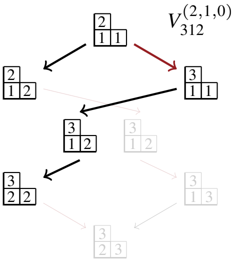
$$\mathfrak{S}_u \mathfrak{S}_v \quad = \quad \sum_w c_{u,v}^w \mathfrak{S}_w$$

# Characters of Demazure modules for $\mathrm{GL}_n$

Finite dimensional irred. representations of $\mathfrak{g}$ decompose into weight spaces $V^\lambda = \bigoplus_a V^\lambda_a$.
The Weyl group acts on extremal weight spaces $\{V^\lambda_{w\cdot\lambda} \mid w \in W\}$, which are all 1-dimensional.

## Definition (Demazure (1974))

The Demazure module $V^\lambda_w$ is the $\mathfrak{b}$-submodule of the irreducible $\mathfrak{g}$-representation $V^\lambda$ generated by the extremal weight space $V^\lambda_{w\cdot\lambda}$. Demazure characters are $\mathrm{char}(V^\lambda_w) = \kappa_{w\cdot\lambda}$




Demazure characters are a $\mathbb{Z}$-basis for the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ whose structure constants are not nonnegative.

## Example (Demazure modules)

- $V^\lambda_{\mathrm{id}} = V^\lambda_\lambda$ is the 1-dim highest wt space
- $V^\lambda_{w_0} = V^\lambda_{\mathrm{rev}(\lambda)} = V^\lambda$ is the full module

For $\mathfrak{gl}_n$, index Demazure modules by

$$(w, \lambda) \mapsto w \cdot \lambda \qquad a \mapsto (w_a, \mathrm{sort}(a))$$

where $w_a$ is the shortest s.t. $w_a \cdot a = \lambda$.

## Example (Demazure characters)

- $\kappa_\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$
- $\kappa_{\mathrm{rev}(\lambda)} = s_\lambda(x_1, \ldots, x_n)$

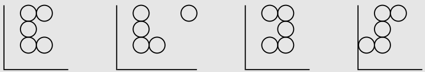
# Kohnert's rule

A diagram is finitely many cells in $\mathbb{N} \times \mathbb{N}$.

The weight is $\mathrm{wt}(D)_i = \#$cells in row $i$.

A key diagram is a left-justified diagram.

**Example (Diagrams of weight $(0, 2, 1, 2)$)**



Axel Kohnert (1962-2013) was a student of Lascoux who devised a simple model for Demazure characters.

**Definition (Kohnert 1991)**

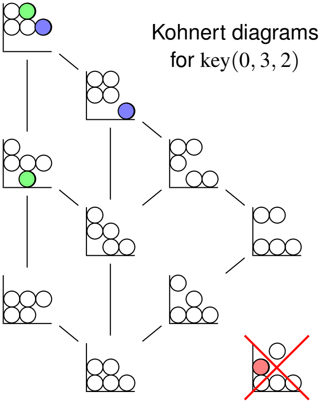
A Kohnert move on a diagram selects the rightmost cell $c$ of a row and moves $c$ to the first available position below, jumping over other cells in its way as needed.

Let $\mathrm{KD}(D)$ be the set of Kohnert diagrams for $D$.

**Theorem (Kohnert 1991)**

*The Demazure character $\kappa_a$ is given by*

$$\kappa_a = \sum_{T \in \mathrm{KD}(\mathrm{key}(a))} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$$



Kohnert diagrams for $\mathrm{key}(0, 3, 2)$

# Kohnert bases

Use this to generalize Demazure characters and Schubert polynomials simultaneously.

**Definition (Assaf–Searles 2019)**

The Kohnert polynomial $\mathfrak{K}_D$ of a diagram $D$ is

$$\mathfrak{K}_D = \sum_{T \in \mathrm{KD}(D)} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}$$

**Example**

- Schur polynomials $\mathfrak{K}_\lambda = s_\lambda(x_1 \cdots x_n)$
- Demazure characters $\mathfrak{K}_{\mathrm{key}(a)} = \kappa_a$
- Schubert polynomials $\mathfrak{K}_{\mathrm{Rothe}(w)} = \mathfrak{S}_w$

Each Kohnert polynomial has a unique leading term minimal in lexicographic order.

**Definition (Assaf–Searles 2019)**

A basis $\{\mathfrak{B}_a\}$ is a Kohnert basis if $\mathfrak{B}_a = \mathfrak{K}_D$ for some diagram $D$ with $\mathrm{wt}(D) = a$.

For each $a$, choose *any* diagram $D$ with $\mathrm{wt}(D) = a$ to construct a Kohnert basis.

**Definition (southwest diagrams)**

A diagram $D$ is southwest if $(r_2, c_1), (r_1, c_2) \in D$ s.t. $r_1 < r_2$ and $c_1 < c_2$ implies $(r_1, c_1) \in D$.

Partition diagrams, key diagrams and Rothe diagrams are all southwest diagrams.

**Conjecture (Assaf–Searles 2019)**

Kohnert polynomials of southwest diagrams are positive sums of Demazure characters.

# Crystal graphs

Schur polynomials are also characters for finite connected normal $\mathfrak{gl}_n$ crystals.

Crystal basis $\mathcal{B}$, weight map $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$

crystal lowering operators $f_i : \mathcal{B} \xrightarrow{i} \mathcal{B} \cup \{0\}$ such that $\mathrm{wt}(b) - \mathrm{wt}(f_i(b)) = \mathbf{e}_i - \mathbf{e}_{i+1}$.

The character of a crystal is

$$\mathrm{char}\,(\mathcal{B}) = \sum_{b \in \mathcal{B}} x_1^{\mathrm{wt}(b)_1} \cdots x_n^{\mathrm{wt}(b)_n}$$

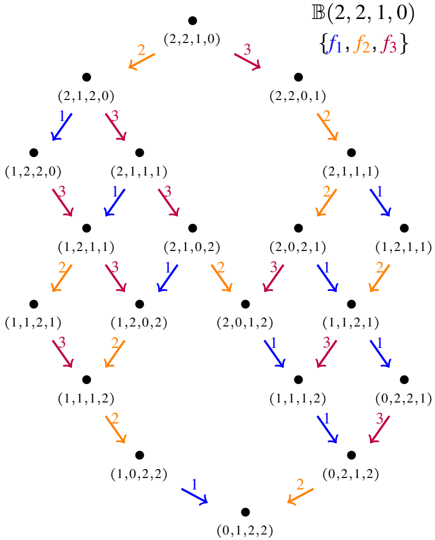
The **standard $\mathfrak{gl}_n$ crystal** has $\mathrm{wt}\left(\boxed{i}\right) = \mathbf{e}_i$

$$\boxed{1} \xrightarrow{f_1} \boxed{2} \xrightarrow{f_2} \boxed{3} \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} \boxed{n}$$

The connected (finite, normal) $\mathfrak{gl}_n$ crystals are indexed by dominant weights (partitions).

For $\mathbb{B}(\lambda)$ is the crystal for the irrep $\mathbb{S}_\lambda(\mathbb{C}^n)$

$$\mathrm{char}(\mathbb{B}(\lambda)) = \mathrm{char}(\mathbb{S}_\lambda(\mathbb{C}^n)) = s_\lambda(x_1, \ldots, x_n)$$



# A crystal on semistandard Young tableaux

Define crystal operators $e_i$ on $\mathrm{SSYT}(\lambda)$ that change an $i+1$ to an $i$ in $T$ by

## Definition (Pairing rule)

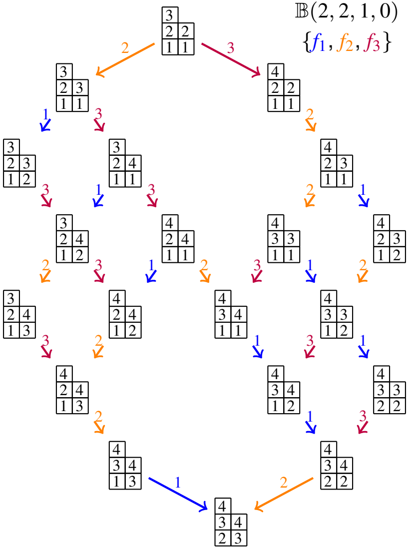
Two cells $\boxed{i}$ and $\boxed{i+1}$ are paired if in the same column or $\boxed{i+1}$ left of $\boxed{i}$ and no unpaired cells $\boxed{i}$ or $\boxed{i+1}$ between.

Compute $e_2$ by changing a 3 to a 2:


## Definition (Crystal raising operators)

For $T \in \mathrm{SSYT}(\lambda)$ and $1 \leq i < n$, the crystal raising operator $e_i$ acts on $T$ by

- $e_i(T) = 0$ if $T$ has no unpaired $\boxed{i+1}$
- change leftmost unpaired $\boxed{i+1}$ to $\boxed{i}$



# Demazure crystals

Define operators $\mathfrak{D}_i$ on subsets $X \subseteq \mathcal{B}$ by

$$\mathfrak{D}_i X = \{b \in \mathcal{B} \mid e_i^k(b) \in X\}$$

For $w = s_{i_k} \cdots s_{i_1}$ reduced expression

$$\mathbb{B}_w(\lambda) = \mathfrak{D}_{i_k} \cdots \mathfrak{D}_{i_1}\{u_\lambda\}$$

where $u_\lambda$ is the highest weight of $\mathbb{B}(\lambda)$.

**Theorem (Kashiwara 1993)**

*The Demazure character $\kappa_a$ is given by*

$$\kappa_a = \mathrm{char}\left(V_w^\lambda\right) = \mathrm{char}\left(\mathcal{B}_w(\lambda)\right)$$

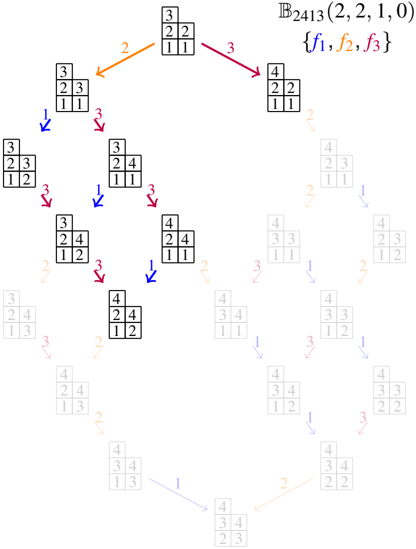
**Example (Compute $\mathbb{B}_{2413}(2,2,1,0)$)**

For $w = 2413$, we may take $w = s_1 s_3 s_2$

$$\kappa_{(1,2,0,2)} = x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_3^2 + x_1x_2^2x_3x_4 + x_1x_2^2x_4^2$$

$\mathbb{B}_{2413}(2,2,1,0)$

$\{f_1, f_2, f_3\}$



# Crystal operators on Kohnert diagrams

Define crystal operators $e_i$ on $\mathrm{KD}(D)$ that moves a cell of $T$ in row $i+1$ down to row $i$

## Definition (Pairing rule)

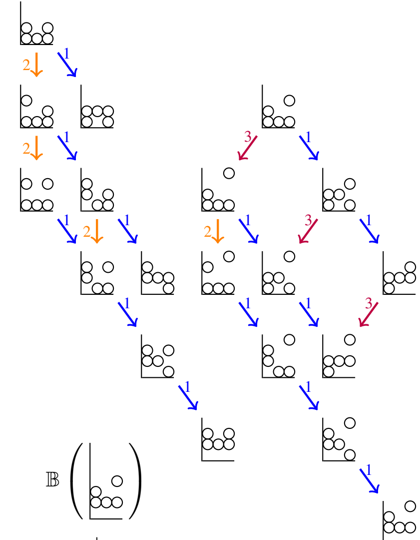
Two cells in rows $i$ and $i+1$ are paired if in the same column or higher is right of lower with no unpaired cells between.

Compute $e_2$ by moving cell in row $3$ to row $2$:


## Definition (Crystal raising operators)

For $T \in \mathrm{KD}(D)$ and $1 \leq i < n$, the crystal raising operator $e_i$ acts on $T$ by

- $e_i(T) = 0$ if no unpaired cell in row $i+1$
- push rightmost unpaired cell in row $i+1$ down to row $i$



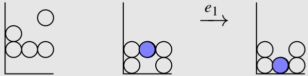

$\kappa_{(0,3,2,0)} + \kappa_{(0,3,1,1)}$

# Kohnert crystal for southwest diagrams

**Theorem (Assaf 2019+)**

*For $D$ a southwest diagram, if $T \in \mathrm{KD}(D)$ and $e_i(T) \neq 0$, then $e_i(T) \in \mathrm{KD}(D)$.*

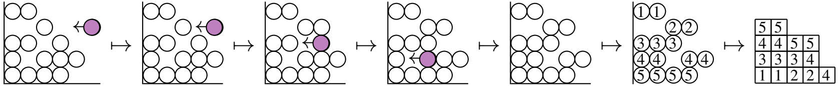
**Example (Closure of crystal operators)**

$D =$ 

In general, crystal operators do not act by Kohnert moves on diagrams.

If $D$ is not southwest, then either $e_i(D) \notin \mathrm{KD}(D)$ or this is not a Demazure crystal.

Define rectification operators that map any diagram to one in $\mathrm{KD}(\mathrm{key}(a))$ then to $\mathrm{SSYT}(\lambda)$ and which commute with crystal operators. This embeds $\mathrm{KD}(D)$ into a normal crystal.



**Theorem (Assaf 2019+)**

*For $D$ a southwest diagram, the rectification map embeds $\mathrm{KD}(D)$ as a Demazure subset of a normal crystal, and so the Kohnert polynomial $\mathfrak{K}_D$ is a positive sum of Demazure characters.*

# Highest weights

A connected crystal has a unique highest weight element $u$ characterized by $e_i(u) = 0$ for all $i$.

$$\mathrm{char}(\mathcal{B}) = \sum_{u \in \mathcal{B} \text{ s.t. } e_i(u)=0 \ \forall\ i} s_{\mathrm{wt}(u)}(x_1, \ldots, x_n)$$

**Theorem (Assaf–Searles 2019)**

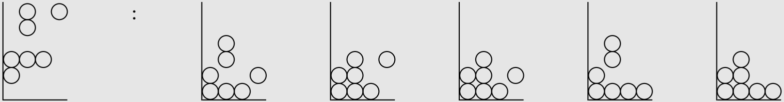
*The Kohnert quasisymmetric functions are well-defined by $\mathcal{K}_D(X) = \lim_{m\to\infty} \mathfrak{K}_{0^m \times D}$, and expand nonnegatively into Gessel's fundamental basis for quasisymmetric functions.*

In particular, for Demazure characters we have $\lim_{m\to\infty} \kappa_{0^m\times a}(x_1, \ldots, x_m, 0, \ldots, 0) = s_{\mathrm{sort}(a)}(X)$

**Corollary**

*For $D$ a southwest diagram, the Kohnert quasisymmetric function $\mathcal{K}_D$ is Schur positive.*

**Example (The five highest weight elements of $\mathrm{KD}(D)$ for $D$ the left diagram)**

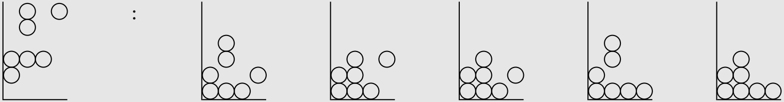


$$\mathcal{K}_D(X) = s_{(3,2,1,1)}(X) + s_{(3,2,2)}(X) + s_{(3,3,1)}(X) + s_{(4,1,1,1)}(X) + s_{(4,2,1)}(X)$$

# Demazure lowest weights

Demazure crystals have unique highest weights but $\mathcal{B}_w(\lambda)$ has highest weight $\lambda$ for every $w$.

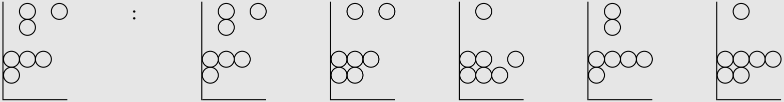
**Example (The five highest weight elements of $\mathrm{KD}(D)$ for $D$ the left diagram)**



$$\lim_{m\to\infty} \mathfrak{K}_{0^m\times D} = s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} + s_{(4,1,1,1)} + s_{(4,2,1)}.$$

We give an explicit algorithm for constructing Demazure lowest weight elements from highest weight elements, and so obtain an explicit formula for the Demazure character expansion.

**Example (The five Demazure lowest weight elements of $\mathrm{KD}(D)$ for $D$ the left diagram)**



$$\mathfrak{K}_D = \kappa_{(0,1,3,0,1,2)} + \kappa_{(0,2,3,0,0,2)} + \kappa_{(0,3,3,0,0,1)} + \kappa_{(0,1,4,0,1,1)} + \kappa_{(0,2,4,0,0,1)}.$$

Open: Is there a simple rule on the diagram $D$ that generates Demazure lowest weights?

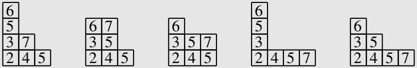
# Schubert polynomial expansions

Stanley (1984) defined symmetric functions $S_w$ using reduced words for a permutation $w$.

**Theorem (Edelman–Greene 1987)**

$$S_w = \sum_{\substack{\rho \in R(w) \\ \exists T \text{ increasing},\ \mathrm{row}(T)=\rho}} s_{\mathrm{Des}(\rho)}$$

**Example (Increasing Young tableaux for $R(13625847)$)**




$$S_{13625847} = s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} + s_{(4,1,1,1)} + s_{(4,2,1)}$$

Macdonald (1991) proved $S_w$ is the stable limit of $\mathfrak{S}_w$, specifically $\lim_{m\to\infty} \mathfrak{S}_{1^m \times w} = S_w$.

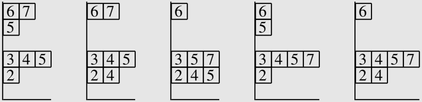
Lascoux–Schützenberger (1990) asserted and Reiner–Shimozono (1995) proved a Demazure expansion for Schubert polynomials based on the Edelman–Greene formula.

**Theorem (Assaf 2019$^+$)**

$$\mathfrak{S}_w = \sum_{\substack{\rho \in R(w) \\ \exists T \text{ increasing},\ \mathrm{row}(T)=\rho}} \kappa_{\mathrm{des}(\mathrm{lift}(\rho))}$$

*where* $\mathrm{lift}(\rho) \in R(w)$ *is explicit.*

**Example (Yamanouchi key tableaux for $R(13625847)$)**




$$\mathfrak{S}_{13625847} = \kappa_{(0,1,3,0,1,2)} + \kappa_{(0,2,3,0,0,2)} + \kappa_{(0,3,3,0,0,1)} + \kappa_{(0,1,4,0,1,1)} + \kappa_{(0,2,4,0,0,1)}$$

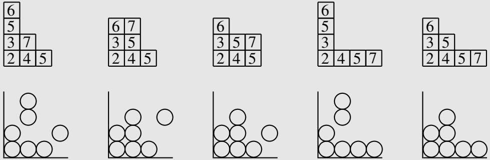
# Schubert polynomials as Kohnert polynomials

The Rothe diagram of a permutation $w$ is $\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\}$.

**Theorem (Assaf 2019$^{+}$)**

*There is a bijection between increasing Young tableaux with reading words in $\mathrm{R}(w)$ and highest weight elements of the Kohnert crystal on $\mathrm{KD}(\mathbb{D}(w))$. In particular, $S_w = \mathcal{K}_{\mathbb{D}(w)}$.*

**Example (Schur expansion bijection for 13625847)**

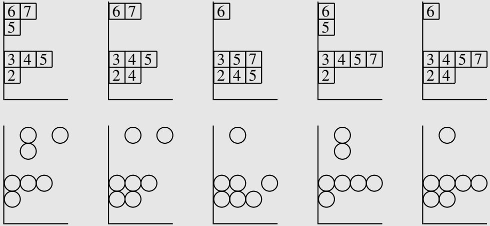


Kohnert asserted $\mathfrak{S}_w = \mathfrak{K}_{\mathbb{D}(w)}$. Winkel (1999, 2002) gave two published proofs of this.

**Theorem (Assaf 2019$^{+}$)**

*There is a bijection between Yamanouchi key tableaux with reading words in $\mathrm{R}(w)$ and Demazure lowest weight elements of the Kohnert crystal on $\mathrm{KD}(\mathbb{D}(w))$. In particular, Kohnert's rule for Schubert polynomials holds: $\mathfrak{S}_w = \mathfrak{K}_{\mathbb{D}(w)}$.*

**Example (Demazure expansion bijection for 13625847)**



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*Merci!*