BPS black holes and generalized error functions

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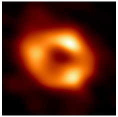
based on work with Rishi Raj, arXiv:2507.08551

+earlier works with S. Alexandrov, S. Banerjee, J. Manschot, A. Sen...

Introduction

- A long standing goal in string theory has been to provide a microscopic explanation of the thermodynamical entropy of black holes in General Relativity *[Bekenstein'72, Hawking'74]*

$S_{BH} = \frac{A}{4G_N}$



$S_{BH} \overset{?}{=} \log \Omega$

- By mid 90s, it was understood how to reproduce the entropy of BPS black holes from D-brane dynamics: 4D black holes arise by wrapping D-branes on $\gamma \times S^1$, where $\gamma \subset X$ is a supersymmetric cycle. The same D-brane on γ gives a 5D black string, described a 2D SCFT. Cardy's formula $S = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24})}$ gives the result.

Sen, Strominger, Vafa, Maldacena, Witten...

Precision counting and wall-crossing

- Since then, the goal post has moved from computing the asymptotics of $\Omega(\gamma)$ as $|\gamma| \to \infty$, to computing the exact BPS indices at finite γ. Mostly *because we can, it's fun*, and it leads to fruitful connections with mathematics.
- Such precision counting was achieved for $X = T^6$, $X = K_3 \times T^2$, and other cases preserving $\mathcal{N} \geq 4$ SUSY. BPS indices arise as Fourier coefficients of suitable (classical, Jacobi or Siegel) modular forms. *[Dijkgraaf Verlinde Verlinde'96, David Sen'05,...]*
- A key lesson was that BPS indices $\Omega_\sigma(\gamma)$ depend on the moduli σ, and jump due to the appearance/decay of multi-centered black hole bound states. In the simplest 'primitive' case, *[Denef Moore'07]*

$$\Omega_{\sigma_+}(\gamma_1 + \gamma_2) = \Omega_{\sigma_-}(\gamma_1 + \gamma_2) + \langle \gamma_1, \gamma_2 \rangle \, \Omega_{\sigma_-}(\gamma_1) \, \Omega_{\sigma_-}(\gamma_2)$$

as predicted from SUSY quantum mechanics of two dyons.

BPS indices and Donaldson-Thomas invariants

- For $\mathcal{N}=4$ vacua, only two-body bound states contribute to the index. Computing the index in the 'attractor' chamber where bound states are absent lead to mock Jacobi forms, with anomalous behavior under modular transformations *[Dabholkar Murthy Zagier'12]*.
- In type IIA string theory on a generic CY3-fold X, bound states involving an arbitrary number of constituents can in principle contribute to the index $\Omega_\sigma(\gamma)$, defined mathematically as the Donalson-Thomas invariant for the derived category $\mathcal{C}$ of coherent sheaves on X, with $\sigma \in \mathsf{Stab}\,\mathcal{C}$. *[Thomas'99, Bridgeland'07]*
- In that context, the general wall-crossing formula of *[Joyce Song'08, Kontsevich Soibelman'08]* can be derived from the SUSY quantum mechanics of n dyons using localization *[Manschot BP Sen'10]*
- We don't know yet how to compute $\Omega_\sigma(\gamma)$ for general γ in any chamber, except for some special cases.

Modularity of D4-D2-D0 indices I

- Nonetheless, viewing Type IIA/X as M-theory/$X \times S^1$, we can make definite predictions for DT invariants on any CY 3-fold X. In particular, D4-D2-D0 black holes arise by wrapping an M5-brane on a divisor $\mathcal{D} \subset X$. The elliptic genus of the resulting SCFT decomposes as

$$\mathrm{Tr}(-1)^F e^{2\pi \mathrm{i}(L_0 - \frac{c_L}{24} + q_a z^a)} = \sum_{\mu \in \Lambda^*/\Lambda} h_{p,\mu}(\tau)\, \Theta(\tau, z^a)$$

  where $h_{p,\mu}(\tau)$ are generating series of BPS indices with fixed D4-brane charge p and D2-brane charge μ (here $\Lambda = (H_4(X,\mathbb{Z})$, equipped with quadratic form $\kappa_{ab} = \kappa_{abc} p^c$).
- If the spectrum was discrete, this would imply that $h_{p,\mu}(\tau)$ should transform as a vector-valued weakly holomorphic modular form.

Mock modularity of rank 0 DT invariants

- Instead, whenever the divisor class $[\mathcal{D}]$ is reducible into a sum of effective divisor classes $\sum_i \mathcal{D}_i$, the spectrum of the SCFT is continuous. After a very long detour *[Alexandrov BP Manschot'16-20]*, we predicted that $h_{p,\mu}(\tau)$ is a mock modular form of higher depth.
- Namely, there exists explicit, universal non-holomorphic theta series $\Theta_n(\{p_i\},\tau,\bar\tau)$ such that (ignoring μ's for simplicity)

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a vector-valued modular form.

Indefinite theta series and generalized error functions

- It would take too long to explain the prescription for $\Theta_n(\{p_i\},\tau,\bar\tau)$. Suffice it to say that it extends Zweger's prescription for the modular completion of indefinite theta series of signature $(n-1,1)$, underlying Ramanujan's mock theta functions,

$$\sum_{k\in\Lambda}[\mathrm{sign}(C,k)-\mathrm{sign}(C',k)]e^{\mathrm{i}\pi\tau k^2}$$
$$\to \sum_{k\in\Lambda}[\mathrm{Erf}(\sqrt{\pi\tau_2}(C,k))-\mathrm{Erf}(\sqrt{\pi\tau_2}(C',k)]e^{\mathrm{i}\pi\tau k^2}$$

- In arbitrary signature $(n-r,r)$ a similar trick works by replacing $\prod_{i=1}^r \mathrm{sign}\left(\mathcal{M}^T\mathbf{x}\right)$ by the generalized error function *[Alexandrov Banerjee Manschot BP'16, Nazaroglu'16]*

$$E_r(\mathcal{M},\mathbf{x})=\int_{\mathbb{R}^r}\mathrm{d}^r\mathbf{z}\, e^{-\pi(\mathbf{x}-\mathbf{z})^T(\mathbf{x}-\mathbf{z})}\prod_{i=1}^r \mathrm{sign}\left(\mathcal{M}^T\mathbf{z}\right)_i$$

This generalizes $E_1(x)=\int_{\mathbb{R}} e^{-\pi(x-x')^2}\,\mathrm{sign}(x')\,\mathrm{d}x'=\mathrm{Erf}(x\sqrt{\pi})$.

Modular completion

This prescription was tested successfully in various set-ups:

- For $X = K_S$, where S is a del Pezzo surface, this prescription is consistent with the holomorphic anomaly equations for Vafa-Witten invariants conjectured by *[Nemeschansky Vafa Warner'98]*. See also *[Dabholkar Putrov Witten'20]*.
- For one-parameter compact CY3-folds such as the quintic X_5, the modularity of $h_{p,\mu}$ for one unit of D4 charge was tested to high level of detail in *[Alexandrov Feyzbakhsh Klemm BP Schimannek'23]*, confirming predictions by *[Gaiotto Strominger Yin'06]*. For X_8 and X_{10}, mock modularity was tested successfully for $[D4] = 2$.

In the remainder of this talk, I shall explain how the non-holomorphic completion naturally arises from the SUSY quantum mechanics of n-dyons, keeping into account the continuum of scattering states.

Multi-black hole quantum mechanics I

- For a system of n dyons with charge γ_i, the dynamics is governed by the Lagrangian with 4 real supercharges

$$L = \sum_{i=1}^{n} \frac{m_i}{2} \left(\dot{\vec{x}}_i^2 + D_i^2 + 2\mathrm{i}\bar{\lambda}_i \dot{\lambda}_i \right) + \sum_{i=1}^{n} \left(\vec{A}_i \cdot \dot{\vec{x}}_i - U_i D_i \right) + \sum_{i,j=1}^{n} \vec{\nabla}_i U_j \cdot \bar{\lambda}_i \vec{\sigma} \lambda_j$$

$$U_i = \sum_{j \neq i} \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} - \vartheta_i, \quad \vec{\nabla}_i U_j = \vec{\nabla}_i \times \vec{A}_j + \vec{\nabla}_j \times \vec{A}_i$$

  Here $\gamma_{ij} := \langle \gamma_i, \gamma_j \rangle$, $m_i = |Z(\gamma_i)|$, $\vartheta_i = \mathrm{Im}(e^{-i\phi} Z(\gamma_i))$ *[Denef 2002]*.
- Eliminating the auxiliary fields D_i, one generates a potential $V = \sum_{i=1}^{n} \frac{U_i^2}{2m_i}$ with supersymmetric vacua $\{\forall i, U_i = 0\}$. These are the same equations which determine the relative positions of the centers in stationary BPS solutions of $\mathcal{N} = 2$ supergravity !

Two-body electron-monopole problem I

- For two mutually non-local particles with $\langle\gamma_1,\gamma_2\rangle := 2q$, for example an electron-monopole system, the Hamiltonian in center of mass frame is

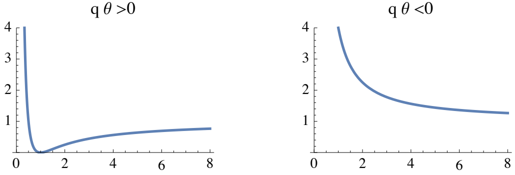
$$\boxed{H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{B}\cdot\vec{\sigma}\otimes(1_2 - \sigma_3) + \frac{1}{2m}\left(\vartheta - \frac{q}{r}\right)^2}$$

$$\vec{\nabla}\wedge\vec{A} = \vec{B} = \frac{\vec{r}}{r^3}\,,\quad m = \frac{|Z_{\gamma_1}||Z_{\gamma_2}|}{|Z_{\gamma_1}|+|Z_{\gamma_2}|}\,,\quad \frac{\vartheta^2}{2m} = |Z_{\gamma_1}| + |Z_{\gamma_2}| - |Z_{\gamma_1+\gamma_2}|$$

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with unusual gyromagnetic ratio $g = 4$. *[D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007]*

Two-body electron-monopole problem I

- Depending on $\mathrm{sign}(q\vartheta)$, there may or may not be SUSY bound states:



- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r}\partial_r^2\, r + \frac{\nu^2 - q^2 - \frac{1}{4}}{r^2} + \left(\vartheta - \frac{q}{r}\right)^2 \right] \Psi(r) = k^2 \Psi\,,$$

where $\nu = j + \frac{1}{2} + h\,, \quad j = |q| + h + \ell\,, \ell \in \mathbb{N}$, much like H-atom !

Two-body electron-monopole problem II

- Supersymmetric bound states exist for $q\vartheta > 0$, $h = -1/2$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac{1}{2}$, with $2j + 1 = |\langle\gamma_1, \gamma_2\rangle|$.

Denef 2002

- The density of states in the continuum $E > \frac{\vartheta^2}{2m}$ can be computed from the S-matrix for partial waves, similar to that for H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)} = e^{2\mathrm{i}\delta_\nu(k)}.$$

BP, arXiv:1501.01643

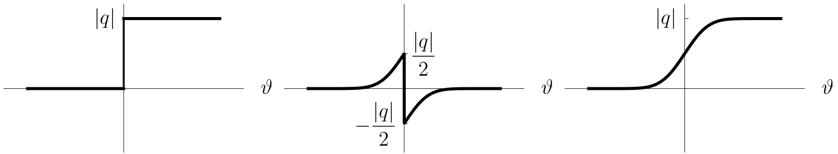
- The contribution of the continuum to $\mathrm{Tr}(-1)^F\, e^{-2\pi R H}$ is thus

$$\sum_{h=0^2,\pm\frac{1}{2}} (-1)^{2h} \sum_{\ell=0}^{\infty} \int_{k=\vartheta}^{\infty} \frac{\mathrm{d}k\,\partial_k}{2\pi\mathrm{i}} \log \frac{\Gamma\left(|q|+\ell+2h+1+\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}{\Gamma\left(|q|+\ell+2h+1-\mathrm{i}\frac{q\vartheta}{\sqrt{k^2-\vartheta^2}}\right)}\, e^{-\frac{\pi R k^2}{m}}$$

Two-body electron-monopole problem III

- After miraculous cancellations, we get

$$\begin{aligned}\mathrm{Tr}(-1)^F\, e^{-2\pi R H} =& |2q|\,\Theta(q\vartheta) + \frac{2q\vartheta}{\pi} \int_{k=|\vartheta|}^{\infty} \frac{\mathrm{d}k}{k\sqrt{k^2-\vartheta^2}} e^{-\frac{\pi R k^2}{m}} \\ =& 2|q|\,\Theta(q\vartheta) - |q|\,\mathrm{sgn}(q\vartheta)\,\mathrm{Erfc}\left(|\vartheta|\sqrt{\frac{\pi R}{m}}\right) \\ =& |q| + q\,\mathrm{Erf}\left(\vartheta\sqrt{\frac{\pi R}{m}}\right) .\end{aligned}$$



BPS index from localization I

- For more than 2 constituents, the quantum mechanics is no longer solvable, but it is still possible to count supersymmetric ground states using supersymmetric localization *[Manschot BP Sen'10]*.
- The space of classical BPS solutions modulo translations has real dimension $2n-2$ and carries a symplectic structure:

$$\mathcal{M}_n(\{\gamma_i,\vartheta_i\}) = \left\{ (\vec{x}_i) \in \mathbb{R}^{3n},\ \forall i \sum_{j\neq i} \frac{\gamma_{ij}}{|\vec{x}_i-\vec{x}_j|} = \vartheta_i \right\} / \mathbb{R}^3$$

$$\omega = \frac{1}{4}\sum_{i<j} \epsilon_{abc} \frac{\gamma_{ij}}{r_{ij}^3}\, \mathrm{x}_{ij}^a\, \mathrm{dx}_{ij}^b \wedge \mathrm{dx}_{ij}^c, \quad \vec{J} = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\frac{\vec{x}_{ij}}{r_{ij}}$$

- BPS states correspond to harmonic spinors on $\mathcal{M}_n(\{\gamma_i,\vartheta_i\})$. The refined index, counting states with fugacity y conjugate to J_3, is then given by the equivariant Dirac index $\mathrm{Ind}((\mathcal{M}_n,\omega),y)$

de Boer El Showk Messamah Van.d.Bleeken'08

BPS index from localization II

- When $\mathcal{M}_n(\{\gamma_i,\vartheta_i\})$ is compact, the Dirac index is computable by localization with respect to $U(1)\subset SO(3)$. Fixed points are collinear configurations subject to 1D version of Denef's eqs,

$$\sum_{j\neq i}\frac{\gamma_{ij}}{|z_i-z_j|}=\vartheta_i \quad\Longleftrightarrow\quad \partial_{z_i}W=0$$

  where $W=-\sum_{i<j}\gamma_{ij}\log|z_j-z_i|-\sum_i\vartheta_i z_i$ [Manschot BP Sen'10-11].
- Solutions are classified by the ordering along the z-axis,

$$\mathrm{Ind}_C(\{\gamma,\vartheta\};\,y)=\frac{(-1)^{\sum_{i<j}\gamma_{ij}-n+1}}{(y-1/y)^{n-1}}\sum_{\sigma\in S_n}F_C(\{\gamma_{\sigma(i)},\vartheta_{\sigma(i)}\})\,y^{\sum_{i<j}\gamma_{\sigma(i)\sigma(j)}}$$

  where the planar Coulomb index $F_C(\{\gamma_i,\vartheta_i\}$ counts the number of solutions with $z_1<z_2<\dots<z_n$, weighted by $\operatorname{sign}\det\partial^2 W$.

BPS index from localization III

- When $\mathcal{M}_n(\{\gamma_i, u_i\})$ is not compact, i.e. in the presence of scaling solutions, there are additional boundary contributions which ensure that $\mathrm{Ind}(\{\gamma, c\};\, y)$ is a symmetric Laurent polynomial in y, hence has a smooth limit as $y \to 1$.
- Alternatively, one can replace $F_{C,n}(\{\gamma_i, \vartheta_i\}$ by the planar tree index $F_{\rm tree}(\{\gamma_i, \vartheta_i\})$ which counts planar attractor flow trees *[Alexandrov BP'18]*. This is the relevant choice for D4-D2-D0 indices at the large volume attractor point.
- For $n = 2$, $\mathcal{M}_2 = (S^2, 2q \sin\theta \mathrm{d}\theta \mathrm{d}\psi)$ is compact and the two procedures agree, leading to (up to overall sign)

$$\mathrm{Ind}_C(\{\gamma, c\};\, y) = \frac{y^{2q} - y^{-2q}}{y - 1/y} H(q\vartheta) \xrightarrow{y \to 1} 2q\, H(q\vartheta)$$

Witten index from localization I

- The Witten index $\mathrm{Tr}(-1)^F e^{-\beta H}$ differs from the BPS index due to contribution of scattering states. Fortunately, it localizes on time-independent configurations *[Girardello Imimbo Mukhi'83]*

$$\mathcal{I}_n = \int \prod_{i=1}^{n-1} \frac{\mathrm{d}^3\vec{x}_i\, \mathrm{d}\bar{\lambda}_i\, \mathrm{d}\lambda_i\, \mathrm{d}D_i}{4\pi^2\beta}\, e^{-\beta\left[\sum_{i=1}^{n} \mathrm{i} U_i D_i + \sum_{i,j}\left(\frac12 M_{ij} D_i D_j + \vec{\nabla}_j U_i \bar{\lambda}_i \vec{\sigma} \lambda_j\right)\right]}$$

where $M_{ij} = m_i \delta_{ij} - \frac{m_i m_j}{m_{\rm tot}}$ is the reduced mass matrix.

- Integrating out the fermions produces

$$\int \prod_{i=1}^{n-1} \mathrm{d}\bar{\lambda}_i\, \mathrm{d}\lambda_i\, e^{-\beta \sum_{i,j=1}^{n-1} \vec{\nabla}_j U_i\, \bar{\lambda}_i \vec{\sigma} \lambda_j} = (\beta^2)^{n-1} \det(\vec{\nabla}_j U_i \otimes \vec{\sigma}),$$

Witten index from localization II

- Key observation: The bosonic configuration space $\mathbb{R}^{3n-3}$ is foliated by the BPS phase spaces $\mathcal{M}_n(\{\gamma_i, u_i\})$ with $u_i = \sum_j \frac{\gamma_{ij}}{|\vec{x}_i - \vec{x}_j|} \in \mathbb{R}^{n-1}$. The flat integration measure on $\mathbb{R}^{3n-3}$ combines with the fermionic determinant to produce

$$\boxed{\prod_{i=1}^{n-1} \mathrm{d}^3\vec{x}_i \, \det(\vec{\nabla}_i U_j \otimes \vec{\sigma}) = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \left(\prod_{i=1}^{n-1} \mathrm{d}u_i \right) \, \omega^{n-1},}$$

Proof: follows from $\det(Q + \mathrm{i}M) = \mathrm{pf}\begin{pmatrix} M & Q \\ -Q & M \end{pmatrix}$.

- For $n=2$, this boils down to $r^2 \mathrm{d}r \mathrm{d}\Omega_2 \times -\frac{q^2}{r^4} = q \, \mathrm{d}\frac{1}{r} \times \frac{1}{2}\kappa \mathrm{d}\Omega_2$.

Witten index from localization III

- Integrating over the auxiliary fields D_i, we get

$$\mathcal{I}_n = \sqrt{\det \tfrac{\beta}{2\pi M}} \int \prod_{i=1}^{n-1} \mathrm{d}u_i \; \mathrm{Vol}(\{\gamma_i, u_i\})\, e^{-\frac{\beta}{2}(u_i-\vartheta_i)M^{-1}_{ij}(u_j-\vartheta_j)}$$

$$\mathrm{Vol}(\{\gamma_i, u_i\}) := \frac{(-1)^{\sum_{i<j}\gamma_{ij}-n+1}}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n(\{\gamma_i,u_i\})} \omega^{n-1} = \lim_{y\to 1} \mathrm{Ind}(\{\gamma_i, u_i\})$$

- The refined index is *expected* to be given by a similar formula, replacing the symplectic volume with the equivariant Dirac index,

$$\mathcal{I}_n(y) = \sqrt{\det \tfrac{\beta}{2\pi M}} \int \prod_{i=1}^{n-1} \mathrm{d}u_i \; \mathrm{Ind}(\{\gamma_i, u_i\}, y)\, e^{-\frac{\beta}{2}(u_i-\vartheta_i)M^{-1}_{ij}(u_j-\vartheta_j)}$$

- How the A-roof genus arises by integrating out fermions remains to be understood...

Witten index from localization IV

- Both $\mathrm{Vol}(\{\gamma_i, u_i\})$ and $\mathrm{Ind}(\{\gamma_i, u_i\}, y)$ are locally constant functions of u_i, away from walls of marginal stability. Thus the Witten index is a linear combination of generalized error functions !
- At zero temperature, this is dominated by $u_i = \vartheta_i$, hence reduces to $\mathrm{Ind}(\{\gamma_i, \vartheta_i\}, y)$ counting supersymmetric bound states.
- For $n = 2$, using $\mathrm{Vol}(\{\gamma_i, u_i\}) = \kappa H(\kappa u)$, with $\kappa = \gamma_{12} = 2q$,

$$\begin{aligned}\mathcal{I}_2 =& \kappa \sqrt{\frac{\beta}{2\pi m}}, \int \mathrm{d}u \, H(\kappa u) \, e^{-\frac{\beta(u-\vartheta)^2}{2m}} \\ =& -\frac{\kappa}{2}\left[\mathrm{sign}(\kappa) + E_1\left(\vartheta\sqrt{\frac{\beta}{2\pi m}}\right)\right]\end{aligned}$$

- This works for any number of centers, and only requires knowing the planar index $F_{\mathrm{tree},n}$, which is computable recursively.

BPS index from localization I

- For three centers, setting $\gamma_{2+3,1} = \gamma_{21} + \gamma_{31}$, $m_{1+2,3} = \frac{m_3(m_1+m_2)}{(m_1+m_2+m_3)}$

$$F_{\mathrm{tree},3} = \frac{1}{4}\left[(\mathrm{sgn}(\vartheta_1) + \mathrm{sgn}(\gamma_{12}))\,(\mathrm{sgn}(\vartheta_1 + \vartheta_2) + \mathrm{sgn}(\gamma_{23}))\right.$$
$$\left. - \left(\mathrm{sgn}(\gamma_{2+3,1}) + \mathrm{sgn}(\gamma_{12})\right)\,\left(\mathrm{sgn}(\gamma_{3,1+2}) + \mathrm{sgn}(\gamma_{23})\right)\right]$$

we find the planar Witten index (to be summed over permutations)

$$\mathcal{J}_3 = \frac{1}{4}\Big[E_2\Big(\sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}};\ \frac{\vartheta_2 m_1 - \vartheta_1 m_2}{\sqrt{m_1 m_2(m_1+m_2)}},\ \frac{\vartheta_3}{\sqrt{m_{1+2,3}}}\Big)$$
$$- \ \mathrm{sgn}(\gamma_{1,2+3})\,\mathrm{sgn}(\gamma_{1+2,3})$$
$$- \ \Big[E_1\Big(\frac{\vartheta_3}{\sqrt{m_{1+2,3}}}\Big) - \mathrm{sgn}(\gamma_{1+2,3})\Big]\,\mathrm{sgn}(\gamma_{12})$$
$$+ \ \Big[E_1\Big(\frac{\vartheta_1}{\sqrt{m_{1,2+3}}}\Big) - \mathrm{sgn}(\gamma_{2+3,1})\Big]\,\mathrm{sgn}(\gamma_{23})\Big].$$

BPS index from localization II

- At the large volume attractor point, only contributions from the continuum of scattering states remain:

$$\mathcal{J}_3^* = \frac{1}{4}\Bigg(M_2\left(\sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}}; -\frac{\sqrt{\tau_2}\left(m_2\gamma_{1,2+3}-m_1\gamma_{2,1+3}\right)}{\sqrt{m_1 m_2(m_1+m_2)}}, -\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}}\right)$$
$$+ M_1\left(\frac{\sqrt{\tau_2}\gamma_{1+2,3}}{\sqrt{m_{1+2,3}}}\right)\left(\mathrm{sgn}\left(m_2\gamma_{1,2+3} - m_1\gamma_{2,1+3}\right) - \mathrm{sgn}\left(\gamma_{12}\right)\right)$$
$$+ M_1\left(\frac{\sqrt{\tau_2}\gamma_{1,2+3}}{\sqrt{m_{1,2+3}}}\right)\left(\mathrm{sgn}\left(m_2\gamma_{1+2,3} - m_3\gamma_{1+3,2}\right) - \mathrm{sgn}\left(\gamma_{23}\right)\right)\Bigg).$$

where M_n are complementary error functions generalizing

$$M_1(x) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}-\mathrm{i}x}\frac{\mathrm{d}z}{z}e^{-\pi z^2-2\pi\mathrm{i}z} = -\,\mathrm{sign}(x)\,\mathrm{Erfc}(|x|\sqrt{\pi})$$

BPS index from localization III

- Setting $\beta = 2\pi\tau_2$, we get precisely the non-holomorphic terms appearing in the indefinite theta series $\Theta_{p,\mu}$...
- ...at least for one-parameter models, ignoring some subtleties such as Kronecker delta contributions supported on walls, and ambiguities in defining $\mathrm{sign}(0)$...
- Once such issues are resolved, this should provide a short cut to, and physical understanding of, the non-holomorphic terms in the modular completion of generating series of DT invariants.
- Recently *[Alexandrov Bendriss'24]* constructed explicit solutions to the modular anomaly equations for arbitrary D4 charge, up to modular ambiguities. Unfortunately, computing the first few terms in the series is still out of reach for $[D4] > 2$, due to our lack of control on higher genus GV invariants. New insights on $Z_{\rm top}$ are needed !

Thanks for your attention !

Back up slide

- Actually, the modular completion for 2 centers includes a Gaussian term which remains mysterious:

$$\frac{1}{2}\left(\gamma_{12}\,M_1\left(\sqrt{\frac{2\tau_2}{m_{12}}}\gamma_{12}\right)+\frac{1}{\pi}\sqrt{\frac{m_{12}}{2\tau_2}}\,e^{-\frac{2\pi\tau_2\gamma_{12}^2}{m_{12}}}\right)$$

- The Gaussian term arises in the limit $y\to 1$ from the refined index

$$\frac{y^{\gamma_{12}}}{y-1/y}\left[E_1\left(\sqrt{\frac{2\tau_2}{m_{12}}}(\gamma_{12}+\eta m_{12})\right)-\mathrm{sign}(\gamma_{12})\right]+(1\leftrightarrow 2)$$

where $\eta=\frac{\mathrm{Re}\log y}{2\pi\tau_2}$, but the origin of this shift is unclear.