Methods for BPS state counting in type II strings on CY3-folds

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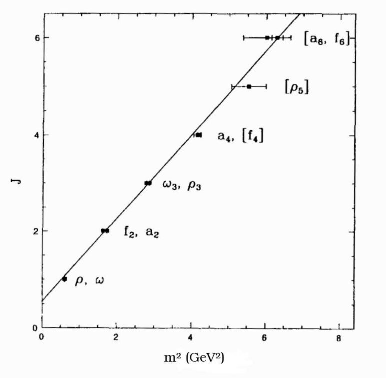
Summer school “Algebra and Geometry meets Quantum Fields and Strings”, DESY / Hamburg, Sep 7-11, 2026

Initial plan for the lectures

1. Overview
2. VM moduli spaces in type II/CY3 and GW/GV invariants
3. BPS indices and Donaldson-Thomas invariants
4. Quiver quantum mechanics and multi-centered black holes
5. BPS spectra on local CY threefolds
6. HM moduli spaces in type II/CY3 (if time permits)

1. A very brief history of string theory

Born as a model of hadronic bound states/resonances *[Veneziano'68]*, reborn as a candidate quantum theory of gravity *[Scherk Schwarz'74]*, String Theory is now (among other virtues) an extraordinarily effective discovery machine for mathematics.



Effective action in II/CY3

- In IIA on X, the moduli space factorizes into vector and hypermultiplets: $\mathcal{M}_4 = \mathcal{M}_V(X) \times \mathcal{M}_H(X)$
  - $\mathcal{M}_V$: special Kähler (Kähler structure + B-field); $\dim_{\mathbb{C}} \mathcal{M}_V = h_{1,1}(X)$,
  - $\mathcal{M}_H$: quaternion-Kähler (complex structure, RR fields, dilaton); $\dim_{\mathbb{H}} \mathcal{M}_H = h_{2,1}(X) + 1$;
- Similarly in IIB/$\widetilde{X}$, $\mathcal{M}_4 = \widetilde{\mathcal{M}}_V(\widetilde{X}) \times \widetilde{\mathcal{M}}_H(\widetilde{X})$ $\dim_{\mathbb{C}} \mathcal{M}_V = h_{2,1}(\widetilde{X})$, $\dim_{\mathbb{H}} \mathcal{M}_H = h_{1,1}(\widetilde{X}) + 1$
- Dilaton sits in $\mathcal{M}_H \Rightarrow$ SK metric on $\mathcal{M}_V$ is exact at tree level, while QK metric on $\mathcal{M}_H$ receives quantum corrections.

Mirror symmetry and enumerative geometry

- Mirror symmetry exchanges

$$h_{1,1}(X) \leftrightarrow h_{2,1}(\widetilde{X}),\ \mathcal{M}_V(X) \leftrightarrow \widetilde{\mathcal{M}}_V(\widetilde{X}),\ \mathcal{M}_H(X) \leftrightarrow \widetilde{\mathcal{M}}_H(\widetilde{X})$$

- Focussing on $\mathcal{M}_V(X)$, *[Candelas de la Ossa Green Parkes'90]* reduced the counting on rational curves on the quintic threefold X_5 to a period integral computation on the mirror quintic $\widetilde{X}_5 = X_5/\mathbb{Z}_5^3$:

$$\begin{aligned} Y_{ttt} =& \kappa + \sum_{k\geq 1} k^3\ \mathrm{GW}_k^{(0)}\, q^d \\ =& 5 + \mathbf{2875}\, \frac{q}{1-q} + \mathbf{609250}\, \frac{2^3 q^2}{1-q^2} + \mathbf{317206375}\, \frac{3^3 q^3}{1-q^3} + \dots , \end{aligned}$$

- $\mathrm{GW}_\beta^{(g)} \in \mathbb{Z}$, related to $\mathrm{GV}_\beta^{(g)} \in \mathbb{Z}$ by multicover formula

$$\sum_{g=0}^{\infty} \sum_{\beta>0} \mathrm{GW}_\beta^{(g)}\, e^{2\pi \mathrm{i} t\cdot\beta} = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\beta>0} \frac{GV_\beta^{(g)}}{k} \left(2 \sin \tfrac{k\lambda}{2}\right)^{2g-2} e^{2\pi \mathrm{i} k\, t\cdot\beta}$$

BPS states and BPS index

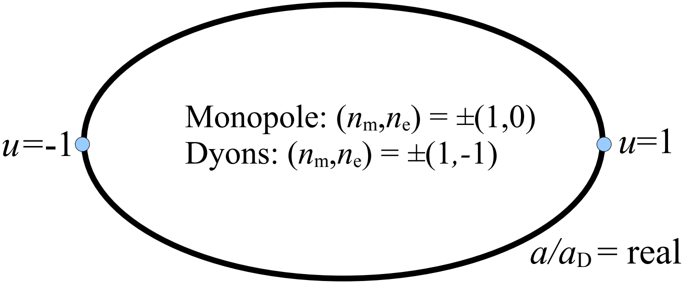
- Due to $\mathcal{N}=2$ SUSY, any state satisfies the BPS bound $M\ell_P \geq |Z_z(\gamma)|$ where γ is the electromagnetic charge vector, valued in the lattice $K(X)$, and Z_z is a linear form on Γ, varying holomorphically with the VM scalars z.
- BPS states saturate the inequality, and form short representation of the SUSY algebra. They can only disappear by pairing up with another short representation, or by decaying into the continuum of multi-particle states.
- The decay $[\gamma_1+\gamma_2] \to [\gamma_1] \oplus [\gamma_2]$ is only possible on codimension 1 walls of marginal stability inside $\mathcal{M}_V(X)$, where the central charges Z_{γ_1} and Z_{γ_2} are aligned, in particular $\Im[Z_{\gamma_1}\overline{Z_{\gamma_2}}]=0$.
- The BPS index $\Omega_z(\gamma) := \mathrm{Tr}'_{z,\gamma}(-1)^{2J_3}$ counting BPS states up to sign is independent of HM moduli, but may jump across walls of marginal stability, as is familiar from Seiberg-Witten theory.

Wall-crossing in Seiberg-Witten theory

u-plane

$u=\infty$

$Z((n_{\rm m},n_{\rm e});u)=a(u)\, n_{\rm e}+a_{\rm D}(u)\, n_{\rm m}$



W-bosons: $(n_{\rm m},n_{\rm e}) = \pm(0,1)$
Dyons: $(n_{\rm m},n_{\rm e}) = \pm(1,n)$

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BPS indices and Donaldson-Thomas invariants

- At weak string coupling, BPS states are represented by Dp-branes wrapped on p-dimensional supersymmetric cycles in the internal space. In IIA, those are those are the D0-brane, D2-brane wrapped on an effective curve C, D4-brane wrapped on an effective divisor $\mathcal{D}$ and D6-brane wrapped on X. In IIB, D3-branes wrapped on special Lagrangian submanifolds of $\widetilde{X}$.
- More generally, multi-particle BPS states are objects in a triangulated category, $\mathcal{C} = D^b\,\mathrm{Coh}(X)$ in IIA/X or $\widetilde{\mathcal{C}} = \mathrm{Fuk}(\widetilde{X})$ in $IIB/\widetilde{X}$. One-particle BPS states correspond to stable objects with respect to a stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}\,\mathcal{C}$ [Bridgeland 2007].
- The mathematical counterpart of the BPS index is the Donaldson-Thomas invariant $\Omega_\sigma(\gamma)$, roughly given by the (weighted) Euler number of the moduli space of semi-stable objects $E \in \mathcal{A}$ of charge γ, where *semi-stable* means that $\arg Z(E') \leq \arg Z(E)$ for any subobject $0 \neq E' \subset E$.

2. Reminder on triangulated categories

- Recall that (topological) branes are objects of a triangulated category. This is an additive category equipped with an isomorphism $\mathcal{E} \mapsto \mathcal{E}[1]$ and a collection of exact triangles

$$\begin{tikzcd} & C \arrow[dl, "c"', "{[1]}"] & \\ A \arrow[rr, "a"] & & B \arrow[ul, "b"'] \end{tikzcd},$$

satisfying various axioms; in particular, If $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ is exact, then the two cyclic permutations are also exact:

$$B \xrightarrow{c} C \xrightarrow{a} A[1] \xrightarrow{-b[1]} B[1]$$
$$C[-1] \xrightarrow{-c[-1]} A \xrightarrow{a} B \xrightarrow{b} C$$

Physically, [1] is CPT and the triangles encode the allowed decays $B \mapsto A + C, C \mapsto B + \overline{A}, A \mapsto B + \overline{C}$.

$D\,\mathrm{Coh}(X)$ is a triangulated category.

- Given an Abelian category $\mathcal{A}$, recall that the (bounded) derived category $D(\mathcal{A})$ has objects given by (finite length) complexes $\mathcal{E} = (E^n)_{n\in\mathbb{Z}}$ of objects $E^n \in \mathcal{A}$, $\mathrm{ch}\,\mathcal{E} := \sum_n (-1)^n \,\mathrm{ch}\, E^n$, and morphisms given by 1) chain maps $f \in \mathrm{Hom}(\mathcal{E},\mathcal{F})$ modulo chain homotopies, i.e. $f \sim g$ for which the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r, "d^{\mathcal{E}}_{n-2}"] & \mathcal{E}^{n-1} \arrow[r, "d^{\mathcal{E}}_{n-1}"] \arrow[d, "f_{n-1}{,}g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \mathcal{E}^{n} \arrow[r, "d^{\mathcal{E}}_{n}"] \arrow[d, "f_n{,}g_n"] \arrow[dl, "h_n"'] & \mathcal{E}^{n+1} \arrow[r, "d^{\mathcal{E}}_{n+1}"] \arrow[d, "f_{n+1}{,}g_{n+1}"] \arrow[dl, "h_{n+1}"'] & \dots \arrow[dl, "h_{n+2}"'] \\
\dots \arrow[r, "d^{\mathcal{F}}_{n-2}"] & \mathcal{F}^{n-1} \arrow[r, "d^{\mathcal{F}}_{n-1}"] & \mathcal{F}^{n} \arrow[r, "d^{\mathcal{F}}_{n}"] & \mathcal{F}^{n+1} \arrow[r, "d^{\mathcal{F}}_{n+1}"] & \dots,
\end{tikzcd}$$

with $f_n - g_n = d^{\mathcal{F}}_{n-1} h_n + h_{n+1} d^{\mathcal{E}}_n$ for all n; and 2) by formally inverting quasi-isomorphisms. The latter are maps $f \in \mathrm{Hom}(\mathcal{E},\mathcal{F})$ such that $f^n_* : H^n(\mathcal{E}) \to H^n(\mathcal{F})$ is an isomorphism.

$D\,\mathrm{Coh}(X)$ is a triangulated category.

- $D(\mathcal{A})$ has a shift functor $[m]$ which shifts complexes by one place to the left.
- Triangles come from the mapping cone construction: Given two objects $\mathcal{E}, \mathcal{F}$ and a morphism $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$, one can construct a new object $\mathcal{G}$, called the mapping cone of f and denoted by $\mathrm{cone}(f)$,

$$\cdots \longrightarrow \begin{array}{c} E^{-1} \\ \oplus \\ F^0 \end{array} \xrightarrow{\begin{pmatrix} d^{\mathcal{E}} & 0 \\ f & d^{\mathcal{F}} \end{pmatrix}} \begin{array}{c} E^{0} \\ \oplus \\ F^1 \end{array} \xrightarrow{\begin{pmatrix} d^{\mathcal{E}} & 0 \\ f & d^{\mathcal{F}} \end{pmatrix}} \begin{array}{c} E^{1} \\ \oplus \\ F^2 \end{array} \longrightarrow \cdots$$

which fits in the exact sequence $\mathcal{E} \xrightarrow{f} \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$. It describes a tachyon condensate on a system of a brane $\mathcal{F}$ and anti-brane $\overline{\mathcal{E}}$, with f being the tachyon vev.

Stability condition on a triangulated category - Def. 1

A pair $(Z, \mathcal{P})$ where $Z : K(\mathcal{C}) \to \mathbb{C}$ is a group homomorphism (called central charge), and $\mathcal{P}(\phi)$ (called slicing) is a full additive subcategory for every $\phi \in \mathbb{R}$ such that

a) for all $E \in \mathcal{P}(\phi)$, $Z(E) = m(E) e^{\mathrm{i}\pi\phi}$ for some $m(E) > 0$

b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$

c) for all $\phi_1 > \phi_2$, $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$

d) for any non-zero object $E \in \mathcal{C}$, there are a finite decreasing sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and a collection of exact triangles (called Harder-Narasimhan filtration)

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & E_{n-1} \arrow[rr] \arrow[dl] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, "{[1]}"] & & A_2 \arrow[ul, "{[1]}"] & & A_3 \arrow[ul, "{[1]}"] & & A_{n-1} \arrow[ul, "{[1]}"] & & A_n \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

such that $A_j \in \mathcal{P}(\phi_j)$ for every j.

Stability condition on a triangulated category - Def. 2

- Equivalently, a pair $(Z, \mathcal{A})$ where $Z : K(\mathcal{C}) \to \mathbb{C}$ is again a group homomorphism and $\mathcal{A}$ is a *heart of a bounded t-structure* such that for any non-zero element $E \in \mathcal{A}$, the central charge $Z(E)$ lies in the upper half plane minus the real positive axis, i.e.

$$\forall 0 \neq E \subset \mathcal{A}, \quad \Im Z(E) > 0 \quad \text{or} \quad (\Im Z(E) = 0 \text{ and } \Re Z(E) < 0)$$

- Recall that a *t*-structure is a full subcategory $\mathcal{F} \subset \mathcal{C}$ such that $\mathcal{F}[1] \subset \mathcal{F}$ and for all $E \in \mathcal{C}$, there exists a distinguished triangle $F \to E \to G$ such that $F \in \mathcal{F}$ and $G \in \mathcal{F}^\perp$, with $\mathcal{F}^\perp := \{G : \mathrm{Hom}(F, G) = 0 \forall F \in \mathcal{F}\}$. The heart of $\mathcal{F}$ is the intersection $\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1]$, and is a full Abelian subcategory.
- The equivalence between slicings $\mathcal{P}$ and hearts $\mathcal{A}$ is via $\mathcal{F} = \mathcal{P}(\mathbb{R}_{>0})$ and $\mathcal{A} = \mathcal{P}((0, 1])$, where $\mathcal{P}(I)$ is the extension-closed subcategory of $\mathcal{C}$ generated by $\{\mathcal{P}(\phi), \phi \in I\}$.

Space of stability conditions and DT invariants

- It turns out that an extra support condition is necessary to ensure a nice wall-crossing formula:
  e) There exists a constant $C > 0$ such that, for all σ-semistable objects $E \in \mathcal{A}$, $\|\gamma(E)\| \leq C\,|Z(E)|$ where $\|\cdot\|$ is any fixed Euclidean norm on $\Gamma \otimes \mathbb{R}$.
- With this extra axiom, any connected component in the space $\mathrm{Stab}\,\mathcal{C}$ is then locally homeomorphic to $\Gamma \otimes \mathbb{C}$, in particular has complex dimension $\dim_{\mathbb{C}} \mathrm{Stab}\,X = \mathrm{rk}\,\Gamma$ *[Bridgeland 2007]*.
- For any generic $\sigma \in \mathrm{Stab}\,\mathcal{C}$ and primitive $\gamma \in \Gamma$, the DT invariant $\Omega_\sigma(\gamma)$ is defined as the (weighted) Euler number of the moduli space $\mathcal{M}_\sigma(\gamma)$ of σ-semistable objects in $\mathcal{A}$ with class γ.
- For non-primitive $\gamma \in \Gamma$,, one first defines a rational invariant $\overline{\Omega}_\sigma(\gamma) \in \mathbb{Q}$ following Joyce (2008), and then sets

$$\Omega_\sigma(\gamma) = \sum_{k|\gamma} \frac{\mu(k)}{k^2}\,\overline{\Omega}_\sigma(\gamma/k),$$

  where $\mu(k)$ is the Möbius function.

Motivic DT invariants

- More generally, one may define motivic rational invariants $\bar\Omega_\sigma(\gamma, y)$ given in favorable cases by the Laurent-Poincaré polynomial of $\mathcal{M}_\sigma(\gamma)$, and their 'integer' counterpart

$$\Omega_\sigma(\gamma, y) = \sum_{k|\gamma} \frac{\mu(k)(y-1/y)}{k(y^k-1/y^k)}\,\bar\Omega_\sigma(\gamma/k, y^k),$$

- Physically, the refinement parameter y is a fugacity for angular momentum. Spatial rotations act on $\mathcal{M}_\sigma(\gamma)$ via Lefschetz action.

Kontsevich-Soibelman wall-crossing formula

- Following KS we introduce the quantum torus algebra A with generators $e_\gamma, \gamma \in \Gamma$ satisfying

$$e_\gamma\, e_{\gamma'} = (-y)^{\langle \gamma,\gamma'\rangle}\, e_{\gamma+\gamma'}$$

- For any ray ℓ in the upper half-plane $\mathcal{H}$, and for σ generic, define the element U_ℓ in the (pronilpotent completion of the) group $G = e^A$

$$U_\sigma(\ell) = \prod_{\gamma: Z_\sigma(\gamma)\in\ell} \exp\left(\frac{\bar\Omega_\sigma(\gamma,y)\, e_\gamma}{y^{-1}-y}\right) = \prod_{\gamma: Z_\sigma(\gamma)\in\ell} \mathrm{Exp}\left(\frac{\Omega_\sigma(\gamma,y)\, e_\gamma}{y^{-1}-y}\right)$$

- For any strict sector $V \subset \mathcal{H}$, define $U_\sigma(V) = \prod_{\ell\in V} U_\sigma(\ell)$ where the product over all active rays ℓ, ordered according to the phase.
- The KS wall-crossing formula states that, for any path $t \in [0,1] \mapsto \sigma(t)$ in $\mathrm{Stab}\, X$ such that no active ray exits or enters the sector V, then $U_{\sigma(0)}(V) = U_{\sigma(1)}(V)$.

Wall-crossing formula - infinitesimal version

- For an infinitesimal path across a wall where the phase of two primitive vectors γ_1, γ_2 become aligned,

$$\prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\downarrow}} U_+(M\gamma_1+N\gamma_2) = \prod_{\substack{M\geq 0, N\geq 0,\\ \gcd(M,N)=1, M/N\uparrow}} U_-(M\gamma_1+N\gamma_2)$$

where

$$U_\pm(\gamma) = \exp\left(\sum_{k\geq 1} \frac{\overline{\Omega}_\pm(k\gamma, y)}{1/y-y} e_{k\gamma}\right)$$

- Using the BCH formula $e^X e^Y e^{-X} = e^{e^X Y e^{-X}}$ and truncating to the subalgebra $A_{M,N}$ spanned by $e_{m\gamma_1+n\gamma_2}$ with $m\leq M, n\leq N$, one can compute $\Delta\Omega(M,N) := \overline{\Omega}_+(M\gamma_1+N\gamma_2) - \overline{\Omega}_-(M\gamma_1+N\gamma_2)$

Wall-crossing formula - exemples

Setting $\kappa(m) = \frac{(-y)^m - (-y)^{-m}}{y - 1/y}$ and $\gamma_{12} = \langle \gamma_1, \gamma_2 \rangle$,

$$
\begin{aligned}
\Delta\overline{\Omega}(1,1) =& \kappa(\gamma_{12})\, \overline{\Omega}_+(0,1)\, \overline{\Omega}_+(1,0) \\
\Delta\overline{\Omega}(1,2) =& \kappa(2\gamma_{12})\, \overline{\Omega}_+(0,2)\, \overline{\Omega}_+(1,0) \\
& + \frac{1}{2}[\kappa(\gamma_{12})]^2\, [\overline{\Omega}_+(0,1)]^2\, \overline{\Omega}_+(1,0) + \kappa(\gamma_{12})\, \overline{\Omega}_+(0,1)\, \overline{\Omega}_+(1,1) \\
\Delta\overline{\Omega}(2,2) =& \kappa(4\gamma_{12})\, \overline{\Omega}_+(0,2)\, \overline{\Omega}_+(2,0) \\
& + \kappa(2\gamma_{12}) \left[\overline{\Omega}_+(1,0)\overline{\Omega}_+(1,2) + \overline{\Omega}_+(0,1)\overline{\Omega}_+(2,1)\right] \\
& + \kappa(\gamma_{12})\, \kappa(2\gamma_{12})\, \overline{\Omega}_+(0,1)\overline{\Omega}_+(1,0)\overline{\Omega}_+(1,1) \\
& + \frac{1}{2}[\kappa(2\gamma_{12})]^2 \left[\overline{\Omega}_+(2,0)\overline{\Omega}_+(0,1)^2 + \overline{\Omega}_+(0,2)\overline{\Omega}_+(1,0)^2\right] \\
& + \frac{1}{4}[\kappa(\gamma_{12})]^2\, \kappa(2\gamma_{12})\, \overline{\Omega}_+(0,1)^2\overline{\Omega}_+(1,0)^2
\end{aligned}
$$

Each contribution has a multi-centered black hole interpretation (see later).

Relation to KS symplectomorphisms in GMN story

- Define $\chi_\gamma := \frac{e_\gamma}{1/y-y}$ and take the limit $y \to 1$: satisfies the Lie algebra of (twisted) Hamiltonian vector fields on a symplectic torus

$$[\chi_\gamma, \chi_{\gamma'}] = (-1)^{\langle\gamma,\gamma'\rangle} \langle\gamma,\gamma'\rangle \chi_{\gamma+\gamma'}$$

- The factor $(-1)^{\langle\gamma,\gamma'\rangle}$ can be absorbed by introducing a quadratic refinement $\psi(\gamma)\psi(\gamma') = (-1)^{\langle\gamma,\gamma'\rangle}\psi_{\gamma+\gamma'}$
- The operators $K_\gamma := \mathrm{Exp}\left(\frac{e_\gamma}{y^{-1}-y}\right)$, in the limit $y \to 1$, acts by

$$\chi_{\gamma'} \mapsto K_\gamma \chi_{\gamma'} K_\gamma^{-1} = \chi_{\gamma'}(1-\chi_\gamma)^{\langle\gamma,\gamma'\rangle}$$

- The operator $U_\sigma(\ell)$ is recognized as $\prod_{\gamma: Z_\sigma(\gamma)\in\ell} K_\gamma^{\Omega_\sigma(\gamma)}$ in GMN construction (see Laura's lectures).

3. Space of stability conditions and physical slice

- The space of stability condition, if not empty, is a complex space of dimension $\dim_{\mathbb{C}} = \mathrm{rk}(\Gamma) = 2\dim_{\mathbb{C}} \mathcal{M}_V(X)$. The physical stability conditions form a complex Lagrangian subspace, determined by the genus zero prepotential (hence by mirror symmetry).
- The mathematical construction of stability conditions $\mathcal{C} = D^b \,\mathrm{Coh}\, X$ for a compact CY3 X is a difficult problem. Weak stability conditions (where the support property is relaxed) are much easier to construct, and still satisfy the wall-crossing formula.
- Until recently, stability conditions on compact CY 3-folds were only known to exist near large volume, by suitably deforming a class of weak stability conditions subject to a certain conjectural inequality *[Bayer Macri Toda 2011]*. This is what I am going to explain now.
- Recently, *[Chuny Li 2006]* has obtained stability conditions near large volume for any smooth projective variety without assuming the BMT inequality. Still, BMT provides very useful empty regions !

Constructing stability conditions for compact CY3

- Let X is a smooth project CY threefold of Picard rank 1, and denote by H the generator of $\mathrm{NS}\,X$. We aim to construct stability condition σ for the large volume central charge

$$Z(E) = -\int_X e^{-(b+\mathrm{i}t)H}\,\mathrm{ch}(E) = \frac{t^2}{2}\,\mathrm{ch}_1^b - \mathrm{ch}_3^b + \mathrm{i}\left(t\,\mathrm{ch}_2^b - \frac{t^3}{6}\,\mathrm{ch}_0^b\right)$$

where $\mathrm{ch}_k^b := \int_X e^{-bH}\,\mathrm{ch}(E).H^{3-k}$, $\mathrm{ch}_k := \mathrm{ch}^0$.

- Start with the Abelian category of coherent sheaves $\mathrm{Coh}\,X$ and introduce, for any $E \in \mathrm{Coh}\,X$ and $b \in \mathbb{R}$, the slope function

$$\mu_b(E) = \frac{\mathrm{ch}_1^b(E)}{\mathrm{ch}_0^b(E)} \text{ if } ch_0(E) \neq 0, \quad +\infty \text{ otherwise}$$

A coherent sheaf E is called slope semi-stable if $\mu_b(E') \leq \mu_b(E)$ for any subsheaf $E' \subset E$. It is known that any slope semistable sheaf satisfies the classical Bogolomov-Gieseker inequality

$$(\mathrm{ch}_1^b(E))^2 - 2\,\mathrm{ch}_0^b(E)\,\mathrm{ch}_2^b(E) = (\mathrm{ch}_1)^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2 \geq 0.$$

Constructing stability conditions for compact CY3

- Let
  - $\mathcal{T}_b \subset \mathrm{Coh}\, X$ be the subcategory generated by slope-semi-stable sheaves E with $\mu_b(E) > 0$,
  - $\mathcal{F}_b \subset \mathrm{Coh}\, X$ be the subcategory generated by slope-semi-stable sheaves E with $\mu_b(E) \le 0$.
- Then $\mathcal{A}_b := \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$ is the heart of a bounded t-structure on $\mathcal{D}^b(C)$ generated by length two complexes of the form $\mathcal{E} = (F \stackrel{d}{\to} T)$ with $\ker d \in \mathcal{T}_b$ and $\mathrm{coker}\, d \in \mathcal{F}_b$.
- For objects in the heart $\mathcal{A}_b$ and $a \in \mathbb{R}_+$, consider the central charge function

$$Z_{b,a}(\mathcal{E}) = -a\,\mathrm{ch}_2^b + \frac{1}{2}\, a^3\, \mathrm{ch}_0^b + \mathrm{i}\, a^2\, \mathrm{ch}_1^b\ .$$

  The resulting pair $(Z_{b,a}, \mathcal{A}_b)$ satisfies the axioms of a weak stability condition but is not an actual stability condition since the central charge of skyscraper sheaves vanishes.

Constructing stability conditions for compact CY3

- For any object $\mathcal{E} \in \mathcal{A}_b$, we define the 'tilt'

$$N_{b,a}(\mathcal{E}) := -\frac{\Re[Z_{b,a}(\mathcal{E})]}{\Im[Z_{b,a}(\mathcal{E})]} = \frac{\mathrm{ch}_2^b(\mathcal{E}) - \frac{1}{2}\, a^2\, \mathrm{ch}_0^b(\mathcal{E})}{a\, \mathrm{ch}_1^b(\mathcal{E})}\,,$$

  with $N_{b,a}(\mathcal{E}) = +\infty$ if $\mathrm{ch}_1^b(\mathcal{E}) = 0$. Then $E \in \mathcal{A}_b$ is semistable with respect to the pair $(Z_{b,a}, \mathcal{A}_b)$ iff for any $0 \neq F \subset E$ in $\mathcal{A}_b$, we have $N_{b,a}(F) \leq N_{b,a}(\mathcal{E})$.
- Again, any such semistable object $E \in \mathcal{A}_b$ satisfies the classical Bogomolov inequality (22). Conjecturally, it also satisfies the following inequality *[Bayer Macri Toda 2011]* involving ch_3,

$$a^2 \left[(\mathrm{ch}_1^b)^2 - 2\, \mathrm{ch}_0^b\, \mathrm{ch}_2^b \right] + 4 (\mathrm{ch}_2^b)^2 - 6\, \mathrm{ch}_1^b\, \mathrm{ch}_3^b \geq 0\,,$$

- The BMT inequality is known to hold true for X_5 *[Li 2019]*.

Constructing stability conditions for compact CY3

- Next define the family of central charges parametrized by $(a, b, \alpha, \beta) \in \mathbb{R}^4$ with $a > 0, \alpha > 0$,

$$Z_{a,b,\alpha,\beta}(\mathcal{E}) = \left(- \operatorname{ch}_3^b + \beta \operatorname{ch}_2^b + \alpha \operatorname{ch}_1^b \right) + \mathrm{i} \left(a \operatorname{ch}_2^b - \frac{1}{2} a^3 \operatorname{ch}_0^b \right)$$

  This includes the LV central charge for $a = \sqrt{\frac{1}{3} t^2}, \alpha = \frac{1}{2} t^2, \beta = 0$.

- Similar to the construction of $\mathcal{A}_b$, define
  - $\mathcal{T}_{b,a} \subset \mathcal{A}_b$ as the subcategory generated by semi-stable objects in $\mathcal{A}_b$ with $N_{b,a}(\mathcal{E}) > 0$,
  - $\mathcal{F}_{b,a} \subset \mathcal{A}_b$ as the subcategory generated by semi-stable objects in $\mathcal{A}_b$ with $N_{b,a}(\mathcal{E}) \leq 0$.

- Then define $\mathcal{A}_{b,a} = \langle \mathcal{F}_{b,a}[1], \mathcal{T}_{b,a} \rangle$. By design, $\Im Z_{a,b,\alpha,\beta}(\mathcal{E}) \geq 0$ for any $E \in \mathcal{A}_{b,a}$. One checks that $(Z_{a,b,\alpha,\beta}, \mathcal{A}_{b,a})$ satisfies the axioms of a (full) stability condition whenever $\alpha > \frac{1}{6} a^2 + \frac{1}{2} a|\beta|$.

Physics of the BMT inequality

- Rewrite the BMT inequality as

$$\frac{1}{2}(\mathrm{ch}_1^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2)(a^2+b^2) + (3\,\mathrm{ch}_0\,\mathrm{ch}_3 - \mathrm{ch}_1\,\mathrm{ch}_2)b + (2\,\mathrm{ch}_2^2 - 3\,\mathrm{ch}_1\,\mathrm{ch}_3) \geq 0$$

- It is automatically satisfied throughout the (a, b) upper half-plane, unless ch satisfies the inequality

$$P_4 := 8\,\mathrm{ch}_0\,\mathrm{ch}_2^3 + 6\,\mathrm{ch}_1^3\,\mathrm{ch}_3 + 9\,\mathrm{ch}_0^2\,\mathrm{ch}_3^2 - 3\,\mathrm{ch}_1^2\,\mathrm{ch}_2^2 - 18\,\mathrm{ch}_0\,\mathrm{ch}_1\,\mathrm{ch}_2\,\mathrm{ch}_3 \geq 0$$

  in which case it is violated in a semi-circle intersecting the axis

$$b_\pm = \frac{\mathrm{ch}_1\,\mathrm{ch}_2 - 3\,\mathrm{ch}_3\,\mathrm{ch}_3 \pm \sqrt{P_4}}{\mathrm{ch}_1^2 + 2\,\mathrm{ch}_0\,\mathrm{ch}_2}$$

- Identifying $p^0 = \mathrm{ch}_0/\kappa, p^1 = \mathrm{ch}_1/\kappa, q_1 = -\mathrm{ch}_2, q_0 = \mathrm{ch}_3$, P_4 is recognized as $-I_4(\gamma)$, the quartic polynomial entering in the entropy $S_{BH} = \pi\sqrt{I_4(\gamma)}$ of spherically symmetric black holes.
- Hence BMT provides an empty chamber whenever single centered BH are ruled out !

DT invariants at large volume

At large volume, some DT invariants can be computed explicitly in terms of GV invariants:

- For point classes (D0-branes), $\Omega_{LV}(0,0,0,k) = -\chi(X)$ $\forall k \neq 0$
- For curve classes (D2-branes), $\Omega_{LV}(0,0,\beta,k) = \mathrm{GV}^{(0)}_{\beta}$ $\forall k$.
- For unit D6-brane charge, $\Omega_{LV}(1,0,-\beta,k)$ coincides with the Pandharipande-Thomas invariant $\mathrm{PT}(\beta,k)$ counting stable pairs $E = (\mathcal{O}_X \stackrel{s}{\to} F)$, where F is a pure one-dimensional sheaf with $[F]=\beta$ and $\chi(F)=k$, and s is a section with zero-dimensional cokernel.
- The generating series $Z_{PT}(t,q) = \sum_{Q,n} \mathrm{PT}(Q,n) e^{2\pi \mathrm{i} Q\cdot t} q^n$ is determined by higher genus GV invariants via *[MNOP06, Pardon23]*

$$Z_{PT}(t,q) = \prod_{\beta>0} \prod_{g} \prod_{m=1-g}^{g-1} \left(1-(-q)^m e^{2\pi \mathrm{i} t\cdot\beta}\right)^{(-1)^{m+1}\binom{2g-2}{g-1-m}\mathrm{GV}^{(g)}_{\beta}}$$

- These properties are clear physically from the M-theory picture.

String theory/M-theory dictionary

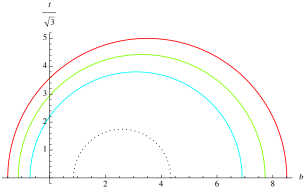
- Type IIA string is secretly 11D SUGRA/M-theory compactified on a circle of radius $R_s = g_s \ell_s$, with 11D Planck scale $\ell_P = g_s^{1/3} \ell_s$.
- D0-branes are KK momenta of the 11D graviton: $\frac{1}{R_s} = \frac{1}{g_s \ell_s}$
- Fundamental strings are wrapped M2-branes: $\frac{1}{\ell_s^2} = \frac{R_s}{\ell_P^3}$
- D2-branes are unwrapped M2: $\frac{1}{g_s \ell_s^3} = \frac{1}{\ell_P^3}$
- D4-branes are wrapped M5-branes: $\frac{1}{g_s \ell_s^5} = \frac{R_s}{\ell_P^6}$
- NS5-branes are unwrapped M5: $\frac{1}{g_s^2 \ell_s^6} = \frac{1}{\ell_P^6}$
- D6-branes are KK/TN monopoles: $\frac{1}{g_s \ell_s^7} = \frac{R_s^2}{\ell_P^9}$

S-duality conjecture

- Rank 0 DT invariants $\Omega_{LV}(0,\mathcal{D},\beta,k)$ physically count D4-D2-D0 brane bound states. Since a D4-brane arises by wrapping an M5-brane on S^1, it is described by a 2D $(0,4)$ SCFT.
- The generating series $\phi_{\mathcal{D}}(v,\tau) \sim \sum_{\beta,k} \Omega(0,\mathcal{D},\beta,k)\, q^k\, e^{2\pi i \beta\cdot v}$ is identified as the elliptic genus of the SCFT, and should transform as a Jacobi form under $SL(2,\mathbb{Z}) : (\tau,z) \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$ *[Maldacena Strominger Witten 1997]*
- This expectation is correct for for $\mathcal{D}$ an ample, irreducible divisor, such that the SCFT has discrete spectrum. Otherwise, it should transform as a mock Jacobi form with specific modular anomaly *[Alexandrov BP Manschot 2019]*.
- Can we test this prediction mathematically ?

Wall-crossing in space of weak stability conditions

- In the space of weak stability conditions $(Z_{b,a}, \mathcal{A}_b)$, the chamber structure is quite simple: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$. Moreover, for $P_4(\gamma) > 0$, the BMT inequality provides an empty chamber.



- Consider a D6-brane with charge $\gamma = (1, 0, -\beta, n)$ such that $P_4(\gamma)) > 0$. At large volume, we know $\Omega_{b,t}(\gamma) = \mathrm{PT}(\beta, m)$. By studying wall-crossing from the BMT chamber where $\Omega_{b,t}(\gamma) = 0$, one can extract the indices of the D4-D2-D0 branes emitted at each wall !

A new explicit formula (S. Feyzbakhsh'23)

Theorem Let (C, H) be a smooth polarised CY threefold with $\mathrm{Pic}(C) = \mathbb{Z}.H$ satisfying the BMT conjecture. There is $f(x)$ such that

- If $\alpha := \frac{m}{\beta \cdot H} > f(\frac{\beta, H}{H})$ then the stable pair invariant $\mathrm{PT}(\beta, m) =$

$$\sum_{(m', \beta')} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} \mathrm{PT}(\beta', m') \, \Omega\left(0, 1, \ \frac{H^2}{2} - \beta' + \beta \ , \ \frac{H^3}{6} + m' - m - \beta'.H\right)$$

  where $\chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12} c_2(X).H$.
- The sum runs over $(\beta', m') \in H_2(C, \mathbb{Z}) \oplus H_0(C, \mathbb{Z})$ such that

$$0 \leq \beta'.H \leq \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

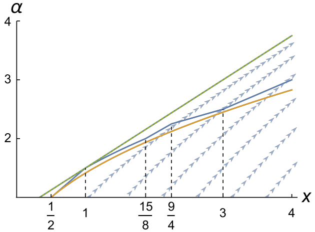
$$-\frac{(\beta'.H)^2}{2H^3} - \frac{\beta'.H}{2} \leq \ m' \ \leq \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{\beta.H + \beta'.H}{2} + m$$

  In particular, $\beta'.H < \beta.H$.

Corollary (Castelnuovo bound): $\mathrm{PT}(\beta, m) = 0$ unless $m \geq -\frac{(\beta.H)^2}{2H^3} - \frac{\beta.H}{2}$

The ad hoc function $f(x)$

$$f(x) := \begin{cases} x+\frac{1}{2} & \text{if } 0<x<1 \\ \sqrt{2x+\frac{1}{4}} & \text{if } 1<x<\frac{15}{8} \\ \frac{2}{3}x+\frac{3}{4} & \text{if } \frac{15}{8}\le x<\frac{9}{4} \\ \frac{1}{3}x+\frac{3}{2} & \text{if } \frac{9}{4}\le x<3 \\ \frac{1}{2}x+1 & \text{if } 3\le x \end{cases}$$



- Green: $\alpha = \frac{3}{4}(x+1)$, above which $PT=0$ by Castelnuovo
- Orange: $\alpha = \sqrt{2x}$ below which BMT provides no empty chamber
- Blue: $\alpha = f(x)$ above which our theorem applies
- Arrows: spectral flow $(\beta.H, m) \mapsto (\beta.H + \kappa k, m - \kappa\beta.H - \frac{1}{2}\kappa k(k+1))$ preserving the discriminant

Tests of modularity for the quintic threefold

In this way, we can compute lots of terms in $\phi_{\mathcal{D}} = \sum_\mu h_\mu(\tau)\,\Theta(\tau, v)$ for $\mathcal{D} = H$ primitive and check modularity, confirming the guess in *[Gaiotto Strominger and Yin 2006]*. Similar results for 1-parameter CY threefolds.

$$
\begin{aligned}
h_0 =& q^{-\frac{55}{24}} \Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4 \\
&+ 28096675153255q^5 + 3756542229485475q^6 \\
&+ 277591744202815875q^7 + 13610985014709888750q^8 + \dots \Big),
\end{aligned}
$$

$$
\begin{aligned}
h_{\pm 1} =& q^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3 \\
&+ 3102750380125q^4 + 577727215123000q^5 + \dots \Big)
\end{aligned}
$$

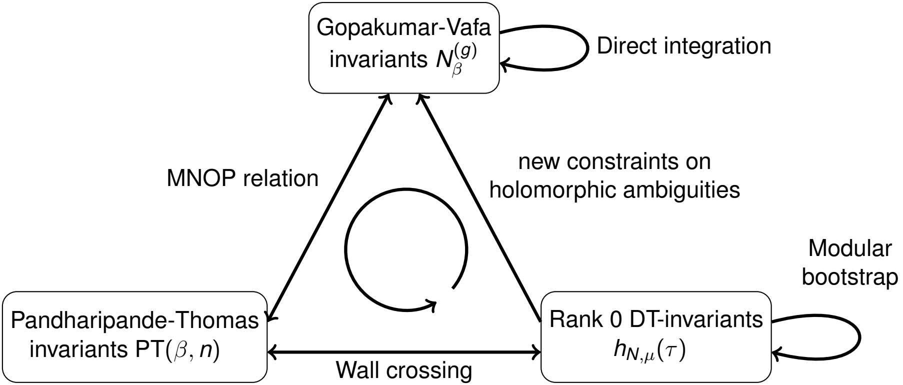
$$
\begin{aligned}
h_{\pm 2} =& q^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 953712511250q^4 \\
&+ 217571250023750q^5 + 22258695264509625q^6 + \dots \Big)
\end{aligned}
$$

Tests of modularity for the quintic threefold

- This confirms the guess by *[Gaiotto Strominger Yin 2006]*:

$$h_\mu = \frac{1}{\eta^{70}} \Big[-\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} + \frac{25\big(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\big)}{53747712}\, D + \frac{25\big(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\big)}{8957952}\, D^2 \Big]\, \vartheta_\mu^{(5)}$$

Quantum geometry from stability and modularity



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

4. Stability conditions for CY3 singularities and quivers

- For non-compact toric CY threefold, the BPS spectrum is under much better control. The derived category $D_c \operatorname{Coh} X$ of compactly supported coherent sheaves is isomorphic to the derived category of representations of a certain quiver with potential (Q, W). This holds also in non-toric case, provided there exist a tilting sequence $T = \bigoplus_{i=1}^{r} T_i$ with $\operatorname{Ext}^k(T, T) = 0 \forall k \neq 0$ generating $D^b \operatorname{Coh} X$.
- $\operatorname{Stab} X$ then contains a region where the heart $\mathcal{A}$ is equivalent to the quiver heart, up to tilt. In this region, $\Omega_z(\gamma)$ coincides with the DT invariant $\Omega_\theta(d)$ associated to the moduli space of θ-stable quiver representations, for a specific dimension vector $d(\gamma)$ and King stability (aka Fayet-Iliopoulos) parameters $\theta(d)$.
- Physically, quiver representations describe vevs of open strings stretched between fractional D-branes. They occur on the Higgs branch of a SUSY gauge theory in $0+1$ dimensions known as quiver quantum mechanics, which also describes the dynamics of BPS black holes !

Quiver quantum mechanics

Consider a supersymmetric gauge theory in $0+1$ dimensions (aka quiver quantum mechanics, or 1D GLSM), with matter content encoded in a quiver with potential (Q, W):

- each node $i \in Q_0$ represents a vector multiplet $(\vec{x}_i, A_i, \lambda_i, D_i)$ transforming in the adjoint of $U(d_i)$.
- each arrow $a : i \to j \in Q_1$ represents a chiral multiplet (Φ^a, ψ^a, F^a) in the bifundamentalr epresentation $(d_i, \bar{d}_j)$ of $U(d_i) \times U(d_j)$
- The (super)potential is a sum $W(\Phi) = \sum_{c \in Q_2} w_c \operatorname{Tr} \prod_{a \in c} \Phi^a$ over oriented closed cycles $c \in Q_2$.
- In addition, one must specify a gauge coupling e_i and Fayet-Iliopoulos (FI) parameter $\zeta_i \in \mathbb{R}$ for each node

Quiver quantum mechanics

The following Lagrangian leads to 4 supersymmetries:

$$\mathcal{L} = \sum_{i\in Q_0} \frac{1}{2e_i^2} \mathrm{Tr}\left[(\nabla_t \vec{x}_i)^2 + (\vec{x}_i \wedge \vec{x}_i)^2 + D_i^2 \right]$$
$$+ \sum_{a\in Q_1} \left[|\nabla_t \Phi^a|^2 + |F^a|^2 - |\vec{x}_j \Phi^a - \Phi^a \vec{x}_i|^2 - \overline{\Phi^a}(D_j\Phi^a - \Phi^a D_i) \right]$$
$$- \sum_{i\in Q_0} \theta_i \,\mathrm{Tr}(D_i) + \sum_{a\in Q_1} \left[\frac{\partial W}{\partial \Phi^a} F^a + c.c. \right]$$

with $\nabla_t \vec{x}_i = \partial_t \vec{x}_i + \mathrm{i}[A_i, \vec{x}_i]$. Eliminating fields D_i, F^a leads to

$$V = \sum_{i\in Q_0} \frac{1}{2e_i^2} \mathrm{Tr}(\vec{x}_i \wedge \vec{x}_i)^2 + \sum_{a\in Q_1} \left\| \vec{x}_k \Phi^a - \Phi^a \vec{x}_i \right\|^2 + \sum_{i\in Q_0} \frac{e_i^2}{2} D_i^2 + \sum_{a\in Q_1} \left\| \frac{\partial W}{\partial \Phi^a} \right\|^2$$

where $D_i = \sum_{a:\cdot i\to j} \overline{\Phi^a}\Phi^a - \sum_{a:j\to i\cdot} \Phi^a \overline{\Phi^a} - \theta_i \,\mathrm{Id}_{d_i}$

Higgs branch and Coulomb branch descriptions

Classical supersymmetric vacua fall into 2 branches:

- The Higgs branch where $\vec{x}_i = 0$ and

$$\forall a \in Q_1, \frac{\partial W}{\partial \Phi^a} = 0 \ \text{ and } \ \forall i \in Q_0, \sum_{a:i\to j} \overline{\Phi^a}\Phi^a - \sum_{a:j\to i} \Phi^a\overline{\Phi^a} = \theta_i \, \mathrm{Id}_{d_i}$$

- The Coulomb branch where $\Phi^a = 0$ and $\vec{x}_i$ are commuting, diagonal matrices.

Unlike in Ben's lectures on 3D mirror symmetry, these are NOT different superselection sectors. Rather two effective, in principle equivalent descriptions where either vector or chiral multiplets have been integrated out. As in 3D, the Coulomb branch has quantum corrections.

Higgs branch and quiver representations

- Observe first that solutions to $D_i = 0$ can only exist on hyperplanes $\theta(d) = \sum_{i \in Q_0} d_i \, \theta_i = 0$. When this holds, the fields Φ^a provide a *representation* of the quiver with potential (Q, W),

$$R(Q,d) = \oplus_{a:i \to j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

satisfying the conditions $\partial W / \partial a = 0$ for all $a \in Q_1$.
- As usual, one can enforce the D-term condition by acting with $G_{\mathbb{C}} = \prod_i GL(d_i, \mathbb{C})$, provided one restricts to *semi-stable* representations Φ such that $\theta(\tilde{d}) \leq \theta(d)$ for any subrep $\tilde{\Phi} \subset \Phi$.
- This is a special case of stability condition on $D^b \, \mathrm{Rep}(Q, W)$ with $Z(d) = -\sum_{i \in Q_0} \theta_i d_i + \mathrm{i} \sum_{i \in Q_0} \rho_i d_i$ for $\rho_i > 0$ and Euler form

$$\chi_Q(d, d') = \sum_{i \in Q_0} d_i d'_i - \sum_{(a:i \to j) \in Q_1} d_i d'_j$$

- For acyclic quivers, $\Omega_\theta(d)$ can be computed by wall-crossing from trivial stability $\theta = 0$ *[Reineke 2002]*. Otherwise, a hard problem.

Coulomb branch

- Restricting to Abelian case $d_i = 1$ first, and integrating out chiral multiplets, one finds the supersymmetric Lagrangian on $\mathbb{R}^{3n}$, $n = |Q_0|$:

$$L = \sum_{i=1}^{n} \frac{1}{2e_i^2} \left(\dot{\vec{x}}_i^2 + D_i^2 \right) + \sum_{i=1}^{n} (-U_i D_i + \vec{A}_i \cdot \dot{\vec{x}}_i) + \text{fermions}$$

where $U_i(\vec{x})$ and $\vec{A}_i(\vec{x})$ are determined via

$$U_i = -\frac{1}{2} \left(\sum_{j \neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} - \theta_i \right) , \quad \vec{\nabla}_i U_j = \vec{\nabla}_j U_i = \frac{1}{2} (\vec{\nabla}_i \times \vec{A}_j + \vec{\nabla}_j \times \vec{A}_i).$$

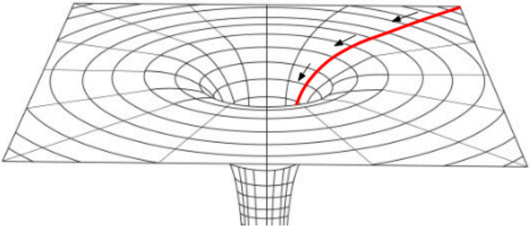
Here $\gamma_i \in \mathbb{Z}^{Q_0}$ are the basis vectors supported on node i.
- Remarkably, this describes the dynamics of n BPS black holes with mass $m_i = 1/e_i^2$, charge γ_i and $\theta_i = 2\Im(e^{-i\phi} Z(\gamma_i))$!

BPS states as BPS black holes

- At strong coupling $g_s \gg 1$, D-branes lose their meaning, but one can represent BPS states as supersymmetric solutions of the low energy effective action.
- Stationary, supersymmetric solutions can be shown to be of the form
$$\mathrm{d}s_4^2 = -e^{2U}(\mathrm{d}t + A)^2 + e^{-2U}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$$
where the scale factor U, 1-form A and VM multiplet scalars are determined by harmonic functions $H : \mathbb{R}^3 \to \Gamma \otimes \mathbb{R}$ *[Denef 200]*,
$$H(\vec{r}) = \sum_{i=1}^{n} \frac{\gamma_i}{|\vec{r} - \vec{r}_i|} + \beta, \qquad \gamma := \sum_{i=1}^{n} \gamma_i$$

BPS states from single-centered black holes

- For a single pole, this describes a spherically symmetric BPS black hole with charge γ, interpolating between $\mathbb{R}^{3,1}$ at $|\vec{r}| = \infty$ and $AdS_2 \times S_2$ at the horizon at $\vec{r} = 0$. VM scalars follow the gradient flow of $|Z_z(\gamma)|$, until they reach a local minimum $z_\star(\gamma)$ at the horizon, called attractor point.

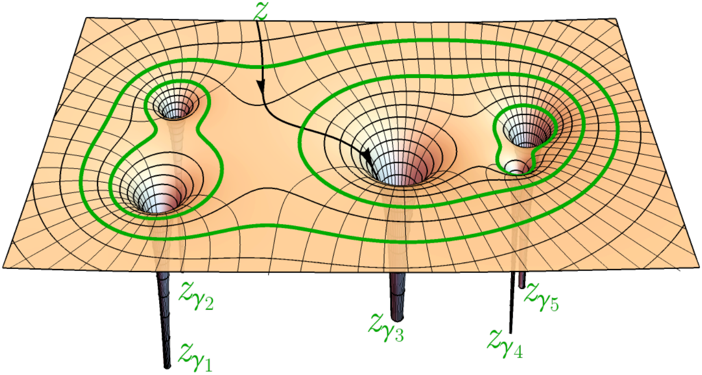


- The Bekenstein-Hawking argument predicts that such a black hole should describe $\mathcal{O}(e^{S_{BH}})$ micro-states of charge γ, with $S_{BH} = \pi |Z_{z_\star(\gamma)}(\gamma)|^2$. A highly non-trivial prediction for DT theory !

BPS states as multi-centered black holes

- For multiple poles, multi-centered solutions exist provided Denef's balance conditions are satisfied,

$$\forall i, \quad \sum_{j\neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{r}_i - \vec{r}_j|} = \theta_i \,, \qquad \vec{J} = \frac{1}{2} \sum_{i<j} \langle \gamma_i, \gamma_j \rangle \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|}$$



Quantizing multi-centered black holes

- For 2 centers, BPS bound states exist iff $\gamma_{12}\theta_1 > 0$. It is straightforward to solve the quantum mechanics of 2 mutually non-local dyons and reproduce the primitive wall-crossing formula (Denef Moore 2007)

$$\Delta\Omega(\alpha_1+\alpha_2) = (-1)^{\langle\alpha_1,\alpha_2\rangle+1} |\langle\alpha_1,\alpha_2\rangle| \, \Omega(\alpha_1) \, \Omega(\alpha_2)$$

- In general, the space $\mathcal{M}_n(\{\gamma_i,\theta_i\})$ of solutions to Denef's equations form a symplectic space of dimension $2n-2$, with an Hamiltonian action of $SO(3)$ generated by $\vec{J}$ *[de Boer, El-Showk, Messamah, van den Bleeken 2008]*.
- Quantum mechanically, BPS states are harmonic spinors on $\mathcal{M}_n(\{\gamma_i,\theta_i\})$. The equivariant Dirac index counting them is easily computed by localization with respect to rotations along the z axis, and reproduces the Kontsevich-Soibelman wall-crossing formula *[Manschot BP Sen 2010]*.

Attractor flow tree formula

- More generally, one can use the black hole picture to express the index $\Omega_z(\gamma)$ in terms of attractor indices $\Omega_\star(\gamma) = \Omega_{z_\star(\gamma)}(\Omega)$.
- In order to properly deal with Bose-Fermi statistics, while ensuring manifest charge conservation, it is useful to introduce the rational DT invariant (note similarity with multicover formula for $\mathrm{GW}_\beta^{(0)}$!)

$$\overline{\Omega}(\gamma_i) := \sum_{d|\gamma_i} \frac{1}{d^2} \Omega(\gamma_i/d)$$

- The formula then takes the form *[Denef Greene Raugas 2001]*,

$$\Omega_z(\gamma) = \sum_n \sum_{\gamma=\gamma_1+\cdots+\gamma_n} \frac{g_z(\{\gamma_i\})}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_{i=1}^n \overline{\Omega}_*(\gamma_i)$$

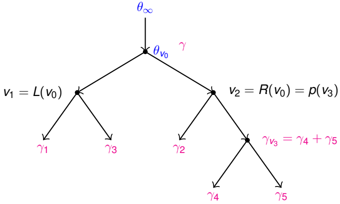
where $g_z(\{\gamma_i\})$ is a sum over attractor flow trees, describing multicentered black holes as iterated two-centered bound states.

Attractor flow tree formula for quivers

- Define the attractor invariant as $\overline{\Omega}_*(\gamma_i) = \overline{\Omega}_{\theta_\star(\gamma)}(\gamma_i)$ for self-stability condition $\theta^i_\star(\gamma) = \langle\gamma,\gamma_i\rangle = \theta^i_\star(d) = -\sum_{a:i\to j} d_j + \sum_{a:j\to i} d_j$.
- For any decomposition $\gamma = \gamma_1 + \cdots + \gamma_n$, consider the set $\mathcal{T}_n$ of rooted binary trees T with n leaves, with charges γ_v and stability parameters at each vertex $v \in V_T$ defined through

$$\begin{aligned}\gamma_v &= \gamma_{L(v)} + \gamma_{R(v)} \\ \theta_v &= \theta_{p(v)} + \frac{\theta_{p(v)}(\gamma_{L(v)})}{\langle\gamma_{L(v)},\gamma_v\rangle}\langle\gamma_v,-\rangle\end{aligned}$$

where $p(v), L(v), R(v)$ are the parent and two descendents of v:



Attractor flow tree formula for quivers

- Define the sum over stable attractor flow trees

$$g_\theta(\{\gamma_i\}) = \sum_{T\in\mathcal{T}_n} \prod_{v\in V_T} \tfrac{1}{2}\langle\gamma_{L(v)},\gamma_{R(v)}\rangle \left[\mathrm{sgn}\,\theta_v(\gamma_{L(v)}) + \mathrm{sgn}\langle\gamma_{L(v)},\gamma_{R(v)}\rangle\right]$$

- The index $\overline{\Omega}_\theta(\gamma)$ is then given for any generic θ by *[Alexandrov BP 2018; Argüz Bousseau 2023]*

$$\overline{\Omega}_z(\gamma) = \sum_{n=1}^{\infty} \sum_{\gamma=\gamma_1+\dots+\gamma_n} \frac{g_\theta(\{\gamma_i\})}{|\,\mathrm{Aut}(\{\gamma_i\})|} \prod_{i=1}^{n} \overline{\Omega}_\star(\gamma_i)$$

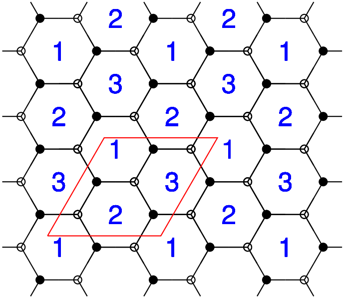
- This formula, along with many others, is implemented in the Mathematica package `CoulombHiggs.m` available on GitHub.
- This reduces the computation of DT invariants for quivers to determining $\Omega_\star(d)$. In general, one can show that $\Omega_\star(\gamma) = 0$ unless γ is a basis vector γ_i or γ has support on an oriented cycle. For quivers associated to CY singularities, attractor invariants are even more restricted.

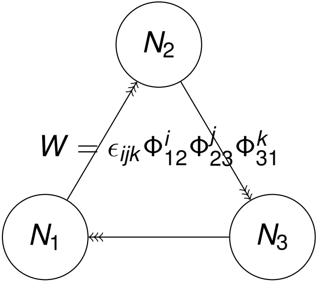
5. BPS spectrum on CY3 singularities

For a non-compact CY3 admitting a tilting sequence $T = \bigoplus_{i=1}^{r} T_i$, $D_c \operatorname{Coh} X$ (where c stands for compact support) is isomorphic to $D \operatorname{Rep}(Q, W)$ for a certain quiver with potential. This includes

- Orbifolds $\mathbb{C}^3/\Gamma$ where Γ is a discrete subgroup of $SU(3)$, and their crepant resolutions. The T_i's are called fractional branes.
- Local surfaces K_S where S a surface admitting a (strong, full, cyclic) exceptional collection of sheaves $\mathcal{E} = (E_1, \dots, E_r)$, Mutations of $\mathcal{E}$ lead to different quivers whose derived categories $D \operatorname{Rep}(Q, W)$ are isomorphic.
- Toric CY3 singularities, for which (Q, W) can be read off from periodic tilings *[Franco, Hanany Kennaway, Vegh and Wecht 2005]*

Orbifold quiver for $K_{\mathbb{P}^2} = \widetilde{\mathbb{C}^3/\mathbb{Z}_3}$





$$
\begin{aligned}
E_1 &= \mathcal{O} & \gamma_1 &= [1, 0, 0] \\
E_2 &= \Omega(1)[1] & \gamma_2 &= [-2, 1, \tfrac{1}{2}] \\
E_3 &= \mathcal{O}(-1)[2] & \gamma_3 &= [1, -1, \tfrac{1}{2}]
\end{aligned}
$$

Attractor conjecture for local CY3 folds

- For (Q, W) associated to a CY3 singularity, the structure of attractor invariants is conjecturally very simple *[Beaujard BP Manschot 2020]*: $\Omega_\theta(\gamma) = 0$ unless γ is one of the basis vectors γ_i, or $\langle\gamma, -\rangle = 0$
- For $\mathbb{C}^3/\mathbb{Z}_3$, this can be argued from the expected dimension $\mathcal{M}_\theta(\gamma)$. For $\gamma = (d_1, d_2, d_3)$, the attractor stability parameters are

$$\theta_1^\star = 3(d_3 - d_2), \quad \theta_2^\star = 3(d_1 - d_3), \quad \theta_3^\star = 3(d_2 - d_1)$$

Depending on the sign of $\theta_i^\star$, either Φ_{31}^α or Φ_{12}^α or Φ_{23}^α must vanish for all $\alpha = 1, 2, 3$. The first case occurs when $\theta_1^\star > 0, \theta_3^\star < 0$:

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{M}_\theta^\star(\gamma) =& 3(d_1 d_2 + d_2 d_3 - d_3 d_1) - d_1^2 - d_2^2 - d_3^2 + 1 \\ =& 1 - Q(d) - \frac{2}{3} d_1 \theta_1^\star + \frac{2}{3} d_3 \theta_3^\star \leq 1\end{aligned}$$

where $Q(d) = d_1^2 + d_2^2 + d_3^2 - d_1 d_2 - d_2 d_3 - d_3 d_1$ is a positive quadratic form with kernel along $(1, 1, 1)$.

BPS spectrum on local CY3 folds

- Assuming the Attractor Conjecture is true, we can compute $\Omega_\theta(\gamma)$ for any (γ, θ) using the attractor flow tree formula.
- For $\sigma = (Z, \mathcal{A}) \in \mathsf{Stab}\, X$, the heart $\mathcal{A}$ coincides with the quiver heart only in regions of $\mathsf{Stab}\, X$ where the objects T_i in the tilting sequence are stable and their central charge $Z_\sigma(T_i)$ lies in a common half plane. In particular, the quiver description is not valid at large volume !
- Scattering diagrams give a way to track down DT invariants across $\mathsf{Stab}\, X$.

Scattering diagrams on triangulated categories

- For a general triangulated category $\mathcal{C}$, define the scattering diagram $\mathcal{D}_\psi(\mathcal{C})$ as the union of codimension-one loci in $\mathsf{Stab}\,\mathcal{C}$,

$$\mathcal{R}_\psi(\gamma) = \{\sigma : \arg Z(\gamma) = \psi + \tfrac{\pi}{2},\ \overline{\Omega}_Z(\gamma) \neq 0\}$$

  equipped with (suitably regularized) automorphism

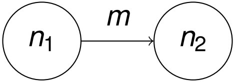
$$\mathcal{U}_\sigma(\gamma) = \exp\left(\frac{\overline{\Omega}_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right) = \mathrm{Exp}\left(\frac{\Omega_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right)$$

- The WCF ensures that the diagram is locally consistent: for any small path $\mathcal{P} : t \in [0,1] \to \mathsf{Stab}\,\mathcal{C}$ around a codimension-two intersection, $\prod_i \mathcal{U}_{\sigma(t_i)}(\gamma_i)^{\epsilon_i} = 1$
- For a given quiver, the scattering diagram $\mathcal{D}$ in the space $\mathbb{R}^+$ of King stability conditions defined by

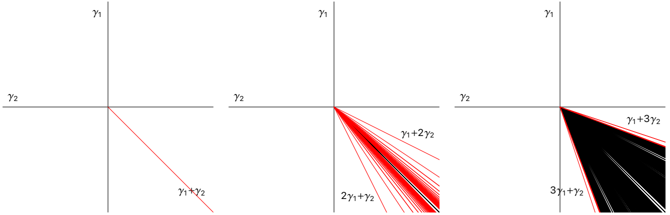
$$\mathcal{R}(\gamma) = \{\theta \in \mathbb{R}^{Q_0} : (\theta, \gamma) = 0,\ \overline{\Omega}_\theta(\gamma) \neq 0\}$$

  is globally consistent, $\prod_i \mathcal{U}_{\theta(t_i)}(\gamma_i)^{\epsilon_i} = 1$ for any path.

Scattering diagram for Kronecker quivers



$$\theta_1 > 0, \theta_2 < 0: \quad \dim \mathcal{M}_\theta(\gamma) = m n_1 n_2 - n_1^2 - n_2^2 + 1$$

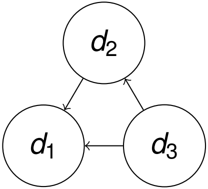


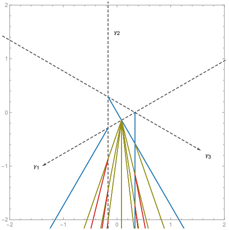
$m = 1$ $\qquad$ $m = 2$ $\qquad$ $m = 3$

K_1 is relevant for (A_1, A_2) AD theory, K_2 for pure Seiberg-Witten theory.

Scattering diagram for Seiberg-Witten with $N_f = 1$

- The BPS spectrum is described by a 3-node acyclic quiver, with nodes associated to the monopole $\gamma_1 = [0, 1; 1/2]$, quark $\gamma_2 = [1, 0; -1]$ and dyon $\gamma_3 = [1, -1; 1/2]$ where the entries denote the electric,magnetic and $U(1)$ flavor charge $[q, p; S]$.
- Since $\langle (1, -1, 1), - \rangle = 0$, taking a slice $\theta_1 - \theta_2 + \theta_3 = 1$,

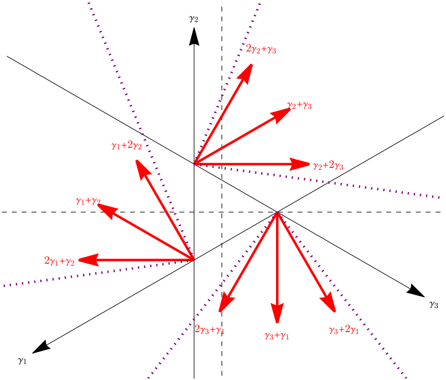




- Outgoing rays are $(1, 1, 1)$ (corresponding to the W-boson with charge $[2, 0; 0]$), $(1, 0, 1)$ (corresponding to the other quark in the $SU(2)$ doublet, with charge $[1, 0; -1]$), and semi-classical dyons $(k + 1, k, k)$,$(k + 1, k + 1, k)$,$(k, k + 1, k + 1)$,$(k, k, k + 1)$ with charge $[q, \pm 1; \pm 1/2]$ for $q \in \mathbb{Z}$.

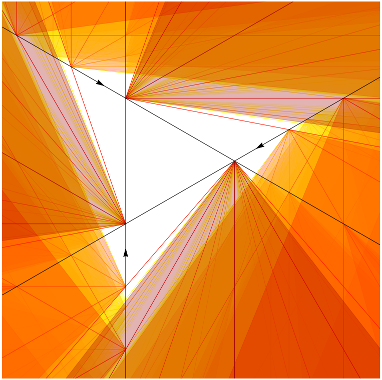
Scattering diagram for the $\mathbb{C}^3/\mathbb{Z}_3$ quiver

Restricting to the hyperplane $\theta_1 + \theta_2 + \theta_3 = 1$, primary scatterings only:



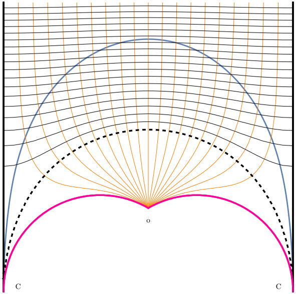
Scattering diagram for the $\mathbb{C}^3/\mathbb{Z}_3$ quiver

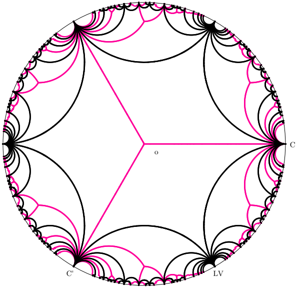
Restricting to the hyperplane $\theta_1 + \theta_2 + \theta_3 = 1$, adding secondary rays:



Kähler moduli space of $X = K_{\mathbb{P}^2}$

- By local mirror symmetry, the Kähler moduli space of $X = K_{\mathbb{P}^2}$ is the quotient $X_1(3) = \mathbb{H}/\Gamma_1(3)$. It admits two cusps LV, C and one orbifold point o of order 3.





- A BPS state on X is a stable object E in the bounded derived category $\mathcal{C}$ of compactly supported sheaves on X, with charge $\gamma(E) = \mathrm{ch}(\pi_*(E)) = [r, d, \mathrm{ch}_2] \sim [D4, D2, D0]$

Central charge as Eichler integral

- $\mathrm{Stab}\, X$ has complex dimension 3. Up to $\mathbb{C}^\times$ action, the central charge can be written as

$$Z_\tau(\gamma) = -rT_D + dT - \mathrm{ch}_2$$

  Orbits of $\widetilde{GL^+}(2,\mathbb{R})$ are labelled by $s = \frac{\Im T_D}{\Im T}$ and $w = -\frac{\Im T \overline{T_D}}{\Im T}$. Geometric stability conditions exist above the parabola $w > \frac{1}{2} s^2$ *[Bayer Macri 2011]*.
- Along the physical slice, isomorphic to $\widetilde{\mathcal{M}_K} = \mathbb{H}$, T_D, T are holomorphic functions of τ, given by periods of a logarithmic one-form on the mirror curve. They satisfyi a Picard-Fuchs equation of degree 3 and admit an integral representation

$$\boxed{\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u)\, \mathrm{d}u}$$

  where $C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9q + \dots$ is a weight 3 Eisenstein series for $\Gamma_1(3)$.

Central charge as Eichler integral

- This provides an computationally efficient analytic continuation of Z_τ throughout $\mathbb{H}$, and gives access to monodromies:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}$$

  where (m, m_D) are period integrals of C from τ_o to $\frac{d\tau_o - b}{a - c\tau_o}$.

- At large volume $\tau \to \mathrm{i}\infty$, using $C = 1 + \mathcal{O}(q)$ one finds

$$T = \tau + \mathcal{O}(q), \quad T_D = \frac{1}{2}\tau^2 + \frac{1}{8} + \mathcal{O}(q)$$

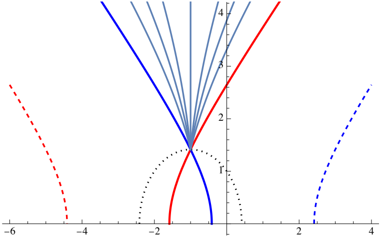
  in agreement with $Z_\tau(\gamma) \sim -\int_S e^{-\tau H} \sqrt{\mathrm{Td}(S)}\, \mathrm{ch}(E)$.

Large volume scattering diagram for $K_{\mathbb{P}^2}$

- Consider the large volume slice with

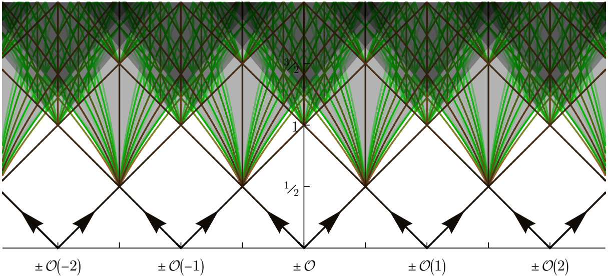
$$Z^{\mathrm{LV}}(\gamma) = -r\frac{1}{2}T^2 + dT - \mathrm{ch}_2, \quad T = s + \mathrm{i}t$$

- Since $\Re Z(\gamma) = \frac{1}{2}r(t^2 - s^2) + ds - \mathrm{ch}_2$ that each ray $\mathcal{R}_0(\gamma)$ is contained in a branch of hyperbola asymptoting to $t = \pm(s - \frac{d}{r})$ for $r \neq 0$, or a vertical line when $r = 0$. Walls of marginal stability $\mathcal{W}(\gamma, \gamma')$ are half-circles centered on real axis.



Large volume scattering diagram

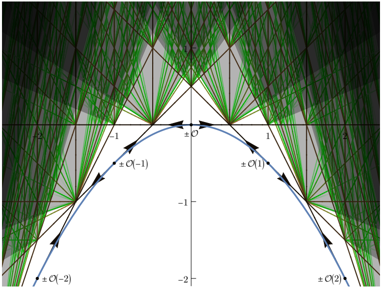
- The objects $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$ are known to be stable throughout the large volume slice *[Arcara Bertram (2013)]*. The corresponding rays are 45 degree lines ending at $s = m$.
- The region of validity of the orbifold exceptional collection and its mutations are valid covers the vicinity of the boundary at $t = 0$, hence there can be no other initial ray. *[Bousseau'19]*.



Scattering diagram in affine coordinates

Actually, Bousseau used different coordinates such that the rays become line segment $rx + dy - \mathrm{ch}_2 = 0$. This works for any ψ:

$$x = \frac{\Re\left(e^{-\mathrm{i}\psi} T\right)}{\cos\psi}\ , \quad y = -\frac{\Re\left(e^{-\mathrm{i}\psi} T_D\right)}{\cos\psi} > -\frac{1}{2}x^2$$

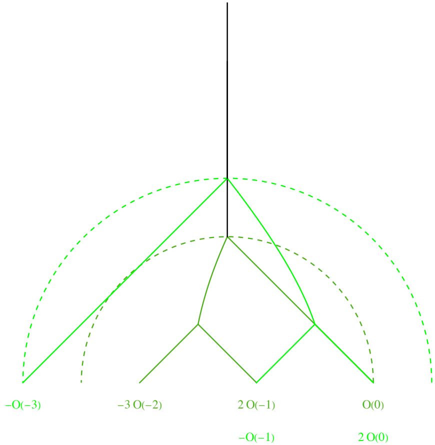


Flow tree formula at large radius

- This implies that all BPS states at large volume must arise as bound states of fluxed D4 and anti D4-branes. But how to find the possible constituents for given γ and (s,t) ?
- Think of $\mathcal{R}(\gamma)$ as the worldline of a fictitious particle of charge r, mass $M^2 = \frac{1}{2}d^2 - r\,\mathrm{ch}_2$ moving in a constant electric field. This makes it clear that constituents must lie in the past light cone.
- Moreover, the 'electric potential' $\varphi_s(\gamma) = 2(d - sr) = 2\Im Z_\gamma / t$ increases along the flow. The first scatterings occur after a time $t \geq \frac{1}{2}$, after each constituent $k_i\mathcal{O}(m_i)$ has moved by $|\Delta s| \geq \frac{1}{2}$, by which time $\varphi_s(\gamma_i) \geq |k_i|$.
- Since $\varphi_s(\gamma)$ is additive at each vertex, this gives a bound on the number and charges of constituents contributing to $\Omega_{(s,t)}(\gamma)$:

$$\sum_i k_i[1, m_i, \frac{1}{2}m_i^2] = \gamma \ , \quad s - t \leq m_i \leq s + t, \quad \sum_i |k_i| \leq \varphi_s(\gamma)$$

Flow trees for $\gamma = [0, 4, 1)$

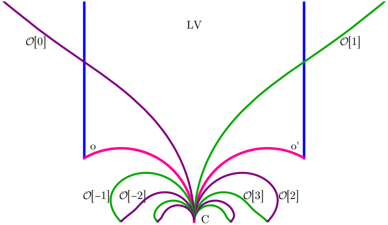


- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$:
  $K_3(2,3)K_{12}(1,1) \to -156$
- $\{-\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$:
  $K_3(1,2)K_{12}(1,1) \to -36$

Total: $\Omega_\infty(\gamma) = -192 = GV_4^{(0)}$

Exact scattering diagram

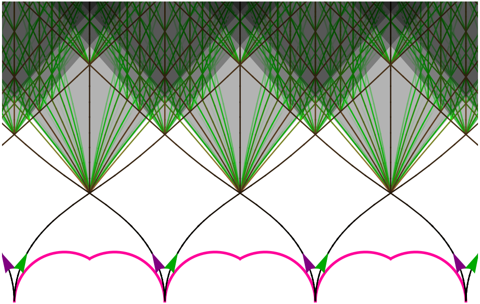
- The scattering diagram $\mathcal{D}^{\Pi}_{\psi}$ along the physical slice should interpolate between $\mathcal{D}^{\rm LV}_{\psi}$ and $\mathcal{D}_o$, and be invariant under $\Gamma_1(3)$.
- Under $\tau \mapsto \frac{\tau}{3n\tau+1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[n]$. Hence there is a doubly infinite family of initial rays emitted at $\tau = 0$, associated to $\mathcal{O}[n]$:



- Similarly, there must be an infinite family of rays emitted from $\tau = \frac{p}{q}$ with $q \neq 0 \mod 3$, corresponding to $\Gamma_1(3)$-images of $\mathcal{O}$.

Exact scattering diagram for small ψ

- For $|\psi|$ small enough, the only rays which reach the large volume region are those associated to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$. Thus, the scattering diagram $\mathcal{D}_{\psi}^{\Pi}$ is isomorphic to $\mathcal{D}_{0}^{\rm LV}$ inside $\cup_n \mathcal{F}(n)$:



Scattering diagram in affine coordinates

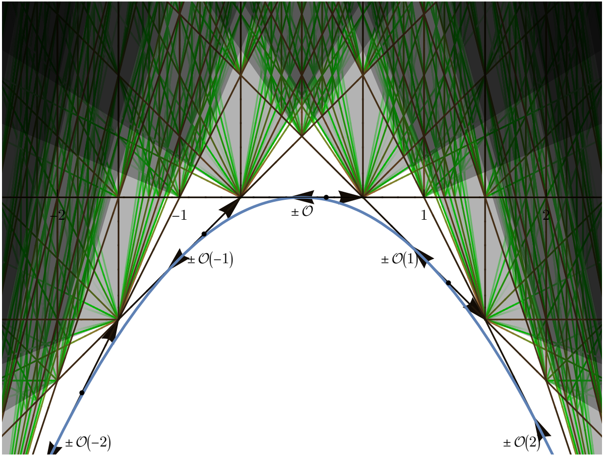
- To see this, map both of them to the (x, y)-plane *[Bousseau'19]*

$$x = \frac{\Re\left(e^{-\mathrm{i}\psi}\,T\right)}{\cos\psi}\,, \quad y = -\frac{\Re\left(e^{-\mathrm{i}\psi}\,T_D\right)}{\cos\psi}$$

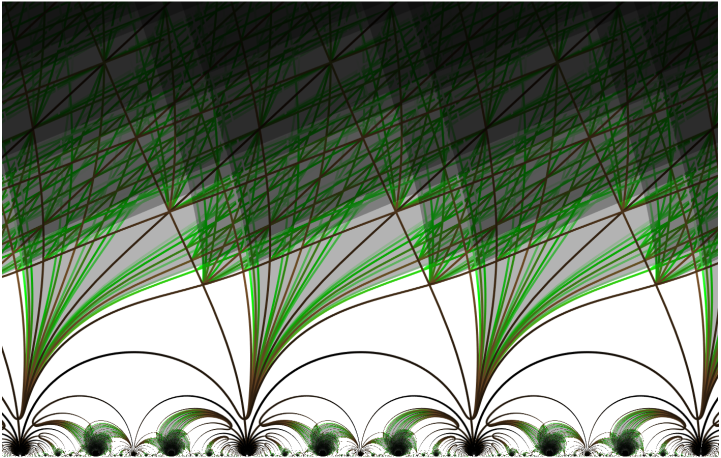
  such that $\mathcal{R}_\psi(\gamma)$ becomes a line segment $rx + dy - \mathrm{ch}_2 = 0$.
- The initial rays $\mathcal{R}_\psi(\mathcal{O}(m))$ are tangent to the parabola $y = -\frac{1}{2}x^2$ at $x = m$, but the origin of each ray is shifted to $x = m + \mathcal{V}\tan\psi$ where $\mathcal{V}$ is the quantum volume

$$\mathcal{V} = \Im T(0) = \frac{27}{4\pi^2}\Im\left[\mathrm{Li}_2(e^{2\pi\mathrm{i}/3})\right] \simeq 0.463$$

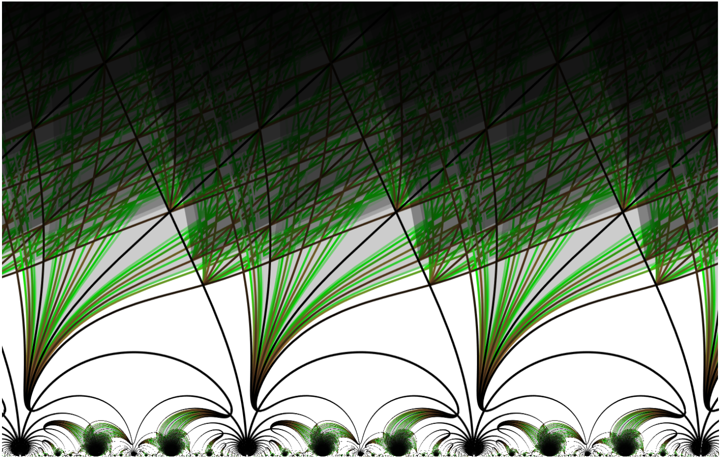
Affine scattering diagram, $\psi = 0$



Exact scattering diagram, $\psi = 0.824$

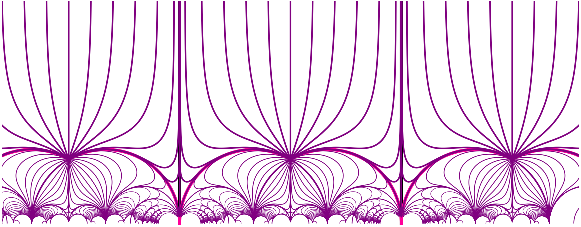


Exact scattering diagram, $\psi = 0.825$



Exact scattering diagram for $\psi = \pm\frac{\pi}{2}$

- For $\psi = \pm\frac{\pi}{2}$, the geometric rays $\{\Im Z_\tau(\gamma) = 0\}$ coincide with lines of constant $s = \frac{\Im T_D}{\Im T} = \frac{d}{r}$, independent of ch_2:



- Hence, there is no wall-crossing between τ_o and $\tau = \mathrm{i}\infty$ when $-1 \leq \frac{d}{r} \leq 0$, explaining why the Gieseker index $\Omega_\infty(\gamma)$ agrees with the quiver index $\Omega_c(\gamma)$ in the anti-attractor chamber.

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Some future directions

- Higher del Pezzo surfaces, higher rank toric singularities
- Framed BPS states and line operators
- D4-D2-D0 indices on multiparameter compact CY3
- Compute attractor invariants in compact CY3
- DT invariants seem to control the resurgent structure of topological strings, why ?
- Exact QK metric on HM moduli space, including NS5/KKM instantons
- …

Thank you for your attention, and mind the wall !