Progress on BPS state counting on compact Calabi-Yau threefolds (part I)

Boris Pioline







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see part II by Sergey Alexandrov

Based on works with S. Alexandrov. S. Feyzbakhsh, A. Klemm, J. Manschot, T. Schimannek, mostly from BC era

Outline

1. Precision counting of BPS states in max/half-max SUSY vacua
2. BPS states in CY3 compactifications and enumerative invariants
3. D4-D2-D0 invariants from GV invariants via wall-crossing
4. Testing the (mock) modularity of MSW invariants
5. Outlook

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Precision counting of BPS states: a brief review

- Understanding the microscopic origin of the Bekenstein-Hawking entropy has been a long standing problem in quantum gravity. It was solved for BPS black holes in string theory in mid 90s, by viewing them as effective, coarse-grained descriptions of D-brane bound states at strong coupling *[Strominger Vafa'96, . . .]*.

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- The black string SCFT is a deformation of the symmetric orbifold X^N/S_N, $N = Q_1 Q_5$, with central charge $c = 6Q_1 Q_5$. The Cardy formula $\log \Omega \sim 2\pi\sqrt{cL_0/6} = \pi\sqrt{Q_1 Q_5 P}$ reproduces the Bekenstein-Hawking entropy with correct numerical factor.

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- General charges $\gamma \in \mathbb{Z}^{27}$ can be reached by applying U-duality, $E_{6(6)}(\mathbb{Z})$ for $X = T^4$, or $O(5, 21, \mathbb{Z})$ for $X = K_3$. In particular, one can map (Q_1, Q_5) to $(Q_1 Q_5, 1)$.

Precision counting of BPS states: 1/4-BPS in $D = 4$

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$$\Omega_{1/4}(\vec{p},\vec{q}) = \int_\Gamma \frac{e^{i\pi(\tau\vec{p}^2+2v\vec{p}\cdot\vec{q}+\sigma\vec{q}^2)}}{\chi_{10}(\tau,v,\sigma)} d\tau dv d\sigma$$

where χ_{10} is a Siegel modular form, obtained as the Borcherds lift of the K3 elliptic genus, $1/\chi_{10} \sim \exp\left(\sum_N e^{2\pi i N\sigma} H_N \cdot \varphi_{K3}(\tau,v)\right)$.

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- Ambiguities in the choice of Γ reflect the wall-crossing phenomenon: two 1/2-BPS states may bind to form a 1/4-BPS dyon *[Sen'07, Cheng Verlinde'07]*.

Precision counting of BPS states: 1/4-BPS in $D = 4, 5$

- A complete derivation of the DVV formula (and generalization to CHL orbifolds) is only reached in '06 in terms of the D1-D5-P-TN system *[David Sen'06]*, using insights from the 4D/5D lift: $Z_N := H_N \cdot \varphi_{K3}$ is nothing but the elliptic genus of the symmetric product $K3^N/S_N$, and $\exp\left(\sum_N p^N Z_N\right)$ is the partition function of a gas of 5D black holes in a Taub-NUT space (times $\mathbb{R}_t \times X$).

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- The origin of the Siegel modular group $Sp(4,\mathbb{Z})$ was elucidated in *[Bossard Cosnier-Horeau BP'18]*: At weak heterotic coupling, the D^2F^4 effective coupling in 3D is perturbatively exact at genus 2, with $f^{(2)}_{D^2F^4} \sim \int_{\mathcal{F}_2} Z^{(2)}_{6,22}/\chi_{10}$.

  Its non-perturbative completion $f_{D^2F^4} \sim \int_{\mathcal{F}_2} Z^{(2)}_{7,23}/\chi_{10}$ receives instanton corrections from 4D black holes winding around the circle, weighted by the same index $\Omega_{1/4}(\vec{p},\vec{q})$.

Precision counting of BPS states: 1/8-BPS in $D = 4, 5$

- Meanwhile, for 1/8-BPS dyons in type II/T^6, by combining the modified 5D elliptic genus *[Maldacena Moore Strominger'96]*, the 4D/5D lift and U-duality, one obtains an exact formula for counting 1/8-BPS dyons *[Shih Strominger Yin'05, BP'05]*:

$$\Omega_{1/8}(\vec{p},\vec{q}) = \int_{0+\mathrm{i}L}^{1+\mathrm{i}L} e^{-2\pi \mathrm{i} I_4(\vec{p},\vec{q})\tau} \frac{\theta_4(2\tau)}{\eta(4\tau)^6} \mathrm{d}\tau \sim e^{\pi\sqrt{I_4(\vec{p},\vec{q})}}$$

  This was reproduced recently from quiver quantum mechanics *[Shi Turiaci'26]*, extending early work by *[Chowdhury Garavuso Mondal Sen'15]*.

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- For vacua with 8 supercharges, such as type IIA or M-theory compactified on a CY 3-fold X, the index counting 1/2-BPS states is no longer encoded in a single automorphic form. Rather it turns out (after many twists and turns) to be determined by the topological string partition function $Z_{\rm top}$...

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D4-D2-D0 bound states and MSW string

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- Assuming that $\mathcal{D}$ is ample, the Cardy formula with $c_L = \mathcal{D}^3 + c_2 \cdot \mathcal{D}$ matches the Bekenstein-Hawking-Wald entropy, including leading R^2 correction. Moreover, $h_{p^a,q_a} := \sum_{q_0} \Omega(0, p^a, q_a, q_0) e^{2\pi i q_0 \tau}$ should transform as a vector-valued modular form under $SL(2,\mathbb{Z})$.

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- Importantly, these forms live in a finite dimension vector space. It suffices to compute the polar terms with $q_0 < \frac{\chi(\mathcal{D})}{24} + \frac{1}{2} p^a q_a - \frac{1}{2} q_a^2$.

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- Assuming modularity and some inspired guesses, *[Gaiotto Strominger Yin'06]* determine the elliptic genus for the quintic X_5 and a few other one-parameter models, with $\mathcal{D}$ the primitive divisor. *[Collinucci van der Heerk Wyder'08]* point out an issue in their guess for X_{10}.

5D index and GV invariants

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- *[Gopakumar Vafa'98]* boldly conjectured that the topological free energy $F_{top}(t,\lambda) := \sum_{g\geq 0} \lambda^{2g-2} F_g(t)$ is given by a Schwinger-type one-loop amplitude with 5D BPS states propagating in a constant anti-self-dual graviphoton background $\mathcal{F}_- \propto \lambda$. This predicts a hidden integral structure behind Gromov-Witten invariants:

$$\sum_{g\geq 0, d\geq 1} \lambda^{2g-2}\, \mathrm{GW}_d^{(g)}\, e^{2\pi i d t} = \sum_{\substack{g\geq 0, d\geq 1,\\ k\geq 1}} \frac{\mathrm{GV}_d^{(g)}}{k} \left(2\sin\tfrac{k\lambda}{2}\right)^{2g-2} e^{2\pi i k d t}$$

where the integers $\mathrm{GV}_d^{(g)}$ are related to the number $N_d^{j_L, j_R} \in \mathbb{Z}_+$ of 5D BPS states of angular momentum (j_L, j_R) by tracing over j_R.

5D index and GV invariants

- More precisely, they are related by a change of basis of $SU(2)_L$ characters which arises naturally from the 1-loop computation,

$$\sum_{j_L,j_R}(-1)^{2j_L+2j_R}(2j_R+1)N_d^{j_L,j_R}\chi_{j_L}(y)=\sum_{g=0}^{g_{\max}(d)}\mathrm{GV}_d^{(g)}\,(2-y-1/y)^g \qquad [*]$$

$$\Omega_{5D}(d,m)=\sum_{g=|m|-1}^{g_{\max}(d)}\mathrm{GV}_d^{(g)}\begin{pmatrix}2g+2\\ g-m+1\end{pmatrix}$$

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- Assuming that 5D BPS black holes dominate in the large charge regime, one expects $\log|\Omega_{5D}(d,m)| \sim \pi\sqrt{d^3-m^2}$. In absence of analytic handle on the growth of $\mathrm{GV}_d^{(g)}$, *[Klemm Marino Tavanfar'05]* give convincing numerical evidence that this holds in the static case, given available data for one-parameter CY threefolds at the time.

Rank 1 DT invariants from GV invariants

- Coming back to 4D, *[Maulik Nekrasov Okounkov Pandharipande'03]* identify the coefficients in the expansion of $Z_{\rm top}$ in powers of $q = e^{{\rm i}\lambda}$ as the Pandharipande-Thomas invariants $PT(d,m)$ counting BPS states with $[D6] = -1, [D2] = d, [D0] = m$:

$$Z_{\rm top} = \prod_{\beta} \prod_{g \geq 0^*} \Big(1 - (-q)^{g-\ell-1} e^{2\pi {\rm i} \beta \cdot t}\Big)^{(-1)^{g+\ell} \binom{2g-2}{\ell} {\rm GV}_\beta^{(g)}}$$
$$= \sum_{d,m} DT(d,m) e^{2\pi {\rm i} dt} q^m := Z_{PT}(t,q)$$

The top line is the partition function of a gas of M2-branes in $\mathbb{R}_t \times TN_k$ with $k = -1$ *[Dijkgraaf Vafa Verlinde'06]*.

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- The Donaldson-Thomas invariants $DT(d,m)$ counting BPS states with $[D6] = 1$ (in same chamber) get contributions from $D6 - D0$ bound states, $Z_{DT}(t,q) = \prod_{n \geq 1} (1 - (-{\rm q})^n)^{-n\chi} Z_{PT}(t,q)$.

4D BPS indices from $Z_{\rm top}$?

- A couple years before, *[Ooguri Strominger Verlinde'04]* proposed that the BPS indices counting 4D BPS black hole microstates can be extracted from the modulus square of $Z_{\rm top}$,

$$\Omega(\vec{p},\vec{q}) \sim \int \mathrm{d}\vec{\phi} \left\| Z_{\rm top}(\vec{\phi}+\mathrm{i}\vec{p}) \right\|^2 e^{\mathrm{i}\vec{\phi}\vec{q}} \sim e^{S_{BH}(\vec{p},\vec{q})}$$

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- This follows from localization on $AdS_2 \times S^2 \times X_*$, as anticipated in *[Beasley et al'06]* and shown recently in *[Bobev et al 2026]*. It suggests that D4-D2-D0 BPS states are $D6 - \overline{D6}$ bound states, but this is in general not true for non-polar states *[Denef Moore'07]*.

Stability conditions and wall-crossing

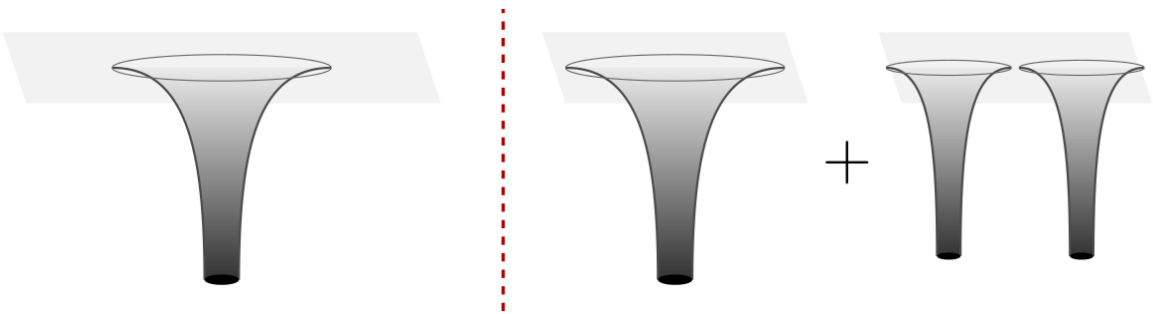
- In general, the Donaldson-Thomas $\Omega_\sigma(\gamma)$ depends on a choice of stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}\, X$ *[Bridgeland'03]*. The physical slice $\Pi = \mathcal{M}_K$ is a middle-dimensional locus inside $\mathrm{Stab}\, X$.

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- DT invariants can jump on codimension one walls $\mathcal{W} \subset \mathrm{Stab}\, X$ where stable objects can decay, $\gamma \to \gamma' \oplus \gamma''$. They satisfy a universal wall-crossing formula *[Kontsevich Soibelman'08, Joyce-Song'08]*, which can be derived physically from the quantum mechanics of multi-centered black holes *[Manschot BP Sen'10]*.

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- For two centers, $\Delta\Omega(\gamma) = \pm\langle\gamma', \gamma''\rangle\, \Omega(\gamma')\, \Omega(\gamma'')$.

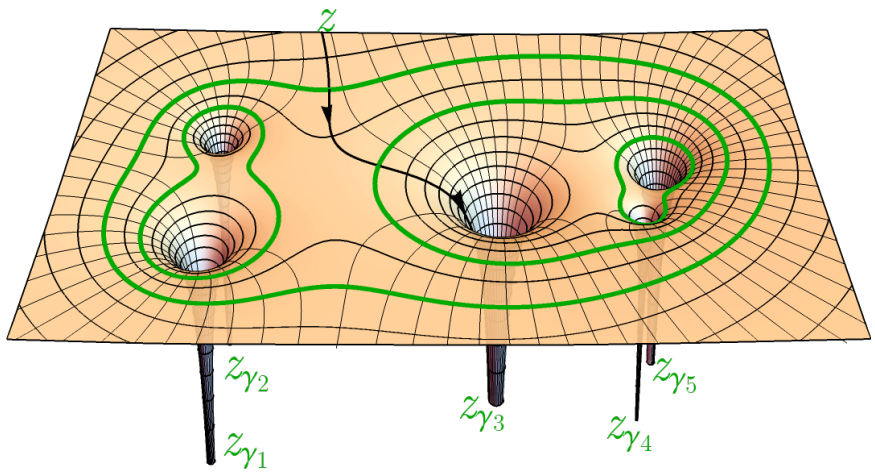


Attractor flow tree formula

- In particular, the index $\Omega_t(\gamma)$ can be expressed in terms of the attractor indices $\Omega_*(\gamma) := \Omega_{t_*}(\gamma)$, where t_* is the endpoint of the gradient flow of $|Z_t(\gamma)|$, through the attractor flow tree formula *[Denef Greene Raugas'01,Alexandrov BP'18]*.

$$\Omega_z(\gamma) = \sum_n \sum_{\gamma=\gamma_1+\cdots+\gamma_n} \frac{g_z(\{\gamma_i\})}{|\mathrm{Aut}(\{\gamma_i\})|} \prod_{i=1}^n \bar{\Omega}_*(\gamma_i)$$

Here $g_z(\{\gamma_i\})$ is a sum over attractor flow trees, describing multicentered black holes as iterated two-centered bound states.



Modular anomaly of D4-D2-D0 indices

- As a byproduct of our study of instanton correction to hypermultiplet moduli spaces *[Alexandrov Banerjee Manschot Persson BP Vandoren Saueressig...08-19]*, we could fix the precise mock modular properties of generating series h_{p^a,q_a} of attractor D4-D2-D0 indices (aka MSW indices) *[Alexandrov Manschot BP'19]*.

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- The key was requiring that the metric on the VM moduli space in 3D, including corrections from 4D BPS states wrapped on S^1 à la *[Gaiotto Moore Neitzke'08]*, admits an isometric action of $SL(2,\mathbb{Z})$:

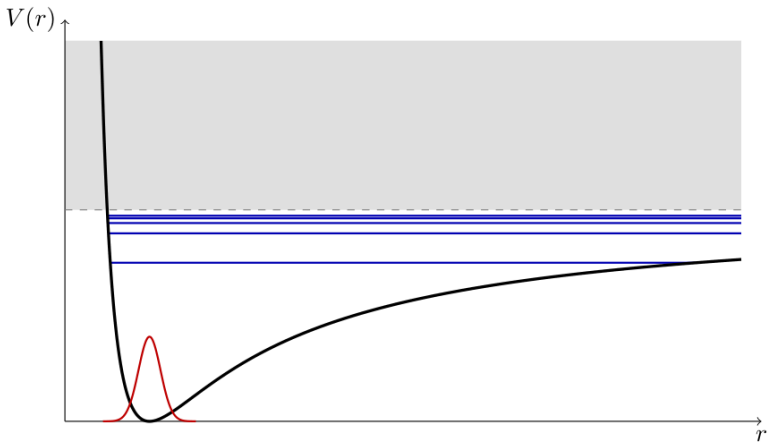
$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^n h_{p_i}(\tau)$$

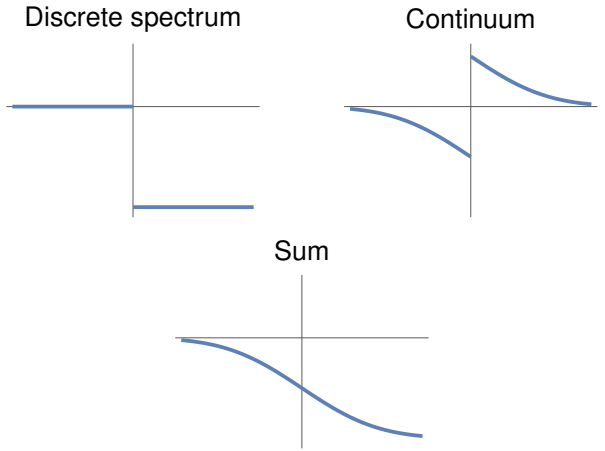
satisfying the holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^n \widehat{h}_{p_i}(\tau,\bar\tau)$$

Non-holomorphic completion from Witten index

- Physically, the non-holomorphic corrections arise from the spectral asymmetry of the continuum of scattering states in the supersymmetric quantum mechanics of n BPS black holes.

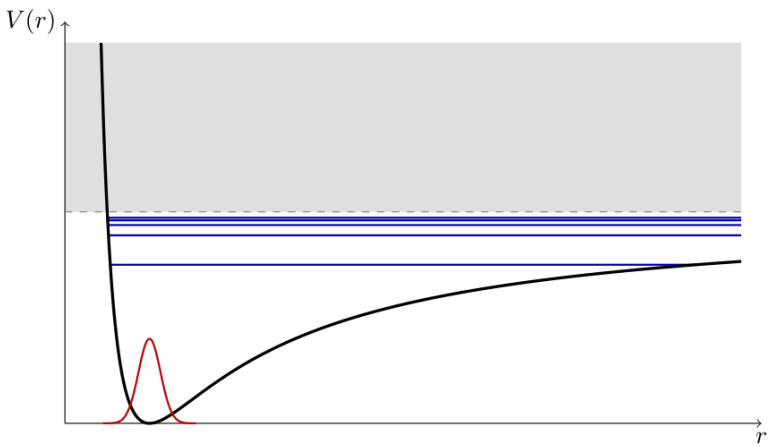


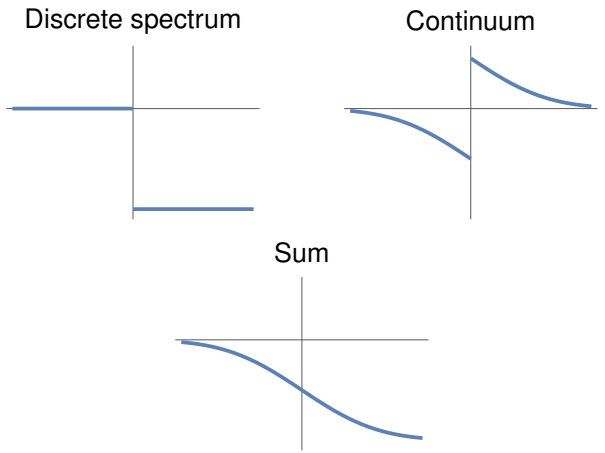


BP 2015; Murthy BP 2018

Non-holomorphic completion from Witten index

- Physically, the non-holomorphic corrections arise from the spectral asymmetry of the continuum of scattering states in the supersymmetric quantum mechanics of n BPS black holes.





BP 2015; Murthy BP 2018

- Using localization, one can actually compute the Witten index for any number of centers and reproduce the full modular completion ! (at least for collinear D4-brane charges) *[BP Raj'25]*

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All DT invariants from GV invariants

- From the math side, *[Feyzbakhsh Thomas'20-22]* prove that for any smooth CY threefold X *(subject to a 'technical condition')*, the DT invariants for any D6-charge r (including $r = 0$) can be expressed at large volume in terms of rank $r = 1$ DT invariants. Those are in turn determined by GV invariants via MNOP.

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- This relies on wall-crossing arguments in the space $\mathrm{Stab}\, X$ away from the physical slice, specifically along the slice of tilt-stability conditions.
- The 'technical condition' is a conjectural inequality on the D0-brane charge of tilt-stable objects due to BMT *[Bayer Macri Toda'11]*, proven for the quintic *[Li 2019]*, and now for any CY3 *[Open AI'26, Li Liu Perry Zhao'26]*.

The slice of tilt-stability conditions

- For any $t \in \mathbb{H}$, consider the central charge obtained by setting to 0 the coefficient of $[D0]$ in the large volume central charge:

$$Z_t(\gamma) = \tfrac{\mathrm{i}}{6} t^3 [D6] - \tfrac{1}{2} t^2 [D4] - \mathrm{i} t [D2] + 0[D0]$$

With a suitable choice of heart (defined with respect to the slope $\frac{[D4]}{[D6]}$), this defines a weak stability condition called tilt-stability.

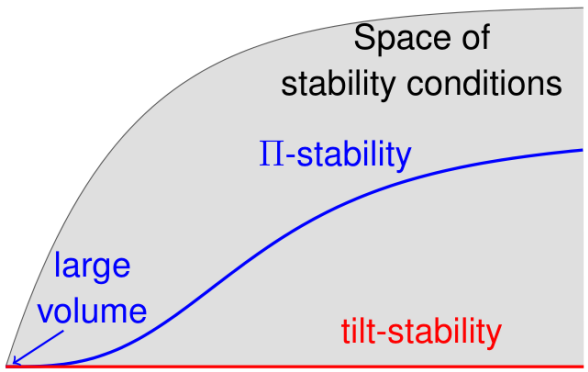
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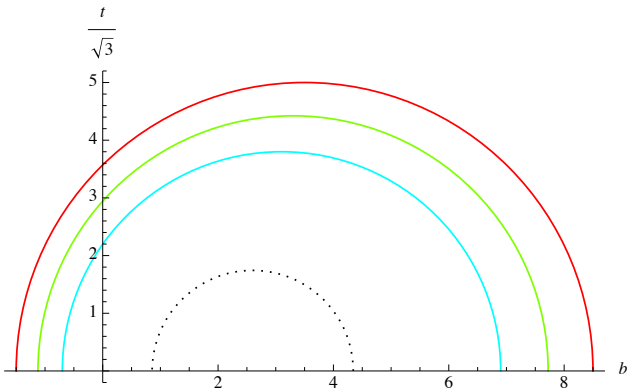
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- Since D0-branes have vanishing central charge, this is not a *bona fide* stability condition, but it serves as a first step in constructing genuine stability conditions near the large volume point *[Bayer Macri Toda'11]*. Moreover the KS/JS wall-crossing formulae holds along this slice.



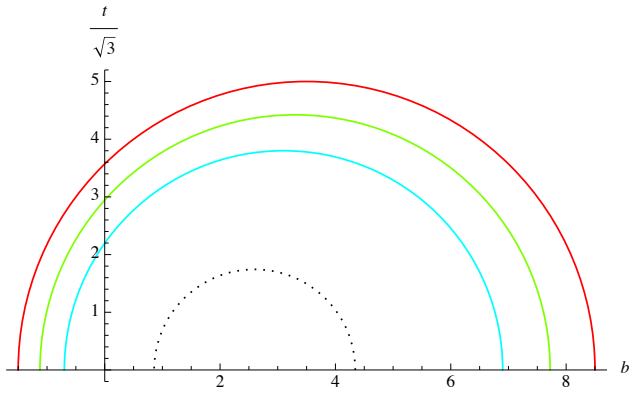
Rank 0 DT invariants from GV invariants

- Tilt stability agrees with physical stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



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- The BMT inequality is the statement that for any tilt-semistable object E, the Chern classes $C_i := \int_X \mathrm{ch}_i(E).H^{3-i}$ must satisfy

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

For suitable C_i, this gives an empty chamber (dotted half-circle above), analogous to the attractor chamber on the physical slice.

Outline

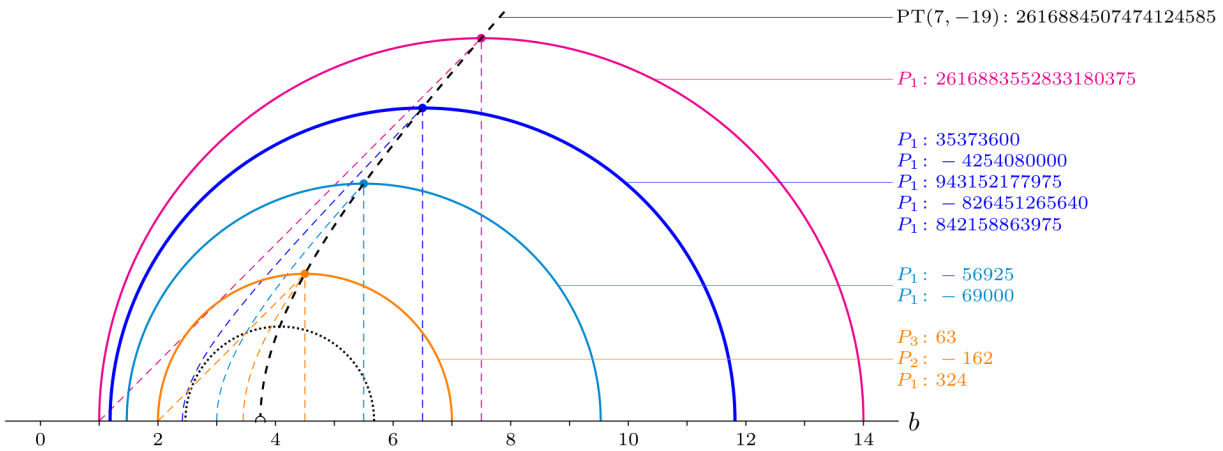
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Testing the (mock) modularity of MSW invariants

- Using these relations, and available data on GV invariants for one-parameter hypergeometric models, *[Alexandrov Feyzbakhsh Klemm BP Schimannek'23 BC]* were able to compute rigorously a large number of coefficients in h_{p^a,q_a} and check (mock) modularity.

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- Actually, in order to compute both polar and non-polar D4-D2-D0 indices, it proved very useful to study the chamber structure for a different charge γ with $[D6] = -1$, and track down its decay by emission of D4-D2-D0 bound states. For e.g. for X_8, $[D4] = 2$,



Modularity for one-modulus compact CY

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1	n_2	C_2
$X_5(1^5)$	-200	5	50	5	7	0	36	1
$X_6(1^4,2)$	-204	3	42	4	4	0	19	1
$X_8(1^4,4)$	-296	2	44	4	4	0	14	1
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0	7	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1	25	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0	5	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1	78	3
$X_{4,2}(1^6)$	-176	8	56	6	15	1	69	3
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3	185	4

Modular predictions for the quintic threefold

- Using known $GV_\beta^{(g\leq 53)}$ we could compute more than 20 terms:

$$h_0 = q^{-\frac{55}{24}}\Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4$$
$$+28096675153255q^5 + 3756542229485475q^6$$
$$+27759174420281 5875q^7 + 1361098501470988 8750q^8 + \dots\Big),$$
$$h_{\pm 1} = q^{-\frac{55}{24}+\frac{3}{5}}\Big(\underline{0 + 8625q} - 1138500q^2 + 3777474000q^3$$
$$+ 310275038012 5q^4 + 577727215123000q^5 + \dots\Big)$$
$$h_{\pm 2} = q^{-\frac{55}{24}+\frac{2}{5}}\Big(\underline{0 + 0q} - 1218500q^2 + 441969250q^3 + 95371251125 0q^4$$
$$+ 217571250023750q^5 + 22258695264509625q^6 + \dots\Big)$$

Mock modularity of rank 2 MSW indices on X_{10}

- For X_{10}, we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523E_4^{12}+70149738E_4^9E_6^2-12112656E_4^6E_6^4-61127530E_4^3E_6^6-2307075E_6^8}{46438023168\eta^{100}}\vartheta_\mu^{(1,2)} + \frac{-10826123E_4^{10}E_6-14574207E_4^7E_6^3+20196255E_4^4E_6^5+5204075E_4E_6^7}{1934917632\eta^{100}}D\vartheta_\mu^{(1,2)} + (-1)^{\mu+1}H_{\mu+1}(\tau)\,h_1(\tau)^2$$

with $h_1 = \frac{203E_4^4+445E_4E_6^2}{216\,\eta^{35}} = q^{-\frac{35}{24}}(3-575q+\dots)$ and $H_\mu(\tau)$ the generating series of Hurwitz class numbers. This predicts

$$h_{2,0}^{(\mathrm{int})} = q^{-\frac{19}{6}}\left(7-1728q+203778q^2-13717632q^3-23922034036q^4+\dots\right)$$

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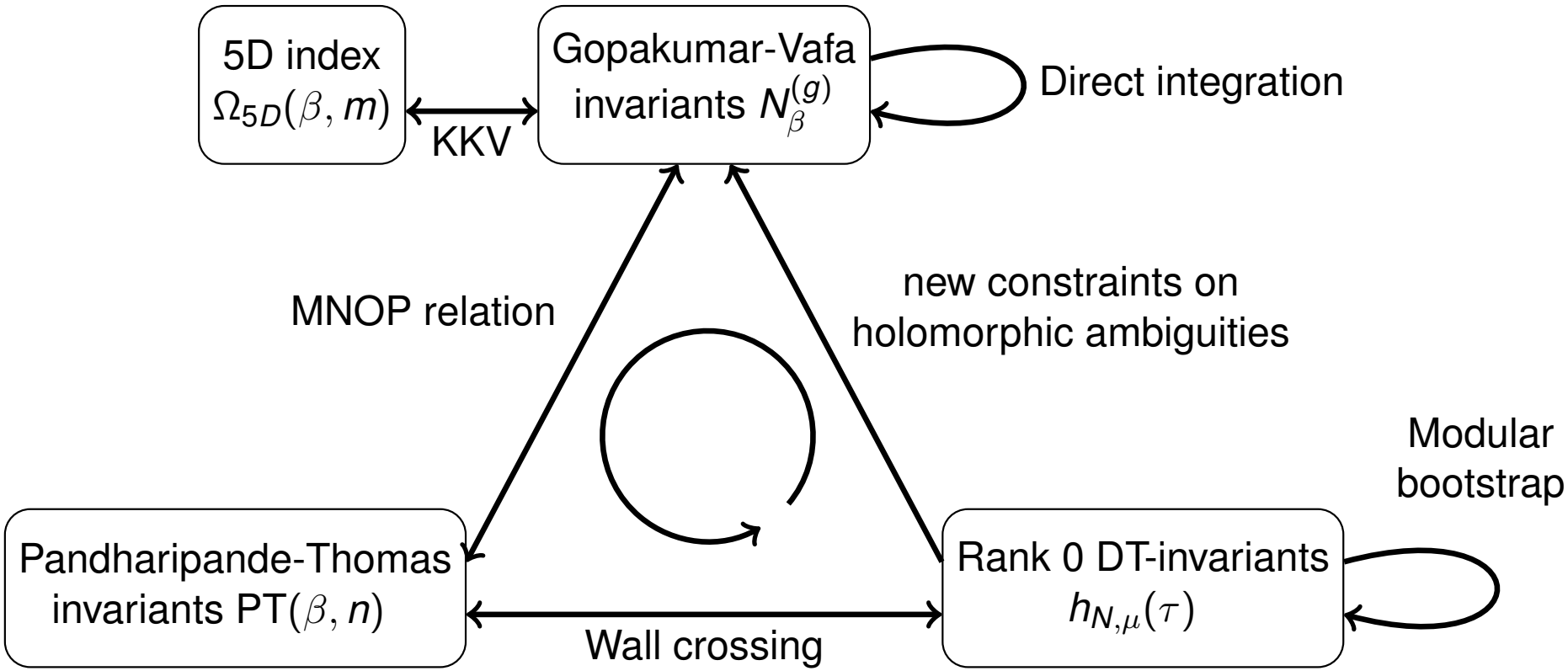
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- Similar results for X_8 *[S. Alexandrov, S. Feyzbakhsh, A. Klemm'23]*

Quantum geometry from stability and modularity



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

Quantum geometry from stability and modularity

X	χ_X	κ	type	g_{integ}	$g^{(1)}_{\text{mod}}$	$g^{(2)}_{\text{mod}}$
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http://www.th.physik.uni-bonn.de/Groups/Klemm/data.php [OBSOLETE]

New results from AD era [Anno Domini, AI Dominates]

Using AI (mostly Claude Code and OpenAI codex), a host of new results was obtained in the last few weeks:

- Direct integration for X_5 pushed to genus 83 using modular predictions
- Modularity checked for rank 1 MSW invariants for all hypergeometric models, and one non-hypergeometric model
- Depth-two mock modularity for rank 3 MSW invariants on X_{10}, reproducing *[Alexandrov Bendriss'24]* on the nose !
- ...

A small math/physics collaboration, CYCLOPS[1], is being set up to digest and write up these results, more generally to combine human expertise and AI tools/insights to push the frontiers of enumerative geometry of CY threefolds.

[1] Calabi–Yau Counts: Links Of Physics and Sheaves

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- Similar wall-crossing arguments also allow to compute higher rank DT invariants. Is there some higher rank version of *[MNOP'03]* ?
- All computations of BPS indices so far rely on wall-crossing from an empty chamber. Can we compute attractor BPS indices using some version of D-brane quiver quantum mechanics, à la *[Chowdhury Garavuso Mondal Sen'14, Turiaci Shi'26]* ?

Thank you for your attention, and mind the wall !

