Classical & quantum glasses

Leticia F. Cugliandolo
Univ. Pierre et Marie Curie – Paris VI
leticia@lpthe.jussieu.fr
www.lpthe.jussieu.fr/~leticia

Slides & useful notes in front page

Cargèse, June 2011
Plan

- First lesson.
  
  Introduction. Overview of disordered systems and methods.

- Second lesson.
  
  Statics of classical and quantum disordered systems.

- Third lesson.
  
  Classical dynamics. Coarsening. Formalism.

- Fourth lesson.
  
  Quantum dynamics. Formalism. Results for mean-field models.
Isolated systems

Dynamics of a classical isolated system.

Foundations of statistical physics.

Question: does the dynamics of a particular system reach a flat distribution over the constant energy surface in phase space?

Ergodic theory, ∈ mathematical physics at present.

Dynamics of a quantum isolated system:

a problem of current interest, recently boosted by cold atom experiments.

Question: after a quantum quench, i.e. a rapid variation of a parameter in the system, are at least some observables described by thermal ones?

When, how, which? but we shall not discuss these issues here.
Dissipative systems

Aim

Our interest is to describe the statics and dynamics of a classical or quantum system coupled to a classical or quantum environment.

The Hamiltonian of the ensemble is

\[ H = H_{\text{syst}} + H_{\text{env}} + H_{\text{int}} \]

The dynamics of all variables are given by Newton or Heisenberg rules, depending on the variables being classical or quantum.

The total energy is conserved, \( E = ct \) but each contribution is not, in particular, \( E_{\text{syst}} \neq ct \), and we’ll take \( E_{\text{syst}} \ll E_{\text{env}} \).
Reduced system

Model the environment and the interaction

E.g., an ensemble of harmonic oscillators and a bi-linear coupling:

\[
H_{env} + H_{int} = \sum_{\alpha=1}^{N} \left[ \frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{m_{\alpha}\omega_{\alpha}^2}{2}q_{\alpha}^2 \right] + \sum_{\alpha=1}^{N} c_{\alpha} q_{\alpha} x
\]

Classically (coupled Newton equations) and quantum mechanically (easier in a path-integral formalism) one can integrate out the oscillator variables. Assuming the environment is coupled to the sample at the initial time, \( T \), and that its variables are characterized by a Gibbs-Boltzmann distribution or density function at inverse temperature \( \beta \) one finds a colored Langevin equation (classical) or a reduced dynamic generating functional \( Z_{red} \) (quantum mechanically). (see later explicit calculations)
What we know

Collective phenomena lead to phase transitions.

*E.g.*, thermal PM - FM transitions in classical magnetic systems.

We understand the nature of the equilibrium phases and the phase transitions. We can describe the phases with mean-field theory and the critical behavior with the renormalization group.

Quantum and thermal fluctuations conspire against the ordered phases.

We understand the equilibrium and out of equilibrium relaxation at the critical point or within the phases. We describe it with the dynamic RG at the critical point or the dynamic scaling hypothesis in the ordered phase.

*E.g.*, growth of critical structures or ordered domains.
What we do not know

In systems with competing interactions there is no consensus upon:

- whether there are phase transitions,
- which is the nature of the putative ordered phases,
- which is the dynamic mechanism.

Examples are:

- systems with quenched disorder;
- systems with geometric frustration;
- glasses of all kinds.

Static and dynamic mean-field theory has been developed – both classically and quantum mechanically – and they yield new concepts and predictions.

Extensions of the RG have been proposed and are currently being explored.
Quenched disorder

Quenched variables are frozen during time-scales over which other variables fluctuate.

**Time scales**

\[ \tau_0 \ll \tau_{exp} \ll \tau_{eq} \]

\( \tau_{eq}^{qd} \) could be the diffusion time-scale for magnetic impurities the magnetic moments of which will be the variables of a magnetic system; or the flipping time of impurities that create random fields acting on other magnetic variables.

**Weak** disorder (modifies the critical properties but not the phases) vs. **strong** disorder (that modifies both).

e.g. random ferromagnets vs. spin-glasses.
The case of spin-glasses

**Magnetic impurities (spins) randomly placed in an inert host**

Quenched random interactions

**RKKY potential**

\[ V(r_{ij}) \propto \frac{\cos 2k_F r_{ij}}{r_{ij}^3} s_i s_j \]

very rapid oscillations about 0

and slow power law decay.

Standard lore: there is a 2nd order static phase transition at \( T_s \)

separating a paramagnetic from a spin-glass phase.

No dynamic precursor above \( T_s \). Glassy dynamics below \( T_s \) with aging, memory effects, etc.
Measurements

FIG. 1. Susceptibility $\chi'_{\text{emu/mole Mn}}$ as a function of reduced temperature $T/T_f$ for various CuMn spin-glass (powder) alloys: (a) Ia (0.23 at. % Mn) $T_f = 2.85$ K, Ib (0.46 at. % Mn) $T_f = 5.00$ K, and Ic (1.48 at. % Mn) $T_f = 12.40$ K; (b) IIa (0.57 at. % Mn) $T_f = 6.00$ K and IIb (0.70 at. % Mn) $T_f = 7.65$ K. The dashed lines represent data in an external magnetic field as labeled. The samples Ia, Ib, Ic, and IIa, IIb, respectively, have been prepared and annealed using different methods, as described in text.

J. Mydosh et al. 81.
Transition

Hérisson & Ocio 02.
Dynamics
Pinning by impurities

Competition between elasticity and quenched randomness

$d$-dimensional elastic manifold in a transverse $N$-dimensional quenched random potential.

Interface between two phases; vortex line in type-II supercond; stretched polymer.

Distorted Abrikosov lattice

Goa et al. 01.
Frustration

\[ H_J[\{s\}] = - \sum_{\langle ij \rangle} J_{ij} s_i s_j \]

Ising model

Disordered

\[ E_{gs}^{frust} > E_{gs}^{FM} \]
\[ S_{gs}^{frust} > S_{gs}^{FM} \]

Geometric

Frustration enhances the ground-state energy and entropy

Disordered example

\[ E_{gs}^{frust} = -2J > E_{gs}^{FM} = -4J \]
\[ S_{gs}^{frust} = \ln 6 > S_{gs}^{FM} = \ln 2 \]

Geometric case

\[ E_{gs}^{frust} = 3J > E_{gs}^{FM} = -3J \]
\[ S_{gs}^{frust} = \ln 3 > S_{gs}^{FM} = \ln 2 \]
Frustration

\[ H_J[\{s\}] = - \sum_{\langle ij \rangle} J_{ij} s_i s_j \]

Ising model

One cannot satisfy all couplings simultaneously if \( \prod_{\text{loop}} J_{ij} < 0 \).

One can expect to have metastable states too.

**Trick**: Avoid frustration by defining a spin model on a tree, with no loops, or in a dilute graph, with loops of length \( \ln N \).

But it could be an over-simplication.
Heterogeneity

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$, contrary to what happens in a ferromagnetic sample, for instance.

Each sample is a priori different but,

do they all have a different thermodynamic and dynamic behavior?

**Homogeneous**

- $h_i = -4J \ \forall \ i.$
- $h_j = 2J$
- $h_k = 0$
- $h_l = -2J.$

**Heterogeneous**
Self-averageness

The disorder-induced free-energy density distribution approaches a Gaussian with vanishing dispersion in the thermodynamic limit:

\[
\lim_{N \to \infty} f_N(\beta, J) = f_{\infty}(\beta)
\]

independently of disorder.

- **Experiments**: all *typical* samples behave in the same way.
- **Theory**: one can perform a (hard) average of disorder,

\[
-\beta f_{\infty}(\beta) = \lim_{N \to \infty} \lim_{n \to 0} \frac{Z_N^n(\beta, J) - 1}{N^n}
\]

**Exercise**: Prove it for the 1d Ising chain; argument for finite \(d\) systems.

Intensive quantities are also self-averaging.

**Replica theory**

\[
-\beta f_{\infty}(\beta) = \lim_{N \to \infty} \lim_{n \to 0} \frac{Z_N^n(\beta, J) - 1}{N^n}
\]
Everyday-life glasses

- **3000 BC** Glass discovered in the Middle East. Luxurious objects.


- By the time of the Crusades glass manufacture had been revived in Venice. Cristallo

- After 1890, the engineering of glass as a material developed very fast everywhere.
What do glasses look like?

Simulation
Molecular (Sodium Silicate)

Confocal microscopy
Colloids (e.g. $d \sim 162$ nm in water)

Experiment
Granular matter

Simulation
Polymer melt
Structural glasses

Characteristics

• Selected variables (molecules, colloidal particles, vortices or polymers in the pictures) are coupled to their surroundings (other kinds of molecules, water, etc.) that act as thermal baths in equilibrium.

• There is no quenched disorder.

• The interactions each variable feels are still in competition, e.g. Lenard-Jones potential, frustration.

• Each variables feels a different set of forces, time-dependent heterogeneity.

Sometimes one talks about self-generated disorder.
The two-time dependent density-density correlation function:

\[ g(r; t, t_w) \equiv \langle \delta \rho(\vec{x}, t) \delta \rho(\vec{y}, t_w) \rangle \quad \text{with} \quad r = |\vec{x} - \vec{y}| \]

The average over different dynamical histories (simulation/experiment) \( \langle \ldots \rangle \) implies isotropy (all directions are equivalent) and invariance under translations of the reference point \( \vec{x} \).

Its Fourier transform:

\[ F(q; t, t_w) = N^{-1} \sum_{i,j=1}^{N} \langle e^{i\vec{q}(\vec{r}_i(t) - \vec{r}_j(t_w))} \rangle \]

The incoherent intermediate or self correlation:

\[ F_s(q; t, t_w) = N^{-1} \sum_{i=1}^{N} \langle e^{i\vec{q}(\vec{r}_i(t) - \vec{r}_i(t_w))} \rangle \]

Hansen & McDonald 06.
No obvious structural change but slowing down!

LJ mixture: \[ V_{\alpha\beta}(r) = 4\epsilon_{\alpha\beta} \left[ \left( \frac{\sigma_{\alpha\beta}}{r} \right)^{12} - \left( \frac{\sigma_{\alpha\beta}}{r} \right)^6 \right] \]

\[ \tau_0 \ll \tau_{\text{exp}} \ll \tau_{\text{eq}} \text{ that changes by } >10 \text{ orders of magnitude!} \]

Time-scale separation & slow non-equilibrium dynamics

Time-scales

Calorimetric measurement of entropy

What is making the relaxation so slow?
Is there growth of static order?
Phase space picture?
Fluctuations

Energy scales

Thermal fluctuations

- irrelevant for granular matter since $mgd \gg k_B T$; dynamics is induced by macroscopic external forces.
- important for magnets, colloidal suspensions, etc.

Quantum fluctuations

- when $\hbar \omega \gtrsim k_B T$ quantum fluctuations are important.
- one can even set $T \to 0$ and keep just quantum fluctuations.

Examples: quantum magnets, Wigner crystals, etc.
Summary

Structural glasses

Crystallization at $T_m$ is avoided by cooling fast enough.

Liquid
- Supercooled liquid
- Glass

Exponential relax
- Non-exponential relax

Equilibrium
- Metastable equilibrium
- Non-equilibrium

Separation of time-scales &
- An exponential number of metastable states!

Stationary
- Aging

Aging means that correlations and responses depend on $t$ and $t_w$

ac susceptibilities depend on $\omega$ and $t_w$

There might be an equilibrium transition to an ideal glass at $T_s$. 
Methods
for classical and quantum disordered systems

Statics

- TAP Thouless-Anderson-Palmer
- Replica theory
- Cavity or Peierls approx.
- Bubbles & droplet arguments
- Functional RG

fully-connected (complete graph)
Gaussian approx. to field-theories
dilute (random graph)
finite dimensions

Dynamics

Generating functional for classical field theories (MSRJD).
Schwinger-Keldysh closed-time path-integral for quantum dissipative models
(the previous is recovered in the $\hbar \to 0$ limit).
Perturbation theory, renormalization group techniques, self-consistent approx.  

1 See P. Le Doussal’s seminar.
References

– Liquids & glass transition

– Spin-glasses
References

– Disorder elastic systems

– Phase ordering kinetics

– Glasses
End of 1st talk
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Simulation
Polymer melt
Correlation functions

Structure and dynamics

The two-time dependent density-density correlation:

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The incoherent intermediate or self correlation:

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Hansen & McDonald 06.
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No obvious structural change but slowing down!

LJ mixture

\[ V_{\alpha\beta}(r) = 4\epsilon_{\alpha\beta} \left[ \left( \frac{\sigma_{\alpha\beta}}{r} \right)^{12} - \left( \frac{\sigma_{\alpha\beta}}{r} \right)^{6} \right] \]

\[ \tau_0 \ll \tau_{exp} \ll \tau_{eq} \]

Time-scale separation & slow non-equilibrium dynamics

Why are they all ‘glasses’?

What do they have in common?

- No obvious spatial order: disorder (differently from crystals).
- Many metastable states
  Rugged landscape
- Slow non-equilibrium relaxation
  \[ \tau_0 \ll \tau_{exp} \ll \tau_{eq} \]
  Time-scale separation
- Hard to make them flow under external forces.

  Pinning, creep, slow non-linear rheology.
PM-FM transition

e.g., up & down spins in a 2d Ising model (IM)

\[ \langle \phi \rangle = 0 \quad \langle \phi \rangle = 0 \quad \langle \phi \rangle \neq 0 \]

\[ T \to \infty \quad T = T_c \quad T < T_c \]

In a canonical setting the control parameter is \( T/J \).
Mean-field theory for PM-FM

Fully connected Ising model

Normalize $J$ by the size of the system $N$ to have $O(1)$ local fields

$$H = -\frac{J}{2N} \sum_{i \neq j} s_is_j - h \sum_i s_i$$

The partition function reads $Z_N = \int_{-1}^{1} du e^{-\beta N f(u)}$ with $Nu = \sum_i s_i$

$$f(u) = -\frac{J}{2} u^2 - hu + T \left[ \frac{1+u}{2} \ln \frac{1+u}{2} + \frac{1-u}{2} \ln \frac{1-u}{2} \right]$$

Energy terms and entropic contribution stemming from $\mathcal{N}(\{s_i\})$ yielding the same $u$ value.

Use the saddle-point, $\lim_{N \to \infty} f_N(\beta J, \beta h) = f(u_{sp})$, with

$$u_{sp} = \tanh (\beta Ju_{sp} + \beta h) = \langle u \rangle$$

Exercise: do these calculations, see notes.
Coarse-grain the spin
\[ \phi(\vec{r}) = \frac{1}{V} \sum_{i \in V} s_i. \]
Set \( h = 0 \).

The partition function is
\[ Z_V = \int \mathcal{D}\phi \, e^{-\beta V f(\phi)} \]
with \( V \) the volume and
\[ f(\phi) = \int d^d r \left\{ \frac{1}{2} \left[ \nabla \phi(\vec{r}) \right]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{\lambda}{4} \phi^4(\vec{r}) \right\}. \]

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around \( \phi \sim 0 \) and symmetry arguments.

**Uniform** saddle point in the \( V \to \infty \) limit:
\[ \phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = \phi_0. \]

The free-energy is
\[ \lim_{V \to \infty} f_V(\beta, J) = f(\phi_{sp}). \]
2nd order phase-transition

bi-valued equilibrium states related by symmetry, e.g. Ising magnets

Ginzburg-Landau free-energy

Scalar order parameter
Features

- **Spontaneous symmetry breaking** below $T_c$.  
- Two equilibrium states related by symmetry $\phi \rightarrow -\phi$.  
- The state is chosen by a **pinning field**.  
- If the partition sum is performed over the whole phase space $\langle \phi \rangle = 0$ (a consequence of the symmetry of the action).  
- **Restricted statistical averages** (**half** phase space) yield $\langle \phi \rangle \neq 0$.  
- Under a magnetic field the free-energy landscape is tilted and one of the minima becomes a **metastable state**.  
- The barrier in the **free-energy landscape** between the two states diverges with the size of the system implying **ergodicity breaking**.  
- With $p > 2$-uplet interactions one finds **first order phase transitions**.

*These results were not fully accepted as realistic at the time.*
Equilibrium configurations

e.g. up & down spins in a $2d$ Ising model (IM)

\[ \langle \phi \rangle = 0 \]
\[ \langle \phi \rangle = 0 \]
\[ \langle \phi \rangle \neq 0 \]

\( T \rightarrow \infty \) \hspace{1cm} \( T = T_c \) \hspace{1cm} \( T < T_c \)

In a canonical setting the control parameter is $T/J$. 
Spin-glasses

e.g. up & down spins in a 3d Edwards-Anderson model (EA)

\[ H_J = \sum_{\langle ij \rangle} J_{ij} s_i s_j \]

\[ [J_{ij}] = 0 \quad [J^2_{ij}] = J^2 \]

\[ \langle \phi \rangle = 0 \]

\[ T \rightarrow \infty \]

\[ T = T_c \]

\[ T < T_c \]

Is there another order-parameter?
MF theory for spin-glasses

Fully connected SG: Sherrington-Kirkpatrick model

\[ H = -\frac{1}{2\sqrt{N}} \sum_{i \neq j} J_{ij} s_i s_j - \sum_{i} h_i s_i \]

with \( J_{ij} \) i.i.d. Gaussian variables, \( [J_{ij}] = 0 \) and \( [J_{ij}^2] = J^2 = \mathcal{O}(1) \).

One finds the naive free-energy landscape

\[ Nf(\{m_i\}) = -\frac{1}{2\sqrt{N}} \sum_{i \neq j} J_{ij} m_i m_j + T \sum_{i=1}^{N} \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2}. \]

and the naive TAP equations

\[ m_{isp} = \tanh(\beta \sum_{j(\neq i)} J_{ij} m_{jsp} + \beta h_i) \]

that determine the restricted averages \( m_i = \langle s_i \rangle = m_{isp} \).

These should be corrected by the Onsager reaction term, that subtracts the self-response of a spin, see notes. Thouless, Anderson, Palmer 77.
MF theory for spin-glasses

A hint on the proof

The more traditional one assumes independence of the spins,

\[ P(\{s_i\}) = \prod_i p_i(s_i) \]

with \( p_i(s_i) = \frac{1+m_i}{2} \delta_{s_i,1} + \frac{1-m_i}{2} \delta_{s_i,-1} \)

and uses this form to express \( \langle H \rangle - T \langle S \rangle \) with \( S = \ln P(\{s_i\}) \); see notes.

A more powerful proof expresses \( f \) as the Legendre transform of \(-\beta F(h_i)\)

with \( m_i = N^{-1} \partial[-\beta F(h_i)]/\partial h_i = \langle s_i \rangle_h \).

Georges & Yedidia 91.

This proof is easier to generalize to dynamics (Biroli 00) and quantum systems (Biroli & LFC 01), see later.
Features

- Saddle-points $m_i \neq 0$ below $T_c$ are heterogeneous.
- The fact that an equilibrium state is determined by $\{m_i\}$ can be made rigorous in MF.
- There are $N$ order parameters, $m_i$, $i = 1, \ldots, N$.
- The TAP equations have an exponential in $N$ number of solutions $\{m_i^\alpha\}$ that are extrema of the TAP free-energy landscape, i.e. saddles of all types, at low temperatures.
- For each solution $\{m_i^\alpha\}$ one also has $\{-m_i^{\alpha_{sp}}\}$ but apart from this trivial doubling, the remaining ones are not related by symmetry.
- One can study the temperature-dependent free-energy landscape and compute, e.g., how many saddles of each kind exist and how many of these at each level of $f$.
- One finds a hierarchy of metastable states, with high degeneracy, separated by diverging barriers (infinite life-time).
The average of a generic observable is
\[ \langle O \rangle = \sum_{\alpha} w_{\alpha} \langle O \rangle_{\alpha} \]

In the FM case, each state \( \langle \phi \rangle = \pm \phi_0 \) has weight \( w_\pm = e^{-\beta N f_\pm} Z_N = 1/2 \) and the sum is \( \langle O \rangle = \frac{1}{2} \langle O \rangle_+ + \frac{1}{2} \langle O \rangle_- \). For instance, the averaged magnetization vanishes if one sums over the \( \pm \) states or it is different from zero if one restricts the sum to one of them.

Within the TAP approach
\[ w^J_\alpha = \frac{e^{-\beta N f^J_\alpha}}{\sum_\gamma e^{-\beta N f^J_\gamma}} \]
and
\[ \langle O \rangle = Z^{-1}(\beta, J) \int df \ e^{-\beta [N f - T \ln N_J(f, \beta)]} O(f) \]
where \( N_J \) is the (possibly exponential in \( N \)) number of solutions to the TAP equations with free-energy density \( f \).
Statistical averages

Consequences

The equilibrium free-energy is given by the saddle-point evaluation of the partition sum that implies

\[ N f = N f_{\text{min}} - T S_c(f_{\text{min}}, \beta) \quad S_c(f, \beta) = \ln \mathcal{N}_J(f, \beta) \]

The rhs is the Landau free-energy of the problem, with \( f \) playing the role of the energy and \( N^{-1} \ln \mathcal{N}_J(f, \beta) \) the one of the entropy.

In the sum we do not distinguish the stability of the TAP solutions.

In some cases higher lying extrema (metastable states) can be so numerous to dominate the partition sum with respect to lower lying ones.

This feature is proposed to describe super-cooled liquids.
The ruggedness of the free-energy landscape increases upon decreasing temperature until a configurational entropy crisis arises (at Kauzmann $T_K$).

**Numerical simulations** : one cannot access $f$ but one can explore $e$ : potential energy landscape.

D. J. Wales 03.
Replica calculation

A sketch

\[ -\beta f = \lim_{N\to\infty} \ln Z_N(\beta, J) = \lim_{N\to\infty} \lim_{n\to 0} \frac{[Z_n^N(\beta, J)] - 1}{Nn} \]

\(Z_N^n\) partition function of \(n\) independent copies of the system: replicas.

Average over disorder: coupling between replicas

\[ \sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \Rightarrow N^{-1} \sum_{i\neq j} \left( \sum_a s_i^a s_j^a \right)^2 \]

Decoupling with the Hubbard-Stratonovitch trick

\[ Q_{ab} N^{-1} \sum_i s_i^a s_i^b - \frac{1}{2} Q_{ab}^2 \]

\(Q_{ab}\) is a 0 \(\times\) 0 matrix but it admits an interpretation in terms of overlaps.

The elements of \(Q_{ab}\) can be evaluated by saddle-point if one exchanges the limits \(N \to \infty\) \(n \to 0\) with \(n \to 0\) \(N \to \infty\).
Replica calculation

Overlaps

Take one sample and run it until it reaches \textit{equilibrium}, measure \{s_i\}.

Re-initialize the same sample (same $J_{ij}$), run it until it reaches \textit{equilibrium}, measure \{\sigma_i\}.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a \textbf{FM system} there are four possibilities

\begin{align*}
q_{s\sigma} &= m^2 \\
\sigma &= -m^2 \\
\sigma &= -m^2
\end{align*}

Many repetitions: $P(q_{s\sigma}) = \frac{1}{2} \delta(q_{s\sigma} - m^2) + \frac{1}{2} \delta(q_{s\sigma} + m^2)$
Replica calculation

Overlaps in disordered systems

Parisi 79-82 prescription for the replica symmetry breaking Ansatz yields

High temperature FM Structural glasses Spin-glasses

Thermodynamic quantities, in particular the equilibrium free-energy density, are expressed in terms of the functional order parameter $P(q)$. The equilibrium free-energy density predicted by the replica theory was confirmed by Guerra & Talagrand 00-04 independent mathematical-physics methods.
Replica calculation

Applications to other problems

Random elastic manifolds, *e.g.* vortex systems, dirty interfaces, *etc.*

Models with self-generated disorder, *e.g.* Lennard-Jones particles.


Tricks are necessary to introduce quenched randomness in the calculation.
Droplet picture

A much simpler viewpoint for finite-\(d\) systems

Just two equilibrium states as in a FM, only that they look spatially disordered.

Compact excitations of linear size \(\ell\) have energy \(E(\ell) \approx \Upsilon \ell^\theta\).

Proposition for \(P(E_\ell)\).

\(P(q)\) with two peaks at \(\pm q_{EA}\).

A series of scaling laws lead to predictions for themodynamics, etc.

Also applicable to random manifold problems.

*In a sense, a more conventional picture.*

Fisher & Huse 87-89.

Long-standing debate, no consensus; very hard to decide.
Plan

• **First lesson.**
  
  Introduction. Overview of disordered systems. Short presentation of methods.

• **Second lesson.**
  
  *Statics* of classical and *quantum disordered systems*.

• **Third lesson.**
  
  Classical dynamics. Coarsening. Formalism.

• **Fourth lesson.**
  
  Quantum dynamics. Formalism and results for mean-field models.
Summary

Classical disordered systems

The statics and structure of metastable states is fully described in MF. Consistent results from TAP, replicas, cavity for dilute systems and formal arguments.

Attn! the statistics of barriers is not fully known.

MF theory predicts a **functional ordered parameter** and three **universality classes**:

- **FM**: 2nd order transition, two states below \( T_c \). Curie-Weiss, GL
- **Structural glasses**: metastable states combine to make the super-cooled liquid, **random first order transition** p-spin.
- **Spin-glasses**: 2nd order transition, many states below \( T_c \). SK.

Different scenario from droplet model.
Quantum spin-glasses

\[ \hat{H}_{\text{syst}} = - \sum_{ij} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Gamma \sum_i \hat{\sigma}_i^x - \sum_i h_i \hat{\sigma}_i^z. \]

\[ \hat{\sigma}_i^a \] with \( a = 1, 2, 3 \) the Pauli matrices, \([\hat{\sigma}^a, \hat{\sigma}^b] = 2i\epsilon_{abc} \hat{\sigma}^c \).

\( J_{ij} \) quenched random., e.g. Gaussian pdf conveniently normalized.

\( \Gamma \) transverse field. It measures quantum fluctuations.

In the limit \( \Gamma \to 0 \) the classical limit should be recovered.

\( h_i \) longitudinal local fields

\( ij \) nearest neighbours on the lattice – finite \( d \)

or fully connected – mean-field
Quantum SG : finite $d$

Phase transitions in $d = 2, 3$

$\Gamma_c$ Quantum critical point. $T_c$ Classical critical point.

Quantum MC. Overviews in Rieger & Young 95, Kawashima & Rieger 03.

Many special features in $d = 1$ obtained with the Dasgupta-Ma RG decimation procedure. Recall E. Altman’s talk. D. S. Fisher 92.
Mean-field methods

TAP method

Legendre transform of \( f(\beta, h) \) with respect to \( \{m_i(\tau)\} \) and \( C(\tau - \tau') \) with \( m_i(\tau) = \langle s_i(\tau) \rangle_h \) and \( C(\tau - \tau') = N^{-1} \sum_i \langle s_i(\tau)s_i(\tau') \rangle_h \).

Biroli & LFC 01.

Replica trick

The partition function is a trace

\[
\mathcal{Z}_N = \text{Tr} \ e^{-\beta \hat{H}} = \int_{\{s_i(\beta \hbar)\}} \int_{\{s_i(0)\}} \mathcal{D}\{s_i(\tau)\} \ e^{-\frac{1}{\hbar} S_{\text{syst}}[\{s_i\}]}
\]

with the Euclidean action
Mean-field methods

\[ S_{syst}^e \{ s_i \} = \int_0^{\beta \hbar} d\tau \left[ \frac{M}{2} \left( \frac{\partial s_i(\tau)}{\partial \tau} \right)^2 + \sum_{ij} J_{ij} s_i(\tau)s_j(\tau) \right] \]

\[ s_i(\beta \hbar) = s_i(0) \] and the ‘mass’ given by \( M = (\hbar \tau_0)/2 \ln[\hbar/(\Gamma \tau_0)] \) as a function of the transverse field \( \Gamma \).

Feynman-Matsubara construction of functional integral over imaginary time.

Overlap matrix-function:

\[ Q_{ab}(\tau, \tau') N^{-1} \sum_i s_i^a(\tau) s_i^b(\tau') - \frac{1}{2} Q_{ab}^2(\tau, \tau') \]

Slightly intricate imaginary-time & replica index structure. Recipes to deal with them

Bray & Moore 80.

Can an argument à la Guerra-Talagrand can be extended to quantum spin-glass models?
1st order phase transitions

Quantum $p$-spin model

Jump in the susceptibility across the dashed part of the critical line.

LFC, Grempel & da Silva Santos 00 ; Biroli & LFC 01.

Many more examples, e.g. with cavity method in a model with a superfluid/glass transition Foini, Semerjian & Zamponi 11 ; see Yu’s poster.
Combinatorial optimization

Optimization problems consist in finding the configuration that renders minimal a cost function, e.g. the road traveled by a salesman to visit each of \( N \) cities once and only once. The most interesting of these problems can be mapped onto a classical spin model on a random (hyper-)graph with the cost function its Hamiltonian. For instance, \( K \)-satisfiability is written in terms of \( p(\leq K) \)-spin models on a random (hyper-)graph.

Quantum annealing

Kadowaki & Nishimori 98
Dipolar spin-glass
G. Aeppli et al. 90s

1st order transitions : trouble for quantum annealing techniques.

Jorg, Krzakala, Kurchan, Maggs & Pujos 09.
Dissipative systems

Aim

Our interest is to describe the statics and dynamics of a classical or quantum system coupled to a classical or quantum environment.

The Hamiltonian of the ensemble is

\[ H = H_{\text{syst}} + H_{\text{env}} + H_{\text{int}} \]

What is the static and dynamic behaviour of the reduced system?

Discuss it step by step:

1) equilibrium classical,
2) equilibrium quantum,
3) classical dynamics, 4) quantum dynamics.
Imagine the ensemble $H = H_{syst} + H_{env} + H_{int}$ is in equilibrium at inverse temperature $\beta^{-1}$:

Model the environment and the interaction

E.g., an ensemble of harmonic oscillators and a bi-linear coupling:

$$H_{env} + H_{int} = \sum_{\alpha=1}^{N} \left[ \frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{m_{\alpha}\omega_{\alpha}^2}{2} q_{\alpha}^2 \right] + \sum_{\alpha=1}^{N} c_{\alpha} q_{\alpha} x$$

Either classical variables or quantum operators.
Statics of a classical system

The partition function of the coupled system is

\[ Z[\eta] = \sum_{osc, syst} e^{-\beta(H - \eta x)} \]

Integrating out the oscillator variables:

\[ Z_{\text{red}}[\eta] = \sum_{syst} e^{-\beta(H_{\text{syst}} + H_{\text{count}} + \eta x - \frac{1}{2} \sum_{a=1}^{N} \frac{e_a \omega_a^2}{m_a} x^2)} \]

Choosing \( H_{\text{count}} \) to cancel the quadratic term in \( x^2 \) one recovers

\[ Z_{\text{red}}[\eta] = Z_{\text{syst}}[\eta] \]

i.e., the partition function of the system of interest.
Reduced description

Statics of a quantum system

The density matrix of the coupled system is

$$\rho(x'', q_a''; x', q_a') = \langle x'', q_a'' | \hat{\rho} | x', q_a' \rangle$$

$$= \frac{1}{Z} \int_{x(0)=x'}^{x(h\beta)=x''} \mathcal{D}x(\tau) \int_{q_a(0)=q_a'}^{q_a(h\beta)=q_a''} \mathcal{D}q_a(\tau) e^{-\frac{1}{\hbar}S^{e}[\eta]}$$

One integrates the oscillator’s degrees of freedom to get the reduced density matrix

$$\rho_{\text{red}}(x'', x') = \frac{1}{Z^{\text{osc}}} \int_{x'}^{x''} \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} \left( S^e_{\text{syst}} - \int_0^{h\beta} d\tau \int_0^{\tau} d\tau' x(\tau)K(\tau-\tau')x(\tau') \right)}$$

The counter-term was chosen to cancel a quadratic term in $x^2(\tau)$, $Z_{\text{red}} = Z / Z_{\text{osc}}$ and $Z_{\text{osc}}$ the partition function of isolated ensemble of oscillators, but a non-local interaction in the imaginary time with kernel

$$K(\tau) = \frac{2}{\pi \hbar \beta} \sum_{n=-\infty}^{\infty} \int_0^\infty d\omega \frac{I(\omega)}{\omega} \frac{\nu_n^2}{\nu_n^2 + \omega^2} \exp(i\nu_n \tau)$$ remains.
1st order phase transition

Dissipative Ising $p$-spin model with $p \geq 3$ at $T \approx 0$

Magnetic susceptibility

Averaged entropy density

$\alpha = 0, 0.5, 1$

LFC, Grempel & da Silva Santos 00;
LFC, Grempel, Lozano, Lozza, da Silva Santos 04.
Localization

The Caldeira-Leggett problem

A quantum particle in a double-well potential coupled to a bath of quantum harmonic oscillators in equilibrium at $T = 0$.

Quantum tunneling for $0 < \alpha < 1/2$

‘Classical tunneling’ for $1/2 < \alpha < 1$

Localization in initial well for $1 < \alpha$

Bray & Moore 82, Leggett et al. 87.

More later.
Summary

Statics of quantum disordered systems

- We introduced quantum spin-glasses.
- We very briefly ‘explained’ that the TAP and replica approaches as well as the cavity method can be applied to them.
- We showed that models in the random first order phase transition class have first order phase transitions in the low temperature limit.
- We gave an argument as to why a quantum environment can have a highly non-trivial effect quantum mechanically.

Similar results for quantum Ising chains with FM and disordered interactions

  LFC, Lozano & Lozza; Chakravarty, Troyer, Voelker, Werner 04 (MC)
  Schehr & Rieger 06 (decimation RG)
It was the end of the 2nd talk
Plan

- **First lesson.**
  
  Introduction. Overview of disordered systems. Short presentation of methods.

- **Second lesson.**
  
  Statics of classical and quantum disordered systems.

- **Third lesson.**
  
  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson.**
  
  Quantum dynamics. Formalism and results for mean-field models.
Phase ordering kinetics

Dynamics across a phase transition

- Equilibrium phases are known on both sides of the transition.
- The dynamic mechanism can be understood.
- Interesting as a theoretical problem, beyond perturbation theory.
- To cfr. the observables to similar ones in problems for which we do not know the equilibrium phases nor the dynamic mechanisms.
- To investigate whether growth phenomena exist in problems with unknown dynamic mechanisms.
- To unveil “generic” features of macroscopic systems out of equilibrium (classical or quantum).

out of equilibrium statistical mechanics or thermodynamics

e.g. glasses
2nd order phase-transition

bi-valued equilibrium states related by symmetry, e.g. Ising magnets

Ginzburg-Landau free-energy

Scalar order parameter
Evolution

The system is in contact with a thermal bath

**Thermal agitation**

Non-conserved order parameter $\langle \phi \rangle (t, T) \neq ct$

e.g. single spin flips with Glauber or Monte Carlo stochastic rules.

**Development of magnetization in a ferromagnet.**

Conserved order parameter $\langle \phi \rangle (t, T) = \langle \phi \rangle (0, T) = ct$

e.g. pair of antiparallel spin flips with stochastic rules.

**Phase separation in binary fluids.**
Evolution

A quench or an annealing across a phase transition

A quench or an annealing across a phase transition

Non-conserved order parameter \( \langle \phi \rangle(t, T) \neq ct \)

Development of magnetization in a ferromagnet after a quench.
The problem

e.g. up & down spins in a $2d$ Ising model (IM)

Question: starting from equilibrium at $T_0 \rightarrow \infty$ or $T_0 = T_c$ how is equilibrium at $T_f = T_c$ or $T_f < T_c$ attained?
Growth kinetics

- At $T_f = T_c$ the system needs to grow structures of all sizes.

  Critical coarsening.

  At $T_f < T_c$: the system tries to order locally in one of the two competing equilibrium states at the new conditions.

  Sub-critical coarsening.

In both cases the linear size of the equilibrated patches increases in time.

In both cases one extracts a growing linear size of equilibrated patches

$$\mathcal{R}(t, g)$$

from

$$C(r, t) = \frac{1}{N} \sum_{i,j=1}^{N} \langle \delta s_i(t) \delta s_j(t) \rangle_{|\vec{r}_i - \vec{r}_j|=r}$$

- The relaxation time $t_r$ needed to reach $\pm|\langle \phi \rangle_{eq}(T)|$ diverges with the size of the system, $t_r(T, L) \to \infty$ when $L \to \infty$ for $T \leq T_c$. 
Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains \( \mathcal{R}(T, t) \), such that the domain structure is (in statistical sense) independent of time when lengths are scaled by \( \mathcal{R}(T, t) \), e.g.

\[
C(r, t) \equiv \langle s_i(t)s_j(t) \rangle_{|\vec{x}_i-\vec{x}_j|=r} \sim m_{eq}^2(T) f \left( \frac{r}{\mathcal{R}(T, t)} \right),
\]

\[
C(t, t_w) \equiv \langle s_i(t)s_i(t_w) \rangle \sim m_{eq}^2(T) f_c \left( \frac{\mathcal{R}(T, t)}{\mathcal{R}(T, t_w)} \right),
\]

etc. when \( r \gg \xi(T), t, t_w \gg t_0 \) and \( C < m_{eq}^2(T) \).

Suggested by experiments and numerical simulations. Proved for

- Ising chain with Glauber dynamics.
- Langevin dynamics of the \( O(N) \) model with \( N \to \infty \), and the spherical ferromagnet. \( \text{Review Bray, 1994.} \)
- Distribution of hull-enclosed areas in 2d curvature driven coarsening.
Space-time correlation

Magnetic model

Scaling regime \[ a \ll r \ll L, \quad \frac{r}{R(t,T)}, \quad R(t,T) \approx \lambda(T)t^{1/z_d} \]

\[
C(r,t) = N^{-1} \sum_{i,j/rij=r} s_i(t)s_j(t) \approx m_{eq}^2(T) f_c \left( \frac{r}{R(t,T)} \right)
\]
Phase separation
Spinodal decomposition in binary mixtures

$A$ species $\equiv$ spin up; $B$ species $\equiv$ spin down

$2d$ Ising model with Kawasaki dynamics at $T$

locally conserved order parameter

50 : 50 composition ; Rounder boundaries
Dynamics in the 2d XY model

**Schrielen pattern**: gray scale according to $\sin^2 2\theta_i(t)$

Defects are vortices (planar spins turn around these points)

After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t, T) \sim \lambda(T) \left\{ \frac{t}{\ln[t/t_0(T)]} \right\}^{1/2}$$

*Yurke et al* 93, *Bray & Rutenberg* 94.
Growing lengths

Universality classes

\[ \mathcal{R}(t, T) \sim \begin{cases} 
\lambda(T) \ t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\
\lambda(T) \ t^{1/3} & \text{scalar COP} \quad z_d = 3 \\
\lambda(T) \left( \frac{t}{\ln t/t_0} \right)^{1/2} & \text{2-comp. vector NCOP in} \quad d = 2 \\
\text{etc.} & 
\end{cases} \]

Defined by the time-dependent. Temperature and other parameters appear in the prefactor.

Super-universality?

Are scaling functions independent of temperature and other parameters?

Review Bray 94
Weak disorder

e.g., random ferromagnets

At short time scales the dynamics is relatively fast and independent of the quenched disorder; domain walls accommodate in places where the disorder is the weakest, thus

\[ \mathcal{R}(t, T) \sim \lambda(T) t^{1/z_d} \]

At longer time scales domain-wall pinning by disorder becomes important.

Assume that a length-dependent barrier \( B(\mathcal{R}) \sim \Upsilon \mathcal{R}^\psi \)

The Arrhenius time needed to go over such a barrier is

\[ t \sim t_0 e^{\frac{B(\mathcal{R})}{k_BT}} \]

This implies

\[ \mathcal{R}(t, T) \sim \left( \frac{k_BT}{\Upsilon \ln t/t_0} \right)^{1/\psi} \]
Weak disorder

Still two ferromagnetic states related by symmetry

\[ \mathcal{R}(t, T) \simeq \begin{cases} 
[\lambda(T)t]^{1/z_d} & \mathcal{R} \ll L_c(T) \quad \text{curvature-driven} \\
L_c(T)\left(\ln t/t_0\right)^{1/\psi} & \mathcal{R} \gg L_c(T) \quad \text{activated}
\end{cases} \]

with \( L_c(T) \) a growing function of \( T \).

Inverting times as a function of length

\[ t \simeq [\mathcal{R}/\lambda(T)]^{z_d} e^{\mathcal{R}/L_c(T)} \]

At short times this equation can be approximated by an effective power law with a \( T \)-dependent exponent:

\[ t \simeq \mathcal{R}^{\bar{z}_d(T)} \quad \bar{z}_d(T) \simeq z_d \left[1 + ct/L_c(T)\right] \]

Bustingorry et al. 09.
End of 3rd lecture
Plan

- **First lesson.**
  Introduction. Overview of disordered systems. Short presentation of methods.

- **Second lesson.** Static methods.

- **Third lesson.**
  Classical dynamics. Coarsening. Formalism.

- **Fourth lesson.**
  Quantum dynamics. Formalism. Mean-field models.
Real-time dynamics

Two-time dependence

\( t = 0 \) initial time \( t_w \) waiting-time \( t \) measuring time.

Correlation

\[
C(t, t_w) = \langle [\hat{O}(t), \hat{O}(t_w)]_+ \rangle
\]

Symmetrized correlator

Linear response

\[
R(t, t_w) = \left. \frac{\delta \langle \hat{O}(t) \rangle}{\delta h(t_w)} \right|_{h=0} = \langle [\hat{O}(t), \hat{O}(t_w)]_- \rangle
\]

Antisymmetrized correlator
The perturbation couples linearly to the observable \( H \rightarrow H - hB(\{\vec{r}_i\}) \)

The linear instantaneous response of another observable \( A(\{\vec{r}_i\}) \) is

\[
R_{AB}(t, t_w) \equiv \left< \frac{\delta A(\{\vec{r}_i\})(t)}{\delta h(t_w)} \right|_{h=0}
\]

The linear integrated response or dc susceptibility is

\[
\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' \ R_{AB}(t, t')
\]
Formalism

- **Path-integral Schwinger-Keldysh** formalism.

\[ \mathcal{T} > t \]

- Closed-time path to allow for \( \langle \hat{O}(t_w) \hat{O}(t) \rangle \) with \( t > t_w \).

- Connect system to reservoirs at time \( t = 0 \), factorize density matrix:

\[ \hat{\rho}(0) = \hat{\rho}_{syst}(0) \otimes \hat{\rho}_{env}(0) \]

and take the reservoir in equilibrium at its own \( \beta \) (and \( \mu \)).

- Integrate out the bath – assumed to be in equilibrium.

- Obtain an effective action

\[ S = S_{syst} + S_{bath-syst} \]
Formalism

Some important technical remarks

- Integration over bath variables \(\Rightarrow\)
  
  Two real-time long-range interactions (recall static cases).

- The vanishing quantum fluctuations limit (\(\hbar \to 0\)) of the Schwinger-Keldysh generating functional is:
  
  if no environment, Newton dynamics.
  
  if environment on, Langevin dynamics with coloured noise.

How to prove it: linear combinations of the forward and backward variables \(x_+(t)\) and \(x_-(t)\) combine into \(x(t)\) and \(i\hat{x}(t)\). For \(\hbar \to 0\) the quantum bath kernels become the ones of a colored classical Langevin process.
Formalism

The rôle played by the initial condition & disorder

• Typical initial conditions: \( \rho_{\text{syst}}(0) = I \) ‘random’ (quench from PM)

  no need of replica trick to average over disorder!

In the classical limit \( Z[\eta = 0] = \int D\xi P[\xi] = 1 \), independently of disorder

  de Dominicis 78.

In the quantum model \( \rho(0) \) is independent of disorder.

  LFC & Lozano 98.

We simply average \( Z \) or \( \rho \).

• Derive Schwinger-Dyson equations for correlations and responses with saddle-point in the large \( N \) limit or a variational approximation.

• Solve these equations.
An example: rotors with pair interactions

\[
S_{\text{syst}} = \sum_{a=\pm} \int dt \left[ \frac{\hbar^2}{2\Gamma} \sum_i (\dot{n}_{ia}(t))^2 + \sum_{i<j} J_{ij} n_{ia}(t)n_{ja}(t) \right].
\]

\[
S_{\text{bath-syst}} = -\frac{1}{2} \sum_{ab=\pm} \int dt dt' \Sigma^B_{ab}(t, t') \sum_i n_{ia}(t)n_{ib}(t'),
\]

\[
S_\lambda = \sum_{a=\pm} \frac{a}{2} \int dt \sum_i \lambda_{ia}(t)(n_{ia}^2(t) - M)
\]

with the bath induced kernels

\[
\Sigma^B_{ab}(t, t')
\]

that take different forms for different baths, e.g. oscillators, leads, etc.
Real-time dynamics

Paramagnetic phase

Dependence on the quantum parameter $\Gamma$ ($T = 0, \alpha$ fixed.)

Real-time dynamics

Dependence on the coupling to the bath

Symmetric correlation  Linear response

Comparison between $\alpha = 0.2$ (PM) and $\alpha = 1$ (SG)

LFC, Grempel, Lozano, Lozza & da Silva Santos 02.
The effect of the bath

Summary

- Classical statics is not altered by the bath.
- Quantum statics is altered by the bath.
- Classical dynamics becomes a coloured noise Langevin equation and, although the evolution can depend on the bath, the target equilibrium does not.
- Quantum dynamics very importantly altered by the bath, e.g. localization and modification of the phase transition line.

Quantum aging

Other examples: SK (Chamon, Kennett & Yu), SU(N) in large N (Biroli & Parcollet), rotors (Rokni & Chandra; Aron et al), Wigner crystals (LFC, Giamarchi & Le Doussal), etc.
Real-time dynamics

Interactions against localization

![Graph showing the relationship between $C(t + t_w, t_w)$ and time $t$ for different values of $\alpha$ and $J$.]

$LFC, Grempel, Lozano, Lozza & da Silva Santos 02$
Dynamic vs static phase diagram

Quantum $p$-spin model

Dynamic evidence of high-lying metastable states!

The relaxational dynamics gets trapped in a region of phase space named **threshold**.
Rue de Fossés St. Jacques et rue St. Jacques
Paris 5ème Arrondissement.
Fluctuation-dissipation

Equilibrium spontaneous ($C$) and induced ($R$) fluctuations

If

$$p(\{\vec{r}\}, t_w) = p_{eq}(\{\vec{r}\})$$

- The dynamics is stationary, $C \to C(t - t_w)$ and $R \to R(t - t_w)$.
- The fluctuation-dissipation th.

$$R(t - t_w) = -\frac{1}{k_B T} \frac{\partial C(t - t_w)}{\partial t} \theta(t - t_w)$$

holds and implies

$$\chi(t - t_w) \equiv \int_{t_w}^{t} dt' R(t, t') = \frac{1}{k_B T} [C(0) - C(t - t_w)] .$$

In glassy systems below $T_g$: breakdown of stationarity & FDT.

$$\chi(t, t_w) \equiv \int_{t_w}^{t} dt' R(t, t')$$

and $C(t, t_w)$ not obviously related.
Fluctuation-dissipation

Solvable cases: $p$ spin-models

Parametric construction:
- $t_w$ fixed
- $t : t_w \to \infty$ or $dt : 0 \to \infty$.

LFC & Kurchan 93.
For non-equilibrium systems, relaxing slowly towards an asymptotic limit (cfr. threshold in p spin models) such that one-time quantities [e.g. the energy-density $\mathcal{E}(t)$] approach a finite value [e.g. $\mathcal{E}_\infty$]

$$\lim_{t_w \to \infty} C(t, t_w) = C$$

LFC & Kurchan 94.

For weakly forced non-equilibrium systems in the limit of small work

$$\lim_{\epsilon \to 0} C(t, t_w) = C$$

$$\lim_{\epsilon \to 0} \chi(t, t_w) = f_{\chi}(C')$$
FDT in relaxing glasses

Experiments and simulations

Spin-glass (thiospinel)

Hérisson & Ocio 02.

Lennard-Jones binary mixture


also in glycerol (Grigera & Israeloff 99) and after this paper many others in colloidal suspensions & polymer glasses (exps.), silica, vortex glasses, dipolar glasses, etc.
Phenomenology

Times scales

In all these systems the dynamics occur

- In quasi-equilibrium \[ \chi = \frac{1}{k_B T} (1 - C) \] when
  \[ t \gtrsim t_w \quad \text{or} \quad q_{ea} \leq C \leq 1. \]

- Clearly out of equilibrium when
  \[ t > t_w \quad \text{or} \quad C \leq q_{ea}. \]

In structural glassy systems one finds

\[ \chi = \frac{1}{k_B T^*} (q_{ea} - C) + \frac{1}{k_B T} (1 - q_{ea}) \]

Interpretation

- In particle systems, rattling within cages vs. structural relaxation.
- In magnetic coarsening, thermal fluctuations within domains vs. domain wall motion.
FDT & effective temperatures

Can one interpret the slope as a temperature?

(1) Measurement with a thermometer with

- Short internal time scale $\tau_0$, fast dynamics is tested and $T$ is recorded.
- Long internal time scale $\tau_0$, slow dynamics is tested and $T^*$ is recorded.

(2) Partial equilibration

(3) Direction of heat-flow

LFC, Kurchan & Peliti 97.
FDT & effective temperatures

Sheared binary Lennard-Jones mixture

Left: The kinetic energy of a tracer particle (the thermometer) as a function of its mass ($\tau_0 \propto \sqrt{m_{tr}}$)

$$\frac{1}{2} m_{tr} \langle v_z^2 \rangle = \frac{1}{2} k_B T_{eff}.$$ 

Right: $\chi_k(C_k)$ plot for different wave-vectors $k$: partial equilibrations.

J-L Barrat & Berthier 00.
The quantum equilibrium FDT

\[ R(t, t_w) = \frac{i}{\hbar} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} e^{-i\omega(t-t_w)} \tanh \left( \frac{\beta \hbar \omega}{2} \right) C(\omega, t_w) \]

becomes

\[ \chi(t, t_w) \approx -\frac{1}{T_{\text{eff}}} C(t, t_w) \quad t \gg t_w \]

if the integral is dominated by \( \omega(t - t_w) \ll 1 \) and \( T \to T_{\text{eff}} > 0 \) such that \( \beta_{\text{eff}} \hbar \omega \to 0 \).

Real-time dynamics

Glassy phase

Symmetric correlation  Linear response

Summary I

- We analyzed **fully-connected classical and quantum spin models with quenched random interactions**.

- The **TAP method** yields the free-energy landscape in phase space.

- The **replica trick** yields complementary information on equilibrium states.

- The **out of equilibrium relaxation** from a random initial condition captures a slow aging decay of the correlation and linear response. Generation of an **effective temperature**.

These methods can be adapted to deal with particles in interaction moving in a continuous space.
Summary II

Suggest the existence of three classes of systems:

- **FM-like** two-equilibrium states, coarsening. Droplet picture of spin-glasses but the response decays very fast.

- **p-spin-like** random first order phase transition. Have many metastable states that block the relaxation (super-cooled liquids, structural glasses). Different dynamic and static transitions. The effective temperature is finite and takes the ‘microcanonic’ value where the configurational entropy is the relevant one.

- **SK-like**. 2nd order phase transition, exponentially large number of equilibrium states, all kinds of overlaps between them, very slow dynamics.

**A many more peculiar features that I have not discussed here!**
Summary III

What is special of quantum models?

- **First order phase transitions** in models of the $p$-psin/random first order kind.

- The phase transition depends on the bath.

- For the out of equilibrium relaxation, interactions go against localization.

- **Effective temperature ‘driven’ decoherence** at long time or spatial scales; e.g., for coarsening systems the time-dependence of the growing length, $R(t; T, \Gamma)$, should be the same as for the classical counterpart; in all these cases the FDT becomes classical.
Challenges

The main one

Get a convincing real-space translation of the complex phase space and real-time information.
Two problems in progress

$T_{eff}$ from FDT

A quantum quench $\Gamma_0 \rightarrow \Gamma$ of the isolated Ising chain

Here: to its critical point $\Gamma = 1$

Dissipative dynamics of the quantum Ising chain

Part of Laura Foini’s PhD project.
The end