## Disorder

Leticia F. Cugliandolo<br>Sorbonne Universités \& Institut Universitaire de France<br>Université Pierre et Marie Curie<br>Laboratoire de Physique Théorique et Hautes Energies<br>4 Place Jussieu, Tour 13, 5ème étage<br>75252 Paris Cedex 05 France<br>France

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## 1 Introduction

These lectures aim at discussing the physics of disordered systems but also to introduce a number of general ideas and analytical methods that should be useful in other contexts of analytic science.

The Lectures will focus on the statics and dynamics of classical spin systems with random, quenched disordered, parameters in the potential energy. Extension to problems with continuous variables are straightforward.

A number of classical textbooks deal with these systems and have already been published. They will be cited in the text.

### 1.1 The meaning of disorder

The word disorder is used in two senses. On the one hand, one applies it to the configurations of the system and uses it to characterise a phase in the phase diagram in which the order parameter vanishes and there is no order of any kind. Clear examples of disordered phases are the paramagnetic one in which the magnetisation vanishes or the gas-liquid one in an atomic system, with no structural order.

On the other hand, one affects the term disorder to the energy function when it contains a finite or infinite number of parameters taken from a probability distribution. It is quite clear that this randomness will have strong effects on the statistical behaviour of the variables that constitute the system.

We will use the term disorder in these two senses in these lectures, making clear when we refer to one or the other. The theme of the course is the description of the equilibrium phases and dynamic behaviour of systems with quenched random potentials, that is to say, potential energies with some or many parameters that are random but do not change in time.

The focus will be to understand the behaviour of the dynamic or fluctuating variables in presence of different kinds of disordered, characterised by the way in which the randomness enters the Hamiltonian and the kind of probabilities distribution functions used to draw the disordered parameters. This is what is called the direct problem and it is the usual focus of statistical mechanics.

### 1.2 Spin systems

Classical spin models give, on the one side, a theoretical description of material properties like, for example, the ferromagnetic - paramagnetic transition [1, 2, 3, 4, 5]. But their interest is not restricted to physical problems. They are also used in other fields of investigation, such as informatics and social sciences, to name just two. The role played by quenched randomness in the statistical behaviour in generic systems can be deeply
understood by the study of their effect on classical spin systems of various kinds. Before discussing the technical aspects of disordered spin models we present, in the rest of this Subsection, a number of relevant examples in different areas of science.

### 1.2.1 Physical examples

In the physical context, the spin variable represents the local magnetic moment that can be a vector with three components (Heisenberg model), a vector constrained to move on a plane (XY model) or a two-valued variable (Ising model). In all cases the modulus of the vector is fixed. The spins typically lie on a lattice, and the geometry of the lattice represents the relevant microscopic arrangement of the magnetic moments in the material in question. Two dimensions appear in the model definition. The one of the vector, that is conventionally called $\mathcal{N}$, and the one the space, that is called $d$. The definition of the model is completed by two facts. The attribution of an energy function, that is a scalar function of the configuration of all the spins in the sample, $H\left(\left\{\vec{s}_{i}\right\}\right)$. For the magnetic models we will discuss, the form of this energy function is

$$
\begin{equation*}
H_{J}\left(\left\{\vec{s}_{i}\right\}\right)=-\sum_{i j} J_{i j} \vec{s}_{i} \cdot \vec{s}_{j}+\sum_{i} \vec{h}_{i} \vec{s}_{i} \tag{1.1}
\end{equation*}
$$

with just pair-wise interactions mediated by the coupling strengths or exchanges $J_{i j}$. The $\vec{h}_{i}$ are external magnetic fields. The choice of a microscopic dynamics for the spins, will determine the time-evolution of the sample at a microscopic and macroscopic level. As there are no 'Newton equations' for classical spin variables, this choice is made phenomenologically. For vector spins, one proposes the Bloch equation

$$
\begin{equation*}
\frac{d \vec{s}_{i}}{d t}=\vec{s}_{i} \wedge \vec{B}_{i} \tag{1.2}
\end{equation*}
$$

with $\vec{B}_{i}$ the sum of the external and the internal magnetic field acting on the spin $i$ (all parameters have been ignored). For the latter $\vec{B}_{i}^{\text {int }}=\delta H_{J} / \delta \vec{s}_{i}$. $\wedge$ denotes the vector product. In the absence of interactions, this equation describes the precession of a spin around the magnetic field. For Ising spins one typically uses stochastic dynamics (with a master equation formalism or Monte Carlo rules) proposing that the spin is coupled to a thermal bath that determines the transition probabilities. The quenched randomness can be in $H_{J}$.

The quenched randomness is then introduced by drawing the interactions and/or fields from probability distributions $P_{J}\left(J_{i j}^{a}\right)$ and $P_{h}\left(h_{i}^{a}\right)$. In this way one constructs random ferromagnets, spin-glasses, random field spin models, etc. apt to describe different physical materials.

### 1.2.2 Social sciences examples

In the social science context, one typically works with Ising spins, and the variable represents the opinion of an elector, for example. This opinion can be yes or no. Extensions to multi-valued cases also exist. The spins are placed on the vertices of a lattice or a random graph.


Figure 1.1: Random graphs with $N=10$ and different probabilities $p$.
In random graphs the disorder is in the choice of one instance of connectivity between the agents. More precisely, a random graph is constructed as follows. First, one decides the number of vertices that the graph will have, say, $N$. Next one establishes links or edges between pairs of vertices with a given probability. For examples, for vertices $i$ and $j$ a link is drawn with probability $p$ and it is not drawn with probability $1-p$.

Once the graph chosen, dynamic rules (with no physical constraint) for the update of the variables are proposed and the evolution of the ensemble is then studied. In some of these models, as the very well-known voter model [6], no energy function exists and all configurations are equivalent in this sense. The 'interaction' between the agents occurs only via the dynamic rule. The simplest possible rule is the following. Pick a voter at random with flat probability $1 / N$, look at one of its neighbours on the graph also chosen at random with equal probability $1 / z$ and $z$ the connectivity of the chosen site, set the variable of the chosen spin to be equal to the one of the neighbour. Then repeat the sequence until full consensus is achieved or some other stationary state is reached, or else just study the dynamic properties of the ensemble. This is a purely dynamic problem.

### 1.2.3 Economy examples

In traditional economy models, a representative agent that takes its decisions in isolation with no interaction with its fellows is used. This kind of modeling has been criticised
and more modern models tend to take into the many-body character of the problem. The latter should include imitation and social pressure effects that lead to trends, fashion and bubbles.

As it is well-known in physics, the collective behaviour of a large ensemble can be very different from the one of isolated individuals. Collective effects can be beneficial but they can also be detrimental to the group. They definitely have to be taken into account.

One such model considers a problem in which $N$ individuals can take two possible states (be them positive or negative opinion, for example), $s_{i}= \pm 1$. The decision of agent $i$ is supposed to depend on three facts: its personal opinion, modelled by a local field $h_{i}$ (that could be time-dependent but we take it to be fixed), the public information (common to all agents) described by a uniform field $F(t)$, and the social pressure exerted by its neighbours, given by $\sum_{j(i)} J_{i j} s_{j}$. Adding these terms together one has the overall incentive on agent $i$. The evolution of the state of agent $i$ is then proposed to be

$$
\begin{equation*}
s_{i}(t)=\operatorname{sign}\left[h_{i}+F(t)+\sum_{j(i)} J_{i j} s_{j}(t-1)\right] \tag{1.3}
\end{equation*}
$$

The opinion of $i$ becomes positive when the argument reaches the threshold 0 . For $J_{i j}>0$ for all $i j$ this is the dynamics of the random field Ising model at zero temperature (with no thermal noise). In physics the spins typically sit on the vertices of a lattice but in this kind of application the model can be defined on any kind of random graph.

### 1.2.4 Biological examples

In the biological context disordered models have been used to describe neural networks, i.e. an ensemble of many neurons (typically $N \sim 10^{9}$ in the human brain) with a very elevated connectivity. Indeed, each neuron is connected to $\sim 10^{4}$ other neurons and receiving and sending messages via their axons. Moreover, there is no clear-cut notion of distance in the sense that axons can be very long and connections between neurons that are far away have been detected. Hebb proposed that the memory lies in the connections and the peculiarity of neural networks is that the connectivity must then change in time to incorporate the process of learning.

The simplest neural network models represent neurons with Boolean variables or spins, that either fire or are quiescent. The interactions link pairs of neurons and they are assumed to be symmetric (which is definitely not true). The memory of an object, action, etc. is associated to a certain pattern of neuronal activity. It is then represented by an $N$-component vector in which each component corresponds to the activity of each neuron (configuration of the spins). Finally, sums over products of these patterns constitute the interactions. One can then study, for example, the number of chosen specific patterns that the network can store and later recall, or one can try to answer questions on average, as how many typical patterns can a network of $N$ neurons store. The models then become
fully-connected or dilute models of spins in interaction, just as in (1.1), with the exchanges

$$
\begin{equation*}
J_{i j}=\frac{1}{p} \sum_{\mu=1}^{p} \xi_{i}^{\mu} \xi_{j}^{\mu} \quad \xi_{i}^{\mu}= \pm 1 \quad \text { with prob } 1 / 2 \tag{1.4}
\end{equation*}
$$

$\xi_{i}^{\mu}$ are the components of an $N$ vector labelled by $\mu$, the $\mu$ th pattern stored by the network. The quenched disorder is in the $J_{i j} \mathrm{~s}$. This is the Hopfield model, based on Hebb's rule.

Protein folding is the physical process by which a protein chain acquires its native 3-dimensional structure, a conformation that is usually biologically functional, in a reproducible manner. The mechanism of protein folding remains a central open problem in molecular biology. The question to answer would be: which is the $3 d$ structure that a given sequence of amino acids will take? Knowing its answer would have obvious important applications in bio technology. The relative slowness of folding has been ascribed to the existence of many minima in a potential energy landscape in which a single point, representing the state of the macromolecule, would evolve.


Figure 1.2: The folding of a protein on a $3 d$ structure.
A protein is a macromolecule made of one or several amino acid chains also called residues. When translated from a sequence of mRNA the protein exists as a linear chain of amino acids, that still has to fold. The amino acids interact with each other, and these interactions decide the structure that the protein will take. The energy landscape describes the folding pathways in which the unfolded protein is able to assume its native state.

A model with quenched random interactions is sometimes used to mimic this process. The Hamiltonian or energy function of the states of the residues is

$$
\begin{equation*}
H\left[\left\{\alpha_{i}\right\}\right]=-\sum_{i} \epsilon_{i}\left(\alpha_{i}\right)-\sum_{i} J_{i, i+1}\left(\alpha_{i}, \alpha_{i+1}\right)-\sum_{i j} J_{i, j}\left(\alpha_{i}, \alpha_{j} ; \vec{r}_{i}, \vec{r}_{j}\right) \tag{1.5}
\end{equation*}
$$

The index $i$ runs over the residues in the molecule. The first term characterises the state of the amino acids themselves. The second term is a nearest-neighbour interaction along the chain. The third term depends on the distance in real space between amino acids on the chain that are not necessarily nearest neighbours along the chain. This is an exceedingly
complex function of $\left\{\alpha_{i}, \overrightarrow{r_{i}}\right\}$ and the study of the statistical mechanics of such a model is very hard.

An alternative strategy is to replace this complex Hamiltonian by a stochastic one (similarly to the way in which random matrix theory was introduced in nuclear physics context). One samples total energies $E$ of a system with $N$ variables $\alpha_{i}$ taking, say, $m$ states, interacting via (1.5) and constructs $P(E, N)$. These are random energy levels and in this way one can construct a random energy model, similar to the one that was introduced as a simpler spin-glass model.

### 1.2.5 Computer science examples

Finally, we will mention the computer science application of spin models [7]. Optimisation problems can usually be stated in a form that requires the minimisation of a cost (energy) function over a large set of variables. Typically these cost functions have a very large number of local minima - an exponential function of the number of variables separated by barriers that scale with $N$ and finding the truly absolute minimum is hardly non-trivial. Many interesting optimisation problems have the great advantage of being defined on random graphs and are then mean-field in nature. The mean-field machinery that we will discuss at length is then applicable to these problems with minor (or not so minor) modifications due to the finite connectivity of the networks.

Let us illustrate this kind of problems with two examples. The graph partitioning problem consists in, given a graph $G(N, E)$ with $N$ vertices and $E$ edges, to partition it into smaller components with given properties. In its simplest realisation the uniform graph partitioning problem is how to partition, in the optimal way, a graph with $N$ vertices and $E$ links between them in two (or $k$ ) groups of equal size $N / 2$ (or $N / k$ ) and the minimal the number of edges between them. Many other variations are possible. This problem is encountered, for example, in computer design where one wishes to partition the circuits of a computer between two chips. More recent applications include the identification of clustering and detection of cliques in social, pathological and biological networks.

Another example, that we will map to a spin model, is k-satisfiability (k-SAT). The problem is to determine whether the variables of a given Boolean formula can be assigned in such a way to make the formula evaluate to 'TRUE'. Equally important is to determine whether no such assignments exist, which would imply that the function expressed by the formula is identically 'FALSE' for all possible variable assignments. In this latter case, we would say that the function is unsatisfiable; otherwise it is satisfiable.

We illustrate this problem with a concrete example. Let us use the convention $x$ for the requirement $x=$ TRUE and $\bar{x}$ for the requirement $x=$ FALSE. For example, the formula $C_{1}: x_{1}$ OR $\bar{x}_{2}$ made by a single clause $C_{1}$ is satisfiable because one can find the values $x_{1}$ $=$ TRUE (and $x_{2}$ free) or $x_{2}=$ FALSE (and $x_{1}$ free), which make $C_{1}: x_{1}$ OR $x_{2}$ TRUE. This formula is so simple that 3 out of 4 possible configurations of the two variables solve
it. This example belongs to the $k=2$ class of satisfiability problems since the clause is made by two literals (involving different variables) only. It has $M=1$ clauses and $N=2$ variables.

Harder to decide formulæ are made of $M$ clauses involving $k$ literals required to take the true value $(x)$ or the false value $(\bar{x})$ each, these taken from a pool of $N$ variables. An example in 3 -SAT is

$$
\mathrm{F}=\left\{\begin{array}{llllll}
C_{1} & : & x_{1} & \text { OR } & \bar{x}_{2} & \text { OR }  \tag{1.6}\\
C_{2} \\
C_{2} & : & \bar{x}_{5} & \text { OR } & \bar{x}_{7} & \text { OR } \\
x_{9} \\
C_{3} & : & x_{1} & \text { OR } & \bar{x}_{4} & \text { OR } \\
x_{7} \\
C_{4} & : & x_{2} & \text { OR } & \bar{x}_{5} & \text { OR }
\end{array} x_{8}\right.
$$

All clauses have to be satisfied simultaneously so the formula has to be read

$$
\begin{equation*}
\text { F: } C_{1} \text { AND } C_{2} \text { AND } C_{3} \text { AND } C_{4} . \tag{1.7}
\end{equation*}
$$

It is not hard to believe that when $\alpha \equiv M / N \gg 1$ the problems typically become unsolvable while many solutions exist for $\alpha \ll 1$. One could expect to find a sharp threshold between a region of parameters $\alpha<\alpha_{c}$ where the formula is satisfiable and another region of parameters $\alpha \geq \alpha_{c}$ where it is not.

In random k-SAT an instance of the problem, i.e. a formula, is chosen at random with the following procedure: first one takes $k$ variables out of the $N$ available ones. Second one decides to require $x_{i}$ or $\bar{x}_{i}$ for each of them with probability one half. Third one creates a clause taking the OR of these $k$ literals. Forth one returns the variables to the pool and the outlined three steps are repeated $M$ times. The $M$ resulting clauses form the final formula.

The Boolean character of the variables in the $k$-SAT problem suggests to transform them into Ising spins, i.e. $x_{i}$ evaluated to TRUE (FALSE) will correspond to $s_{i}=1(-1)$. The requirement that a formula be evaluated TRUE by an assignment of variables (i.e. a configuration of spins) will correspond to the ground state of an adequately chosen energy function. In the simplest setting, each clause will contribute zero (when satisfied) or one (when unsatisfied) to this cost function. There are several equivalent ways to reach this goal. For instance $C_{1}$ above can be represented by a term $\left(1-s_{1}\right)\left(1+s_{2}\right)\left(1-s_{3}\right) / 8$. The fact that the variables are linked together through the clauses suggests to define $k$-uplet interactions between them. We then choose the interaction matrix to be

$$
J_{a i}=\left\{\begin{array}{rll}
0 & \text { if neither } x_{i} \text { nor } & \bar{x}_{i} \in C_{a}  \tag{1.8}\\
1 & \text { if } & x_{i} \in C_{a} \\
-1 & \text { if } & \bar{x}_{i} \in C_{a}
\end{array}\right.
$$

and the energy function as

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=\sum_{a=1}^{M} \delta\left(\sum_{i=1}^{N} J_{a i} s_{i},-k\right) \tag{1.9}
\end{equation*}
$$

where $\delta(x, y)$ is a Kronecker-delta that equals one when the arguments are identical and zero otherwise. This cost function is easy to understand. The Kronecker delta contributes one to the sum only if all terms in the sum $\sum_{i=1}^{N} J_{a i} s_{i}$ are equal to -1 . This can happen when $J_{a i}=1$ and $s_{i}=-1$ or when $J_{a i}=-1$ and $s_{i}=1$. In both cases the condition on the variable $x_{i}$ is not satisfied. Since this is required from all the variables in the clause, the clause itself and hence the formula are not satisfied.

The energy (1.9) can be rewritten in a way that resembles strongly physical spin models,

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=\frac{M}{2^{K}}+\sum_{R=1}^{K}(-1)^{R} \sum_{i_{1}<\cdots<i_{R}} J_{i_{1} \ldots i_{R}} s_{i_{1}} \ldots s_{i_{R}} \tag{1.10}
\end{equation*}
$$

and

$$
\begin{equation*}
J_{i_{1} \ldots i_{R}}=\frac{1}{2^{K}} \sum_{a=1}^{M} J_{a i_{1}} \ldots J_{a i_{R}} \tag{1.11}
\end{equation*}
$$

These problems are "solved" numerically, with algorithms that do not necessarily respect physical rules. Thus, one can use non-local moves in which several variables are updated at once - as in cluster algorithms of the Swendsen-Wang type used to beat critical slowing down close to phase transitions - or one can introduce a temperature to go beyond cost-function barriers and use dynamic local moves that do not, however, satisfy a detail balance. The problem is that with hard instances of the optimization problem none of these strategies is successful. Indeed, one can expect that glassy aspects, such as the proliferation of metastable states separated by barriers that grow very fast with the number of variables, can hinder the resolutions of these problems in polynomial time, that is to say a time that scales with the system size as $N^{\zeta}$, for any algorithm. These are then hard combinatorial problems.

### 1.3 Inverse problems

The inverse problem consists in finding the set of interaction parameters that are compatible with a set of observations of independent configurations of a system composed of, say, $N$ binary units, typically in the limit of $N$ large [8]. The name is opposed to the direct problem which the usual procedure in statistical mechanics, the one having a Hamiltonian, studying the properties of the ensemble of interacting variables under different conditions.

This problem is very timely. The impressive increase in computational capacities allows the scientific community to collect enormous amounts of experimental or observational data, with no detailed knowledge of which is the underlying 'Hamiltonian' that produces it.

In concrete terms, assuming that the variables observed are of a given kind, say, Ising ones, and observing a large number of sets $\left(s_{1}, \ldots, s_{N}\right)$, the question is which are the $J_{i j}$ in the Hamiltonian (1.1) that would generate these data. Note that the form of the interactions, pair-wise in this case, is also assumed.

Big data are produced in many different areas of science. Two examples outside physics are the following. The financial transactions are recorded with resolutions well below one second, and the dynamics of networks of, e.g., social, economics, neural can be tracked in real-time. These two examples, among many others, involve a large amount of data with no underlying satisfactory model.

With the inverse problem one does not try to understand the reason for the interactions between the units but, instead, one extracts accurate statistical models from empirical datasets, and then uses the models to predict new data sets on the basis of the older ones. The interactions between the units that are inferred are not all equal and therefore the models derived are of the disordered kind.


Figure 1.3: A layered neural network or the perceptron.
An application of this approach is given by the study of neural networks, in which one would like to understand how the brain works (e.g., the presence of collective states of the network, the possibility to store and retrieve informations) by processing data describing the behaviour of a huge set of elementary units (the neurons). It is a kind of reverse engineering. This way of reasoning is also characterised as statistical learning and it is at the basis of the field of deep learning that is just the training by examples of a neural networks with a layered architecture.


Figure 1.4: A two dimensional random walk.

Of course, inverse problems are not restricted to problems with discrete variables of Ising spin kind. For example, one can also observe a random walk and try to infer the random potential in which the particle moves.

### 1.4 Field theory

In the statistical physics treatment of spin models, the Ginzburg-Landau method transforms the discrete sum over spin variables in the partition function into a functional integral over a continuous field that represents the coarse-grained order parameter. Functional field-theoretical methods can then be used to study the critical phenomenon around the phase transition.

Landau proposed an extension of Weiss mean-field theory for ferromagnets that has a much wider range of application, includes space and allows to predict when it applies and when it fails. In a few words, in the Landau theory one first identifies the order parameter for the phase transition, that is to say, a quantity with zero average in the disordered phase and non-zero average on the ordered side. Next, one proposes a field theory for a coarse-grained field that represents the averaged relevant variable - giving rise to the order parameter - over a mesoscopic scale $\ell$ that is, by definition, much larger than the interatomic distance $a$.

A field theory for the spin problem can be rather simply derived by coarse-graining the spins over a coarse-graining length $\ell$. This simply amounts to computing the averaged spin on a box of linear size $\ell$ and volume $v_{x}=\ell^{d}$ around the chosen point $\vec{x}$,

$$
\begin{equation*}
\vec{\phi}(\vec{x}) \equiv \frac{1}{\ell^{d}} \sum_{j \in v_{\vec{x}}} \vec{s}_{j} \tag{1.12}
\end{equation*}
$$

In the limit $\ell \gg a$ where $a$ is the lattice spacing many spins contribute to the sum. The field $\vec{\phi}$ is a continuous vector variable taking real values. An Ising bimodal variable is instead transformed into a continuous real scalar variable taking values in $[-1,1]$. (One can construct a field $\phi$ that takes a different value per lattice site using overlapping coarsegraining volumes in which case the coordinates of the space variable $\vec{x}$ vary by steps of $a$, the lattice spacing. Instead, one can use non-overlapping coarse-graining volumes in which case the coordinates of the space variable $\vec{x}$ vary by steps of $\ell$, the coarse-graining linear size.) Studying the problem at long distances with respect to $\ell$ (or else taking a continuum spatial limit) the problem transforms into a statistical field theory $[3,4,5]$. In general, we do not know how to compute $F(\phi)$. Landau's proposal is to expand $F(\phi)$ in powers of $\phi$ and its gradients $\vec{\nabla} \phi$ and then determine, depending on the problem at hand, which terms vanish and which among the non-vanishing ones are the most relevant. The first question is answered using symmetry arguments.

In the field theory setting the quenched randomness can also be introduced as random potentials, $V(\vec{\phi})$, or random fields $\vec{h}$.

Having set the problem in terms of a field theory is handy since it allows one to use all the techniques available from quantum field theory (perturbation theory, self-consistent approximations, instanton methods, renormalisation group) with minimal changes. However, in the presence of quenched randomness, these have to be extended and complemented by special methods that we will discuss in this Course.

### 1.5 The lectures

These lectures are organised as an introduction to the field of disordered systems, a name intended to describe the models and materials in which there are random parameters in the energy functional. These lead to strange features in the static and dynamic features of these models that we will explain.

We will start with a short review of the main concepts of phase transitions that will serve as a basis for the analysis of the static behaviour of disordered models. We will discuss in detail some of the effects of frustration. Next we will add quenched disorder to the Hamiltonian, or random interactions and fields, and see how these affect their static behaviour. Finally, we will introduce dynamic methods and explain the dynamic properties of these systems. We complete these notes with a number of appendices that contain useful formulæ.

## 2 Phase transitions

Take a piece of material in contact with an external reservoir. The material will be characterised by certain observables, energy, magnetisation, etc.. To characterise macroscopic systems it is convenient to consider densities of energy, magnetisation, etc, by diving the macroscopic value by the number of particles (or the volume) of the system. The external environment will be characterised by some parameters, like the temperature, magnetic field, pressure, etc. In principle, one is able to tune the latter and the former will be a function of them.

Sharp changes in the behaviour of macroscopic systems at critical points (lines) in parameter space have been observed experimentally. These correspond to phase transitions [1, 2, 3, 4, 5], a non-trivial collective phenomenon appearing in the thermodynamic limit. Phase diagrams as the one in Fig. 2.1 are used a visual help to identify the global behaviour of a system according to the value of the control parameters.


Figure 2.1: A quite generic phase diagram.
The models that we discussed in the introduction may have static and dynamic phase transitions. The former are the usual ones studied with statistical physics methods, that is to say, by looking for non-analyticities of the free-energy density as a function of the control parameters, say just $\beta=1 /\left(k_{B} T\right)$,

$$
\begin{equation*}
-\beta f(\beta)=N^{-1} \ln \mathcal{Z}(\beta) \quad \text { with } \quad \mathcal{Z}(\beta)=\sum_{C} e^{-\beta H(C)} \tag{2.1}
\end{equation*}
$$

where $C$ represents all the system configurations. (We will always work in a canonical setting.) One is interested in identifying the order parameter (in some cases this is easy,
in others it is not), finding the critical parameters, studying the critical phenomenon that is to say the behaviour of the order parameter and other properties close to the phase transition, etc.

Let us now discuss some important concepts, order parameters, pinning fields, broken ergodicity and broken symmetry $[1,2,3,4,5]$, with the help of a concrete example, the Ising model

$$
\begin{equation*}
H_{J}\left(\left\{s_{i}\right\}\right)=-J \sum_{\langle i j\rangle} s_{i} s_{j} \tag{2.2}
\end{equation*}
$$

with $s_{i}= \pm 1$. The discussion is however much more general.
The dynamic phase transitions occur in the properties of the system's evolution. We will not discuss them in this Section.

### 2.1 Order parameters

An order parameter is generically defined as a quantity - the average of an observable - that typically vanishes in one phase and is different from zero in another one (or other ones). One must notice though that the order parameter is not unique (any power of an order parameter is itself an order parameter) and that there can exist transitions without an order parameter as in the topological Kosterlitz-Thouless transition in the $2 d \mathrm{XY}$ model. In the rest of this course we focus on problems that do have an order parameter defined as the thermal average of some observable.

In the ferromagnetic Ising model the order parameter is the magnetisation density

$$
\begin{equation*}
m=\frac{1}{N} \sum_{i=1}^{N}\left\langle s_{i}\right\rangle \quad \text { and } \quad\left\langle s_{i}\right\rangle=\mathcal{Z}^{-1} \sum_{C} s_{i} e^{-\beta H(C)} \tag{2.3}
\end{equation*}
$$

where $N$ is the total number of spins and the angular brackets represent the thermal average.

### 2.2 Thermodynamic limit

The abrupt change in the order parameter at a particular value of the external parameters, say temperature and magnetic field $(T, h)$, is associated to the divergence of some derivative of the free-energy with respect to one of these parameters. The partition function is a sum of positive terms. In a system with a finite number of degrees of freedom (as, for instance, in an Ising spin model where the sum has $2^{N}$ terms with $N$ the number of spins) such a sum is an analytic function of the parameters. Thus, no derivative can diverge. One can then have a phase transition only in the thermodynamic limit in which the number of degrees of freedom diverges.

### 2.3 Pinning field

In the absence of a magnetic field for pair interactions the energy is an even function of the spins, $H\left(\left\{s_{i}\right\}\right)=H\left(\left\{-s_{i}\right\}\right)$ and, consequently, the equilibrium magnetisation density computed as an average over all spin configurations with their canonical weight, $e^{-\beta H(C)}$, vanishes at all temperatures.

At high temperatures, $m=0$ characterises completely the equilibrium properties of the system since there is a unique paramagnetic state with vanishing magnetisation density. At low temperatures instead if we perform an experiment we do observe a net magnetisation density. In practice, what happens is that when the experimenter takes the system through the transition he/she cannot avoid the application of tiny external fields - the experimental set-up, the Earth... - and there is always a small pinning field that actually selects one of the two possible equilibrium states, with positive or negative magnetisation density, allowed by symmetry. In the course of time, the experimentalist should see the full magnetisation density reverse, to ensure $m=0$ in equilibrium. However, this is not seen in practice since astronomical time-scales would be needed. We shall see this phenomenon at work when solving mean-field models exactly.

### 2.4 Broken ergodicity

Introducing dynamics into the problem ${ }^{1}$, ergodicity breaking can be stated as the fact that the temporal average over a long (but finite) time window

$$
\begin{equation*}
\bar{A}_{t}=\lim _{\tau \rightarrow \infty} \frac{1}{\tau} \int_{t-\tau}^{t+\tau} d t^{\prime} A\left(t^{\prime}\right) \tag{2.4}
\end{equation*}
$$

is different from the statical one, with the sum running over all configurations with their associated Gibbs-Boltzmann weight:

$$
\begin{equation*}
\bar{A}_{t} \neq\langle A\rangle . \tag{2.5}
\end{equation*}
$$

In practice the temporal average is done in a long but finite interval $\tau<\infty$. During this time, the system is positively or negatively magnetized depending on whether it is in "one or the other degenerate equilibrium states" (see Fig. 2.2). Thus, the temporal average of the orientation of the spins, for instance, yields a non-vanishing result $\bar{A}_{t}=m \neq 0$. If, instead, one computes the statistical average summing over all configurations of the spins, the result is zero, as one can see using just symmetry arguments, explained in Sec. 2.5. The reason for the discrepancy is that with the time average we are actually summing over half of the available configurations of the system. If time $\tau$ is not as large as a function of $N$, the trajectory does not have enough time to visit all configurations in phase space. One can reconcile the two results by, in the statistical average, summing only over the

[^0]configurations with positive (or negative) magnetization density and recovering in this way a non-vanishing result. We shall see this at work in a concrete calculation below.

Note that ergodicity breaking is a statement about the dynamics of a system.


Figure 2.2: Time dependence of the global magnetization.

### 2.5 Spontaneous broken symmetry

In the absence of an external field the Hamiltonian is symmetric with respect to the simultaneous reversal of all spins, $s_{i} \rightarrow-s_{i}$ for all $i$. The phase transition corresponds to a spontaneous symmetry breaking between the states of positive and negative magnetization. One can determine the one that is chosen when going through $T_{c}$ either by applying a small pinning field that is taken to zero only after the thermodynamic limit, or by imposing adequate boundary conditions like, for instance, all spins pointing up on the borders of the sample. Once a system sets into one of the equilibrium states this is completely stable in the $N \rightarrow \infty$ limit. The mathematical statement of spontaneous symmetry breaking is then

$$
\begin{equation*}
\lim _{h \rightarrow 0^{+}}\left\langle s_{i}\right\rangle=-\lim _{h \rightarrow 0^{-}}\left\langle s_{i}\right\rangle \neq 0 \tag{2.6}
\end{equation*}
$$

Ergodicity breaking necessarily accompanies spontaneous symmetry breaking but the reverse is not true; an example is provided by systems with quenched disorder. Indeed, spontaneous symmetry breaking generates disjoint ergodic regions in phase space, related by the broken symmetry, but one cannot prove that these are the only ergodic components in total generality. Mean-field spin-glass models provide a counterexample of this implication.

### 2.6 Energy vs entropy

Let us first use a thermodynamic argument to describe the high and low temperature phases of a magnetic system.

The free energy of a system is given by $F=U-T S$ where $U$ is the internal energy, $U=\langle H\rangle$, and $S$ is the entropy. Here and in the following we measure temperature in units of $k_{B}$ and then set $k_{B}=1$. The equilibrium state may depend on temperature and it is such that it minimises its free-energy $F$. A competition between the energetic contribution and the entropic one may then lead to a change in phase at a definite temperature, i.e. a different group of micro-configurations, constituting a state, with different macroscopic properties dominate the thermodynamics at one side and another of the transition.

At zero temperature the free-energy is identical to the internal energy $U$. In a system with ferromagnetic couplings between magnetic moments, the magnetic interaction is such that the energy is minimised when neighbouring moments are parallel. Thus the preferred configuration is such that all moments are parallel, the system is fully ordered and $U=-\#$ pairs.

Switching on temperature thermal agitation provokes the reorientation of the moments and, consequently, misalignments. Let us then investigate the opposite, infinite temperature case, in which the entropic term dominates and the chosen configurations are such that entropy is maximised. This is achieved by the magnetic moments pointing in random independent directions. For example, for a model with $N$ Ising spins, the entropy at infinite temperature is $S \sim N \ln 2$.

Decreasing temperature magnetic disorder becomes less favourable. The existence or not of a finite temperature phase transitions depends on whether long-range order, as the one observed in the low-temperature phase, can remain stable with respect to fluctuations, or the reversal of some moments, induced by temperature. Up to this point, the discussion has been general and independent of the dimension $d$.

The competition argument made more precise allows one to conclude that there is no finite temperature phase transition in $d=1$ while it suggests there is one in $d>1$. Take a one dimensional ferromagnetic Ising model with closed boundary conditions (the case of open boundary conditions can be treated in a similar way),

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=-J \sum_{i=1}^{N} s_{i} s_{i+1} \tag{2.7}
\end{equation*}
$$

and $s_{N+1}=s_{1}$. At zero temperature it is ordered and its internal energy is just

$$
\begin{equation*}
U_{o}=-J N \tag{2.8}
\end{equation*}
$$

with $N$ the number of links and spins. Since there are two degenerate ordered configurations (all spins up and all spins down) the entropy is

$$
\begin{equation*}
S_{o}=\ln 2 \tag{2.9}
\end{equation*}
$$

The internal energy is extensive while the entropy is just a finite number. At temperature $T$ the free-energy of the completely ordered state is then

$$
\begin{equation*}
F_{o}=U_{o}-T S_{o}=-J N-T \ln 2 . \tag{2.10}
\end{equation*}
$$

Adding a domain of the opposite order in the system, i.e. reversing $n$ spins, two bonds are unsatisfied and the internal energy becomes

$$
\begin{equation*}
U_{2}=-J(N-2)+2 J=-J(N-4), \tag{2.11}
\end{equation*}
$$

for any $n$. Since one can place the misaligned spins anywhere in the lattice, there are $N$ equivalent configurations with this internal energy. The entropy of this state is then

$$
\begin{equation*}
S_{2}=\ln (2 N) \tag{2.12}
\end{equation*}
$$

The factor of 2 inside the logarithm arises due to the fact that we consider a reversed domain in each one of the two ordered states. At temperature $T$ the free-energy of a state with one reversed spin and two domain walls is

$$
\begin{equation*}
F_{2}=U_{2}-T S_{2}=-J(N-4)-T \ln (2 N) \tag{2.13}
\end{equation*}
$$

The variation in free-energy between the ordered state and the one with one domain is

$$
\begin{equation*}
\Delta F=F_{2}-F_{o}=4 J-T \ln N \tag{2.14}
\end{equation*}
$$

Thus, even if the internal energy increases due to the presence of the domain wall, the increase in entropy is such that the free-energy of the state with a droplet in it is much lower, and therefore the state much more favorable, at any finite temperature $T$. We conclude that spin flips are favorable and order is destroyed at any finite temperature. The ferromagnetic Ising chain does not have a finite temperature phase transition.

A similar argument in $d>1$ suggests that one can have, as indeed happens, a finite temperature transition in these cases (see, e.g. [2]).

## 3 Frustration

In this Section we discuss the phenomenon of frustration and its main consequences. We focus on spin models.

### 3.1 Definition and properties

A system is said to be frustrated [9] whenever it cannot minimize its total classical energy by minimizing the interaction energy between each group of interacting degrees of freedom. For two-body interactions, the minimisation should be done pair by pair but this is sometimes impossible, leading to frustration.

Frustration arises in many physical systems but it is most popular in the magnetic context, where the geometry of the lattice [10] and/or the nature of the interactions make the simultaneous minimisation of each term contributing to the energy impossible.

We will now focus on Ising models, $s_{i}= \pm 1$, with nearest-neighbour interactions mediated by coupling strengths $J_{i j}$ on a finite dimensional lattice

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=-\sum_{\langle i j\rangle} J_{i j} s_{i} s_{j} \tag{3.1}
\end{equation*}
$$

The sum runs over all pairs of nearest-neghbours on the lattice.
For the Ising Hamiltonian (3.1) with generic $J_{i j}$ (positive or negative) the minimum possible energy is obtained from the requirements

$$
\begin{equation*}
J_{i j} s_{i} s_{j}>0 \quad \text { for all } i j \tag{3.2}
\end{equation*}
$$

but it is not always possible to find a configuration that satisfies all these contraints.
Property 3.1 frustration Define the Ising model on a triangular lattice, see Fig. 3.1left. Focus on one plaquette, see Fig. 3.1-right. Imagine that one exchange is negative, $J_{12}=-J$ and two are positive, $J_{23}=J_{13}=J$ with $J>0$, all with the same magnitude. Taking into account the overall minus sign in Eq. (3.1), $J_{12}$ favours anti-parallel alignment (anti-ferromagnetism, AF) while $J_{13}$ and $J_{23}$ favour the same orientation (ferromagnetism, FM). We list all possible spin configurations in Table 3.1, where we also give their energy and the name of the links that are broken or unsatisfied. There is no configuration with all links satisfied.

Property 3.2 energy spectrum The energy spectrum of the single triangular plaquette with $J_{12}=-J$ and $J_{23}=J_{13}=J$ is very simple. We add the contribution of each pair of spins only once. There are six degenerate ground states with energy $E_{0}=-J$ and just two excited states with energy $E_{1}=3 J$.

## Property 3.3 degeneracy \& entropy

The six ground states are trivially divided in two classes related to each other by a global spin flip (1. and 8.; 2. and 7.; 3. and 6.). The members of each pair of states share the same broken bond. The two excited states are also related by the global reversal of all spins, and the broken link is the same in the two configurations of a pair.

Property 3.4 excess energy \& excess entropy Compared to a triangular plaquette with no frustration, that is to say, one in which all links have positive strength $J$, the ground state energy is increased by frustration. In the non-frustrated case $E_{0}=-3 J$ while in the frustrated case $E_{0}=-J$. The ground state degeneracy is also increased by frustration. While in the non-frustrated case there are just two ground states related by symmetry, in the frustrated case there are six. One can pair these six states in groups of two via the reversal of all spins.

| $\#$ | $s_{1}$ | $s_{2}$ | $s_{3}$ | $E$ | broken $_{\langle i j\rangle}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1. | 1 | 1 | 1 | $-J$ | 12 |
| 2. | -1 | 1 | 1 | $-J$ | 13 |
| 3. | 1 | -1 | 1 | $-J$ | 23 |
| 4. | 1 | 1 | -1 | $3 J$ | $12,23,13$ |
| 5. | -1 | -1 | 1 | $3 J$ | $12,23,13$ |
| 6. | -1 | 1 | -1 | $-J$ | 23 |
| 7. | 1 | -1 | -1 | $-J$ | 13 |
| 8. | -1 | -1 | -1 | $-J$ | 12 |

Table 3.1: Ising spin configurations and their energy $E=-\sum_{\langle i j\rangle} J_{i j} s_{i} s_{j}$ (the contribution of each pair of spins is added only once) on a triangular plaquette with $J_{12}=-J$ (AF) and two positive, $J_{23}=J_{13}=J(\mathrm{FM})$ exchanges. $J>0$. The last column names the link that is broken or unsatisfied.

The property just described with an example can be generalised. We first note that $J_{12} J_{23} J_{13}<0$ in the example. Take now the Ising model (3.1) with pairwise interactions on any lattice or graph. Any loop of connected (nearest neighbour on the lattice or graph) spins is frustrated if the product of the interaction strengths on the loop is negative,

$$
\begin{equation*}
\prod_{(i j) \in \text { loop }} J_{i j}<0 \tag{3.3}
\end{equation*}
$$

On such a loop, no choice of spin values minimises the contribution to the energy of all terms in the sum $\sum_{(i j)} J_{i j} s_{i} s_{j}$.

In frustrated models, the ground state is the configuration that minimises the number of broken bonds or unsatisfied interactions.

Exercise 3.1 Take an Ising model on a square lattice in two dimensions and choose the couplings in such a way that (a) some plaquettes are frustrated, (b) all plaquettes are frustrated.

The existence of frustration depends on the lattice geometry, the interactions and the dimension of the variables. For example, the Ising AF model in which all exchanges are $J_{i j}<0$ is not frustrated on the $2 d$ square lattice but it is on the triangular lattice. The "amount" of frustration depends on the spin variables as well. For instance, the XY AF on the triangular lattice is less frustrated than the Ising one. Indeed, the energy of an equilateral triangular plaquette is minimised by a configuration with the spins pointing in directions at $120^{\circ}$, and it equals $E_{0}=3 J \cos 120^{\circ}=-3 / 2 J$, to be compared to $-J$ in the Ising case.

With this example at hand, one can also interpret the origin of frustration as constraints, or the impossibility of the microscopic variables to take all their possible values. The Ising spins are more constrained than the XY ones, since they cannot rotate on the plane but just take only two orientations.

Many interesting classes of classical and quantum magnetic systems are highly frustrated. This is the field of constrained magnetism [11, 12].

Anti-ferromagnets on a planar triangular lattice, spin-ice samples, the fully frustrated model and an Ising ferromagnet frustrated by Coulomb interactions are frustrated magnetic materials that we discuss below.

### 3.2 Gauge invariance of Ising models

The gauge transformation

$$
\begin{equation*}
\sigma_{i}=\epsilon_{i} s_{i}, \quad J_{i j}^{\prime}=\epsilon_{i} J_{i j} \epsilon_{j}, \quad \text { with } \quad \epsilon_{i}= \pm 1 \tag{3.4}
\end{equation*}
$$

leaves the energy and the partition function of an Ising spin model with two-body interactions invariant:

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=H_{J^{\prime}}\left[\left\{\sigma_{i}\right\}\right] \quad Z_{J}=Z_{J^{\prime}} \tag{3.5}
\end{equation*}
$$

This invariance means that all thermodynamic quantities are independent of the particular choice of the interactions.

Whenever it exists a set of $\epsilon_{i} \mathrm{~S}$ such that frustration can be eliminated from all loops in the model, the effects of quenched disorder are less strong than in trully frustrated cases, as happens, for example, in the case of the Mattis model:

$$
\begin{equation*}
H_{\text {Mattis }}=-\sum_{i j} J_{i j} s_{i} s_{j} \quad \text { with } \quad J_{i j}=J\left(\left|\vec{r}_{i j}\right|\right) \xi_{i} \xi_{j} \tag{3.6}
\end{equation*}
$$

$J\left(\left|\vec{r}_{i j}\right|\right)>0$, and the $\xi_{i}$ taken from a probability distribution with a bimodal form, such that $\xi_{i}= \pm 1$. It is straightforward to see that under the transformation to new Ising variables

$$
\begin{equation*}
\sigma_{i}=\xi_{i} s_{i} \quad \forall i \tag{3.7}
\end{equation*}
$$

the Hamiltonian (3.6) becomes

$$
\begin{equation*}
H_{\text {Mattis }}=-\sum_{i j} J\left(\vec{r}_{i j} \mid\right) \sigma_{i} \sigma_{j} \tag{3.8}
\end{equation*}
$$

and it is no longer frustrated. (Note that the partition sum runs, in both representations, over Ising variables, $s_{i}= \pm 1$ or $\sigma_{i}= \pm 1$.)

### 3.3 Extensive entropy of the ground state

Frustration makes the energy of the ground states be higher than the one of similar unfrustrated models. Moreover, frustration also enforces a large multiplicity of ground states, often associated with an excess entropy at zero temperature. We show how these two features arise in a number of celebrated statistical mechanics models. We also exhibit two other phenomena generated by frustration: the existence of critical phases and the one of modulated phases.

### 3.3.1 The antiferromagnet on a triangular lattice

The model is defined as

$$
\begin{equation*}
H=-J \sum_{\langle i j\rangle} s_{i} s_{j} \tag{3.9}
\end{equation*}
$$

with $s_{i}= \pm 1, J<0$ and the sum running over first neighbours on the triangular lattice, see Fig. 3.1.

The minimal energy of any triangular plaquette, say $\epsilon_{0}$, is reached by configurations such that there are two pairs of anti-parallel spins and one pair of parallel spins, $\epsilon_{0}^{\mathrm{AF}}=$ $-2 J+J=-J$. A triangular plaquette with ferromagnetic interactions would have $\epsilon_{0}^{\mathrm{FM}}=-3 J$. An increment of $2 J$ per plaquette is caused by frustration.

A very crude counting of the number of ground states, $\Omega_{0}$, just considers the plaquettes as independent. The argument is originally due to Pauling to estimate the water ice zeropoint entropy [17] and goes as follows:

- By simple inspection one can first reckon that, in the thermodynamic limit, there are as many plaquettes pointing up as spins (we avoid subtleties linked to boundary effects by taking $N \rightarrow \infty$ ):

$$
\begin{equation*}
\# \text { spins }=\# \text { up-plaquettes }=N \tag{3.10}
\end{equation*}
$$

- The total number of (unconstrained) spin configurations in a system with $N$ spins is $2^{N}$. However, not all these configurations are energy minima.
- An approximation for the total number of configurations that are energy minima is given by the multiplication of this total number by a deflation weight factor equal to

$$
\begin{equation*}
\text { deflation weight per plaquette }=\left(\frac{\# \text { ground states }}{\# \text { states }}\right) \tag{3.11}
\end{equation*}
$$



Figure 3.1: Left: the triangular lattice and a ground-state configuration. Each spin has six neighbours. The central one (surrounded in green) has zero local field $h_{i}^{\text {loc }}=\sum_{j(\mathrm{nn} i)} s_{j}=0$. Right: a triangular plaquette with anti-ferromagnetic interactions.
on each up-plaquette. This factor has to be taken to the power of the total number of plaquettes.
This counting yields

$$
\begin{equation*}
\Omega_{0}=(\# \text { single spin states })^{\# \text { spins }}\left(\frac{\# \text { ground states on a plaquette }}{\# \text { states on a plaquette }}\right)^{\# \text { up-plaquettes }} \tag{3.12}
\end{equation*}
$$

Say there are $N$ spins on the lattice. A plaquette has three spins and three pair interactions. The total number of configurations of the three Ising spins on a triangular plaquette is $2^{3}=8$. In the lowest energy configuration two pairs of spins are anti-parallel and one is parallel, see Fig. 3.1. The lowest energy configuration is obtained in six possible ways $=3$ (which is the 'parallel' spin) times 2 (global spin reversal). The fraction of minimal energy per all configurations of a plaquette is then $6 / 8$. Then,

$$
\begin{equation*}
\Omega_{0}=2^{N}(6 / 8)^{N} \tag{3.13}
\end{equation*}
$$

and the entropy of the ground states is

$$
\begin{equation*}
S_{0}=k_{B} \ln \Omega_{0}=k_{B} N \ln 3 / 2 \simeq 0.405 k_{B} N \tag{3.14}
\end{equation*}
$$

Such extensive residual zero temperature entropy is associated to a massive level of configurational spin disorder, that also gives rise to no phase transition down to zero temperature $[13,14]$. One classifies the frustration in this example as being geometric: it is the regular periodic structure of the (triangular) lattice that inhibits the long-range antiferromagnetic order.

The exact solution of this model was given by Wannier in the 50s [14], following the steps of Onsager's solution to the square lattice ferromagnetic model. He obtained the residual entropy, $S_{0}=0.3383 R$, he showed that there is no order at any finite temperature
and that the zero-temperature behaviour is "critical" in the sense that the correlation function decays algebraically, $C(r) \simeq r^{-1 / 2}$.

Exercise 3.2 Take a square lattice in two dimensions and choose the couplings in such a way that all plaquettes are frustrated. Compare the minimal energy of the frustrated square plaquette to the one of an elementary unit with the same magnitude of coupling strengths but signs such that it is not frustrated. Compare also their entropy.

Exercise 3.3 Look at the simple arguments in [15] used to evaluate the zero-point entropy of an antiferromagnet in a magnetic field.

### 3.3.2 The six vertex model

In vertex models the degrees of freedom (Ising spins, $q$-valued Potts variables, etc.) sit on the edges of a lattice. The interactions take place on the vertices and involve the spins of the neighbouring edges.

Take an $L \times L$ square lattice with periodic boundary conditions. Label the coordinates of the lattice sites $(m, n)$. This lattice is bipartite, namely, it can be partitioned in two sub-lattices $A_{1}$ and $A_{2}$ such that the sites with $m+n$ even belong to $A_{1}$, those with $m+n$ odd belong to $A_{2}$, and each edge connects a site in $A_{1}$ to one in $A_{2}$. The degrees of freedom sit on the links between two sites or, in other words, on the ÒmedialÓ lattice whose sites are placed on the midpoints of the links of the original lattice. The midpoints are hence labeled by $(m+1 / 2, n)$ and $(m, n+1 / 2)$. Thus, the degrees of freedom are arrows aligned along the edges of a square lattice, which can be naturally mapped into Ising spins, say $s_{m+1 / 2, n}= \pm 1$. Without loss of generality, we choose a convention such that $s=+1$ corresponds to an arrow pointing in the right or up direction, depending on the orientation of the link, and conversely $s=-1$ corresponds to arrows pointing down or left.

In the six-vertex model (or $2 d$ spin-ice) arrows (or Ising spins) are constrained to satisfy the two-in two-out rule. Each node on the lattice has four spins attached to it with two possible directions, as shown in Fig. 3.2.

The model was proposed to describe ferroelectric systems by giving different statistical weights to different vertices: $w_{k} \propto e^{-\beta \epsilon_{k}}$ with $\epsilon_{k}$ the energy of each of the $k=1, \ldots, 6$ vertices. $\beta=1 /\left(k_{B} T\right)$ is the inverse temperature. Spin reversal symmetry naturally imposes $w_{1}=w_{2}=a$ for the first pair of ferromagnetic (FM) vertices, $w_{3}=w_{4}=b$ for the second pair of FM vertices, and $w_{5}=w_{6}=c$ for the anti-ferromagnetic (AF) ones, see Fig. 3.2. We have here introduced the conventional names $a, b$, and $c$ of the Boltzmann weights of the three kinds of vertices. It is customary to parametrize the phase diagram and equilibrium properties in terms of $a / c$ and $b / c$, as shown in the phase diagram in Fig. 3.3. It is important to note, however, that in the context of experiments in artificial spin-ice type samples, vertex energies are fixed and the control parameter is the temperature.

The free-energy density of the model with periodic boundary conditions can be com-


Figure 3.2: The six vertex configurations of the six vertex model and a configuration with the arctic curve.


Figure 3.3: The phase diagram of the six vertex model.
puted with the transfer matrix technique and the Bethe Ansatz to solve the eigenvalue problem [18, 19]. The effect of the boundary conditions is very subtle in these systems, as some thermodynamic properties depend upon them, in contrast to what happens in usual short-range statistical physics models. (These systems present macroscopic phase separation in real space. A frozen region fixed by the boundary conditions is separated from a fluctuating phase by the so-called arctic curve. A similar phenomenon exists in crystal growth [16].)

## Excess ice entropy

Take the case $a=b=c$ in which all vertices are equivalent, the so-called spin-ice point. A simple counting argument was put forward by Pauling to estimate the number of configurations that satisfy the two-in two-out constraint [17]. It is very similar to the one already exposed for the planar antiferromagnet on the triangular lattice. Take a system with $N$ vertices. On the square lattice each vertex has $z=4$ neighbouring vertices attached to it. The number of links is $z N / 2$ as each link is shared by two vertices. In principle, each link can take two configurations for Ising spins, therefore, ignoring the constraint, there are $2^{z N / 2}$ possible arrow configurations. The constraint definitely diminishes the number of allowed configurations but it cannot be taken into account


Figure 3.4: Left: Entropy of a spin-ice sample [20]. Right: the tetrahedra cell in water and spin ice. In the former case, the large open circles represent the O atoms and the small filled ones the H atoms. In the latter case, the arrows represent the moments of the magnetic atoms.
exactly in this kind of simplified argument. Pauling's proposal was to consider it in a kind of mean-field way, by simple proposing that the number of configurations is reduced by a factor given by the ratio between the allowed vertex configurations and all vertex configurations (6/16) for each vertex. Finally,

$$
\begin{equation*}
\Omega_{0}=2^{z N / 2}(6 / 16)^{N} \quad \Rightarrow \quad S_{0}=k_{B} N \ln \frac{3}{2} \simeq 0.405 k_{B} N \tag{3.15}
\end{equation*}
$$

The exact solution of Lieb yields $S_{0}=-3 N / 2 k_{B} \ln 3 / 4 \simeq 0.431 k_{B} N[18,19]$.
A comparison between Pauling's estimate for the zero-point entropy and its actual measurement via

$$
\begin{equation*}
\Delta S=S\left(T_{2}\right)-S\left(T_{1}\right)=\int_{T_{1}}^{T_{2}} d T^{\prime} \frac{C\left(T^{\prime}\right)}{T^{\prime}} \tag{3.16}
\end{equation*}
$$

in a spin-ice sample is shown in Fig. 3.4. This is a magnetic material with a similar crystalline structure to the one of water ice, thus its name.

## Critical phases

The statistical properties of the model can be studied for generic values of the parameters $a, b$ and $c$. A phase diagram with two FM ordered phases (large values of $a$ or $b$ with respect to $c$ ), one AF phase (large value of $c$ ) and one disordered phase (similar values of all the parameters) are found. The disordered phase is very peculiar as it is critical meaning that the spatial correlations decay algebraically with distance:

$$
\begin{equation*}
C(r) \simeq r^{-\eta} \tag{3.17}
\end{equation*}
$$

with $\eta$ a function of the control parameters $a / b$ and $a / c$. This is also called a Coulomb phase.


Figure 3.5: Left: Sketch of artificial spin-ice preparation. Right: A typical configuration.

### 3.4 The fully frustrated model \& the dimer models

The Hamiltonian is the Ising usual one

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=-\sum_{\langle i j\rangle} J_{i j} s_{i} s_{j} \tag{3.18}
\end{equation*}
$$

with nearest neighbour interactions on the lattice. The exchanges $J_{i j}$ have all the same modulus $\left|J_{i j}\right|=J$ and their product over 'elementary polygons' is negative. The elementary polygons are four-bond-squares for the square and simple cubic lattice, triangles for the FCC lattice, and hexagons for the diamond lattice [21]. By exploiting the transformation $\sigma_{i}=\epsilon_{i} s_{i}$ and $J_{i j}^{\prime}=\epsilon_{i} \epsilon_{j} J_{i j}$ with $\epsilon_{i}= \pm 1$, the values of the new exchanges $J_{i j}^{\prime}$ can be rendered periodic. For a square lattice with periodic boundary conditions, there are many ways to achieve the latter. More precisely, there are four equivalence classes, corresponding to whether the product of the $J$ s along a loop winding around a direction of the lattice is +1 or -1 .

Exercise 3.4 Take the square lattice in two dimensions. Show that there is a choice of $\left\{\epsilon_{i}\right\}$ such that the interactions can be taken to be ferromagnetic on all rows, and ferromagnetic and anti-ferromagnetic on alternating columns.

As the interactions can be arranged regularly on the lattice, the free energy can be computed with standard methods. This model can be solved with the transfer matrix technique in $2 d$. Villain showed that it has no thermodynamic instability at any finite $T$. However, the model is special at $T=0$ as it has a high degeneracy of the ground state. The configurations of minimal energy have only one unsatisfied bond, such that $J_{i j}^{\prime} \sigma_{i} \sigma_{j}<0$. Each bond is shared by two plaquettes so the question of counting the number of ground states is equivalent to counting in how many ways on can place the broken bonds on the lattice with the constraint of having only one per plaquette. This is the so-called dimer model and its degeneracy was discussed in [22]. On the square lattice, $S_{0}=\ln \Omega_{0}=N C / \pi$ with $C$ the Catalan number. The ground state energy is $E_{0}=-N J$ as exactly one quarter of the links are unsatisfied. In the thermodynamic limit, the low
temperature expansion of the free-energy density reads

$$
\begin{equation*}
\beta f(\beta)=\beta e_{0}-s_{0}+c_{1} \beta e^{-4 \beta J} \tag{3.19}
\end{equation*}
$$

The equilibrium correlation length is expected to diverge as $\xi_{\mathrm{eq}} \simeq e^{2 \beta J} / 2$. The model has no long-range order of any kind, even at $T=0$.

On other lattices, or for other kinds of spins, a finite temperature transition can be found.

These examples showed the phenomenon

$$
\text { Fluctuations } \mapsto \text { degeneracies } \mapsto \text { zero-point entropy }
$$

This result seems to violate the third law of thermodynamics, that states $S(T \rightarrow 0)=0$. This is however not so, as quantum fluctuations will come to help at sufficiently low temperatures and restate this limit.

### 3.4.1 Other cases

Exercise 3.5 An Ising spin model with $N=5$ spins and two body interactions $J_{i j}$ is defined on a random graph. The exchanges

$$
\begin{array}{lll}
J_{12}=-1.00 & J_{25}=0.4 & J_{23}=-0.1 \\
J_{35}=0.81 & J_{34}=-0.7 & J_{45}=0.03
\end{array}
$$

couple the spins labeled with $i$ and $j$.

1. Give a schematic representation of this system.
2. Is it frustrated? Justify the answer.
3. We will call this system A. Take now another system, that we will call B, in which all the interactions are ferromagnetic, with the same absolute values. Which ground state energy is lower, the one of A or the one of B ?
4. Which ground state entropy is lower?
5. Are the results of questions (c) and (d) generic? Why?

### 3.4.2 The Coulomb frustrated Ising ferromagnet - periodic order

In many systems a short-range tendency to order is opposed by a long-range force that frustrates this order. The low-temperature phase of these systems is characterised by stripe of checkerboard order. A model in this class is an Ising ferromagnet with shortrange exchanges frustrated by antiferromagnetic Coulomb interactions. More precisely,
the model is described by the Hamiltonian [23]

$$
\begin{equation*}
H_{J, Q}\left[\left\{s_{i}\right\}\right]=-\frac{J}{2} \sum_{\langle i j\rangle} s_{i} s_{j}+\frac{Q}{2} \sum_{i \neq j} v\left(\vec{r}_{i j}\right) s_{i} s_{j}, \tag{3.20}
\end{equation*}
$$

where, $J, Q>0$ are the ferromagnetic and antiferromagnetic coupling strengths between the Ising spin variables $s_{i}= \pm 1$ placed on the sites of a three-dimensional cubic lattice. The first sum is restricted to nearest neighbours (each pair is added twice) while the second one runs over all pairs of spins on the lattice. $\vec{r}_{i j}$ is the vector joining sites $i$ and $j$, and $v(\vec{r})$ is a Coulomb-like interaction term with $v(\vec{r}) \sim 1 /|\vec{r}|$ for $|\vec{r}| \rightarrow \infty$.

Because of the Coulomb interaction, the existence of the thermodynamic limit requires that the total magnetization of the system be zero. As a consequence, the ferromagnetic order is forbidden at all temperatures and for any nonzero value of the frustration parameter $Q / J$. In three dimensions, the system has an order-disorder transition at finite temperature with a complex frustration-temperature phase diagram with a variety of modulated phases.

The Hamiltonian (3.20) is quadratic in the spin variables. The interactions are not local in real space but they depend only on the distance between the two spins involved in each term. $H$ can be written as

$$
\begin{equation*}
H_{J, Q}\left[\left\{s_{i}\right\}\right]=-\sum_{i \neq j} V\left(\vec{r}_{i j}\right) s_{i} s_{j} \quad \text { with } \quad V\left(\vec{r}_{i j}\right)=\frac{J}{2} \delta\left(\vec{r}_{i j}, a \hat{e}_{i j}\right)-\frac{Q}{2} v\left(\vec{r}_{i j}\right), \tag{3.21}
\end{equation*}
$$

$a$ the lattice spacing, $\hat{e}_{i j}$ the unit vector linking the site $i$ to the site $j$ and $\delta(x, y)$ the Kronecker delta-function. It is then convenient to diagonalise the Hamiltonian by going to Fourier space

$$
\begin{equation*}
H_{J, Q}\left[\left\{s_{i}\right\}\right]=\frac{J}{2} \sum_{\vec{k}} \hat{V}(\vec{k})|\hat{s}(\vec{k})|^{2} \tag{3.22}
\end{equation*}
$$

where $\hat{s}(\vec{k})$ is the lattice Fourier transform of the Ising spin variables $s_{i}$ placed on the lattice, see App. A.1. We now choose the Fourier representation of the Coulomb-like interaction to be

$$
\begin{equation*}
v(\vec{r})=\frac{4 \pi}{N} \sum_{\vec{k}} \frac{\exp (-i \vec{k} \vec{r})}{2 \sum_{\alpha=x, y, z}\left(1-\cos k_{\alpha}\right)}, \tag{3.23}
\end{equation*}
$$

where $N$ is the number of lattice sites and the sum over $\vec{k}=\left(k_{x}, k_{y}, k_{z}\right)$ is restricted to the first Brillouin zone. Then

$$
\begin{equation*}
\hat{V}(\vec{k})=-2 \sum_{\alpha=x, y, z} \cos k_{\alpha}+\frac{4 \pi Q}{J}\left[\frac{1}{2 \sum_{\alpha=x, y, z}\left(1-\cos k_{\alpha}\right)}-v(\vec{r}=\overrightarrow{0})\right] \tag{3.24}
\end{equation*}
$$

where the subtraction of $v(\vec{r}=\overrightarrow{0})$ serves to cure any problems at $\vec{r}=\overrightarrow{0}$, see [23] for more details.

For $Q=0$ the model reduces to the standard Ising ferromagnet, and the ground state is ferromagnetically ordered. Oppositely, for $J=0$ the model is equivalent to a Coulomb lattice gas (with the mapping between spin and occupation number variables, $\left.n_{i}=\left(1-s_{i}\right) / 2\right)$ and the ground-state is a Néel antiferromagnetic state. When $Q \neq 0$, the Coulomb interaction prevents the existence of a ferromagnetic phase, and in the thermodynamic limit, the total magnetisation (charge) is constrained to be zero. Instead, phases with modulated order, with periodic patterns of "up" and "down" spins subject to the constraint of zero magnetisation, are formed.

## Mean-field approximation

In order to describe phases with spatial modulation one needs to keep the local character of the local magnetisation at site $i, m_{i}=\left\langle s_{i}\right\rangle$, in the mean-field approximation recalled in App. B. The free-energy density is

$$
\begin{equation*}
f\left(\left\{m_{i}\right\}\right)=-\frac{J}{N} \sum_{i \neq j} V\left(r_{i j}\right) m_{i} m_{j}-\sum_{i}\left[\frac{1+m_{i}}{2} \ln \frac{1+m_{i}}{2} \frac{1-m_{j}}{2} \ln \frac{1-m_{j}}{2}\right] . \tag{3.25}
\end{equation*}
$$

Calling $\hat{m}(\vec{k})$ its Fourier transform, the mean-field free energy density $f$ reads

$$
\begin{equation*}
f=-\frac{J}{2 N} \sum_{\vec{k} \neq 0} \hat{V}(\vec{k})|\hat{m}(\vec{k})|^{2}-\frac{k_{B} T}{N} \sum_{i} \ln \left(2 \cosh \left(\beta H_{i}\right)\right) . \tag{3.26}
\end{equation*}
$$

The local mean-field equations read

$$
\begin{equation*}
m_{i}=\tanh \left(\beta H_{i}\right) \quad \text { with } \quad H_{i}=-J \sum_{j \neq i} V_{i j} m_{j} \tag{3.27}
\end{equation*}
$$

the effective field on site $i$. In the case $Q=0, H_{i}=-J \sum_{j(n n i)} m_{j}-h_{i}$ with $h_{i}$ an external local magnetic field, if there is one.

Close to a second-order phase transition the magnetisations are expected to be small and (3.27) can be linearised,

$$
\begin{equation*}
m_{i} \approx \beta H_{i} \quad \Rightarrow \hat{m}(\vec{k}) \approx \beta \hat{H}(\vec{k}) \quad \text { and } \quad H(\vec{k})=-J \hat{V}(\vec{k}) \hat{m}(\vec{k}) \tag{3.28}
\end{equation*}
$$

This equation has two kinds of solutions. The paramagnetic one, $\hat{m}(\vec{k})=0$ for all $\vec{k}$ and non-trivial ones, $\hat{m}(\vec{k}) \neq 0$, at least for some $\vec{k}$. The former one is the high-temperature solution, and the latter appear at the critical temperature $T_{c}$.

For a given value of the frustration parameter, $Q$, the critical temperature $T_{c}(J, Q)$ that is then given by

$$
\begin{equation*}
T_{c}(J, Q)=-J \min _{\vec{k}} \hat{V}(\vec{k})=-J V_{c}(Q / J) \tag{3.29}
\end{equation*}
$$

where the minimum of $\hat{V}(\vec{k}), V_{c}(Q / J)$, is attained for a set of nonzero wave-vectors $\left\{\vec{k}_{c}(Q / J)\right\}$ that characterise the ordering at $T_{c}(J, Q)$.


Figure 3.6: Examples of lamellar (a), (b) and (d) and columnar (c) order in block-polymer systems [24]. The mean-field and Monte Carlo phase diagrams of the model discussed in this Section [23] (note the different normalisation (factor 2 between the two phase diagrams).

For the inverse lattice Laplacian expression of the long-range frustrating interaction in $d=3$, the $\vec{k}_{c}(Q / J)^{\prime}$ 's vary continuously with $Q / J$ :

$$
\begin{array}{llc}
\vec{k}_{c}=( \pm \arccos (1-\sqrt{\pi Q / J}), 0,0) & \text { for } & 0 \leq Q / J<4 / \pi \\
\vec{k}_{c}=(\pi, \pm \arccos (3-\sqrt{\pi Q / J}), 0) & \text { for } & 4 / \pi \leq Q / J<16 / \pi \\
\vec{k}_{c}=(\pi, \pi, \pm \arccos (5-\sqrt{\pi Q / J})) & \text { for } & 6 / \pi \leq Q / J<36 / \pi  \tag{3.30}\\
\vec{k}_{c}=(\pi, \pi, \pi) & \text { for } & 36 / \pi \leq Q / J
\end{array}
$$

and all permutations of the $x, y, z$ coordinates in Eqs. (3.30). The above ordering wavevectors correspond, respectively, to lamellar (full FM order in two directions, for small Coulomb AF frustration), tubular (full FM order in only one direction, for slightly larger Coulomb AF frustration), orthorhombic, and cubic or Néel (for still large AF long-range frustration) phases, see Fig. 3.6. The corresponding critical temperature $T_{c}(Q)$ is then given by

$$
\begin{array}{lc}
T_{c}(Q / J)=J(6-4 \sqrt{\pi Q / J}+4 \pi Q v(0)) & \text { for } 0 \leq Q / J \leq 36 / \pi \\
T_{c}(Q / J)=J(-6+4 \pi Q / J(v(0)-1 / 12)) & \text { for } 36 / \pi \leq Q / J \tag{3.32}
\end{array}
$$

The mean-field approximation gives a line of second-order phase transition from the disordered to the modulated phases. For vanishing small frustrations, the critical temperature goes continuously to $T_{c}^{0}$, the critical temperature of the pure Ising ferromagnet. More details on this problem can be found in [23].

The mean-field analysis of this model showed the existence of a complex phase diagram and very different kinds of order depending on the coupling strengths $J$ and $Q$ and their relative value. The effect of frustrations exhibited in this model can be summarised as

$$
\text { Frustration } \mapsto \text { complex phase diagram with fancy phases }
$$

### 3.5 Order by disorder

Fluctuations tend to disorder typical systems. However, in a class of magnetic systems order is induced by fluctuations, be then thermal or quantum. The order by disorder phenomenon was introduced in [28] and discussed in general in [29]. It arises in systems with infinitely degenerate rather than unique classical ground state. This property can be due to the spatial geometry of the lattice, or the peculiar organisation of the coupling strengths and it is intimately related to frustration. The principal effect of frustration is to ensure that the classical ground state manifold is of higher symmetry than the underlying Hamiltonian. Quantum or thermal fluctuations can dynamically break this additional symmetry, restoring that of the Hamiltonian.

### 3.5.1 The domino model

The domino model is an Ising model on a rectangular lattice with two kinds of atoms placed along alternating columns and nearest-neighbour interactions. The tree interaction strengths are: $J_{A A}>0$ between nearest-neighbour A atoms on a vertical A column, $J_{B B}<0$ between nearest-neighbour B atoms on a vertical B column, and $J_{A B}>0$ between nearest-neighbour A and B atoms on a horizontal row. One can easily check that no spin configuration minimises all pair contributions to the energy of any square plaquette. This model is fully frustrated since all plaquettes are.

Periodic boundary conditions are imposed along the two directions. The system has $N$ spins, $N^{\prime \prime}$ columns and $N^{\prime}=N / N^{\prime \prime}$ sites per column or lines.

The condition $0<J_{A B}<\left|J_{B B}\right|<J_{A A}$ will be required from the exchanges.
Exercise 3.6 Compare the energy of all possible spin configurations on a square plaquette.

Ferromagnetic and anti-ferromagnetic order are favoured along the A and B columns, respectively. There are two possibilities for each. In contrast, no special order in the horizontal direction will be selected at $T=0$. The number of configurations that satisfy these requirements is $\Omega_{0}=2^{N^{\prime \prime}}$ and the zero-point entropy ( $k_{B}=1$ )

$$
\begin{equation*}
S_{0}=\ln \Omega_{0}=N^{\prime \prime} \ln 2 \tag{3.33}
\end{equation*}
$$

is sub-extensive (as $N^{\prime \prime} / N=1 / N^{\prime} \rightarrow 0$ in the thermodynamic two-dimensional limit; for example, for a square lattice $N^{\prime}=N^{\prime \prime}=L$ and $N=L^{2}$ ). The average magnetisation and staggered magnetisation vanish in the thermodynamic limit, as one can easily see from the typical ground state configuration in Fig. 3.7-left. The ground state (equilibrium at $T=0$ ) is therefore globally disordered.

The free-energy density can be computed with the transfer matrix method [30, 31],


Figure 3.7: Left: The domino model and one of its ground states. Ferromagnetic bonds are full lines, antiferromagnetic bonds are dashed lines. Right: the magnetisation as a function of temperature. From [28].
and it has a singularity at a critical temperature $T_{c}$ given by

$$
\begin{equation*}
\sinh \left(2 \beta_{c}\left|J_{A A}+J_{B B}\right|\right) \sinh \left(4 \beta_{c} J_{A B}\right)=1 \tag{3.34}
\end{equation*}
$$

The low- $T$ phase can be analysed from this exact solution or it can be be characterised with a low temperature decimation. Basically, an effective model,

$$
\begin{equation*}
H_{J^{\prime}}^{\mathrm{eff}}\left[\left\{s_{i}^{A}\right\}\right]=\sum_{\langle i j\rangle} J_{i j}^{\prime} s_{i}^{A} s_{j}^{A} \tag{3.35}
\end{equation*}
$$

is obtained by summing over the spins in the B columns, $s_{i}^{B}$,

$$
\begin{equation*}
Z=\sum_{\left\{s_{i}^{A}\right\},\left\{s_{i}^{B}\right\}} e^{-\beta H_{J}\left[\left\{s_{i}^{A}, s_{i}^{B}\right\}\right]} \approx \sum_{\left\{s_{i}^{A}\right\}} e^{-\beta H_{J^{\prime}}^{\mathrm{eff}}\left[\left\{s_{i}^{A}\right\}\right]}=Z_{\mathrm{eff}} \tag{3.36}
\end{equation*}
$$

This calculation is not done exactly, but it is approximated by considering only the very low energy states. More precisely, given a column B sandwiched between two columns A, the two possible ground state orientations of the A columns are considered (parallel, FM or anti-parallel, AF) and the very low excitations in the B columns are summed over in the two cases. The ratio $Z_{F M} / Z_{A F}$ is then compared to $Z_{\text {eff }}$ for a $H_{J^{\prime}}^{\text {eff }}$ with effective horizontal couplings. The new coupling strengths $J_{i j}^{\prime}$ are expressed in terms of the original ones and they are positive in both horizontal and vertical direction when

$$
\begin{equation*}
0<J_{A B}<\left|J_{B B}\right|<J_{A A} \tag{3.37}
\end{equation*}
$$

Therefore, the system of A spins orders ferromagnetically at low temperatures, with $m_{A} \rightarrow$ 1 for $T \rightarrow 0$. On the other hand the B spins have vanishing (ferromagnetic or staggered) magnetisation, $m_{B}=0$, for $T \rightarrow 0$. In conclusion, $m=\left(m_{A}+m_{B}\right) / 2 \rightarrow 1 / 2$ for $T \rightarrow 0$, contrary to $m=0$ at $T=0$.

Order by disorder is graphically represented as follows. At $T=0$ the manifold of ground states in phase space has volume $\Omega_{0}=2^{N^{\prime \prime}}$. Among all these ground states there
are two states with global ferromagnetic up or down order on the A columns. These two very special states are selected at $T=\epsilon$ by the entropic contribution to the free-energy.

The phenomenon of order by disorder in frustrated magnetic systems is not restrained to thermal fluctuations. Quantum, and even quenched noise may sometimes increase ordering in systems where energetics ensure a nontrivially degenerate classical disordered ground state.

$$
\text { Frustration } \mapsto \text { order by disorder }
$$

### 3.6 Discussion

A theoretical approach to the physics of real glasses is based on the concept of frustration, which in this context describes an incompatibility between the extension of the locally preferred order in a liquid and tiling of the whole space. The real glass problem concerns the understanding of the behaviour of an ensemble of, say, particles in interaction which have been cooled fast enough to avoid nucleation (via a first order phase transition) into a stable crystal.

The ground state of four identical particles interacting via an isotropic potential is a perfect tetrahedron with the particles sitting at the vertices. However, it is not possible to fill space with such tetrahedra, see Fig. 3.8. The atom near the gap is frustrated because it cannot simultaneously sit in the minima provided by the pair potentials of its near neighbours. This frustration implies that there is no regular lattice of perfect tetrahedra which fills ordinary three-dimensional space. Familiar close-packed regular lattices, like the fcc structure, contain octahedra as well as tetrahedra. The octahedra are necessary to obtain a global tesselation of space, even though they do not minimise the energy locally.


Figure 3.8: Frustration in a tetrahedral packing. Figure taken from [26].
This idea was put forward by D. Nelson [25] and its development over the years is discussed in [26] and [27].

## 4 Percolation

The understanding of fluid flow in porous media needs, as a first step, the understanding of the static geometry of the connected pores. The typical example, that gave the name to the problem, is coffee percolation, where a solvent (water) filter or trickle through the permeable substance that is the coffee grounds and in passing picks up soluble constituents (the chemical compounds that give coffee its color, taste, and aroma).

Another problem that needs the comprehension of a static random structure is the one of conduction across a disordered sample. Imagine that one mixes randomly a set of conducting and insulating islands. Whether the mix can conduct an electric current from one end to the other of the container is the question posed, and the answer depends on the structure formed by the conducting islands.


Figure 4.1: A measurement of the topography (left) and local current (right) in an inhomogeneous mixture of good and bad conducting polymers [36]. The brighter the zone the more current passing through it. Several grains are contoured in the left image.

Percolation [32, 33, 34, 35] is a simple geometric problem with a critical threshold. It is very helpful since it allows one to become familiar with important concepts of critical phenomena such as fractals, scaling, and renormalisation group theory in a very intuitive way. Moreover, it is not just a mathematical model, since it is at the basis of the understanding of the two physical problems mentioned above among many others.

In this Section we describe its simplest setting as well as some of its variants.

### 4.1 Dilution

Site dilute lattices with missing vertices are intimately related to the site percolation problem. Imagine that one builds a lattice by occupying a site with probability $p$ (and not occupying it with probability $1-p$ ). For $p=0$ the lattice will be completely empty while for $p=1$ is will be totally full. For intermediate values of $p$, on average, order $p L^{d}$ sites will be occupied, with $L$ the linear size of the lattice. Site percolation theory is about
the geometric and statistical properties of the structures thus formed. In particular, it deals with the behaviour of the clusters of nearest neighbour occupied sites.

Similarly, one can construct bond dilute lattices and compare them to the bond percolation problem.

The site percolation problem describes, for example, a binary alloy or dilute ferromagnetic crystal, also called a doped ferromagnet. The question in this context is how much dilution is needed to destroy the ferromagnetic order in the sample at a given temperature. The bond percolation problem corresponds to a randomly blocked maze through which the percolation of a fluid can occur. Many other physical problems can be set in terms of percolation: the distribution of grain size in sand and photographic emulsions, the vulcanisation of rubber and the formation of cross-linked gels, the propagation of an infection, etc.

The main interest lies on characterising the statistical and geometric properties of the clusters on a lattice of linear size $L$ as a function of the probability $p$. The clusters are connected ensembles of nearest neighbour sites. Their easiest geometric property is their size, defined as the number of sites that compose them. Other geometric properties are also interesting and we will define them below.

The percolation problem is specially interesting since it has a threshold phenomenon, with a critical value $p_{c}$ at which a first spanning cluster that goes from one end of the lattice to the opposite in at least one of the Cartesian directions appears. For $p<p_{c}$ there are only finite clusters, for $p>p_{c}$ there is a spanning cluster as well as finite clusters.

The first natural question is whether the value $p_{c}$ depends on the particular sample studied or not, that is to say, whether it suffers from sample-to-sample fluctuations. All samples are different as the sites erased or the links cut are not the same. The threshold value is therefore a random variable and it does not take the same value for different samples. The 'surprise' is that the mean-square deviations of $p_{c}$ from its mean value vanish as a power law with the system size,

$$
\begin{equation*}
\delta_{p_{c}}^{2}(N) \equiv \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}}\left(p_{c}^{(k)}-\bar{p}_{c}\right)^{2} \simeq C^{2} N^{-\nu}, \quad \quad \bar{p}_{c} \equiv \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} p_{c}^{(k)} \tag{4.1}
\end{equation*}
$$

with $k$ labelling different measurements and $\mathcal{N}$ counting its total number. $N$ the number of sites in the sample. ( $C$ turns out to be 0.54 and $\nu=1.3$ in $d=2$.) In the infinite system size limit, $p_{c}$ does not fluctuate from sample to sample.

One can then count the number of sites belonging to the largest cluster and compare this number to the total number of sites in the sample:

$$
\begin{equation*}
r_{L}(p) \equiv \frac{N_{\max }(p)}{N} \tag{4.2}
\end{equation*}
$$

This is, again, a fluctuating quantity that, in the infinite system size limit does no longer fluctuate and defines

$$
\begin{equation*}
r_{\infty} \equiv \lim _{L \rightarrow \infty} r_{L}(p) \tag{4.3}
\end{equation*}
$$

| Lattice | $n$ | $p_{c}^{\text {site }}$ | $p_{c}^{\text {bond }}$ |
| :---: | :---: | :---: | :---: |
| Honeycomb | 3 | 0.70 | 0.65 |
| Square | 4 | 0.59 | 0.50 |
| Kagomé | 4 | 0.65 |  |
| Bowtie-a | 5 | 0.55 |  |
| Triangular | 6 | $1 / 2$ | 0.35 |



Table 4.2: Connectivities, $n$, and critical thresholds for percolation, $p_{c}$, for several two dimensional lattices shown in the figure on the right (the honeycomb lattice is missing). (In the case of the bow-tie lattice, $n=5$ is the average between the connectivities of the sites with connectivities 4 and 6.)

The precise definition of the critical threshold $p_{c}$ involves the infinite size limit and it can be given by

$$
r_{\infty}(p)=\lim _{L \rightarrow \infty} r_{L}(p)=\left\{\begin{array}{lll}
0 & \text { for } & p<p_{c}  \tag{4.4}\\
>0 & \text { for } & p>p_{c}
\end{array}\right.
$$

where $r_{\infty}(p)$ denotes the fraction of sites belonging to the largest cluster in the finite lattice with linear size $L$. In the magnetic application of percolation, this means that the magnetisation vanishes for $p<p_{c}$ and it takes the value that the magnetisation takes on the largest cluster for $p>p_{c}$ (as in both cases the magnetisation on the finite clusters is independent and averages to zero).

An equivalent definition of the critical threshold $p_{c}$ is given by

$$
P_{\infty}(p)=\lim _{L \rightarrow \infty} P_{L}(p)=\left\{\begin{array}{lll}
0 & \text { for } p<p_{c}  \tag{4.5}\\
1 & \text { for } p>p_{c}
\end{array}\right.
$$

where $P_{L}(p)$ denotes the probability of there being a percolating cluster in the finite lattice with linear size $L$.

The percolation threshold $p_{c}$ depends on the lattice geometry and its dimensionality. Moreover, it is not the same for bond percolation and site percolation. Exact results are known for special lattices as the Cayley tree. Examples of how these results are found are given in [32]. Numerical data for finite dimensional lattices are complemented by rigorous upper and lower bounds and the outcome of series expansions for the mean cluster value. Harris showed that $p_{c} \geq 1 / 2$ for the bond percolation problem on a planar square lattice and the numerics suggests $p_{c}=1 / 2$. Fisher put several bounds on $p_{c}$ on various $2 d$ lattices for the site and bond problem. In particular, $p_{c} \geq 1 / 2$ for site percolation on planar regular lattices with no crossings. The values of $p_{c}$ for some bidimensional lattices are given in Table 4.2.

The value of the percolation threshold, decreases for increasing spatial dimensionality. For cubic lattices, $p_{c}=1$ in $d=1, p_{c} \simeq 0.59$ in $d=2$ and $p_{c} \simeq 0.32$ in $d=3$. This implies that the case $p=1 / 2$ is below threshold in $d=2$ for all the lattices in Table 4.2 but above threshold in the cubic lattice in $d=3$.

If one conducts a laboratory or a numerical experiment in a finite size array, the majority of the threshold values measured fall in a window centred at $p_{c}$ of width $\delta(N)$. The infinite size limit is estimated by first taking the average at finite $N$ and then performing an infinite size extrapolation

$$
\begin{equation*}
\bar{p}_{c}(N)=p_{c}+C N^{-\gamma} . \tag{4.6}
\end{equation*}
$$

As, in general, the value $p_{c}=p_{c}(N \rightarrow \infty)$ cannot be computed analytically, numerical simulations are used to obtain it. The best algorithm to find the largest cluster is given in [37].

The problem is quantitative characterised by a number of observables that depend on the control parameter $p$. The critical threshold $p_{c}$ plays a role similar to the one of a thermodynamic transition in a physical problem. Several observables vanish and behave as order parameters and others diverge and behave as susceptibilities when the control parameter $p$ approaches its critical value. Moreover, they do algebraically as in usual critical phenomena. Indeed, there is a large degree of universality in random percolation models: $p_{c}$ is model dependent but the critical exponents depend only on the spatial dimension of the lattice. Microscopic details do not influence the behaviour close to $p_{c}$.

The cluster interfaces are also an interesting characterisation of the geometry of the clusters. The domain wall of a spin cluster is its external and internal contour, constructed as follows. One first generates a dual lattice by placing a site at the centre of each plaquette of the original lattice. Next, the links on the dual lattice that cross broken bonds on the original lattice are joined together. In this way, one finds a closed loop on the dual lattice that runs along the external, and possibly also internal, boundary of a spin cluster. The hull of a cluster is restricted to the external part of the contour, that is to say, one excludes the contribution of the holes of the cluster. The hull-enclosed area is the area, i.e. the total number of sites, inside the hull (the holes within the domains are thus filled). The lengths of the different contours existing on the dual lattice are computed by counting the number of broken bonds crossed by the boundary. The external perimeter is built by closing the narrow gates of the hull, making in this way a smoother version of the contour by eliminating the deep fjords.

Take a geometric object with a given property (area, length, etc.) generically called $x$ that takes values $X$. For large but not necessarily infinite $L$, close to criticality, its probability distribution per lattice site or number density, $n_{x}(X)$, takes the form

$$
\begin{equation*}
n_{x}(X) \simeq X^{-\alpha_{x}} f_{x}\left(X^{\sigma_{x}}\left(p-p_{c}\right)\right)+n_{\mathrm{fs}}\left(X / L^{D_{x}}\right) \tag{4.7}
\end{equation*}
$$

The second term in the rhs are the finite size corrections while the first term is the only one remaining in the infinite size limit $L \rightarrow \infty$. The scaling function $f_{x}$ is required to decay fast (it could be an exponential) for large arguments and to approach a constant for small arguments. At $p_{c}$ the decay is purely algebraic. $\left(p-p_{c}\right)^{1 / \sigma_{x}}$ acts as cut-off and controls the crossover from a behaviour of "critical clusters" (power-law distributed) to that of non-critical clusters. A way to check this scaling form is to search data collapse by plotting

$$
\begin{equation*}
X^{\alpha_{x}} n_{x}(X) \quad \text { vs } \quad X^{\sigma_{x}}\left(p-p_{c}\right) \tag{4.8}
\end{equation*}
$$



Figure 4.2: A percolating cluster highlighted in black and the hull of a cluster. The domain wall will add to the red line the internal interface and the external perimeter will close the two fjords.
for the best choices of $\alpha_{x}$ and $\sigma_{x}$. The master curve yields the scaling function $f_{x}$ that depends on the observable $x$ as also do the exponents $\alpha_{x}$ and $\sigma_{x}$. For example, $n_{x}(X)$ can be the number of clusters with area $s=S, N_{s}(S)$, normalised by the number of lattice sites, $n_{s}(S)=N_{s}(S) / N$.

Exercise 4.1 Set the percolation problem on a one dimensional lattice. Show that $p_{c}=1$. Demonstrate Eq. (4.7) for $x$ being the cluster size $s$ in the infinite system size limit, $L \rightarrow \infty$. Show that $\sigma_{s}=1$ and find the scaling function $f_{s}(y)$. Identify an exponential cut-off and compute the characteristic length $s_{\xi}$ from its decay.

The probability that a site belong to any cluster is, on the one side, equal to $p$ and, on the other side, it can be written in terms of $n_{s}(S)$. For $p<p_{c}$ there are only finite size clusters and the sum rule

$$
\begin{equation*}
\sum_{S=1}^{\infty} S n_{s}(S)=p \tag{4.9}
\end{equation*}
$$

holds.
Exercise 4.2 Verify the sum rule (4.9) in $d=1$.
One can prove that the average size of the cluster diverges when approaching $p_{c}$ from below, see [34] for an explicit calculation, and the way in which it does is also characterised by a critical exponent

$$
\begin{equation*}
\langle S\rangle(p) \simeq\left|p_{c}-p\right|^{-\gamma_{s}} \tag{4.10}
\end{equation*}
$$

On the other side, one can use the definition and scaling Ansatz for $n_{s}(S)$ to show that $\langle S\rangle(p) \simeq\left|p-p_{c}\right|^{\left(\alpha_{s}-3\right) / \sigma_{s}}$. Therefore, the equivalence of these two expressions requires

$$
\begin{equation*}
\gamma_{s}=\frac{3-\alpha_{s}}{\sigma_{s}} \tag{4.11}
\end{equation*}
$$

Note that $\gamma_{s}>0$ implies $\alpha_{s}<3$.

The pair connectedness correlation function $g(\vec{r})$ is defined as the probability that a site at distance $\vec{r}$ from an occupied site belong to the same cluster. The correlation length can be computed from

$$
\begin{equation*}
\xi^{2}=\frac{\sum_{\vec{r}} r^{2} g(\vec{r})}{\sum_{\vec{r}} g(\vec{r})} \tag{4.12}
\end{equation*}
$$

Close to the threshold the correlation length diverges as

$$
\begin{equation*}
\xi \simeq\left|p-p_{c}\right|^{-\nu} \tag{4.13}
\end{equation*}
$$

Exercise 4.3 Show that in $d=1, g(r)=p^{r}$ and $\xi=-1 / \ln p \simeq(1-p)^{-1}$ close to $p_{c}$.
Another sum rule that one can easily show is $\sum_{\vec{r}} g(\vec{r})=\langle S\rangle$.
The strength of the percolating cluster, $r(p)$, measures the proportion of sites on the lattice that belong to the infinite cluster and plays the role of an order parameter for the percolation transition. It vanishes as

$$
\begin{equation*}
r(p) \simeq\left(p-p_{c}\right)^{\beta} \quad \text { for } \quad p \gtrsim p_{c} \tag{4.14}
\end{equation*}
$$

and the transition is continuous. For $p>p_{c}$ an occupied site can belong to the infinite cluster or it can be in one of the many finite clusters. Therefore, the sum rule (4.9) is modified to

$$
\begin{equation*}
r(p)+\sum_{S=1}^{\infty} S n_{s}(S)=p \tag{4.15}
\end{equation*}
$$

where the sum runs only on finite clusters. From (4.14) and (4.15) with the scaling form of $n_{s}(S)$ one shows

$$
\begin{equation*}
r(p) \simeq\left|p-p_{c}\right|^{\left(\alpha_{s}-2\right) / \sigma_{s}} \quad \Rightarrow \quad \beta=\frac{\alpha_{s}-2}{\sigma_{s}} \tag{4.16}
\end{equation*}
$$

Several geometric properties of the clusters can also be studied. Take a site at $\vec{r}_{i}$ in a cluster. The centre of mass of a cluster with mass $S$ and the radius of gyration of the same cluster are

$$
\begin{equation*}
\vec{r}_{\mathrm{cm}}=S^{-1} \sum_{j=1}^{S} \vec{r}_{j} \quad R_{g}^{2}=S^{-1} \sum_{j=1}^{S}\left|\vec{r}_{j}-\vec{r}_{\mathrm{cm}}\right|^{2} \tag{4.17}
\end{equation*}
$$

Exercise 4.4 Prove that $R_{g}^{2}=\left(2 S^{2}\right)^{-1} \sum_{i j}\left|\vec{r}_{i}-\vec{r}_{j}\right|^{2}$ with $\vec{r}_{i}$ and $\vec{r}_{j}$ the positions of two sites $i$ and $j$ on the cluster.

The mass of the percolating cluster, that is to say the number of sites $S$ in it, depends on $L$ the linear size of the system. One can write it as $M_{L}(p)=L^{d} P_{L}(p)$ where $P_{L}(p)$ was defined above in the infinite size limit and its definition is now extended to finite system
size. At $p_{c}$ the correlation length diverges and the percolating cluster is a fractal object. Accordingly, $M_{L}\left(p_{c}\right)$ does not scale as $L^{d}$ but, instead, as

$$
\begin{equation*}
M_{L} \simeq L^{D_{s}} \quad \text { for } \quad L \ll \xi \tag{4.18}
\end{equation*}
$$

with $D_{s}$ a fractal dimension. If, instead, the linear size of the system is larger than the correlation length, one can divide the full system in cubic boxes of linear size $\ell \approx \xi$. Inside each box, the cluster is fractal and its mass scales as $M_{\ell} \approx \ell^{D_{s}}=\xi^{D_{s}}$. Adding together the independent contributions from all boxes,

$$
\begin{equation*}
M_{L} \simeq(L / \xi)^{d} \ell^{D_{s}} \quad \text { for } \quad L \gg \xi \tag{4.19}
\end{equation*}
$$

These two limits can be joined into a single equation

$$
M_{L}(\xi) \simeq L^{D_{s}} m\left(\frac{L}{\xi}\right) \quad \text { with } \quad m(y) \simeq \begin{cases}\text { const } & y \ll 1  \tag{4.20}\\ y^{d-D_{s}} & y \gg 1\end{cases}
$$

These relations imply that large clusters at $p \neq p_{c}$ appear fractal at length scales smaller than $\xi$ and regular at longer length scales.

For generic clusters of size $S$ and radius of gyration $R_{g}$ the relation (4.18) generalises to

$$
\begin{equation*}
S \simeq R_{g}^{D_{s}} \tag{4.21}
\end{equation*}
$$

Equation (4.20) is an example of finite size scaling, that for a generic observable $X$ takes the form

$$
\begin{equation*}
X(p, L)=\xi^{-\beta_{x} / \nu} F_{x}\left[\left(p-p_{c}\right) L^{1 / \nu}\right]=\xi^{-\beta_{x} / \nu} F_{x}\left[(L / \xi)^{1 / \nu}\right] \tag{4.22}
\end{equation*}
$$

where $\beta_{x}$ is a quantity-dependent critical exponent and $\nu$ is the exponent of the power law divergence of the incipient cluster size. The scaling function $F_{x}$ should have the limits $F_{x}(y \ll 1)=1$ and $F_{x}(y \gg 1)=y^{-\beta_{x} / \nu}$.

Many quantities diverge at $p_{c}$ and they can be expressed as a sum over clusters of all sizes. The main contribution to these sums comes from the size scale $s_{\xi} \simeq\left|p-p_{c}\right|^{-1 / \sigma}$, as can be read from the scaling form of $n_{s}(S)$. Since one also expects this size to be given by $s_{\xi} \simeq \xi^{D}$ in terms of the fractal dimension $D$, one has

$$
\begin{equation*}
\xi^{D_{s}} \simeq \xi^{1 /\left(\nu \sigma_{s}\right)} \quad \Rightarrow \quad D_{s}=1 /\left(\nu \sigma_{s}\right) \tag{4.23}
\end{equation*}
$$

As in standard phase transitions, the exponents $\alpha_{x}, \sigma_{x}, \beta, \nu$ depend on the dimension of space but do not depend on the lattice geometry. In dimension one and two they take specially simple values. They are integer in $d=1$ and fractions in $d=2$. In particular, the fractal dimension of the clusters areas is $D_{s}=1$ in $d=1$ and $D_{s}=91 / 48$ in $d=2$. The interfaces (hulls, external perimeters, etc.) also have fractal properties that can be characterised with fractal exponents.

Physical phase transitions also admit a geometric description. In the 70s the surprise was that the clusters that characterise the phase transition are not the geometric ones.


Fig. 26. Directed white bonds are shown by the solid lines with arrows; black bonds are shown by the thin lines. Flame propagates from left to right along path 1 , and cannot propagate along path 2. Two segments of path 2 on which the fire would have to move against the wind are marked by two dashes.

Figure 4.3: Two paths of occupied bonds in the directed percolation problem, one along which a fire propagates from left to right and another one that does not allow propagation.

The idea was to study the clusters of the reversed fluctuations in the Ising model. While these percolate at $T_{c}$ in $2 d$, they do at a $T_{p}$ that is strictly below $T_{c}$ in $3 d$. This indicates that the geometric clusters are not the correct one to describe criticality. Fortuin and Kasteleyn proposed to erase or keep the links between points belonging to the clusters with a probability distribution that depends on temperature. Having done this, the clusters reduce their size and while they still percolate at $T_{c}$ in $2 d$ they do at a higher temperature in $3 d$ that coincides with $T_{c}$. The analysis of their critical exponents $(\alpha, \sigma, \nu, \ldots)$ proved that they yield the correct thermodynamic exponents $(\beta, \gamma, \delta, \ldots)$.

### 4.2 Directed percolation

A figure extracted from [32] shows the definition of the directed percolation problem.

### 4.3 Bootstrap \& diffusion percolation

In the bootstrap percolation model the sites of a lattice are initially occupied with a probability $p$. This configuration is taken as the initial state for the dynamics based on some simple local rules, in which the occupation status of a point is updated according to the configuration of its neighbours. This model is used to mimic the spreading of diseases, for example.

In its most common setting the dynamic rule is the following. Empty sites remain empty for ever. The sites that do not have at least $m$ occupied neighbours are emptied. This procedure goes on until no occupied site can be further removed or the lattice is void of particles. This stage corresponds to the infinite time limit. The critical $p_{c}$ for bootstrap percolation is the value of the parameter $p$ below which no infinite cluster is found in the
infinite time limit and for an infinite size system. For all lattices $p_{c}^{\mathrm{BP}}(m)$ is greater than or equal to the $p_{c}$ of random site percolation. If $m=0$, no particle extraction is possible, and $p_{c}^{\mathrm{BP}}(0)=p_{c}$. The existence of a spanning cluster in a given configuration does not ensure that this cluster cannot be broken by the dynamic rule.

An alternative definition of bootstrap percolation states that it is a cellular automaton with a discrete time deterministic dynamics in which at each time unit the configuration is updated according to a local and translation invariant rule.

In the dual diffusion percolation model the sites of the lattice are also occupied initially with a given probability $p$. In successive time steps empty sites with at least $n$ occupied neighbours are occupied at their turn. Occupied sites remain occupied. In this problem, the percolation threshold $p_{c}^{\mathrm{DP}}(n)$ is the value of $p$ at which an infinite cluster of occupied sites appears after the site-addition procedure, in the thermodynamic limit. It is also clear that for all lattices $p_{c}^{\mathrm{DP}}(n)$ is lower than or equal to the $p_{c}$ of random percolation. If $n>z$, the coordination of the lattice, no particle inclusion is possible, and $p_{c}^{\mathrm{BP}}(n>z)=p_{c}$.

These two problems are dual to each other if one identifies removed sites with less than $m$ neighbours with added sites with at least $z+1-m$ occupied neighbours.

There are choices of $m$ or $n, d$ and the lattice structure such that the BP or DP percolation transitions are of first order, meaning that the density of occupied sites at the threshold is finite (instead of zero as in a continuous transition).

Rigorous analysis of the, say, diffusion percolation problem on the hypercubic lattice establish that one can always fill the lattice for any choice of $n>0$. In particular, for $n=2$, and an initial condition chosen with $p=c / \ln L$ the probability of filling the lattice is one if $c>\pi^{2} / 18$ and zero otherwise. Note the very special finite size effects in this problem.

### 4.4 Away from the lattice

It is also possible to define percolation problems on the continuous space. This can be done, for example, but setting circles of equal radius $R$ randomly on a plane. Two disks are considered to be nearest-neighbours if the centre of one falls into the other one. The path linking such centres then identified. Remote disks can be then connected via these paths and the question is whether the longest path runs from one border of the sample to an opposite one. In this problem there are, a priori, two independent parameters: the concentration of the disks, i.e. their number over the size of the plane, $N / L^{2}$, and their radius, $R$. The control parameter turns out to be the dimensional quantity $N R^{2} / L^{2}$. This model is used to describe conduction in impurity semiconductors [32].

### 4.5 Mapping to the Potts model

The bond percolation problem can be easily mapped to the Potts model, a simple
generalisation of the Ising model,

$$
\begin{equation*}
H_{J}\left[s_{i}\right]=\sum_{\langle i j\rangle} \delta_{s_{i} s_{j}} \tag{4.24}
\end{equation*}
$$

with spins $s_{i}$ that take $q$ values, $s_{i}=1, \ldots, q$ and $\delta_{a b}$ the Kronecker delta function, $\delta_{a b}=1$ if $a=b$ and $\delta_{a b}=0$ otherwise. The case $q=2$ corresponds to the Ising model. In the limit $q \rightarrow 1$ the bond percolation properties are recovered in the way that we will describe below. This results is due to Kasteleyn and Fortuin [38, 39]. This connection between these two apparently different problems illustrates the use of an analytical continuation from integer $q$ to real $q$ as a tool to solve one problem (percolation) using a non-physical extension of another one (the Potts model). A similar mapping, though slightly more complicated, exists for the site percolation problem.

Each configuration of the bond percolation problem is associated to a sub-graph $G^{\prime}$ of the embedding graph $G$ on which the problem is defined, that appears with probability

$$
\begin{equation*}
\pi\left(G^{\prime}\right)=p^{E\left(G^{\prime}\right)}(1-p)^{M-E\left(G^{\prime}\right)} \tag{4.25}
\end{equation*}
$$

where $E\left(G^{\prime}\right)$ is the number of edges in $G^{\prime}$ and $M$ is the total number of edges in the original graph. Each sub-graph $G^{\prime}$ is not necessarily formed by a single component, so we will later call $c\left(G^{\prime}\right)$ the number of clusters in the sub-graph $G^{\prime}$. (A cluster is defined as the ensemble of edges that are occupied and joined by a vertex on the lattice.) Any averaged property of the bond percolation problem can then be computed as

$$
\begin{equation*}
\langle A\rangle=\sum_{G^{\prime}} A\left(G^{\prime}\right) \pi\left(G^{\prime}\right)=\sum_{G^{\prime}} A\left(G^{\prime}\right) p^{E}\left(G^{\prime}\right)(1-p)^{M-E\left(G^{\prime}\right)} \tag{4.26}
\end{equation*}
$$

$A$ could be, for example, the number of clusters.
The partition function of the Potts model reads

$$
\begin{align*}
Z_{J} & =\sum_{\left\{s_{i}\right\}} e^{-\beta H_{J}\left[\left\{s_{i}\right\}\right]}=\sum_{\left\{s_{i}\right\}} e^{\beta J \sum_{\langle i j\rangle} \delta_{s_{i} s_{j}}}=\sum_{\left\{s_{i}\right\}} \prod_{\langle i j\rangle} e^{\beta J \delta_{s_{i} s_{j}}} \\
& =\sum_{\left\{s_{i}\right\}} \prod_{\langle i j\rangle}\left(1+v \delta_{s_{i} s_{j}}\right) \tag{4.27}
\end{align*}
$$

with

$$
\begin{equation*}
v=e^{\beta J}-1 \tag{4.28}
\end{equation*}
$$

The last identity can be readily checked, by considering the two possible outcomes for each factor in the product:

$$
\begin{array}{lll}
s_{i}=s_{j} & \Rightarrow & e^{\beta J} \\
s_{i} \neq s_{j} & \Rightarrow & 1 \tag{4.30}
\end{array}
$$

Note that if one took $q=1$, that is to say if $s_{i}=1$ for all $i$, the partition function is identical to one, $Z_{J}=1$.

The product $\prod_{\langle i j\rangle}\left(1+v \delta_{s_{i} s_{j}}\right)$ under the sum $\sum_{\left\{s_{i}\right\}}$ can now be expanded. Each term in the resulting sum is represented by a subgraph on the lattice in which all its edges coincide with the $v \delta_{s_{i} s_{j}}$ factors in the term. Let us explain how this works listing the first terms in the sum for a square lattice:

- the first term is just the product of number-of-edges=M 1s and equals 1. The sum over the spin configurations is fully unconstrained and yields a factor $q^{N}$.
- the second kind of term is the product of $M-11 \mathrm{~s}$ and a factor $v \delta_{s_{i} s_{j}}$ and equals $v \delta_{s_{i} s_{j}}$. There are $M$ ways of choosing this non-trivial factor. Moreover, it will give a non-vanishing result only if $s_{i}=s_{j}$ and there are $q$ possibilities for this to happen. All these terms can be associated to isolated clusters with only two sites on the lattice. All the other spins on the graph are unconstrained and the sum over their configurations yields a factor $q^{N-2}$. Therefore, these terms are accompanied by an overall $q^{N-1}$ factor.
- the third kind of term is the product of $M-21 \mathrm{~s}$ and two factors $v \delta_{s_{i} s_{j}}$ and $v \delta_{s_{k} s_{l}}$. There are now three options for the two edges $\langle i j\rangle$ and $\langle k l\rangle$. They can be separated, when all indices are different, yielding two isolated clusters with two sites each. They can be joined together by one site, when e.g. $i \neq j=k \neq l$, and form a single cluster with three sites. They cannot be such that $i=k$ and $j=l$ since there is no double counting of links. The Kronecker delta imposes that all the spins on each individual cluster in the graph take the same value (out of the $q$ possible ones) while the spins that are not on vertices belonging to the clusters are free to take any of the $q$ possibilities.
The construction goes on along these lines.
The product is then represented as a sum over all possible subgraphs on the original graph. Each edge on each sub-graph is accompanied by a factor $v$. The sum over the spin configurations yields a factor $q$ for each cluster in the sub-graph and a factor $q^{n}$ with $n$ the number of sites that do not belong to any cluster. In short,

$$
\begin{equation*}
Z_{J}=\sum_{G^{\prime}} v^{E\left(G^{\prime}\right)} q^{c\left(G^{\prime}\right)} \tag{4.31}
\end{equation*}
$$

where we called $E\left(G^{\prime}\right)$ the number of edges and $c\left(G^{\prime}\right)$ the number of clusters in the sub-graph $G^{\prime}$, including as independent single-site clusters the sites that are alone. This model, for generic $q$ is called the random cluster model.

The free-energy density of the Potts model is

$$
\begin{equation*}
-\beta f_{J}=N^{-1} \ln Z_{J}=N^{-1} \ln \sum_{G^{\prime}} v^{E\left(G^{\prime}\right)} q^{c\left(G^{\prime}\right)} \equiv N^{-1} \ln \left\langle\left\langle q^{c\left(G^{\prime}\right)}\right\rangle\right\rangle \tag{4.32}
\end{equation*}
$$

where we introduced the symbol $\langle\langle\ldots\rangle\rangle$ to indicate the average over the subgraphs with weight $v^{E\left(G^{\prime}\right)}$. Now, we approximate this average in the $q \simeq 1$ limit:

$$
\begin{equation*}
\ln \left\langle\left\langle q^{c\left(G^{\prime}\right)}\right\rangle\right\rangle=\ln \left\langle\left\langle e^{\ln q^{c\left(G^{\prime}\right)}}\right\rangle\right\rangle=\ln \left\langle\left\langle e^{c\left(G^{\prime}\right) \ln q}\right\rangle\right\rangle \tag{4.33}
\end{equation*}
$$

Using

$$
\begin{equation*}
\ln q=\ln [1+(q-1)] \simeq(q-1)+O\left((q-1)^{2}\right) \tag{4.34}
\end{equation*}
$$

then

$$
\begin{equation*}
\ln \left\langle\left\langle q^{c\left(G^{\prime}\right)}\right\rangle\right\rangle \simeq \ln \left\langle\left\langle e^{c\left(G^{\prime}\right)(q-1)}\right\rangle\right\rangle \simeq \ln \left[1+\left\langle\left\langle c\left(G^{\prime}\right)(q-1)\right\rangle\right\rangle\right] \simeq(q-1)\left\langle\left\langle c\left(G^{\prime}\right)\right\rangle\right\rangle \tag{4.35}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
-\beta f_{J}=N^{-1} \ln Z_{J} \simeq(q-1)\left\langle\left\langle c\left(G^{\prime}\right)\right\rangle\right\rangle=(q-1) \sum_{G^{\prime}} v^{E\left(G^{\prime}\right)} c\left(G^{\prime}\right) \tag{4.36}
\end{equation*}
$$

The remaining average will be then average number of clusters in the percolation problem provided the statistical weight $v^{E\left(G^{\prime}\right)}$ in the Potts model be proportional to the one in the percolation problem (apart from an irrelevant constant $(1-p)^{M}$ that does not depend on the sub-graph configuration)

$$
\begin{equation*}
\left(\frac{p}{1-p}\right)^{E\left(G^{\prime}\right)}=v^{E\left(G^{\prime}\right)} \tag{4.37}
\end{equation*}
$$

that implies

$$
\begin{equation*}
v=\frac{p}{1-p} \quad \text { or } \quad p=1-e^{-\beta J} \tag{4.38}
\end{equation*}
$$

## 5 Random fields, random interactions

No material is perfectly homogeneous: impurities of different kinds are distributed randomly throughout the samples. In ultra-cold atom systems, so much studied nowadays, disorder can be realized, for example, using speckle laser light.

A natural effect of disorder should be to lower the critical temperature. Much attention has been payed to the effect of weak disorder on phase transitions, that is to say, situations in which the nature of the ordered and disordered phases is not modified by the impurities but the critical phenomenon is. On the one hand, the critical exponents of second order phase transitions might be modified by disorder, on the other hand, disorder may smooth out the discontinuities of first order phase transitions rendering them of second order. Strong disorder instead changes the nature of the low-temperature phase and before discussing the critical phenomenon one needs to understand how to characterize the new ordered 'glassy' phase.

In this Section we shall discuss several types of quenched disorder and models that account for it. We shall also overview some of the theoretical methods used to deal with the static properties of models with quenched disorder, namely, scaling arguments and the droplet theory, mean-field equations, and the replica method.

### 5.1 Quenched and annealed disorder

First, one has to distinguish between quenched or frozen-in and annealed disorder. Imagine that one mixes some random impurities in a melt and then very slowly cools it down in such a way that the impurities and the host remain in thermal equilibrium. If one wants to study the statistical properties of the full system one then has to compute the full partition function in which one sums over all configurations of original components and impurities. This is called annealed disorder. In the opposite case in which upon cooling the host and impurities do not equilibrate but the impurities remain blocked in random fixed positions, one talks about quenched disorder. Basically, the relaxation time associated with the diffusion of the impurities in the sample is so long that these remain trapped:

$$
\begin{equation*}
\tau_{o} \sim 10^{-12}-10^{-15} \mathrm{sec} \ll t_{\mathrm{obs}} \sim 10^{4} \mathrm{sec} \ll t_{\mathrm{diff}} \tag{5.1}
\end{equation*}
$$

where $\tau_{o}$ is the microscopic time associated to the typical scale needed to reverse a spin.
The former case is easier to treat analytically but is less physically relevant. The latter is the one that leads to new phenomena and ideas that we shall discuss next.

Quenched disorder is static. Instead, in annealed disorder the impurities are in thermal equilibrium in the experimental time-scales, and they can simply be included in the statistical mechanic description of the problem.


Figure 5.1: A frustrated (left) and an unfrustrated (center) square plaquette. A frustrated triangular plaquette (right).

### 5.2 Properties

### 5.2.1 Lack of homogeneity

It is clear that the presence of quench disorder, in the form of random interactions, fields, dilution, etc. breaks spatial homogeneity and renders single samples heterogenous. Homogeneity is recovered though, if one performs an average of all possible realizations of disorder, each weighted with its own probability.

### 5.2.2 Frustration

Depending on the value of the distance $r_{i j}$ the numerator in Eq. (5.10) can be positive or negative implying that both ferromagnetic and antiferromagnetic interactions exist. This leads to frustration, which means that in any configuration some two-body interactions remain unsatisfied. In other words, there is no spin configuration that minimizes all terms in the Hamiltonian. An example with four sites and four links is shown in Fig. 5.1-left, where we took three positive exchanges and one negative one all, for simplicity, with the same absolute value, $J$. Four configurations minimize the energy, $E_{f}=-2 J$, but none of them satisfies the lower link. One can easily check that any closed loop such that the product of the interactions takes a negative sign is frustrated. Frustration naturally leads to a higher energy and a larger degeneracy of the number of ground states. This is again easy to grasp by comparing the number of ground states of the frustrated plaquette in Fig. 5.1-left to its unfrustrated counterpart shown on the central panel. Indeed, the energy and degeneracy of the ground state of the unfrustrated plaquette are $E_{u}=-4 J$ and $n_{u}=2$, respectively.

Frustration may also be due to pure geometrical constraints. The canonical example is an antiferromagnet on a triangular lattice in which each plaquette is frustrated, see Fig. 5.1-right.

In short, frustration arises when the geometry of the lattice and/or the nature of the interactions make impossible to simultaneously minimize the energy of all pair couplings between the spins. Any loop of connected spins is said to be frustrated if the product of the signs of connecting bonds is negative. In general, energy and entropy of the ground states increase due to frustration.

### 5.2.3 Self-averageness

If each sample is characterized by its own realization of the exchanges, should one expect a totally different behavior from sample to sample? Fortunately, many generic static and dynamic properties of spin-glasses (and other systems with quenched disorder) do not depend on the specific realization of the random couplings and are self-averaging. This means that the typical value is equal to the average over the disorder:

$$
\begin{equation*}
A_{J}^{\mathrm{typ}}=\left[A_{J}\right] \tag{5.2}
\end{equation*}
$$

in the thermodynamic limit. Henceforth, we use square brackets to indicate the average over the random couplings. More precisely, in self-averaging quantities sample-to-sample fluctuations with respect to the mean value are expected to be $\mathrm{O}\left(N^{-a}\right)$ with $a>0$. Roughly, observables that involve summing over the entire volume of the system are expected to be self-averaging. In particular, the free-energy density of models with shortranged interactions is expected to be self-averaging in this limit.

An example: the disordered Ising chain
The meaning of this property can be grasped from the solution of the random bond Ising chain defined by the energy function $H_{J}\left[\left\{s_{i}\right\}\right]=-\sum_{i} J_{i} s_{i} s_{i+1}$ with spin variables $s_{i}= \pm$, for $i=1, \ldots, N$ and random bonds $J_{i}$ independently taken from a probability distribution $P\left(J_{i}\right)$. For simplicity, we consider periodic boundary conditions. The disorder-dependent partition function reads

$$
\begin{equation*}
Z_{J}=\sum_{\left\{s_{i}= \pm 1\right\}} e^{\beta \sum_{i} J_{i} s_{i} s_{i+1}} \tag{5.3}
\end{equation*}
$$

and this can be readily computed introducing the change of variables $\sigma_{i} \equiv s_{i} s_{i+1}$. (Note that these new variables are not independent, since they are constrained to satisfy $\prod_{i} \eta_{i}=$ 1. This constraint is irrelevant in the thermodynamic limit.) One finds

$$
\begin{equation*}
Z_{J}=\prod_{i} 2 \cosh \left(\beta J_{i}\right) \quad \Rightarrow \quad-\beta F_{J}=\sum_{i} \ln \cosh \left(\beta J_{i}\right)+N \ln 2 \tag{5.4}
\end{equation*}
$$

The partition function is a product of i.i.d. random variables and it is itself a random variable with a log-normal distribution. The free-energy density instead is a sum of i.i.d. random variables and, using the central limit theorem, in the large $N$ limit becomes a Gaussian random variable narrowly peaked at its maximum. The typical value, given by the maximum of the Gaussian distribution, coincides with the average, $\lim _{N \rightarrow \infty}\left(f_{J}^{\text {typ }}-\right.$ $\left.\left[f_{J}\right]\right)=0$ with $f_{J}=F_{J} / N$.

## General argument

A simple argument justifies the self-averageness of the free-energy density in generic finite dimensional systems with short-range interactions. Let us divide a, say, cubic system of volume $V=L^{d}$ in $n$ subsystems, say also cubes, of volume $v=\ell^{d}$ with $V=n v$.

If the interactions are short-ranged, the total free-energy is the sum of two terms, a contribution from the bulk of the subsystems and a contribution from the interfaces between the subsystems: $-\beta F_{J}=\ln Z_{J}=\ln \sum_{\mathrm{conf}} e^{-\beta H_{J}(\mathrm{conf})}=\ln \sum_{\mathrm{conf}} e^{-\beta H_{J}(\mathrm{bulk})-\beta \mathrm{H}_{\mathrm{J}}(\text { surf })} \approx$ $\ln \sum_{\text {bulk }} e^{-\beta H_{J}(\text { bulk })}+\ln \sum_{\text {surf }} e^{-\beta H_{J}(\text { surf })}=-\beta F_{J}^{\text {bulk }}-\beta F_{J}^{\text {surf }}$ (we neglected the contributions from the interaction between surface and bulk). If the interaction extends over a short distance $\sigma$ and the linear size of the boxes is $\ell \gg \sigma$, the surface energy is negligible with respect to the bulk one and $-\beta F_{J} \approx \ln \sum_{\text {bulk }} e^{-\beta H_{J}(\text { bulk })}$. In the thermodynamic limit, the disorder dependent free-energy is then a sum of $n=(L / \ell)^{d}$ random numbers, each one being the disorder dependent free-energy of the bulk of each subsystem: $-\beta F_{J} \approx \sum_{k=1}^{n} \ln \sum_{\text {bulk }_{k}} e^{-\beta H_{J}\left(\text { bulk }_{k}\right)}$. In the limit of a very large number of subsystems ( $L \gg \ell$ or $n \gg 1$ ) the central limit theorem implies that the total free-energy is Gaussian distributed with the maximum reached at a value $F_{J}^{\text {typ }}$ that coincides with the average over all realizations of the randomness $\left[F_{J}\right]$. Morever, the dispersion about the typical value vanishes in the large $n$ limit, $\sigma_{F_{J}} /\left[F_{J}\right] \propto \sqrt{n} / n=n^{-1 / 2} \rightarrow 0$. Similarly, $\sigma_{f_{J}} /\left[f_{J}\right] \sim O\left(n^{-1 / 2}\right)$ where $f_{J}=F_{J} / N$ is the intensive free-energy. In a sufficiently large system the typical $F_{J}$ is then very close to the averaged $\left[F_{J}\right]$ and one can compute the latter to understand the static properties of typical systems.

Lack of self-averageness in the correlation functions
Once one has $\left[F_{J}\right]$, one derives all disordered average thermal averages by taking derivatives of the disordered averaged free-energy with respect to sources introduced in the partition function. For example,

$$
\begin{align*}
& {\left[\left\langle s_{i}\right\rangle\right]=-\left.\frac{\partial\left[F_{J}\right]}{\partial h_{i}}\right|_{h_{i}=0}}  \tag{5.5}\\
& {\left[\left\langle s_{i} s_{j}\right\rangle-\left\langle s_{i}\right\rangle\left\langle s_{j}\right\rangle\right]=-\left.T \frac{\partial^{2}\left[F_{J}\right]}{\partial h_{i} h_{j}}\right|_{h_{i}=0}} \tag{5.6}
\end{align*}
$$

with $H_{J} \rightarrow H_{J}-\sum_{i} h_{i} s_{i}$. Connected correlation functions, though, are not self-averaging quantities. This can be seen, again, studying the random bond Ising chain. Take $i<j$. One can easily check that

$$
\begin{equation*}
\left\langle s_{i} s_{j}\right\rangle_{J}-\left\langle s_{i}\right\rangle_{J}\left\langle s_{j}\right\rangle_{J}=Z_{J}^{-1} \frac{\partial}{\partial \beta J_{j-1}} \ldots \frac{\partial}{\partial \beta J_{i}} Z_{J}=\tanh \left(\beta J_{i}\right) \ldots \tanh \left(\beta J_{j}\right) \tag{5.7}
\end{equation*}
$$

where we used $\left\langle s_{i}\right\rangle=0$ (valid for a distribution of random bonds with zero mean) and the dots indicate all sites on the chain between the ending points $i$ and $j$, i.e. $i+1 \leq k \leq j-1$. The last expression is a product of random variables and it is not equal to its average (5.6) - not even in the large separation limit $\left|\vec{r}_{i}-\vec{r}_{j}\right| \rightarrow \infty$.

### 5.2.4 Annealed disorder

The thermodynamics of a system with annealed disorder is obtained by averaging the partition function over the impurity degrees of freedom,

$$
\begin{equation*}
Z=\left[Z_{J}\right] \tag{5.8}
\end{equation*}
$$

since one needs to do the partition sum over the disorder degrees of freedom as well.

### 5.3 Models

### 5.3.1 Bethe lattices and random graphs

The Bethe lattice is a tree, in which each site has $z$ neighbours and each branch gives rise to $z-1$ new branches. Two important properties of these lattices are:

- there are no closed loops.
- the number of sites on the border is of the same order of magnitude as the total number of sites on the lattice.
- All sites on the lattice are equivalent, there is no notion of a central site.

Exercise 5.1 Show that the total number of sites on the Bethe lattice with $z=3$ and $g$ generations (or the distance from the site designed as the central one) is $n_{\text {tot }}=32^{g}-1$ and the number of sites on the border is $n_{\text {border }}=32^{g-1}$. The surface to volume ratio tends to $1 / 2$.

Exercise 5.2 Take a hypercubic lattice in dimensions and estimate the surface to volume ratio. Show that this ratio tends to a finite value only if $d \rightarrow \infty$.

A random graph is obtained by starting with a set of $n$ isolated vertices and adding successive edges between them at random. A popular ensemble is the one denoted $G(n, p)$, in which every possible edge occurs independently with probability $0<p<1$. Random graphs with fixed connectivity are also commonly used.

Random graphs are used in social sciences modeling (nodes representing individuals and edges the friendship relationship), technology (interconnections of routers in the Internet, pages of the WWW, or production centers in an electrical network), biology (interactions of genes in a regulatory network) [40, 41]. Disordered systems are usually defined on random graphs, especially the ones motivated by the combinatorial optimisation.

### 5.3.2 Dilute spin models

Lattice models with site or link dilution are

$$
\begin{equation*}
H_{J}^{\text {site dil }}=-J \sum_{\langle i j\rangle} s_{i} s_{j} \epsilon_{i} \epsilon_{j}, \quad H_{J}^{\text {link dil }}=-J \sum_{\langle i j\rangle} s_{i} s_{j} \epsilon_{i j} \tag{5.9}
\end{equation*}
$$

with $P\left(\epsilon_{i}=1,0\right)=p, 1-p$ in the first case and $P\left(\epsilon_{i j}=1,0\right)=p, 1-p$ in the second. These models are intimately related to Percolation theory. Physically, dilution is realised by vacancies or impurity atoms in a crystal.

### 5.3.3 Spin-glass models

Spin-glasses are alloys in which magnetic impurities substitute the original atoms in positions randomly selected during the chemical preparation of the sample [43, 45, 46]. The interactions between the impurities are of RKKY type:

$$
\begin{equation*}
V_{\mathrm{rkky}}=-J \frac{\cos \left(2 k_{F} r_{i j}\right)}{r_{i j}^{3}} s_{i} s_{j} \tag{5.10}
\end{equation*}
$$

with $r_{i j}=\left|\vec{r}_{i}-\vec{r}_{j}\right|$ the distance between them and $s_{i}$ a spin variable that represents their magnetic moment. Clearly, the initial location of the impurities varies from sample to sample. The time-scale for diffusion of the magnetic impurities is much longer than the time-scale for spin flips. Thus, for all practical purposes the positions $\vec{r}_{i}$ can be associated to quenched random variables distributed according to a uniform probability distribution that in turn implies a probability distribution of the exchanges. This is called quenched disorder.

In early 70s Edwards and Anderson proposed a rather simple model that should capture the main features of spin-glasses. The interactions (5.10) decay with a cubic power of the distance and hence they are relatively short-ranged. This suggests to put the spins on a regular cubic lattice model and to trade the randomness in the positions into random nearest neighbour exchanges taken from a Gaussian probability distribution:

$$
\begin{equation*}
H_{J}^{\mathrm{ea}}=-\sum_{\langle i j\rangle} J_{i j} s_{i} s_{j} \quad \text { with } \quad P\left(J_{i j}\right)=\left(2 \pi \sigma^{2}\right)^{-\frac{1}{2}} e^{-\frac{J_{i j}^{2}}{2 \sigma^{2}}} \tag{5.11}
\end{equation*}
$$

The precise form of the probability distribution of the exchanges is supposed not to be important, though some authors claim that there might be non-universality with respect to it.

Another natural choice is to use bimodal exchanges

$$
\begin{equation*}
P\left(J_{i j}\right)=p \delta\left(J_{i j}-J_{0}\right)+(1-p) \delta\left(J_{i j}+J_{0}\right) \tag{5.12}
\end{equation*}
$$

with the possibility of a bias towards positive or negative interactions depending on the parameter $p$. A tendency to non-zero average $J_{i j}$ can also be introduced in the Gaussian pdf.

A natural extension of the EA model in which all spins interact has been proposed by Sherrington and Kirkpatrick

$$
\begin{equation*}
H_{J}^{\mathrm{SK}}=-\sum_{i \neq j} J_{i j} s_{i} s_{j}-\sum_{i} h_{i} s_{i} \tag{5.13}
\end{equation*}
$$

and it is called the $S K$ model. The interaction strengths $J_{i j}$ are taken from a Gaussian pdf and they scale with $N$ in such a way that the thermodynamic limit is non-trivial:

$$
\begin{equation*}
P\left(J_{i j}\right)=\left(2 \pi \sigma_{N}^{2}\right)^{-\frac{1}{2}} e^{-\frac{J_{i j}^{2}}{2 \sigma_{N}^{2}}} \quad \quad \sigma_{N}^{2}=\sigma^{2} N \tag{5.14}
\end{equation*}
$$

The first two-moments of the exchange distribution are $\left[J_{i j}\right]=0$ and $\left[J_{i j}^{2}\right]=J^{2} /(2 N)$. This is a case for which a mean-field theory is expected to be exact.

### 5.3.4 Glass models

A further extension of the EA model is called the $p$ spin model

$$
\begin{equation*}
H_{J \mathrm{p}-\text { spin }}=-\sum_{i_{1}<\cdots<i_{p}} J_{i_{1} \ldots i_{p}} s_{i_{1}} \ldots s_{i_{p}}-\sum_{i} h_{i} s_{i} \tag{5.15}
\end{equation*}
$$

with $p \geq 3$. The sum can also be written as $\sum_{i_{1}<i_{2}<\cdots<i_{p}}=1 / p!\sum_{i_{1} \neq i_{2} \neq i_{p}}$. The exchanges are now taken from a Gaussian probability distribution

$$
\begin{equation*}
P\left(J_{i j}\right)=\left(2 \pi \sigma_{N}^{2}\right)^{-\frac{1}{2}} e^{-\frac{J_{i j}^{2}}{2 \sigma_{N}^{N}}} \quad \quad \sigma_{N}^{2}=J^{2} p!/\left(2 N^{p-1}\right) \tag{5.16}
\end{equation*}
$$

with $\left[J_{i_{1} \ldots i_{p}}\right]=0$ and $\left[J_{i_{1} \ldots i_{p}}^{2}\right]=\frac{J^{2} p!}{2 N^{p-1}}$. Indeed, an extensive free-energy is achieved by scaling $J_{i_{1} \ldots i_{p}}$ with $N^{-(p-1) / 2}$. This scaling can be justified as follows. The 'local field' $h_{i}=1 /(p-1)!\sum_{i i_{2} \neq i_{p}} J_{i i_{2} \ldots i_{p}} m_{i_{2}} \ldots m_{i_{p}}$ should be of order one. At low temperatures the $m_{i}$ 's take plus and minus signs. In particular, we estimate the order of magnitude of this term by working at $T=0$ and taking $m_{i}= \pm 1$ with probability $\frac{1}{2}$. In order to keep the discussion simple, let us take $p=2$. In this case, if the strengths $J_{i j}$, are of order one, $h_{i}$ is a sum of $N$ i.i.d. random variables, with zero mean and unit variance ${ }^{2}$, and $h_{i}$ has zero mean and variance equal to $N$. Therefore, one can argue that $h_{i}$ is of order $\sqrt{N}$. To make it finite we then chose $J_{i j}$ to be of order $1 / \sqrt{N}$ or, in other words, we impose $\left[J_{i j}^{2}\right]=J^{2} /(2 N)$. The generalization to $p \geq 2$ is straightforward.

We classify this model in the "glass" class since it has been shown that its behaviour mimics the one of so-called fragile glasses.

### 5.3.5 Vector spins

Extensions to vector spins with two (XY), three (Heisenberg) or $N$ components also exist. In the former cases can be relevant to describe real samples. One usually keeps the modulus of the spins fixed to be 1 in these cases.

But there is another way to extend the spin variables and it is to use a spherical constraint,

$$
\begin{equation*}
-\infty \leq s_{i} \leq \infty \quad \sum_{i=1} s_{i}^{2}=N \tag{5.17}
\end{equation*}
$$

In this case, the spins $s_{i}$ are the components of an $N$-dimensional vector, constrained to be an $N$-dimensional sphere.

[^1]
### 5.3.6 Optimization problems

Cases that find an application in computer science are defined on random graphs with fixed or fluctuating finite connectivity. In the latter case one places the spins on the vertices of a graph with links between couples or groups of $p$ spins chosen with a probability $c$. These are dilute spin-glasses on graphs (instead of lattices).

### 5.3.7 Random bond ferromagnets

Let us now discuss some, a priori simpler cases. An example is the Mattis random magnet with generic nergy (5.15) in which the interaction strengths are given by

$$
\begin{equation*}
J_{i_{1} \ldots i_{p}}=\xi_{i_{1}} \ldots \xi_{i_{p}} \quad \text { with } \quad \xi_{j}= \pm \text { with } \text { prob }=1 / 2 \tag{5.18}
\end{equation*}
$$

for any $p$ and any kind of graph. In this case a simple gauge transformation, $\eta_{i} \equiv \xi_{i} s_{i}$, allows one to transform the disordered model in a ferromagnet, showing that there was no true frustration in the system.

Random bond ferromagnets (RBFMs) are systems in which the strengths of the interactions are not all identical but their sign is always positive. One can imagine such a exchange as the sum of two terms:

$$
\begin{equation*}
J_{i j}=J+\delta J_{i j}, \quad \text { with } J>0 \quad \text { and } \quad \delta J_{i j} \quad \text { small and random } \tag{5.19}
\end{equation*}
$$

There is no frustration in these systems either.
As long as all $J_{i j}$ remain positive, this kind of disorder should not change the two bulk phases with a paramagnetic-ferromagnetic second-order phase transition. Moreover the up-down spin symmetry is not broken by the disorder. The disorder just changes the local tendency towards ferromagnetism that can be interpreted as a change in the local critical temperature. Consequently, this type of disorder is often called random- $T_{c}$ disorder, and it admits a Ginzburg-Landau kind of description, with a random distance from criticality, $\delta u(\mathbf{x})$,

$$
\begin{equation*}
F[m(\vec{r})]=\int d^{d} r\left\{-h m(\vec{r})+[r+\delta r(\mathbf{x})] m^{2}(\vec{r})+(\nabla m(\vec{r}))^{2}+u m^{4}(\vec{r})+\ldots\right\} \ldots \tag{5.20}
\end{equation*}
$$

The disorder couples to the $m^{2}$ term in the free-energy functional. In quantum field theory, this term is called the mass term and, therefore, random- $T_{c}$ disorder is also called random-mass disorder. (In addition to random exchange couplings, random-mass disorder can also be realized by random dilution of the spins.)

### 5.3.8 Random field ferromagnets

Link randomness is not the only type of disorder encountered experimentally. Random fields, that couple linearly to the magnetic moments, are also quite common; the classical
model is the ferromagnetic random field Ising model (RFIM):

$$
\begin{equation*}
H_{J}^{\mathrm{rfim}}=-J \sum_{\langle i j\rangle} s_{i} s_{j}-\sum_{i} s_{i} h_{i} \quad \text { with } \quad P\left(h_{i}\right)=\left(2 \pi \sigma^{2}\right)^{-\frac{1}{2}} e^{-\frac{h_{i}^{2}}{2 \sigma^{2}}} \tag{5.21}
\end{equation*}
$$

The dilute antiferromagnet in a uniform magnetic field is believed to behave similarly to the ferromagnetic random field Ising model. Experimental realizations of the former are common and measurements have been performed in samples like $\mathrm{Rb}_{2} \mathrm{Co}_{0.7} \mathrm{Mg}_{0.3} \mathrm{~F}_{4}$.

Note that the up-down Ising symmetry is not preserved in models in the RFIMm and any spin model such that the disorder couples to the local order parameter.

In the Ginzburg-Landau description this model reads

$$
\begin{equation*}
F[m(\vec{r})]=\int d^{d} r\left\{-h(\mathbf{x}) m(\vec{r})+r m^{2}(\vec{r})+(\nabla m(\vec{r}))^{2}+u m^{4}(\vec{r})+\ldots\right\} \tag{5.22}
\end{equation*}
$$

where $h(\vec{r})$ is the local random variable that breaks the up-down spin symmetry. Whether or not the symmetry is broken globally depends on the probability distribution of the random fields. A particularly interesting situation arises if the distribution is even in $h$ such that the up-down symmetry is globally preserved in the statistical sense.

Random-field disorder is generally stronger than random-mass disorder.
The random fields give rise to many metastable states that modify the equilibrium and non-equilibrium behaviour of the RFIM. In one dimension the RFIM does not order at all, in $d=2$ there is strong evidence that the model is disordered even at zero temperature, in $d=3$ it there is a finite temperature transition towards a ferromagnetic state. Whether there is a glassy phase near zero temperture and close to the critical point is still and open problem.

The RFIM at zero temperature has been proposed to yield a generic description of material cracking through a series of avalaches. In this problem one cracking domain triggers others, of which size, depends on the quenched disorder in the samples. In a random magnetic system this phenomenon corresponds to the variation of the magnetization in discrete steps as the external field is adiabatically increased (the time scale for an avalanche to take place is much shorter than the time-scale to modify the field) and it is accessed using Barkhausen noise experiments. Disorder is responsible for the jerky motion of the domain walls. The distribution of sizes and duration of the avalanches is found to decay with a power law tail and cut-off at a given size. The value of the cut-off size depends on the strength of the random field and it moves to infinity at the critical point.

### 5.3.9 Random manifolds

Once again, disorder is not only present in magnetic systems. An example that has received much attention is the so-called random manifold. This is a $d$ dimensional directed elastic manifold moving in an embedding $N+d$ dimensional space under the effect of a quenched random potential. The simplest case with $d=0$ corresponds to a particle
moving in an embedding space with $N$ dimensions. If, for instance $N=1$, the particle moves on a line, if $N=2$ it moves on a plane and so on and so forth. If $d=1$ one has a line that can represent a domain wall, a polymer, a vortex line, etc. The fact that the line is directed means it has a preferred direction, in particular, it does not have overhangs. If the line moves in a plane, the embedding space has $(N=1)+(d=1)$ dimensions. One usually describes the system with an $N$-dimensional coordinate, $\vec{\phi}$, that locates in the transverse space each point on the manifold, represented by the internal $d$-dimensional coordinate $\vec{r}$,

The elastic energy is $H_{\text {elas }}=\gamma \int d^{d} x \sqrt{1+(\nabla \phi(\vec{r}))^{2}}$ with $\gamma$ the deformation cost of a unit surface. Assuming the deformation is small one can linearise this expression and get, upto an additive constant, $H_{\text {elas }}=\frac{\gamma}{2} \int d^{d} r(\nabla \phi(\vec{r}))^{2}$.

Disorder is introduced in the form of a random potential energy $V(\vec{\phi}(\vec{r}), \vec{r})$ characterised by its pdf.

The random manifold model is then

$$
\begin{equation*}
H_{V}(\vec{\phi})=\int d^{d} r\left[\frac{\gamma}{2}(\nabla \phi(\vec{r}))^{2}+V(\vec{\phi}(\vec{r}), \vec{r})\right] . \tag{5.23}
\end{equation*}
$$

If the random potential is the result of a large number of impurities, the central limit theorem implies that its probability density is Gaussian. Just by shifting the energy scale one can set its average to zero, $[V]=0$. As for its correlations, one typically assumes, for simplicity, that they exist in the transverse direction only:

$$
\begin{equation*}
\left[V(\vec{\phi}(\vec{r}), \vec{r}) V\left(\vec{\phi}^{\prime}\left(\vec{r}^{\prime}\right), \vec{r}^{\prime}\right)\right]=\delta^{d}\left(\vec{r}-\vec{r}^{\prime}\right) \mathcal{V}\left(\vec{\phi}, \vec{\phi}^{\prime}\right) \tag{5.24}
\end{equation*}
$$

If one further assumes that there is a statistical isotropy and translational invariance of the correlations, $\mathcal{V}\left(\vec{\phi}, \overrightarrow{\phi^{\prime}}\right)=W / \Delta^{2} \mathcal{V}\left(\left|\vec{\phi}-\overrightarrow{\phi^{\prime}}\right| / \Delta\right)$ with $\Delta$ a correlation length and $\left(W \Delta^{d-2}\right)^{1 / 2}$ the strength of the disorder. The disorder can now be of two types: short-ranged if $\mathcal{V}$ falls to zero at infinity sufficiently rapidly and long-range if it either grows with distance or has a slow decay to zero. An example involving both cases is given by the power law $\mathcal{V}(z)=(\theta+z)^{-\gamma}$ where $\theta$ is a short distance cut-off and $\gamma$ controls the range of the correlations with $\gamma>1$ being short-ranged and $\gamma<1$ being long-ranged.

This model also describes directed domain walls in random systems. One can derive it in the long length-scales limit by taking the continuum limit of the pure Ising part (that leads to the elastic term) and the random part (that leads to the second disordered potential). In the pure Ising model the second term is a constant that can be set to zero while the first one implies that the ground state is a perfectly flat wall, as expected. In cases with quenched disorder, the long-ranged and short-ranged random potentials mimic cases in which the interfaces are attracted by pinning centres ('random field' type) or the phases are attracted by disorder ('random bond' type), respectively. For instance, random bond disorder is typically described by a Gaussian pdf with zero mean and delta-correlated $\left[V(\vec{\phi}(\vec{r}), \vec{r}), V\left(\vec{\phi}^{\prime}\left(\vec{r}^{\prime}\right), \vec{r}^{\prime}\right)\right]=W \Delta^{d-2} \delta^{d}\left(\vec{r}-\vec{r}^{\prime}\right) \delta\left(\vec{\phi}-\vec{\phi}^{\prime}\right)$.

### 5.4 Properties of finite dimensional disordered systems

Once various kinds of quenched disorder introduced, a number of questions on their effect on the equilibrium and dynamic properties arise. Concerning the former:

- Are the equilibrium phases qualitatively changed by the random interactions?
- Is the phase transition still sharp, or is it smeared because different parts of the system undergo the transition independently?
- If there is still a phase transition, does its order (first order vs. continuous) change?
- If the phase transition remains continuous, does the critical behavior, i.e., the values of the critical exponents, change?

Now, for the latter:

- Is the dynamic behaviour of the system modified by the quenched randomness?

In the following we explain a series of classical results in this field: the Harris criterium, the proof of non-analyticity of the free-energy of quenched disordered systems below their critical temperature given by Griffiths, the analysis of droplets and their domain wall stiffness, and the derivation of some exact results using the gauge invariance.

We first focus on impurities or defects that lead to spatial variations with respect to the tendency to order but do not induce new types of order, that is to say, no changes are induced in the two phases at the two sides of the transition. Only later we consider the spin-glass case.

### 5.4.1 The Harris criterium

The first question to ask is how does the average disorder strength behave under coarsegraining or, equivalently, how is it seen at long distances. This is the question answered by the Harris argument.

The Harris' criterion [51] states that if the specific-heat of a pure system

$$
\begin{equation*}
C^{\text {pure }}(T) \simeq\left|T-T_{c}^{\text {pure }}\right|^{-\alpha} \tag{5.25}
\end{equation*}
$$

presents a power-like divergence with

$$
\begin{equation*}
\alpha_{\text {pure }}>0 \tag{5.26}
\end{equation*}
$$

the disorder may induce a new universality class. Otherwise, if $\alpha_{\text {pure }}<0$, disorder is irrelevant in a renormalisation group sense and the critical behaviour of the model remains unchanged. The criterium does not decide in the marginal case $\alpha_{\text {pure }}=0$ case. Note that


Figure 5.2: Left: scheme of the Harris construction. The disordered system is divided into cells with linear length $\xi_{\text {dis }}$, its correlation length. Right: a typical configuration of the dilute Ising ferromagnet. Figures taken from [50].
the Harris criterium is a necessary condition for a change in critical behaviour but not a sufficient one.

The hyper-scaling relation $2-d \nu_{\text {pure }}=\alpha_{\text {pure }}$ allows to rewrite the Harris criterium as

$$
\text { critical behaviour }=\left\{\begin{array}{l}
\text { unchanged if } \nu_{\text {pure }}>2 / d  \tag{5.27}\\
\text { changed } \\
\text { if } \nu_{\text {pure }}<2 / d
\end{array}\right.
$$

where $\nu_{\text {pure }}$ is the correlation length exponent

$$
\begin{equation*}
\left\langle s_{0} s_{\vec{r}}\right\rangle \simeq e^{-r / \xi_{\text {pure }}} \quad \text { and } \quad \xi \simeq\left|T-T_{c}^{\text {pure }}\right|^{-\nu_{\text {pure }}} \tag{5.28}
\end{equation*}
$$

of the pure system.


Figure 5.3: A sketch of critical temperatures.
The proof of the Harris result is rather simple and illustrates a way of reasoning that is extremely useful $[51,50]$. Take the full system with frozen-in disorder at a temperature $T$ slightly above its critical temperature $T_{c}^{\text {dis }}$. Divide it into equal pieces with linear size $\xi_{\text {dis }}$, the correlation length at the working temperature. By construction, the spins within each of these blocks behave as a super-spin since they are effectively parallel. Because of disorder, each block $k$ has its own local critical temperature $T_{c}^{(k)}$ determined by the interactions (or dilution) within the block. Harris proposes to compare the fluctuations in the local critical temperatures $\Delta T^{\mathrm{loc}} \equiv T_{c}^{(k)}-T_{c}^{\text {dis }}$ with respect to the global critical one $T_{c}^{\text {dis }}$, with the distance from the critical point $\Delta T_{c} \equiv T-T_{c}^{\text {dis }}>0$, taken to be positive:

- If $\Delta T_{c}^{\text {loc }}<\Delta T_{c}$, all blocks have critical temperature below the working one, $T_{c}^{(k)}<$ $T$, and the system is 'uniform' with respect to the phase transition.
- If $\Delta T_{c}^{\text {loc }}>\Delta T_{c}$, some blocks are in the disordered (paramagnetic) phase and some are in the ordered (ferromagnetic) phase, making a uniform transition impossible. The inhomogeneity in the system may then be important.

Require now $\Delta T_{c}^{\text {loc }}<\Delta T_{c}$ to have an unmodified critical behaviour. Use also that an unmodified critical behaviour implies $\xi_{\text {dis }}=\xi_{\text {pure }}=\xi$ and, consequently, $\nu_{\text {dis }}=\nu_{\text {pure }}$.
$\Delta T_{c}^{\text {loc }}$ can be estimated using the central limit theorem. Indeed, as the local $T_{c}^{\text {loc }}$ is determined by an average of a large number of random variables in the block (e.g., the random $J_{i j}$ in the Hamiltonian), its variations decay as the square root of the block volume, $\Delta T_{c}^{\text {loc }} \simeq \xi^{-d / 2}$. On the other hand, $\Delta T_{c} \simeq \xi^{-1 / \nu_{\text {pure }}}$. Therefore,

$$
\begin{equation*}
\Delta T_{c}^{\mathrm{loc}}<\Delta T_{c} \quad \Rightarrow \quad d \nu_{\mathrm{pure}}>2 \tag{5.29}
\end{equation*}
$$

The interpretation of this inequality is the following. If the Harris criterion $d \nu_{\text {pure }}>2$ is fulfilled, the ratio $\Delta T_{c}^{\mathrm{loc}} / \Delta T_{c}$ goes to zero as the critical point is approached. The system looks less and less disordered on larger length scales, the effective disorder strength vanishes right at criticality, and the disordered system features the same critical behaviour as the clean one. An example of a transition that fullfills the Harris criterion is the ferromagnetic transition in a three-dimensional classical Heisenberg model. Its clean correlation length exponent is $\nu_{\text {pure }} \approx 0.69>2 / d=2 / 3$.

In contrast, if $d \nu_{\text {pure }}<2$, the ratio $\Delta T_{c}^{\mathrm{loc}} / \Delta T_{c}$ increases upon approaching the phase transition. The blocks differ more and more on larger length scales. Eventually, some blocks are on one side of the transition while other blocks are on the other side. This makes a uniform sharp phase transition impossible. The clean critical behavior is unstable and the phase transition can be erased or it can remain continuous but with different critical behaviour. More precisely, the disordered system can be in a new universality class featuring a correlation length exponent that fullfills the inequality $d \nu^{\text {dis }}>2$. Many phase transitions in classical disordered systems follow this scenario, for example the threedimensional classical Ising model. Its clean correlation length exponent is $\nu_{\text {pure }} \approx 0.63$ which violates the Harris criterion. In the presence of random-mass disorder, the critical behavior changes and $\nu^{\text {dis }} \approx 0.68$.

In the marginal case $d \nu_{\text {pure }}=2$, more sophisticated methods are required to decide the stability of the clean critical point.

Chayes et al. [52] turned this argument around to show rigorously that for all the continuous phase transitions in presence of disorder, the correlation-length critical exponent of the disordered system, $\nu_{\text {dis }}$ verifies $\nu_{\text {dis }} \geq 2 / d$, independently of whether or not the critical behaviour is the same as in the uniform system and even when the system does not have a uniform analogue.

Finally, note that the Harris criterion $d \nu_{\text {pure }}>2$ applies to uncorrelated or short-range correlated disorder. If the disorder displays long-range correlations in space, the inequality
needs to be modified because the central-limit theorem estimate of $\Delta T_{c}^{\text {loc }}$ changes. Longrange correlated disorder is especially important in quantum phase transitions.

### 5.4.2 The Griffiths phase

The critical temperature of a spin system is usually estimated from the high temperature expansion and the evaluation of its radius of convergence (see App. C.1). However, Griffiths showed that the temperature at which the free-energy of models with quenched disorder starts being non-analytical falls above the critical temperature where the order parameter detaches from zero [49]. The argument applies to models with second order phase transitions.

Griffiths explained his argument using the dilute ferromagnetic Ising model. First, he argued that the critical temperature of the disordered model should decrease for increasing $p$, the probability of empty sites. This is 'intuitively obvious' since no spontaneous magnetisation can occur at a finite temperature if the probability of occupied sites is less than the critical percolation probability at which an 'infinite cluster' first appears. See Fig. 5.4 where the phase diagram of the dilute Ising ferromagnet is shown.


Figure 5.4: The phase diagram of the dilute ferromagnetic Ising model. $p$ is the probability of empty sites in this figure, taken from [50]. With increasing dilution the ordered phase is eventually suppressed.

In the following paragraph we sketch Griffiths' argument and we use his notation in which $p$ is the probability of occupying a site. For any concentration $p<1$ the magnetisation $m$ is not an analytic function of $h$ at $h=0$ at any temperature below $T_{c}^{\text {pure }}$, the critical temperature of the regular Ising model $p=1$. As he explains, this fact is most easily explained for $p<p_{c}$. The magnetisation $m$ per lattice site in the thermodynamic limit has the form

$$
\begin{equation*}
m=\frac{1}{N} \sum_{i=1}^{N}\left\langle s_{i}\right\rangle=\sum_{c} P(c) m(c) \tag{5.30}
\end{equation*}
$$

where $P(c)$ is the probability that a particular site on the lattice belongs to a cluster $c$ that is necessarily finite for $p<p_{c}$, and $m(c)$ is the magnetisation density of the cluster $c$, that is to say $m(c)=N^{-1}(c) \sum_{i \in c}\left\langle s_{i}\right\rangle$ with $N(c)$ the number of sites in the cluster.

Griffiths uses the Yang-Lee theorem, see App. C, to express $m(c)$ as

$$
\begin{equation*}
m(c)=1+\frac{2 z}{N(c)} \sum_{i \in c} \frac{1}{\xi_{i}-z} \quad \text { with } \quad z=e^{-2 \beta h} \tag{5.31}
\end{equation*}
$$

and $\xi_{i}, i=1, \ldots, N(c)$, complex numbers with $\left|\xi_{i}\right|=1$. The total magnetisation density is then of the same form

$$
\begin{equation*}
m=1+z f(z) \quad f(z)=\sum_{i} \eta_{i}\left(\xi_{i}-z\right)^{-1} \tag{5.32}
\end{equation*}
$$

with $\eta_{i}=2 P(c) / N(c)$. He then argues that this form is analytic for $z<1$ but non-analytic at $z=1$ that corresponds to $h=0$.

A more intuitive understand of what is going on in the temperature region above the critical temperature of the disordered model, $T_{c}^{\text {dis }}$, and below the critical temperature the pure one, $T_{c}^{\text {pure }}$, can be reached as follows [50]. The effects of quenched disorder show up already in the paramagnetic phase of finite dimensional systems. Below the critical point of the pure case (no disorder) finite regions of the system can order due to fluctuations in the couplings or, in a dilute ferromagnetic model, they can be regions where all sites are occupied, as shown in Fig. 5.2. As such rare regions are finite-size pieces of the clean system, their spins align parallel to each other below the clean critical temperature $T_{c}^{\text {pure }}$. Because they are of finite size, these regions cannot undergo a true phase transition by themselves, but for temperatures between the actual transition temperature $T_{c}^{\text {dis }}$ and $T_{c}^{\text {pure }}$ they act as large superspins.

Note that using the ideas of percolation theory, one can estimate the scaling of $P(c)$ with its size. Recall the one dimensional case. Take a segment of length $L+2$ on the lattice. A cluster of size $L$ will occupy the internal sites with empty borders with probability $p^{L}(1-p)^{2}$. This is because one needs $L$ contiguous sites to be occupied and its boundary sites be empty. In larger dimensions, this probability will be approximately $p^{L^{d}}(1-p)^{L^{d-1}}$ with the first factor linked to the filled volume and the second to the empty surface. In the large $L$ limit one can make a harsh approximation and use $\simeq \exp \left\{\ln \left[p^{L^{d}}(1-p)^{L^{d-1}}\right]\right\}=$ $\exp \left[\ln p^{L^{d}}+\ln (1-p)^{L^{d-1}}\right] \simeq \exp \left[-c(p) L^{d}\right]$ with $c(p)=\ln 1 / p$.

The sum in eq. (5.30) is made of two contributions. On the one hand, there are the large clusters that are basically frozen at the working temperature. On the other, there are the free spins that belong to small clusters and are easy to flip at the working temperature. Let us focus on the former. Their magnetic moment is proportional to their volume $m(c) \simeq \mu L^{d}$. The energy gain due to their alignment with the field is $\Delta E(c)=-h m(c)=-h \mu L^{d}$ where $h$ is a small uniform field applied to the system, say to measure its susceptibility.


Figure 5.5: Rare regions in a random ferromagnet, figure taken from [50]. On the left, a ferromagnetically ordered region in the paramagnetic bulk ( $T>T_{c}^{\text {dis }}$ ). On the right, a paramagnetic band in a system that is ordered ferromagnetically in a patchwork way ( $T<T_{c}^{\text {dis }}$ ).

The separation of the clusters in the two groups is then controlled by $\Delta E(c)$ : the small clusters with $|\Delta E(c)|<k_{B} T$ can be flipped by thermal fluctuations, and the large clusters with $|\Delta E(c)|>k_{B} T$ and are frozen.

Then effect of the frozen clusters for which $|\Delta E(c)|>k_{B} T$ is then

$$
\begin{equation*}
m(T, h) \approx \sum_{|\Delta E(c)|>k_{B} T} P(c) m(c) \approx \int_{L_{c}}^{\infty} d L e^{-c(p) L^{d}} \mu L^{d} \tag{5.33}
\end{equation*}
$$

and $L_{c}^{d} \approx k_{B} T /(\mu h)$. This integral can be computed by the saddle-point method, see App. A.7, and it is dominated by the lower border. The result is

$$
\begin{equation*}
m(T, h) \approx e^{-\mathrm{c}(p) L_{c}^{d}}=e^{-\mathrm{c}(p) k_{B} T /(\mu h)} \tag{5.34}
\end{equation*}
$$

and this contribution has an essential singularity in the $h \rightarrow 0$ limit.
It is important to note that the clusters that contribute to this integral are rare regions since they occur with probability $P(c) \simeq e^{-c(p) L^{d}}$ that is exponentially small in their volume. Still they are the cause of the non-analytic behaviour of $m(h)$.

The magnetic susceptibility $\chi$ can be analyzed similarly. Each locally ordered rare region makes a Curie contribution $m^{2}(c) / k_{B} T$ to $\chi$. The total rare region susceptibility can therefore be estimated as

$$
\begin{equation*}
\chi(T, h) \sim \int_{L_{c}}^{\infty} d L e^{-\mathrm{c}(p) L^{d}} \mu^{2} L^{2 d} /\left(k_{B} T\right) \approx e^{-\mathrm{c}(p) k_{B} T /(\mu h)} \tag{5.35}
\end{equation*}
$$

This equation shows that the susceptibility of an individual rare region does not increase fast enough to overcome the exponential decay of the rare region probability with increasing size $L$. Consequently, large rare regions only make an exponentially small contribution to the susceptibility.

Rare regions also exist on the ordered side of the transition $T<T_{c}$. One has to consider locally ordered islands inside holes that can fluctuate between up and down because they are only very weakly coupled to the bulk ferromagnet outside the hole, see Fig. 5.5. This
conceptual difference entails a different probability for the rare events as one needs to find a large enough vacancy-rich region around a locally ordered island.

There are therefore slight differences in the resulting Griffiths singularities on the two sides of the transition. In the site-diluted Ising model, the ferromagnetic Griffiths phase comprises all of the ferromagnetic phase for $p>0$. The phase diagram of the dilute ferromagnetic Ising model is sketched in Fig. 5.4.

### 5.4.3 Scenario for the phase transitions

The argument put forward by Harris is based on the effect of disorder on average over the local critical temperatures. The intuitive explanation of the Griffiths phase shows the importance of rare regions on the behaviour of global observables such as the magnetisation or the susceptibility, The analysis of the effect of randomness on the phase transitions should then be refined to take into account the effect of rare regions (tails in the distributions). Different classes of rare regions can be identified according to their dimension $d_{\mathrm{rr}}$. This leaves place for three possibilities for the effect of (still weak in the sense of not having frustration) disorder on the phase transition.

- The rare regions have dimension $d_{\mathrm{rr}}$ smaller than the lower critical dimension of the pure problem, $d_{\mathrm{rr}}<d_{L}$; therefore the critical behaviour is not modified with respect to the one of the clean problem.
- When the rare regions have dimension equal to the lower-critical one, $d_{\mathrm{rr}}=d_{L}$, the critical point is still of second order with conventional power law scaling but with different exponents that vary in the Griffiths phase. At the critical point the Harris criterium is satisfied $d \nu_{\text {dis }}>2$.
- Infinite randomness strength, appearing mostly in problems with correlated disorder, lead to a complete change in the critical properties, with unconventional activated scaling. This occurs when $d_{\mathrm{rr}}>d_{L}$.

In the derivation of this scenario the rare regions are supposed to act independently, with no interactions among them. This picture is therefore limited to systems with shortrange interactions.

### 5.4.4 Domain-wall stiffness and droplets

Let us now just discuss one simple argument that is at the basis of what is needed to derive the results of the droplet theory for spin-glasses without entering into the complications of the calculations.

At very high temperature the configurations are disordered and one does not see large patches of ordered spins.

Close but above the critical temperature $T_{c}$ finite patches of the system are ordered (in all possible low-temperature equilibrium states) but none of these include a finite fraction of the spins in the sample and the magnetization density vanishes. However, these patches are enough to generate non-trivial thermodynamic properties very close to $T_{c}$ and the richness of critical phenomena.

At criticality one observes ordered domains of the two equilibrium states at all length scales - with fractal properties.

Below the critical temperature thermal fluctuations induce the spin reversal with respect to the order selected by the spontaneous symmetry breaking. It is clear that the structure of droplets, meaning patches in which the spins point in the opposite direction to the one of the background ordered state, plays an important role in the thermodynamic behaviour at low temperatures.
M. Fisher and others developed a droplet phenomenological theory for critical phenomena in clean systems. Later D. S. Fisher and D. Huse extended these arguments to describe the effects of quenched disorder in spin-glasses and other random systems; this is the so-called droplet model.

## Domain-wall stiffness

Ordered phases resist spatial variations of their order parameter. This property is called stiffness or rigidity and it is absent in high-temperature disordered phases.

More precisely, in an ordered phase the free-energy cost for changing one part of the system with respect to another part far away is proportional to $k_{B} T$ and usually diverges as a power law of the system size. In a disordered phase the information about the reversed part propagates only a finite distance (of the order of the correlation length, see below) and the stiffness vanishes.

Concretely, the free-energy cost of installing a domain-wall in a system, gives a measure of the stiffness of a phase. The domain wall can be imposed by special boundary conditions. Compare then the free-energy of an Ising model with linear length $L$, in its ordered phase, with periodic and anti-periodic boundary conditions on one Cartesian direction and periodic boundary conditions on the $d-1$ other directions of a $d$-dimensional hypercube. The $\pm$ boundary conditions forces an interface between the regions with positive and negative magnetisations. At $T=0$, the minimum energy interface is a $d-1$ flat hyper-plane and the energy cost is

$$
\begin{equation*}
\Delta E(L) \simeq \sigma L^{\theta} \quad \text { with } \quad \theta=d-1 \tag{5.36}
\end{equation*}
$$

and $\sigma=2 J$ the interfacial energy per unit area or the interfacial tension of the domain wall.

## Droplets - generalisation of the Peierls argument

In an ordered system at finite temperature domain walls, surrounding droplet fluctuations, or domains with reversed spins with respect to the bulk order, are naturally generated by thermal fluctuations. The study of droplet fluctuations is useful to establish
whether an ordered phase can exist at low (but finite) temperatures. One then studies the free-energy cost for creating large droplets with thermal fluctuations that may destabilise the ordered phase, in the way usually done in the simple Ising chain (the Peierls argument).

Indeed, temperature generates fluctuations of different size and the question is whether these are favourable or not. These are the droplet excitations made by simply connected regions (domains) with spins reversed with respect to the ordered state. Because of the surface tension, the minimal energy droplets with linear size or radius $L$ will be compact spherical-like objects with volume $L^{d}$ and surface $L^{d-1}$. The surface determines their energy and, at finite temperature, an entropic contribution has to be taken into account as well. Simplifying, one argues that the free-energy cost is of the order of $L^{\theta}$, that is $L^{d-1}$ in the ferromagnetic case but can be different in disordered systems.

Summarising, in system with symmetry breaking the free-energy cost of an excitation of linear size $L$ is expected to scale as

$$
\begin{equation*}
\Delta F(L) \simeq \sigma(T) L^{\theta} \tag{5.37}
\end{equation*}
$$

The sign of $\theta$ determines whether thermal fluctuations destroy the ordered phase or not. For $\theta>0$ large excitations are costly and very unlikely to occur; the order phase is expected to be stable. For $\theta<0$ instead large scale excitations cost little energy and one can expect that the gain in entropy due to the large choice in the position of these excitations will render the free-energy variation negative. A proliferation of droplets and droplets within droplets is expected and the ordered phase will be destroyed by thermal fluctuations. The case $\theta=0$ is marginal and its analysis needs the use of other methods.

As the phase transitions is approached from below the surface tension $\sigma(T)$ should vanish. Moreover, one expects that the stiffness should be independent of length close to $T_{c}$ and therefore, $\theta_{c}=0$.

Above the transition the stiffness should decay exponentially

$$
\begin{equation*}
\Delta F(L) \simeq e^{-L / \xi} \tag{5.38}
\end{equation*}
$$

with $\xi$ the equilibrium correlation length.

### 5.4.5 Stability of ordered phases

## A ferromagnet under a magnetic field

Let us study the stability properties of an equilibrium ferromagnetic phase under an applied external field that tends to destabilize it. If we set $T=0$ the free-energy is just the energy. In the ferromagnetic case the free-energy cost of a spherical droplet of radius $R$ of the equilibrium phase parallel to the applied field embedded in the dominant one (see Fig. 5.6-left) is

$$
\begin{equation*}
\Delta F(R)=-2 \Omega_{d} R^{d} h m_{\mathrm{eq}}+\Omega_{d-1} R^{d-1} \sigma_{0} \tag{5.39}
\end{equation*}
$$

where $\sigma_{0}$ is the interfacial free-energy density (the energy cost of the domain wall) and $\Omega_{d}$ is the volume of a $d$-dimensional unit sphere. We assume here that the droplet has a regular surface and volume such that they are proportional to $R^{d-1}$ and $R^{d}$, respectively. The excess free-energy reaches a maximum

$$
\begin{equation*}
\Delta F_{c}=\frac{\Omega_{d}}{d} \frac{\Omega_{d-1}^{d}}{\Omega_{d}^{d}}\left(\frac{d-1}{2 d h m_{\mathrm{eq}}}\right)^{d-1} \sigma_{0}^{d} \tag{5.40}
\end{equation*}
$$

at the critical radius

$$
\begin{equation*}
R_{c}=\frac{(d-1) \Omega_{d-1} \sigma_{0}}{2 d \Omega_{d} h m_{\mathrm{eq}}}, \tag{5.41}
\end{equation*}
$$

see Fig. 5.6-right $\left(h>0\right.$ and $m_{\text {eq }}>0$ here, the signs have already been taken into account). The free-energy difference vanishes at

$$
\begin{equation*}
\Delta F\left(R_{0}\right)=0 \quad \Rightarrow \quad R_{0}=\frac{\Omega_{d-1} \sigma_{0}}{2 \Omega_{d} h m_{\mathrm{eq}}} \tag{5.42}
\end{equation*}
$$

Several features are to be stressed:

- The barrier vanishes in $d=1$; indeed, the free-energy is a linear function of $R$ in this case.
- Both $R_{c}$ and $R_{0}$ have the same dependence on $h m_{\text {eq }}$ : they monotonically decrease with increasing $h m_{\mathrm{eq}}$ vanishing for $h m_{\mathrm{eq}} \rightarrow \infty$ and diverging for $h m_{\mathrm{eq}} \rightarrow 0$.
- In dynamic terms that we shall discuss later, the passage above the barrier is done via thermal activation; as soon as the system has reached the height of the barrier it rolls on the right side of 'potential' $\Delta F$ and the favorable phase nucleates.
- As long as the critical size $R_{c}$ is not reached the droplet is not favorable and the system remains positively magnetized.


## The Imry-Ma argument for the random field Ising model

Take a ferromagnetic Ising model in a random field, defined in eq. (5.21). In zero applied field and low enough temperature, if $d>1$ there is a phase transition between a ferromagnetic and a paramagnetic phase at a critical value of the variance of the random fields, $\sigma_{h}^{2}=\left[h_{i}^{2}\right] \propto h^{2}$, that sets the scale of the values that these random fields can take. Under the effect of a random field with very strong typical strength, the spins align with the local external fields that point in both directions and the system is paramagnetic. It is, however, non-trivial to determine the effect of a relatively weak random field on the ferromagnetic phase at sufficiently low temperature. The long-range ferromagnetic order could be preserved or else the field could be enough to break up the system into large but finite domains of the two ferromagnetic phases.


Figure 5.6: Left: the droplet. Right: the free-energy density $f(R)$ of a spherical droplet with radius $R$.

A qualitative argument to decide whether the ferromagnetic phase survives or not in presence of the external random field is due to Imry and Ma [53]. Let us fix $T=0$ and switch on a random field. If a compact domain $\mathcal{D}$ of the opposite order (say down) is created within the bulk of the ordered state (say up) the system pays an energy due to the unsatisfied links lying on the boundary that is

$$
\begin{equation*}
\Delta E_{\text {border }} \sim 2 J R^{d-1} \tag{5.43}
\end{equation*}
$$

where $R$ is the radius of the domain and $d-1$ is the dimension of the border of a domain embedded in $d$ a dimensional volume, assuming the interface is not fractal. By creating a domain boundary the system can also gain a magnetic energy in the interior of the domain due to the external field:

$$
\begin{equation*}
\Delta E_{\text {random field }} \sim-h R^{d / 2} \tag{5.44}
\end{equation*}
$$

since there are $N \propto R^{d}$ spins inside the domain of linear scale $R$ (assuming now that the bulk of the domain is not fractal) and, using the central limit theorem, $-h \sum_{j \in \mathcal{D}} s_{i} \sim$ $-h \sqrt{N} \propto-h R^{d / 2} . h \approx \sigma_{h}$ is the width of the random field distribution.

Dimension lower than two. In $d=1$ the energy difference is a monotonically decreasing function of $R$ thus suggesting that the creation of droplets is very favourable and there is no barrier to cross to do it. Indeed, for any $d<2$, the random field energy increases faster with $R$ than the domain wall energy. Even for weak random fields, there will be a critical $R$ beyond which forming domains that align with the local random field becomes favourable. Consequently, the uniform ferromagnetic state is unstable against domain formation for arbitrary random field strength. In other words, in dimensions $d<2$ random-field disorder prevents spontaneous symmetry breaking.

Dimension larger than two. The functional form of the total energy variation $\Delta E=$ $\Delta E_{\mathrm{border}}+\Delta E_{\text {random field }}$ as a function of $R$ is characterised by $\Delta E \rightarrow 0$ for $R \rightarrow 0$ and
$\Delta E \rightarrow \infty$ for $R \rightarrow \infty$. The function has a minimum at

$$
\begin{equation*}
R_{c} \sim\left(\frac{h d}{4 J(d-1)}\right)^{2 /(d-2)} \tag{5.45}
\end{equation*}
$$

and crosses zero at $R_{0}$ to approach $\infty$ at $R \rightarrow \infty$. The comparison between these two energy scales yields

$$
\begin{equation*}
2 J R_{0}^{d-1} \sim h R_{0}^{d / 2} \quad \Rightarrow \quad R_{0} \sim\left(\frac{h}{2 J}\right)^{\frac{2}{d-2}} \tag{5.46}
\end{equation*}
$$

This equation clearly shows a change in $d=2$, with

$$
\lim _{h / J \rightarrow 0} R_{0}(h / J)=\left\{\begin{array}{lll}
0 & \text { if } & d>2  \tag{5.47}\\
\infty & \text { if } & d<2
\end{array}\right.
$$

Therefore, in $d>2$ the energy difference first decreases from $\Delta E(R=0)=0$ to reach a negative minimum at $R_{c}$, and then increases back to pass through zero at $R_{0}$ and diverge at infinity. This indicates that the creation of domains at zero temperature is not favourable in $d>2$. Just domains of finite length, up to $R_{0}$ can be created. Note that $R_{0}$ increases with $h / J$ in $d>2$. Therefore, a higher field tends to generate larger droplets and thus disorder more the sample.

The marginal case $d=2$ is more subtle and more powerful techniques are needed to decide.

With this argument one cannot show the existence of a phase transition at $h_{c}$ nor the nature of it. The argument is such that it suggests that order can be supported by the system at zero temperature and small fields in $d>2$.

Again, we stress that these results hold for short-range correlated disorder.
There are rigorous proofs that random fields destroy long-range order (and thus prevent spontaneous symmetry breaking) in all dimensions $d \leq 2$ for discrete (Ising) symmetry and in dimensions $d \leq 4$ for continuous (Heisenberg) symmetry. The existence of a phase transition from a FM to a PM state at zero temperature in $3 d$ was shown in [54].

An elastic line in a random potential
The interfacial tension, $\sigma$, will tend to make an interface, forced into a system as flat as possible. However, this will be resisted by thermal fluctuations and, in a system with random impurities, by quenched disorder.

Let us take an interface model of the type defined in eq. (5.23) with $N=1$. If one assumes that the interface makes an excursion of longitudinal length $L$ and transverse length $\phi$ the elastic energy cost is

$$
\begin{equation*}
E_{\text {elast }}=\frac{c}{2} \int d^{d} x(\nabla \phi(\vec{x}))^{2} \quad \Rightarrow \quad \Delta E_{\text {elast }} \sim c L^{d}\left(L^{-1} \phi\right)^{2}=c L^{d-2} \phi^{2} \tag{5.48}
\end{equation*}
$$

Ignore for the moment the random potential. Thermal fluctuations cause fluctuations of the kind shown in Fig. 5.7. The interfaces roughens, that is to say, it deviates from


Figure 5.7: Illustration of an interface modeled as a directed manifold. In the example, the domain wall separates a region with positive magnetisation (above) from one with negative magnetisation (below). The line represents a lowest energy configuration that deviates from a flat one due to the quenched randomness. An excitation on a length-scale $L$ is shown with a dashed line. The relative displacement is $\delta h \equiv \delta \phi \simeq L^{\alpha}$ and the excitation energy $\Delta E(L) \simeq L^{\theta}$. Figure taken from [55].
being flat. Its mean-square displacement between two point $\vec{x}$ and $\vec{y}$, or its width on a scale $L$ satisfies

$$
\begin{equation*}
\left\langle[\phi(\vec{x})-\phi(\vec{y})]^{2}\right\rangle \simeq T|\vec{x}-\vec{y}|^{2 \zeta_{T}} \tag{5.49}
\end{equation*}
$$

with $\zeta_{T}$ the roughness exponent.
The elastic energy cost of an excitation of length $L$ is then

$$
\begin{equation*}
\Delta E_{\text {elast }}(L) \simeq c L^{d-2} \phi^{2}(L) \simeq c T L^{d-2} L^{2 \zeta_{T}} \tag{5.50}
\end{equation*}
$$

and this is of order one if

$$
\begin{equation*}
\zeta_{T}=\frac{2-d}{2} . \tag{5.51}
\end{equation*}
$$

In the presence of quenched randomness, the deformation energy cost competes with gains in energy obtained from finding more optimal regions of the random potential. Naively, the energy gain due to the randomness is

$$
\begin{equation*}
\int d^{d} x V \simeq\left[W^{2} L^{d}\right]^{1 / 2} \simeq W L^{d / 2} \tag{5.52}
\end{equation*}
$$

and the balance with the elastic cost, assumed to be the same as with no disorder, yields

$$
\begin{equation*}
c T L^{d-2} L^{2 \zeta_{D}} \simeq W L^{d / 2} \quad \Rightarrow \quad \zeta_{D}=\frac{4-d}{2} \tag{5.53}
\end{equation*}
$$

This result turns out to be an upper bound of the exponent value [55]. It is called the Flory exponent for the roughness of the surface. One then concludes that for $d>4$ disorder is


Figure 5.8: The interface width and the roughness exponent in a magnetic domain wall in a thin film. The value measured $\zeta_{D} \simeq 0.6$ is compatible with the Flory value $2 / 3$ expected for a one dimensional domain wall in a two dimensional space ( $N=1$ and $d=1$ in the calculations discussed in the text.) [56].
irrelevant and the interface is flat $(\phi \rightarrow 0$ when $L \rightarrow \infty)$. Since the linearization of the elastic energy [see the discussion leading to eq. (5.23)] holds only if $\phi / L \ll 1$, the result (5.53) may hold only for $d>1$ where $\alpha<1$.

Destruction of first order phase transitions under randomness
A first order phase transition is characterized by macroscopic phase coexistence at the transition point. For example, at the liquid-gas phase transition of a fluid, a macroscopic liquid phase coexists with a macroscopic vapour phase. Random-mass disorder locally favors one phase over the other. The question is whether the macroscopic phases survives in the presence of disorder or the system forms domains (droplets) that follow the local value of the random-mass.

Consider a single domain or droplet (of linear size $L$ ) of one phase embedded in the other phase. The free energy cost due to forming the surface is

$$
\begin{equation*}
\Delta F_{\text {surf }} \sim \sigma L^{d-1} \tag{5.54}
\end{equation*}
$$

where $\sigma$ is the surface energy between the two phases. The energy gain from the randommass disorder can be estimated via the central limit theorem, resulting in a typical magnitude of

$$
\begin{equation*}
\left|\Delta F_{\mathrm{dis}}\right| \sim W^{1 / 2} L^{d / 2} \tag{5.55}
\end{equation*}
$$

where $W$ is the variance of the random-mass disorder.
The macroscopic phases are stable if $\left|\Delta F_{\text {dis }}\right|<\Delta F_{\text {surf }}$, but this is impossible in dimensions $d \leq 2$ no matter how weak the disorder is. In dimensions $d>2$, phase coexistence is possible for weak disorder but will be destabilized for sufficiently strong disorder.

We thus conclude that random-mass disorder destroys first-order phase transitions in dimensions $d \leq 2$. In many examples, the first-order transition is replaced by ('rounded to') a continuous one, but more complicated scenarios cannot be excluded.

## The 3d Edwards-Anderson model in a uniform magnetic field

A very similar reasoning is used to argue that there cannot be spin-glass order in an Edwards-Anderson model in an external field [69, 70]. The only difference is that the domain wall energy is here assumed to be proportional to $L^{y}$ with an a priori unknown $d$-dependent exponent $y$ that is related to the geometry of the domains.

## Comments

These arguments are easy to implement when one knows the equilibrium states (or one assumes what they are). They cannot be used in models in which the energy is not a slowly varying function of the domain wall position.

### 5.4.6 Consequences of the gauge invariace

H. Nishimori used the gauge transformation explained in Sec. 3.2 to derive a series of exact results for averaged observables of finite dimensional disordered systems [46].

The idea follows the steps by which one easily proves, for example, that the averaged local magnetization of a ferromagnetic Ising model vanishes, that is to say, one applies a transformation of variables within the partition sum and evaluates the consequences over the averaged observables. For example,

$$
\begin{equation*}
\left\langle s_{i}\right\rangle=\sum_{\left\{s_{j}\right\}} s_{i} e^{\beta J \sum_{i j} s_{i} s_{j}}=\left\langle s_{i}\right\rangle=\sum_{\left\{s_{j}\right\}}\left(-s_{i}\right) e^{\beta J \sum_{i j} s_{i} s_{j}}=-\left\langle s_{i}\right\rangle . \tag{5.56}
\end{equation*}
$$

This immediately implies $\left\langle s_{i}\right\rangle=0$ and, more generally, the fact that the average of any odd function under $\left\{s_{i}\right\} \rightarrow\left\{-s_{i}\right\}$ vanishes exactly.

In the case of disordered systems, one is interested in observables that are averaged over the random variables weighted with their probability distribution. The gauge transformation that leaves the Hamiltonian unchanged involves a change of spins accompanied by a transformation of the exchanges:

$$
\begin{equation*}
\bar{s}_{i}=\eta_{i} s_{i} \quad \bar{J}_{i j}=\eta_{i} \eta_{j} J_{i j} \tag{5.57}
\end{equation*}
$$

with $\eta_{i}= \pm 1$. The latter affects their probability distribution as this one, in general, is not gauge invariant. For instance, the bimodal pdf $P\left(J_{i j}\right)=p \delta\left(J_{i j}-J\right)+(1-p) \delta\left(J_{i j}+J\right)$ can be rewritten as

$$
\begin{equation*}
P\left(J_{i j}\right)=\frac{e^{K_{p} J_{i j} / J}}{2 \cosh K_{p}} \quad \text { with } \quad e^{2 K_{p}}=\frac{p}{1-p} \tag{5.58}
\end{equation*}
$$

as one can simply check. $\tau_{i j} \equiv J_{i j} / J$ are just the signs of the $J_{i j}$. Under the gauge transformation $P\left(J_{i j}\right)$ transforms as

$$
\begin{equation*}
\bar{P}\left(\bar{J}_{i j}\right) d \bar{J}_{i j}=P\left(J_{i j}\right) d J_{i j} \quad \Rightarrow \quad \bar{P}\left(\bar{J}_{i j}\right)=P\left(J_{i j}\left(\bar{J}_{i j}\right)\right) \frac{d J_{i j}}{d \bar{J}_{i j}} \tag{5.59}
\end{equation*}
$$

that implies

$$
\begin{equation*}
\bar{P}\left(\bar{J}_{i j}\right)=\frac{e^{K_{p} \bar{J}_{i j} /\left(\eta_{i} \eta_{j} J\right)}}{2 \cosh K_{p}} \frac{1}{\eta_{i} \eta_{j}} \quad \Rightarrow \quad \bar{P}\left(\bar{J}_{i j}\right)=\eta_{i} \eta_{j} \frac{e^{K_{p} \bar{J}_{i j} \eta_{i} \eta_{j} / J}}{2 \cosh K_{p}} \tag{5.60}
\end{equation*}
$$

For instance, applying the gauge transformation to the internal energy of an Ising spinglass model with bimodal disorder, after a series of straightforward transformations one finds

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=-N_{B} J \tanh K_{p} \tag{5.61}
\end{equation*}
$$

with $N_{B}$ the number of bonds in the lattice, under the condition $\beta J=K_{p}$. This relation holds for any lattice. The constraint $\beta J=K_{p}$ relates the inverse temperature $J /\left(k_{B} T\right)$ and the probability $p=\left(\tanh K_{p}+1\right) / 2$. The curve $\beta J=K_{p}$ connects the points $(p=1, T=0)$ and $(p=1 / 2, T \rightarrow \infty)$ in the $(p, T)$ phase diagram and it is called the Nishimori line.

The proof of the relation above goes as follows. The full pdf of the interactions is

$$
\begin{equation*}
P\left(\left\{J_{i j}\right\}\right)=\prod_{\langle i j\rangle} P\left(J_{i j}\right) \tag{5.62}
\end{equation*}
$$

and the average of any disorder dependent quantity is expressed as

$$
\begin{equation*}
\left[A_{J}\right]=\sum_{\left\{J_{i j}= \pm J\right\}} \prod_{\langle i j\rangle} P\left(J_{i j}\right) A_{J} \tag{5.63}
\end{equation*}
$$

The disorder average Hamiltonian reads

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=\sum_{\left\{J_{i j}\right\}} \frac{e^{K_{p} \sum_{\langle i j\rangle} J_{i j} / J}}{\left(2 \cosh K_{p}\right)^{N_{B}}} \frac{\sum_{\left\{s_{i}\right\}}\left(-\sum_{i j} J_{i j} s_{i} s_{j}\right) e^{\beta \sum_{\langle i j\rangle} J_{i j} s_{i} s_{j}}}{\sum_{\left\{s_{i}\right\}} e^{\beta \sum_{\langle i j\rangle} J_{i j} s_{i} s_{j}}} \tag{5.64}
\end{equation*}
$$

with $N_{B}$ the number of bonds in the graph or lattice. Performing the gauge transformation

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=\sum_{\left\{J_{i j}\right\}} \frac{e^{K_{p} \sum_{\langle i j\rangle} J_{i j} \eta_{i} \eta_{j} / J}}{\left(2 \cosh K_{p}\right)^{N_{B}}} \frac{\sum_{\left\{s_{i}\right\}}\left(-\sum_{i j} J_{i j} s_{i} s_{j}\right) e^{\beta \sum_{\langle i j\rangle} J_{i j} s_{i} s_{j}}}{\sum_{\left\{s_{i}\right\}} e^{\beta \sum_{\langle i j\rangle} i_{j j} s_{i} s_{j}}} \tag{5.65}
\end{equation*}
$$

where gauge invariance of the Hamiltonian has been used and the spins and interactions have been renamed $J_{i j}$ and $s_{i}$. As this is independent of the choice of the parameters $\left\{\eta_{i}\right\}$
used in the transformation, one can sum over all possible $2^{N}$ choices and divide by this number keeping the result unchanged:

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=\frac{1}{2^{N}} \sum_{\left\{J_{i j}\right\}} \frac{\sum_{\left\{\eta_{i}\right\}} e^{K_{p} \sum_{\langle i j\rangle} J_{i j} \eta_{i} \eta_{j} / J}}{\left(2 \cosh K_{p}\right)^{N_{B}}} \frac{\sum_{\left\{s_{i}\right\}}\left(-\sum_{i j} J_{i j} s_{i} s_{j}\right) e^{\beta \sum_{\langle i j\rangle} J_{i j} s_{i} s_{j}}}{\sum_{\left\{s_{i}\right\}} e^{\beta \sum_{\langle i j\rangle} i_{i j} s_{i} s_{j}}} \tag{5.66}
\end{equation*}
$$

If $\beta$ is chosen to be $\beta=K_{p} / J$ the sum over the spins in the denominator (the partition sum in the normalisation) cancels out the sum over the parameters $\eta_{i}$ introduced via the gauge transformation. The sum over $J_{i j}$ and the remaining sum over the spin configurations can be rewritten

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=\frac{1}{2^{N}} \frac{1}{\left(2 \cosh K_{p}\right)^{N_{B}}}\left(-\frac{\partial}{\partial \beta}\right) \sum_{\left\{s_{i}\right\}} \prod_{\langle i j\rangle} \sum_{\left\{J_{i j}= \pm J\right\}} e^{\beta J_{i j} s_{i} s_{j}} \tag{5.67}
\end{equation*}
$$

Changing now variables in the sum over $J_{i j}= \pm J$ to $\tau_{i j}=J_{i j} s_{i} s_{j}= \pm J$,

$$
\begin{align*}
{\left[\left\langle H_{J}\right\rangle\right]_{J} } & =\frac{1}{2^{N}} \frac{1}{\left(2 \cosh K_{p}\right)^{N_{B}}}\left(-\frac{\partial}{\partial \beta}\right) \sum_{\left\{s_{i}\right\}} \prod_{\langle i j\rangle} \sum_{\tau_{i j}= \pm J} e^{\beta \tau_{i j}} \\
& =\frac{1}{2^{N}} \frac{1}{\left(2 \cosh K_{p}\right)^{N_{B}}}\left(-\frac{\partial}{\partial \beta}\right) 2^{N}\left(2 \cosh K_{p}\right)^{N_{B}} \tag{5.68}
\end{align*}
$$

where the sum over the spin configurations yields the $2^{N}$ factor and the sum over the independent $\tau_{i j}$ configurations yields the last factor. Finally, taking the derivative with respect to $\beta$ :

$$
\begin{equation*}
\left[\left\langle H_{J}\right\rangle\right]_{J}=-N_{B} J \tanh K_{p} \tag{5.69}
\end{equation*}
$$

with $K_{p}=\beta J$, defining the Nishimori line in the phase diagram.
For Gaussian distributed quenched randomness there also exists a Nishimori line and the averaged internal energy can also be computed exactly on this line.

Many other relations of this kind exist and are explained in [46].

## 6 Solvable disordered models

There are few solvable disordered models. Even if rather far from describing realistic systems in detail, these models are of great help to test several features of disordered systems that we expect to find in more realistic cases.

In this Section we describe the static properties of a family of solvable models that include the spherical ferromagnet and spin-glass. These models illustrate a mechanism for slow relaxation that is due to the existence of saddles and flat directions in phase space.

### 6.1 Spherical spin models

Spherical spin models are not very realistic but have the advantage of rendering the models easy to solve analytically. In the spherical approximation the Ising constraint is relaxed and the individual spins are taken to be unbounded continuous variables $-\infty \leq$ $s_{i} \leq \infty$ subject to the global constraint $\sum_{i=1}^{N} s_{i}^{2}=N$ that is imposed on average. One can then represent the configuration of the system with an $N$-dimensional vector, $\vec{s}=$ $\left\{s_{1}, \ldots, s_{N}\right\}$, pointing on an $N$-dimensional sphere with radius $\sqrt{N}$. The spherical model with generic two-body interactions in a local magnetic field is defined by the quadratic Hamiltonian

$$
\begin{equation*}
H_{J}=-\frac{1}{2} \sum_{i \neq j} J_{i j} s_{i} s_{j}-\sum_{i} h_{i} s_{i} \tag{6.1}
\end{equation*}
$$

The first sum is over all distinct pairs of spins and the interactions $J_{i j}$ are symmetric but otherwise arbitrary.

The spherical constraint is enforced by an adding an extra term to the energy

$$
\begin{equation*}
H_{J} \rightarrow H_{J}+\frac{z}{2}\left(\sum_{i=1}^{N} s_{i}^{2}-N\right) \tag{6.2}
\end{equation*}
$$

with $z$ a complex Lagrange multiplier. In this way, the constrained is enforced on average and not strictly, as in the partition sum one sums over all configurations of the spins and not only over the ones on the sphere.

We shall see below that the density of eigenvalues of the interaction matrix $J_{i j}$ determines the phase transition and most of the static and dynamic properties of the spherical models. All density of states with a finite support, $\rho\left(\lambda_{\mu}\right) \neq 0$ in $\left[\lambda_{\text {min }}, \lambda_{\text {max }}\right]$ lead to similar static and dynamic behaviours while the ones with long tails yield a rather different phenomenology.

One can now distinguish ordered and disordered spherical spin models. The spherical ferromagnet introduced by Berlin and Kac [58] is such that the spins lie on the vertices
of a cubic $d$ dimensional lattice with lattice spacing $a$ that one usually sets to one. The interactions are ferromagnetic nearest-neighbour couplings with strength, say, unity. In the limit $N \rightarrow \infty$ the density of eigenvalues $\lambda_{\mu}$ of the corresponding interaction matrix $J_{i j}$ is

$$
\begin{equation*}
\rho\left(\lambda_{\mu}\right)=\pi^{-1} \int_{0}^{\infty} d y \cos \left(\lambda_{\mu} y\right)\left[J_{o}(2 y)\right]^{d} \theta\left(2 d-\left|\lambda_{\mu}\right|\right) \tag{6.3}
\end{equation*}
$$

where $J_{o}(y)$ is the zero-th order Bessel function. The definition of spherical antiferromagnets is slightly more complicated but is is also possible.

In the disordered case the interactions $J_{i j}$ are taken from a probability distribution. Since one is usually interested in describing the spin-glass state its average, $\left[J_{i j}\right]$, is set to zero. The scaling of its variance, $\left[J_{i j}^{2}\right]$, is chosen in such a way to have a sensible thermodynamic limit.

If the model is fully connected, meaning that all entries $J_{i j}$ are typically different from zero, the variance scales as $J^{2} /(2 N)$. One such model is the one with a Gaussian distribution of exchanges and it was introduced by Kosterlitz, Thouless and Jones [59] as a spherical spin-glass (although, as we will see later, it is not really a spin-glass). When $N \rightarrow \infty$ the eigenvalues of a typical member of this Gaussian orthogonal ensemble, that we call $\lambda_{\mu}$, with $\mu=1, \ldots, N$, are distributed according to the Wigner semi-circle law $[60]^{3}$,

$$
\begin{equation*}
\rho\left(\lambda_{\mu}\right)=\frac{1}{2 \pi J} \sqrt{4 J^{2}-\lambda_{\mu}^{2}} \theta\left(2 J-\left|\lambda_{\mu}\right|\right) . \tag{6.4}
\end{equation*}
$$

In the following we measure temperature in units of the interaction strength $J$ and thus we set $J=1$.

A dilute system in which each spin interacts with only a finite fraction of other ones in the sample is modelled with

$$
\begin{equation*}
P\left(J_{i j}\right)=(1-p / N) \delta\left(J_{i j}\right)+p / N \rho\left(J_{i j}\right) . \tag{6.5}
\end{equation*}
$$

One can visualize this model as one with the spins occupying the vertices of a random graph with average connectivity $p$. When $p \rightarrow N$ one recovers the complete graph and the fully-connected case. If $\rho\left(J_{i j}\right)$ has support on positive values of $J_{i j}$ only one has a dilute random ferromagnet. If $\rho\left(J_{i j}\right)$ is Gaussian centred in zero one has a dilute spin-glass. In this case the density of eigenvalues has a symmetric central band in $\left[-\lambda_{c}(p), \lambda_{c}(p)\right]$, a crossover extending beyond $\left|\lambda_{c}(p)\right|$ that is not known in detail, and two tails that vanish as $\rho\left(\lambda_{\mu}\right) \sim \exp \left[-p \lambda_{\mu}^{2} \ln \lambda_{\mu}^{2}\right]$ when $\lambda_{\mu} \rightarrow \pm \infty$. The tails are due to large fluctuations of the local connectivity. For $k \gg 2 p$ a site with $k$ neighbors gives rise to an eigenvalue $\lambda_{\mu} \sim \sqrt{k / p}$ with a localized eigenvector $\vec{v}_{\mu}$ on it. When $p \rightarrow N \rightarrow \infty, \lambda_{c}(p) \rightarrow \lambda_{\max }=$ $\max \left\{\lambda_{1}, \ldots, \lambda_{N}\right\}$ and the tails disappear [62].

The magnetic field $h_{i}$ might be quenched and random, uniform and stationary, or time-dependent.

[^2]
### 6.1.1 The potential energy landscape

Let us label the eigenvalues of $J_{i j}$ in such a way that they are ordered: $\lambda_{1} \leq \lambda_{2} \leq \cdots \leq$ $\lambda_{N}$. We call their associated eigenvectors $\pm \vec{v}_{\mu}$ with $\mu=1, \ldots, N$. (We take orthonormal eigenvectors such that $v_{\mu}^{2}=1$.) In the absence of a magnetic field, all eigenstates of the interaction matrix are stationary points of the energy hyper-surface,

$$
\left.\frac{\partial H_{J}}{\partial s_{i}}\right|_{\vec{s}^{*}}=-\sum_{j(\neq i)}^{N} J_{i j} s_{j}+\left.z s_{i}\right|_{\vec{s}^{*}}=0 \quad \forall i, \quad \Rightarrow \quad \vec{s}^{*}= \pm \sqrt{N} \vec{v}_{\mu}, z^{*}=\lambda_{\mu}, \forall \mu
$$

These stationary points are the metastable states in the models and their number is linear in $N$, the number of spins.

The Hessian of the potential energy surface on each stationary point is

$$
\begin{equation*}
\left.\frac{\partial H_{J}}{\partial s_{i} \partial s_{j}}\right|_{\vec{S}^{*}}=-J_{i j}+\left.z \delta_{i j}\right|_{\vec{s}^{*}}=-J_{i j}+\lambda_{\mu} \delta_{i j} \tag{6.6}
\end{equation*}
$$

This matrix can be easily diagonalized, one finds $D_{\nu \eta}=\left(-\lambda_{\nu}+\lambda_{\mu}\right) \delta_{\nu \eta}$. Thus, on the stationary point, $\vec{s}^{*}= \pm \vec{v}_{\mu}$, the Hessian has one vanishing eigenvalue (when $\nu=\mu$ ), $\mu-1$ positive eigenvalues (when $\nu<\mu$ ), and $N-\mu$ negative eigenvalues (when $\nu>\mu$ ). Positive (negative) eigenvalues of the Hessian indicate stable (unstable) directions. This implies that each saddle point labeled by $\mu$ has one marginally stable direction, $\mu-1$ stable directions and $N-\mu$ unstable directions. (In other words, the number of stable directions plus the marginally stable one is given by the index $\mu$ labelling the eigenvalue associated to the stationary state.) In conclusion, there are two maxima, $\vec{s}^{*}= \pm \sqrt{N} \vec{v}_{1}$, in general two saddles $\vec{s}^{*}= \pm \sqrt{N} \vec{v}_{I}$ with $I=\mu-1$ stable directions and $N-I$ unstable ones, with $I$ running with $\mu$ as $I=\mu-1$ and $\mu=2, \ldots, N$ and finally two (marginally stable) minima, $\vec{s}^{*}= \pm \sqrt{N} \vec{v}_{N}$. In the large $N$ limit the density of eigenvalues of the Hessian at each metastable state $\mu$ is a translated semi circle law.

The energy of a generic configuration under no applied field is

$$
\begin{equation*}
H_{J}=-\frac{1}{2} \sum_{\mu}\left(\lambda_{\mu}-z\right) s_{\mu}^{2}-\frac{z^{*}}{2} N . \tag{6.7}
\end{equation*}
$$

The zero-temperature energy-density of each stationary point is

$$
\begin{equation*}
H_{J}^{*}=-\frac{1}{2}\left(\lambda_{\mu}-z^{*}\right) s^{* 2}-\frac{z^{*}}{2} N=-\frac{1}{2} \lambda_{\mu} N . \tag{6.8}
\end{equation*}
$$

The energy difference between the minima and the lowest saddles depends on the distribution of eigenvalues.

A magnetic field reduces the number of stationary points from a macroscopic number to just two. Indeed, the stationary state equation now reads

$$
\left.\frac{\partial H_{J}}{\partial s_{i}}\right|_{\vec{s}^{*}}=-\sum_{j(\neq i)}^{N} J_{i j} s_{j}+z s_{i}-\left.h_{i}\right|_{\vec{s}^{*}}=0, \forall i, \quad \Rightarrow \quad s_{i}^{*}=\left(z^{*}-J\right)_{i j}^{-1} h_{j}
$$

and $z^{*}$ is fixed by imposing the spherical constraint on $\vec{s}^{*}$. One then finds two solutions for the Lagrange multiplier that lie outside the interval of variation of the eigenvalues of the $J_{i j}$ matrix: $\left|z^{*}\right|>\lambda_{N}$. The stability analysis shows that the stationary points are just one fully stable minimum and a fully unstable maximum. The elimination of the saddle-points has important consequences on the dynamics of the system.

### 6.1.2 The free-energy density

The partition function reads

$$
Z_{J}=\prod_{i=1}^{N} \int_{-\infty}^{\infty} d s_{i} e^{\beta / 2 \sum_{i \neq j} J_{i j} s_{i} s_{j}+\beta \sum_{i} h_{i} s_{i}} \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} d z e^{-\frac{\beta z}{2}\left(\sum_{i=1}^{N} s_{i}^{2}-N\right)}
$$

where $c$ is a real constant to be fixed below.
It is convenient to diagonalise the matrix $J_{i j}$ with an orthogonal transformation and write the exponent in terms of the projection of the spin vector $\vec{s}$ on the eigenvectors of $J_{i j}, s_{\mu} \equiv \vec{s} \cdot \vec{v}_{\mu}$ This operation can be done for any particular realisation of the interaction matrix. In the disordered case this means that one uses a fixed realisation of the random exchanges. The new variables $s_{\mu}$ are also continuous and unbounded and the partition function can be recast as

$$
\begin{equation*}
Z_{J}=\prod_{\mu=1}^{N} \int_{-\infty}^{\infty} d s_{\mu} \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} d z e^{\sum_{\mu=1}^{N} \beta\left(\lambda_{\mu}-z\right) s_{\mu}^{2} / 2+\beta \sum_{\mu=1}^{N} h_{\mu} s_{\mu}+\beta z N / 2} \tag{6.9}
\end{equation*}
$$

with $h_{\mu} \equiv \vec{h} \cdot \vec{v}_{\mu}$ and $\vec{h}=\left(h_{1}, \ldots, h_{N}\right)$. Assuming that one can exchange the quadratic integration over $s_{\mu}$ with the one over the Lagrange multiplier, and that $c$ is such that the influence of eigenvalues $\lambda_{\mu}>c$ is negligible, one obtains

$$
\begin{equation*}
Z_{J}=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} d z e^{-N\left[-\beta z / 2+(2 N)^{-1} \sum_{\mu} \ln \left[\beta\left(z-\lambda_{\mu}\right) / 2\right]-\beta N^{-1} \sum_{\mu}\left(z-\lambda_{\mu}\right)^{-1} h_{\mu}^{2}\right] .} \tag{6.10}
\end{equation*}
$$

In the saddle-point approximation the Lagrange multiplier is given by

$$
\begin{equation*}
1=\left\langle\left\langle k_{B} T\left(z_{s p}-\lambda_{\mu}\right)^{-1}+h_{\mu}^{2}\left(z_{s p}-\lambda_{\mu}\right)^{-2}\right\rangle\right\rangle \tag{6.11}
\end{equation*}
$$

and this equation determines the different phases in the model. We indicate with double brackets the sum over the eigenvalues of the matrix $J_{i j}$ that in the limit $N \rightarrow \infty$ can be traded for an integration over its density:

$$
\begin{equation*}
\frac{1}{N} \sum_{\mu=1}^{N} g\left(\lambda_{\mu}\right)=\int d \lambda_{\mu} \rho\left(\lambda_{\mu}\right) g\left(\lambda_{\mu}\right)=\left\langle\left\langle g\left(\lambda_{\mu}\right)\right\rangle\right\rangle \tag{6.12}
\end{equation*}
$$

Let us first discuss the problem in the absence of a magnetic field. The high temperature solution can be smoothly continued to lower temperatures until the critical temperature,

$$
\begin{equation*}
\left(k_{B} T_{c}\right)^{-1}=\left\langle\left\langle\left(z_{s p}-\lambda_{\mu}\right)^{-1}\right\rangle\right\rangle \tag{6.13}
\end{equation*}
$$

is reached where $z_{s p}$ reaches the value of the maximum eigenvalue and it sticks to this value for all $T<T_{c}$ :

$$
\begin{equation*}
z_{s p}=\lambda_{\max } \quad T \leq T_{c} \tag{6.14}
\end{equation*}
$$

A magnetic field with a component on the largest eigenvalue, $\vec{h} \cdot \vec{v}_{\text {max }} \neq 0$, acts as an ordering field and erases the phase transition.

The thermal average of the projection of the spin vector on each eigenvalue vanishes in the high temperature phase and reads

$$
\left\langle s_{\mu}\right\rangle= \begin{cases}{\left[N\left(1-T / T_{s}\right)\right]^{\frac{1}{2}}} & \lambda_{\mu}=\lambda_{\max }  \tag{6.15}\\ 0 & \lambda_{\mu}<\lambda_{\max }\end{cases}
$$

below the phase transition. The configuration condenses onto the eigenvector associated to the largest eigenvalue of the exchange matrix that carries a weight proportional to $\sqrt{N}$. The mean magnetization per site is zero at all temperatures but the thermal average of the square of the local magnetisation (that will define the Edwards-Anderson parameter) is not when $T<T_{c}$ :

$$
\begin{equation*}
\left\langle m_{i}^{2}\right\rangle=1-T / T_{c} \Rightarrow q_{E A} \equiv\left[\left\langle m_{i}^{2}\right\rangle\right]_{J}=1-T / T_{c} . \tag{6.16}
\end{equation*}
$$

The condensation phenomenon occurs for any distribution of exchanges with a finite support. If the distribution has long tails the energy density diverges and the behaviour is more subtle.

The order parameters $m$ (for the ferromagnets) or $q_{E A}$ (for the spin-glass) vanish at $T_{c}$. The static transition is of second order. All thermodynamic properties can be computed from the free-energy density. In particular, the low temperature asymptotic energy-density is

$$
\begin{equation*}
\mathcal{E}_{e q}=\frac{1}{2}\left(k_{B} T-z_{s p}\right)=\frac{1}{2}\left(k_{B} T-\lambda_{\max }\right), \tag{6.17}
\end{equation*}
$$

and the entropy diverges at low temperatures as $\ln T$, just as for the classical ideal gas, as usual in classical continuous spin models.

The disordered averaged free-energy density can also be computed using the replica trick that we will introduce later. When $N \rightarrow \infty$ a replica symmetric Ansatz yields a marginally stable solution with identical physical properties to the ones discussed above. For this model the replica approach is exact.

### 6.2 The $\mathrm{O}(\mathcal{N})$ model

In this case the spins are generalised to have $\mathcal{N}$ components and the large $\mathcal{N}$ limit is taken. More precisely, the Hamiltonian is given by

$$
\begin{equation*}
H_{J}=-\sum_{\langle i j\rangle} J_{i j} \vec{s}_{i} \cdot \vec{s}_{j}-\sum_{i} \vec{h}_{i} \vec{s}_{i} \tag{6.18}
\end{equation*}
$$

where the spins

$$
\begin{equation*}
\vec{s}_{i}=\left(s_{i}, \ldots, s_{i}^{\mathcal{N}}\right) \tag{6.19}
\end{equation*}
$$

have $\mathcal{N}$ components and length $\mathcal{N}^{1 / 2}$

$$
\begin{equation*}
\sum_{a=1}^{\mathcal{N}}\left(s_{i}^{a}\right)^{2}=\mathcal{N} \tag{6.20}
\end{equation*}
$$

In the ferromagnetic finite $d$ case, this procedure defines the celebrated $\mathrm{O}(\mathcal{N})$ model, that becomes fully solvable in the large $\mathcal{N} \rightarrow \infty$ limit.

The large $\mathcal{N}$ limit is usually taken in the field theoretical (coarse-grained GinzburgLandau) representation of the free-energy

$$
\begin{equation*}
F[\phi]=\int d^{d} x\left[\frac{1}{2}(\vec{\nabla} \vec{\phi})^{2}+\frac{m_{0}}{2}(\vec{\phi})^{2}+\lambda_{0}\left((\vec{\phi})^{2}\right)^{2}\right] \tag{6.21}
\end{equation*}
$$

where $\vec{\phi}=\left(\phi_{1}, \ldots, \phi_{\mathcal{N}}\right)$,

$$
\begin{align*}
\vec{\phi}^{2} & =\sum_{\alpha=1}^{\mathcal{N}} \phi_{\alpha}^{2} \\
(\vec{\nabla} \vec{\phi})^{2} & =\sum_{a=1}^{d} \sum_{\alpha=1}^{\mathcal{N}} \frac{\partial \phi_{\alpha}}{\partial x_{a}} \frac{\partial \phi_{\alpha}}{\partial x_{a}} \tag{6.22}
\end{align*}
$$

and the vector position in $d$ dimensions is $\vec{x}=\left(x_{1}, \ldots, x_{d}\right)$. We have not included randomness here. This can be done by including a random potential $V[\vec{\phi}]$, for example. The cases $\mathcal{N}=2$ and $\mathcal{N}=3$ correspond to the XY and Heisenberg models, respectively. The mean field theory for this model yields critical exponents which are independent of $\mathcal{N}$, but the renormalisation group below $d=4$ gives $\mathcal{N}$-dependent results.

In the $\mathcal{N} \rightarrow \infty$ limit the model becomes exactly solvable. Note that by counting powers of $\mathcal{N}$ one easily remarks that the last quartic term is of higher order than the two previous ones. This will be cured with a special scaling of the parameter $\lambda_{0}$.

The simplest way to see that this model is solvable is to notice that, by the central limit theorem, the random variable $\phi^{2}=\sum_{\alpha=1}^{\mathcal{N}} \phi_{\alpha}^{2}$ is a sum over a large number of terms, that by symmetry should be identically distributed, and have then a normal (Gaussian) distribution. This means, in particular, that the fourth cumulant $\left(\phi^{2}\right)^{2}-3\left\langle\phi^{2}\right\rangle \phi^{2}$ vanishes, so we can replace the last term in the action by $3 \lambda_{0}\left\langle\phi^{2}\right\rangle \phi^{2}$. This makes the free-energy Gaussian, with an effective mass

$$
\begin{equation*}
m_{\mathrm{eff}}^{2}=\xi_{\mathrm{eq}}^{-2}=m_{0}^{2}+6 \lambda_{0}\left\langle\phi^{2}\right\rangle . \tag{6.23}
\end{equation*}
$$

In a spatially homogeneous state, the average $\left\langle\phi^{2}\right\rangle$ should be independent of the space point on which is it is measured and, in particular, it should be identical to its value at
the origin, $\left\langle\phi^{2}(\vec{x})\right\rangle=\left\langle\phi^{2}(\overrightarrow{0})\right\rangle$. Moreover, since we argued that $\phi^{2}$ is Gaussian distributed, its average can be readily computed, and for any of its components,

$$
\begin{equation*}
\left\langle\phi_{\alpha}^{2}(\overrightarrow{0})\right\rangle=\int_{|k|<\Lambda} \frac{d^{d} k}{(2 \pi)^{d}} \frac{1}{k^{2}+m_{\mathrm{eff}}^{2}} \tag{6.24}
\end{equation*}
$$

where we have included a (ultra-violet) cut-off $\Lambda$ that can be related to the inverse lattice spacing of the original microscopic theory. Replacing in (6.23)

$$
\begin{equation*}
m_{\mathrm{eff}}^{2}=m_{0}^{2}+6 \lambda_{0} \mathcal{N} \int_{|k|<\Lambda} \frac{d^{d} k}{(2 \pi)^{d}} \frac{1}{k^{2}+m_{\mathrm{eff}}^{2}} \tag{6.25}
\end{equation*}
$$

and the factor $\mathcal{N}$ is due to the sum over $\alpha$. This equation admits a non-trivial result for $\lambda=6 \lambda_{0} \mathcal{N}$ finite, that is to say, $\lambda_{0} \propto \mathcal{N}^{-1}$.

This problem can be studied statically within the canonical formalism. If the volume $V$ is kept finite the equilibrium order parameter probability distribution is given by the Gibbs state [64]

$$
\begin{equation*}
P_{\mathrm{eq}}[\vec{\phi}(\vec{k})]=\frac{1}{Z} \exp \left(-\frac{1}{2 k_{B} T V} \sum_{\vec{k}}\left(k^{2}+\xi_{\mathrm{eq}}^{-2}\right) \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k})\right) \tag{6.26}
\end{equation*}
$$

where $\xi_{\mathrm{eq}}$ is the correlation length

$$
\begin{equation*}
\xi_{\mathrm{eq}}^{-2}=-m_{0}^{2}+\frac{\lambda}{\mathcal{N}}\left\langle\vec{\phi}^{2}(\vec{x})\right\rangle_{\mathrm{eq}} \tag{6.27}
\end{equation*}
$$

with $\langle\cdots\rangle_{\text {eq }}$ standing for the average taken with (6.26). (In this expression we have not distinguished one vector direction to signal the symmetry breaking [65] but we considered the symmetric measure in which one sums over all such states.) Note that this is, indeed, a Gaussian measure.

In order to analyze the properties of $P_{\text {eq }}[\vec{\phi}(\vec{k})]$ it is necessary to extract from (6.27) the dependence of $\xi_{\text {eq }}^{-2}$ on $T, m_{0}, \lambda_{0}$ and $V$. Evaluating the average, the above equation yields

$$
\begin{equation*}
\xi_{\mathrm{eq}}^{-2}=-m_{0}^{2}+\frac{\lambda}{V} \sum_{\vec{k}} \frac{k_{B} T}{k^{2}+\xi_{\mathrm{eq}}^{-2}} . \tag{6.28}
\end{equation*}
$$

The solution of this equation is well known [19] and here we summarize the main features, as presented in [64]. Separating the $\vec{k}=0$ term under the sum, for very large volume we may rewrite

$$
\begin{equation*}
\xi_{\mathrm{eq}}^{-2}=-m_{0}^{2}+\lambda k_{B} T B\left(\xi_{\mathrm{eq}}^{-2}\right)+\frac{\lambda k_{B} T}{V \xi_{\mathrm{eq}}^{-2}} \tag{6.29}
\end{equation*}
$$

where

$$
\begin{equation*}
B\left(\xi_{\mathrm{eq}}^{-2}\right)=\lim _{V \rightarrow \infty} \frac{1}{V} \sum_{\vec{k}} \frac{1}{k^{2}+\xi_{\mathrm{eq}}^{-2}}=\int \frac{d^{d} k}{(2 \pi)^{d}} \frac{e^{-\frac{k^{2}}{\Lambda^{2}}}}{k^{2}+\xi_{\mathrm{eq}}^{-2}} \tag{6.30}
\end{equation*}
$$

regularising the integral by introducing the high momentum (ultra-violet) cutoff $\Lambda$. The function $B(x)$ is a non negative monotonically decreasing function with the maximum value at $x=0$

$$
\begin{equation*}
B(0)=\int \frac{d^{d} k}{(2 \pi)^{d}} \frac{e^{-\frac{k^{2}}{\Lambda^{2}}}}{k^{2}}=(4 \pi)^{-\frac{d}{2}} \frac{2}{d-2} \Lambda^{d-2} . \tag{6.31}
\end{equation*}
$$

By graphical analysis one can easily show that (6.29) admits a finite solution for all $k_{B} T$. However, there exists the critical value of the temperature $T_{c}$ defined by

$$
\begin{equation*}
-m_{0}^{2}+\lambda k_{B} T_{c} B(0)=0 \tag{6.32}
\end{equation*}
$$

such that for $T>T_{c}$ the solution is independent of the volume, while for $T \leq T_{c}$ it depends on the volume. Using

$$
\begin{equation*}
B(x)=(4 \pi)^{-\frac{d}{2}} x^{\frac{d}{2}-1} e^{\frac{x}{\Lambda^{2}}} \Gamma\left(1-\frac{d}{2}, \frac{x}{\Lambda^{2}}\right) \tag{6.33}
\end{equation*}
$$

where $\Gamma\left(1-\frac{d}{2}, \frac{x}{\Lambda^{2}}\right)$ is the incomplete gamma function, for $0<\frac{T-T_{c}}{T_{c}} \ll 1$ one finds $\xi_{\text {eq }} \sim$ $\left(\frac{T-T_{c}}{T_{c}}\right)^{-\nu}$, i.e. close but above $T_{c}$, where $\nu=1 / 2$ for $d>4$ and $\nu=1 /(d-2)$ in $d<4$, with logarithmic corrections for $d=4$. At $T_{c}$ one has $\xi_{e q} \sim V^{\bar{\lambda}}$ with $\bar{\lambda}=1 / 4$ for $d>4$ and $\bar{\lambda}=1 / d$ for $d<4$, again with logarithmic corrections in $d=4$. Finally, below $T_{c}$ one finds $\xi_{\text {eq }}^{2}=\frac{M^{2} V}{k_{B} T}$ where $M^{2}=\phi_{0}^{2}\left(\frac{T_{c}-T}{T_{c}}\right)$ and $\phi_{0}^{2}=m_{0}^{2} / \lambda$.

Let us now see what are the implications for the equilibrium state. As Eq. (6.26) shows, the individual Fourier components are independent random variables, with a Gaussian distribution with zero average. The variance is given by

$$
\begin{equation*}
\frac{1}{\mathcal{N}}\langle\vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k})\rangle_{\mathrm{eq}}=V S(\vec{k}) \tag{6.34}
\end{equation*}
$$

where

$$
\begin{equation*}
S(\vec{k})=\frac{k_{B} T}{k^{2}+\xi_{\mathrm{eq}}^{-2}} \tag{6.35}
\end{equation*}
$$

is the equilibrium structure factor. For $T>T_{c}$, all $\vec{k}$ modes behave in the same way, with the variance growing linearly with the volume. For $T \leq T_{c}$, instead, $\xi_{\text {eq }}^{-2}$ is negligible with respect to $k^{2}$ except at $\vec{k}=0$, yielding

$$
S(\vec{k})= \begin{cases}\frac{T_{c}}{k^{2}}\left(1-\delta_{\vec{k}, 0}\right)+c V^{2 \bar{\lambda}} \delta_{\vec{k}, 0} & \text { for } T=T_{c}  \tag{6.36}\\ \frac{T}{k^{2}}\left(1-\delta_{\vec{k}, 0}\right)+M^{2} V \delta_{\vec{k}, 0} & \text { for } T<T_{c}\end{cases}
$$

where $c$ is a constant. This produces a volume dependence in the variance of the $\vec{k}=0$ mode growing faster than linear. Therefore, for $T \leq T_{c}$ the $\vec{k}=0$ mode behaves differently from all the other modes with $\vec{k} \neq 0$. For $T<T_{c}$ the probability distribution (6.26) takes the form

$$
\begin{equation*}
P_{\mathrm{eq}}[\vec{\phi}(\vec{k})]=\frac{1}{Z} e^{-\frac{\dot{\phi}^{2}(0)}{2 M^{2} V^{2}}} e^{-\frac{1}{2 k_{B} T V} \sum_{\vec{k}} k^{2} \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k})} \tag{6.37}
\end{equation*}
$$

Therefore, crossing $T_{c}$ there is a transition from the usual disordered high temperature phase to a low temperature phase characterized by a macroscopic variance in the distribution of the $\vec{k}=0$ mode. The distinction between this phase and the mixture of pure states, obtained below $T_{c}$ when $\mathcal{N}$ is kept finite can be discussed but we will not do it here.

Although the effective Hamiltonian is 'almost' quadratic, the phase transition in the form of a Bose-Einstein-like condensation on the $\vec{k}=\overrightarrow{0}$ mode is due to the self-consistent constraint.

### 6.3 Connection between the two models

The behaviour of the large $\mathcal{N} O(\mathcal{N})$ model is very similar to what derived for the spherical spin-glass model [59]. Why is this so? The reason is that the behaviour of both models are pseudo-quadratic models, for which the behaviour is controlled by the way in which the distribution of modes decays to zero. In the field theory these are the wavevectors modulii $k$ while in the spherical model these are the eigenvalues of the random interaction matrix close to the edge of their distribution.

More precisely, requiring

$$
\begin{equation*}
\rho\left(\bar{\lambda}_{\mu}\right) d \bar{\lambda}_{\mu}=\varrho(k) d k \tag{6.38}
\end{equation*}
$$

with $\bar{\lambda}_{\mu}=2 J-\lambda_{\mu}$,

$$
\begin{align*}
\rho\left(\bar{\lambda}_{\mu}\right) & =\frac{1}{2 \pi J} \sqrt{4 J^{2}-\lambda_{\mu}^{2}} \simeq \frac{1}{\pi J} \sqrt{J \bar{\lambda}_{\mu}} \\
\varrho(k) & =k^{d-1} \tag{6.39}
\end{align*}
$$

and $k^{2}=\bar{\lambda}_{\mu}$ from the equivalence between the quadratic term in the Hamiltonian of the spherical model and the free-energy of the field theory, then this implies

$$
\begin{equation*}
k^{2} \propto k^{d-1} \tag{6.40}
\end{equation*}
$$

and the two models are equivalent, in this sense, in $d=3$.

## 7 Spin-glasses

Let us now discuss a problem in which disorder is so strong as to modify the nature of the low temperature phase $[42,43,44,45,47,48]$. If this is so, one needs to define a new order parameter, capable of identifying order in this phase.

### 7.1 Static properties

### 7.1.1 The ferromagnetic order parameter

The paramagnetic-ferromagnetic transition in a model with no quench randomness is characterised by the local magnetisation, $m_{i}=\left\langle s_{i}\right\rangle$, or the global magnetisation density, $m=N^{-1} \sum_{i=1}^{N} m_{i}$, that detach from zero at $T_{c}$ if the thermodynamic average $\langle\ldots\rangle$ is computed on 'half' phase space to counteract the global spin reversal symmetry of the Hamiltonian. Otherwise, both quantities are identical to zero at all temperatures.

In finite size systems $m$ is distributed around the (two) equilibrium infinite-size limit values, with peaks that get narrower and narrower for larger and larger system sizes. The local magnetisations $m_{i}$ are also distributed around the (two) equilibrium infinite-size limit $m$ values.

### 7.1.2 The spin-glass order parameter

The spin-glass equilibrium phase is one in which spins 'freeze' in randomly-looking configurations. In finite dimensions these configurations are spatially irregular. A snapshot looks statistical identical to a high temperature paramagnetic configuration in which spins point in any direction (two if the spins are Ising like). However, while at high temperatures the spins flip rapidly and another snapshot taken immediately after would look completely different from the previous one, at low temperatures two snapshots taken at close times are highly correlated. Similarly, two snapshots taken at the same very long time but on different realisations of the same experiment, that is to say, after quenching the same sample in the same way, are also very similar.

Let us use the language of Ising models in the following.
In a spin-glass state the local magnetisation is expected to take a non-zero value, $m_{i}=\left\langle s_{i}\right\rangle \neq 0$, where the average is interpreted in the restricted sense introduced in the discussion of ferromagnets, that we shall call here within a pure state. ${ }^{4}$ Instead, the total magnetisation density, $m=N^{-1} \sum_{i=1}^{N} m_{i}$, vanishes since one expects to have as many averaged local magnetisation pointing up $\left(m_{i}>0\right)$ as pointing down ( $m_{i}<0$ ) with each

[^3]

Figure 7.1: A spin configuration in a Heisenberg spin-glass and in an Ising ferromagnet.
possible value of $\left|m_{i}\right|$. Therefore,

$$
\begin{equation*}
m_{i} \neq 0 \quad \text { but } \quad m=0 \tag{7.1}
\end{equation*}
$$

Thus, the total magnetisation density, $m$, of a spin-glass vanishes at all temperatures and it is not a good order parameter.

The spin-glass transition is characterised by a finite peak in the linear magnetic susceptibility and a diverging non-linear magnetic susceptibility. Let us discuss the former first and show how it yields evidence for the freezing of the local magnetic moments. For a generic magnetic model such that the magnetic field couples linearly to the Ising spin, $H_{J}\left[\left\{s_{i}\right\}\right] \rightarrow H_{J}\left[\left\{s_{i}\right\}\right]-\sum_{i} h_{i} s_{i}$, the linear susceptibility is related, via the static fluctuation-dissipation theorem to the correlations of the fluctuations of the magnetisation:

$$
\begin{equation*}
\left.\chi_{i j} \equiv \frac{\partial\left\langle s_{i}\right\rangle_{h}}{\partial h_{j}}\right|_{h=0}=\beta\left\langle\left(s_{i}-\left\langle s_{i}\right\rangle\right)\left(s_{j}-\left\langle s_{j}\right\rangle\right)\right\rangle \tag{7.2}
\end{equation*}
$$

The averages in the rhs are taken without perturbing field. This relation is proven by using the definition of $\left\langle s_{i}\right\rangle_{h}$ and simply computing the derivative with respect to $h_{j}$.

Exercise 7.1 Prove Eq. (7.2).
In particular,

$$
\begin{equation*}
\chi_{i i}=\beta\left\langle\left(s_{i}-\left\langle s_{i}\right\rangle\right)^{2}\right\rangle=\beta\left(1-m_{i}^{2}\right) \geq 0, \tag{7.3}
\end{equation*}
$$

with $m_{i}=\left\langle s_{i}\right\rangle$. The total susceptibility measured experimentally is $\chi \equiv N^{-1} \sum_{i j} \chi_{i j}$. On the experimental side we do not expect to see $O(1)$ sample-to-sample fluctuations in this global quantity. On the analytical side one can use a similar argument to the one presented in Sect. 5.2.3 to argue that $\chi$ should be self-averaging (it is a sum over the entire volume of site-dependent terms). Thus, the experimentally observed susceptibility of sufficiently large samples should be given by

$$
\begin{equation*}
\chi=[\chi]=N^{-1} \sum_{i j}\left[\chi_{i j}\right] \approx N^{-1} \sum_{i}\left[\chi_{i i}\right]=N^{-1} \sum_{i} \beta\left(1-\left[m_{i}^{2}\right]\right) \tag{7.4}
\end{equation*}
$$

since we can expect that cross-terms cancel under the disorder average. ${ }^{5}$ The fall of $\chi$ at low temperatures with respect to its value at $T_{c}$, i.e. the cusp observed experimentally, signals the freezing of the local magnetizations, $m_{i}$, in the non-zero values that are more favourable thermodynamically. Note that this argument is based on the assumption that the measurement is done in equilibrium. The linear ac susceptibility of a spin-glass sample is shown in the left panel in Fig. 7.2.

Thus, the natural global order parameter that characterises the spin-glass transition is

$$
\begin{equation*}
q \equiv N^{-1} \sum_{i}\left[m_{i}^{2}\right] \tag{7.5}
\end{equation*}
$$

as proposed in the seminal 1975 Edwards-Anderson paper [72]. $q$ vanishes in the high temperature phase since all $m_{i}$ are zero but it does not in the low temperature phase since the square power takes care of the different signs. Averaging over disorder eliminates the site dependence. Thus, $q$ is also given by

$$
\begin{equation*}
q=\left[m_{i}^{2}\right] . \tag{7.6}
\end{equation*}
$$

These definitions, reasonable as they seem at a first glance, hide a subtle distinction that we discuss below.

### 7.1.3 Two or many pure states

Let us keep disorder fixed and imagine that once the global spin inversion symmetry has been taken into account there still remain more than one pure or equilibrium states in the selected sample. Consider the disorder-dependent quantity

$$
\begin{equation*}
q_{J}=N^{-1} \sum_{i} m_{i}^{2} \tag{7.7}
\end{equation*}
$$

where the $m_{i}$ depend on the realisation of the exchanges but we do not write the subindex $J$ explicitly to lighten the notation. Then, two possibilities for the statistical average in $m_{i}=\left\langle s_{i}\right\rangle$ have to be distinguished:

- If we interpret it in the same restricted sense as the one discussed in the paramagnetic - ferromagnetic transition of the usual Ising model, i.e. under a pinning field that selects one chosen pure state, in (7.7) we define a disorder dependent Edwards-Anderson parameter,

$$
\begin{equation*}
q_{J E \mathrm{~A}}^{\alpha}=N^{-1} \sum_{i}^{N}\left(m_{i}^{\alpha}\right)^{2} \tag{7.8}
\end{equation*}
$$

[^4]


Figure 7.2: Left: The ac-susceptibility of $\mathrm{Fe}_{0.5} \mathrm{Mn}_{0.5} \mathrm{~T}$ iO $\mathrm{O}_{3}$ at logarithmically evenly spaced frequencies from 0.017 Hz to 1.7 kHz (top to bottom) [47]. Right: Temperature dependence of $-\chi_{3}$ (vertical axis) above $T_{c}$ measured at 10 Hz in static fields of 0 (open circles) and 90 G (solid circles) as a function of reduced temperature $\tau$ (lower axis). The slope is $-\gamma$. Plot of the susceptibility ratios $-\chi_{5}^{\prime} h^{2} / \chi_{3}^{\prime},-\chi_{7}^{\prime} h^{2} / \chi_{5}^{\prime}$ (top axis) as a function of $-\chi_{3}$ (vertical axis) in zero field. The slope is $1+\beta / \gamma[67,68]$.
where we label $\alpha$ the selected pure state. Although $q_{J E A}^{\alpha}$ could depend on $\alpha$ it turns out that in all known cases it does not and the $\alpha$ label in $q_{J E A}^{\alpha}$ is superfluous. In addition, $q_{J_{E A}}$ could fluctuate from sample to sample since the individual $m_{i}$ 's do. It turns out that in the thermodynamic limit $q_{J_{E A}}$ does not fluctuate. Therefore, later we will use

$$
\begin{equation*}
q_{\mathrm{EA}}=q_{\mathrm{JEA}} . \tag{7.9}
\end{equation*}
$$

- If, instead, the statistical average in $m_{i}^{\alpha}$ runs over all possible equilibrium states (on half the phase space, that is to say, eliminating spin-reversal) the quantity (7.7) has non-trivial contributions from overlaps between different states. Imagine each state has a probability weight $w_{\alpha}^{J}$ (in the ferromagnetic phase of the Ising model one has only one (two) pure states with $w_{1}=w_{2}=1 / 2$ ) then

$$
\begin{equation*}
q_{J}=N^{-1} \sum_{i=1}^{N}\left(\sum_{\alpha} w_{\alpha}^{J} m_{i}^{\alpha}\right)^{2} \tag{7.10}
\end{equation*}
$$

In the ferromagnetic transition $q=q_{\mathrm{EA}}=m^{2}$, and $q_{\mathrm{EA}}$ and $q$ are identical order parameters.

In the disorder case, $q_{J}{ }_{\mathrm{EA}}^{\alpha}$ takes the same value on all equilibrium states independently of there being only two (as in the usual ferromagnetic phase) or more (as we shall see appear in fully-connected spin-glass models). Therefore it does not allow us to distinguish between the two-state and the many-state scenarii. Instead, $q_{J}$ does.

It is important to note that which are the pure states in the model depends on the quenched disorder realization.

The parameter $q$ in Eq. (7.5), that involves a further average over quenched disorder, is then

$$
\begin{equation*}
q=\left[q_{J}\right] . \tag{7.11}
\end{equation*}
$$

Having defined a disorder-dependent order parameter, $q_{J}$, and its disorder average, $q$, that explains the decay of the susceptibility below $T_{c}$, we still have to study whether this order parameter characterises the low temperature phase completely. It will turn out that the knowledge of the disorder-averaged $q$ is not enough, at least in fully-connected and dilute spin-glass models. Indeed, one needs to consider the disorder-dependent probability distribution of the fluctuating $q_{J}, P_{J}\left(q_{J}\right)$, see Fig. 7.3. The more pertinent definition of an order parameter as being given by such a probability distribution allows one to distinguish between the simple, two-state, and the complex, many-state, scenarii.

In practice, a way to compute the probability distribution of the order parameter is by using an overlap - or correlation - between two spin configurations, say $\left\{s_{i}\right\}$ and $\left\{\sigma_{i}\right\}$, defined as

$$
\begin{equation*}
q_{s \sigma}^{J}=N^{-1} \sum_{i}\left\langle s_{i} \sigma_{i}\right\rangle \tag{7.12}
\end{equation*}
$$

where $\langle\ldots\rangle$ is an unrestricted thermal average. $q_{s \sigma}^{J}$ takes values between -1 and 1 . It equals one if $\left\{s_{i}\right\}$ and $\left\{\sigma_{i}\right\}$ differ in a number of spins that is smaller than $O(N)$, it equals -1 when the two configurations are totally anti-correlated - with the same proviso concerning a number of spins that is not $O(N)$ - and it equals zero when $\left\{s_{i}\right\}$ and $\left\{\sigma_{i}\right\}$ are completely uncorrelated. Other values are also possible. Note that the self-overlap of a configuration with itself is identically one for Ising spins.

The overlap can be computed by running a Monte Carlo simulation, equilibrating a sample and recording many equilibrium configurations. With them one computes the overlap and should find a histogram with two peaks at $q_{\mathrm{EA}}$ and $-q_{\mathrm{EA}}$ (the values of the overlap when the two configurations fall in the same pure state or in the sign reversed ones) and, in cases with many different pure states, other peaks at other values of $q_{s \sigma}^{J}$. This is observed in the $3 d$ EA model as exemplified in Fig. 7.3. Note that $q_{s \sigma}^{J}$ is related to the $q$ definition above. A related definition is the one of the Hamming distance:

$$
\begin{equation*}
d_{s \sigma}^{J}=N^{-1} \sum_{i=1}^{N}\left\langle\left(s_{i}-\sigma_{i}\right)^{2}\right\rangle=2\left(1-q_{s \sigma}^{J}\right) . \tag{7.13}
\end{equation*}
$$

Figure 7.3 shows the probability distribution $P_{J}(q)$ obtained from a MC simulation of the $3 d$ EA model. The external peaks are at $q_{\text {EA }}$, cases in which the two copies are taken in the same equilibrium state. In the first panel there are only two states, one and its reversed. In the other figures other peaks appear associated with the existence of more than one state and the overlap between them. They are sampled differently in the various panels since the temperature of the simulation is changed.


Figure 7.3: Monte Carlo simulations of the $3 d$ Edwards-Anderson model. The disorderdependent overlap probability distribution function, $P_{J}(q)$, for different choices of the random couplings. Figure taken from [77].


Figure 7.4: Monte Carlo simulations of the $3 d$ Edwards-Anderson model. The disorder averaged overlap distribution function, $\left[P_{J}(q)\right]$, for different system sizes given in the key. Figure taken from [78].

Instead, Fig. 7.4 displays the disorder averaged $P(q)$ for a $3 d$ Edwards-Anderson model at low temperatures. The dotted line is the theoretical prediction for the SherringtonKirkpatrick model that we will discuss below. It has a sharp peak at $q_{\text {EA }}$ and a nonvanishing continuous weight at all values of $q<q_{\text {EA }}$. The various lines represent numerical data for different system sizes. The questions is whether the intermediate part will remain non-vanishing in the infinite size limit or whether it will eventually vanish.

### 7.1.4 Pinning fields

In the discussion of the ferromagnetic phase transition we established that one of the two equilibrium states, related by spin reversal symmetry, is chosen by a small pinning field that is taken to zero after the thermodynamic limit, $\lim _{h \rightarrow 0} \lim _{N \rightarrow \infty}$.

In a problem with quenched disorder it is no longer feasible to choose and apply a magnetic field that is correlated to the statistical averaged local magnetization in a single pure state since this configuration is not known! Moreover, the remanent magnetic field that might be left in any experience will not be correlated with any special pure state of the system at hand.

Which is then the statistical average relevant to describe experiments? We shall come back to this point below.

### 7.1.5 Divergent susceptibility

In a pure magnetic system with a second-order phase transition the susceptibility of
the order parameter to a field that couples linearly to it diverges when approaching the transition from both sides. In a paramagnet, one induces a local magnetisation with a local field

$$
\begin{equation*}
m_{i}=\left\langle s_{i}\right\rangle=\sum_{j=1}^{N} \chi_{i j} h_{j} \tag{7.14}
\end{equation*}
$$

with $\chi_{i j}$ the linear susceptibilities, the magnetic energy given by $E=E_{0}-\sum_{i} s_{i} h_{i}$, and the field is set to zero at the end of the calculation. Using this expression, the order parameter in the high temperature phase becomes

$$
\begin{equation*}
q=q_{\mathrm{EA}}=\frac{1}{N} \sum_{i=1}^{N}\left[m_{i}^{2}\right]=\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N}\left[\chi_{i j} \chi_{i k} h_{j} h_{k}\right] \tag{7.15}
\end{equation*}
$$

If the applied fields are random and taken from a probability distribution such that $\overline{h_{j} h_{k}}=\sigma^{2} \delta_{j k}$ one can replace $h_{j} h_{k}$ by $\sigma^{2} \delta_{j k}$ and obtain

$$
\begin{equation*}
q=\frac{1}{N} \sum_{i=1}^{N}\left[m_{i}^{2}\right]=\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N}\left[\chi_{i j}^{2}\right] \sigma^{2} \equiv \chi_{S G} \sigma^{2} \tag{7.16}
\end{equation*}
$$

$\sigma^{2}$ acts as a field conjugated to the order parameter $q_{\mathrm{EA}}$. (One can also argue that a uniform field looks random to a spin-glass sample and therefore the same result holds. It is more natural though to use a trully random field since a uniform one induces a net magnetization in the sample.) The spin-glass susceptibility is then defined as

$$
\begin{equation*}
\chi_{S G} \equiv \frac{1}{N} \sum_{i j}\left[\chi_{i j}^{2}\right]=\frac{\beta^{2}}{N} \sum_{i j}\left[\left(\left\langle s_{i} s_{j}\right\rangle-\left\langle s_{i}\right\rangle\left\langle s_{j}\right\rangle\right)^{2}\right]=\frac{\beta^{2}}{N} \sum_{i j}\left[\left\langle s_{i} s_{j}\right\rangle^{2}\right] \tag{7.17}
\end{equation*}
$$

and one finds that it diverges as $T \rightarrow T_{c}^{+}$as expected in a second-order phase transition. (Note that there is no cancelation of crossed terms because of the square.) Indeed, the divergence of $\chi_{S G}$ is related to the divergence of the non-linear magnetic susceptibility that is measurable experimentally and numerically. An expansion of the total mangnetization in powers of a uniform field $h$ acting as $E \rightarrow E-h \sum_{i} s_{i}$ is

$$
\begin{equation*}
M_{h}=\chi h-\frac{\chi^{(3)}}{6} h^{3}+\ldots \tag{7.18}
\end{equation*}
$$

and the first non-linear susceptibility is then given by

$$
\begin{equation*}
-\left.\chi^{(3)} \equiv \frac{\partial^{3} M_{h}}{\partial h^{3}}\right|_{h=0}=-\left.\beta^{-1} \frac{\partial^{4} \ln Z_{h}}{\partial h^{4}}\right|_{h=0}=-\frac{\beta^{3} N}{3}\left\langle\left(\sum_{i} s_{i}\right)^{4}\right\rangle_{c} \tag{7.19}
\end{equation*}
$$

with the subindex $c$ indicating that the quartic correlation function is connected. Above $T_{c}, m_{i}=0$ at zero field,

$$
\begin{equation*}
\chi^{(3)}=\beta^{3} \sum_{i j k l}\left(\left\langle s_{i} s_{j} s_{k} s_{l}\right\rangle-3\left\langle s_{i} s_{j}\right\rangle\left\langle s_{k} s_{l}\right\rangle\right)=\frac{\beta^{3}}{N} 3\left(4 N-6 \sum_{i j}\left\langle s_{i} s_{j}\right\rangle^{2}\right) \tag{7.20}
\end{equation*}
$$

and one can identify $\chi_{S G}$ when $i=k$ and $j=l$ plus many other terms that we assume are finite. Then,

$$
\begin{equation*}
\chi^{(3)}=\beta\left(\chi_{S G}-\frac{2}{3} \beta^{2}\right) . \tag{7.21}
\end{equation*}
$$

This quantity can be accessed experimentally. A careful experimental measurement of $\chi^{(3)}, \chi^{(5)}$ and $\chi^{(7)}$ demonstrated that all these susceptibilities diverge at $T_{c}[67,68]$, see the right panel in Fig. 7.2.

### 7.1.6 Phase transition and scaling

Having identified an order parameter, the linear and the non-linear susceptibility one can now check whether there is a static phase transition and, if it is of second order, whether the usual scaling laws apply. Many experiments have been devoted to this task. It is by now quite accepted that Ising spin-glasses in $3 d$ have a conventional second order phase transition. Still, the exponents are difficult to obtain and there is no real consensus about their values. There are two main reasons for this: one is that as $T_{c}$ is approached the dynamics becomes so slow that equilibrium measurements cannot really be done. Critical data are thus restricted to $T>T_{c}$. The other reason is that the actual value of $T_{c}$ is difficult to determine and the value used has an important influence on the critical exponents. Possibly, the most used technique to determine the exponents is via the scaling relation for the non-linear susceptibility:

$$
\begin{equation*}
\chi_{n l}=t^{\beta} f\left(\frac{h^{2}}{t^{\gamma+\beta}}\right) \tag{7.22}
\end{equation*}
$$

with $t=\left|T-T_{c}\right| / T_{c}$ and one finds, approximately, the values given in Table 7.3 to be compared with the values for the ferromagnetic transitions summarized in Table 7.3.

|  | $d$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha$ | $\nu$ | $\eta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| FM | $\infty$ | 1 | 1 | 2 | -1 | $1 / 2$ | 0 |
| FM | 3 | 0.326 | 1.237 | 4.790 | 0.110 | 0.630 | 0.036 |
| SK | $\infty$ | 1 | 1 | 2 | -1 | $1 / 2$ | 0 |
| Exp | 3 | 1 | 2.2 | 3.1 | x |  |  |

Table 7.3: Critical exponents in the Ising ferromagnetic and spin-glass transitions. $d \rightarrow \infty$ corresponds to the mean-field results and the Sherrington-Kirkpatrick model. Experiments measuring the critical exponents of an Ising spin-glass were reported in [80], for example, and are given in the last row.

No cusp in the specific heat of spin-glasses is seen experimentally. Since one expects a second order phase transition this means that the divergence of this quantity must be very weak.

| authors | year | $T_{c}$ | $\nu$ | $\eta$ |
| :---: | :---: | :---: | :---: | :---: |
| Ogielski, Morgenstern | 1985 | 1.20(5) | 1.2(1) |  |
| Ogielski | 1985 | 1.175(25) | 1.3(1) | -0.22(5) |
| Singh, Chakravarty | 1986 | 1.2(1) | 1.3(2) |  |
| Bhatt, Young | 1985 | 1.2(2) | 1.3(3) | -0.3(2) |
| Kawashima, Young | 1996 | 1.11(4) | 1.7 (3) | -0.35(5) |
| Bernardi et al | 1996 | 1.165(10) |  | -0.245(20) |
| Berg, Janke | 1998 | 1.12(1) |  | -0.37(4) |
| Palassini, Caracciolo | 1999 | 1.156(15) | 1.8(2) | -0.26(4) |
| Mari, Campbell | 1999 | 1.20(1) |  | -0.21(2) |
| Ballesteros et al. | 2000 | 1.138(10) | 2.15(15) | -0.337(15) |
| Mari, Campbell | 2001 | 1.190(15) |  | -0.20(2) |
| Mari, Campbell | 2002 | 1.195(15) | 1.35(10) | -0.225(25) |
| Nakamura et al | 2003 | 1.17(4) | 1.5(3) | -0.4(1) |
| Pleimling, Campbell | 2005 | 1.19(1) |  | -0.22(2) |
| Katzgraber et al. | 2006 | 1.120(4) | 2.39(5) | -0.395(17) |
| our result | 2007 | 1.101(5) | 2.53(8) | -0.384(9) |

Figure 7.5: List of critical exponents of the $3 d$ EA model. The last results for $\nu$ and $\eta$ were determined with numerical simulations using system sizes $L=3-28$ [79].

The critical exponents satisfy the usual relations

$$
\begin{array}{ll}
\gamma=\nu(2-\eta) & \alpha=2-\nu d \\
\beta=(2+\alpha-\gamma) / 2 & 2 \beta \delta=2+\alpha+\gamma \tag{7.23}
\end{array}
$$

### 7.1.7 The droplet theory

The droplet theory is a phenomenological model that assumes that the low temperature phase of a spin-glass model has only two equilibrium states related by an overall spin flip. It is then rather similar to a ferromagnet, only that the nature of the order in the two equilibrium states is not easy to see, it is not just most spins pointing up or most spins pointing down with some thermal fluctuations within. At a glance, one sees a disordered paramagnetic like configuration and a more elaborate order parameter has to be measured to observe the order. The spin-glass phase is then called a disguised ferromagnet and a usual spontaneous symmetry breaking (between the two equilibrium states related spin reversal symmetry) leading to usual ergodicity breaking is supposed to take place at $T_{c}$.

Once this assumption has been done, renormalisation group arguments are used to describe the scaling behaviour of several thermodynamic quantities. The results found are then quantitatively different from the ones for a ferromagnet but no novelties appear.

### 7.2 The TAP approach

Disordered models have quenched random interactions. Due to the fluctuating values of the exchanges, one expects that the equilibrium configurations be such that in each equilibrium state the spins freeze in different directions. The local averaged magnetizations
need not be identical, on the contrary one expects $\left\langle s_{i}\right\rangle=m_{i}$ and, if many states exist, each of them can be identified by the vector $\left(m_{1}, \ldots, m_{N}\right)$.

Let us focus on the Sherrington-Kirkpatrick model, defined by

$$
\begin{equation*}
H_{J}^{\mathrm{SK}}=-\frac{1}{2} \sum_{i \neq j} J_{i j} s_{i} s_{j}-\sum_{i} h_{i}^{\mathrm{ext}} s_{i} \tag{7.24}
\end{equation*}
$$

with interaction strengths $J_{i j}$ taken from a Gaussian pdf and scaled with $N$ in such a way that the thermodynamic limit is non-trivial:

$$
\begin{equation*}
P\left(J_{i j}\right)=\left(2 \pi \sigma_{N}^{2}\right)^{-\frac{1}{2}} e^{-\frac{J_{i j}^{2}}{2 \sigma_{N}^{N}}} \tag{7.25}
\end{equation*}
$$

and external applied field $h_{i}^{\text {ext }}$. The first two-moments of the exchange distribution are $\left[J_{i j}\right]=0$ and $\left[J_{i j}^{2}\right]=\sigma_{N}^{2}=J^{2} /(2 N)$.

One may try to use the naive mean-field equations (B.2.21), generalised to local variational parameters $m_{i}$, to characterise the low temperature properties of these models at fixed quenched disorder:

$$
m_{i}=\tanh \left(\beta h_{i}^{\mathrm{loc}}\right)=\tanh \left(\sum_{j(\neq i)} \beta J_{i j} m_{j}+\beta h_{i}^{\mathrm{ext}}\right)
$$

and determine then the different $\left\{m_{i}^{\alpha}\right\}=\left(m_{1}^{\alpha}, \ldots, m_{N}^{\alpha}\right)$ values from them, with the label $\alpha$ indicating the possibility of there being many solutions to these equations. It is important to reckon that, in this discussion, the $m_{i}=\left\langle s_{i}\right\rangle$ are assumed to be average in each thermodynamic state (with no mixture between them).

It has been shown by Thouless-Anderson-Palmer (TAP) [74] that these equations are not completely correct even in the fully-connected disordered case: a term which is called the Onsager reaction term is missing. This term represents the reaction of the spin $i$ : the magnetisation of the spin $i$ produces a field $h_{j(i)}^{\prime}=J_{j i} m_{i}=J_{i j} m_{i}$ on spin $j$; this field induces a magnetisation $m_{j(i)}^{\prime}=\chi_{j j} h_{j(i)}^{\prime}=\chi_{j j} J_{i j} m_{i}$ on the spin $j$. This magnetisation, in turn, produces a field $h_{i(j)}^{\prime}=J_{i j} m_{j(i)}^{\prime}=J_{i j} \chi_{j j} J_{i j} m_{i}=\chi_{j j} J_{i j}^{2} m_{i}$ on the site $i$. The equilibrium fluctuation-dissipation relation between susceptibilities and connected correlations implies $\chi_{j j}=\beta\left\langle\left(s_{j}-\left\langle s_{j}\right\rangle\right)^{2}\right\rangle=\beta\left(1-m_{j}^{2}\right)$ and one then has $h_{i(j)}^{\prime}=\beta\left(1-m_{j}^{2}\right) J_{i j}^{2} m_{i}$. The idea of Onsager - or cavity method - is that one has to study the ordering of the spin $i$ in the absence of its own effect on the rest of the system. Thus, the total field produced by the sum of $h_{i(j)}^{\prime}=\beta\left(1-m_{j}^{2}\right) J_{i j}^{2} m_{i}$ over all the spins $j$ with which it can connect, has to be subtracted from the mean-field created by the other spins in the sample, i.e.

$$
\begin{equation*}
h_{i}^{\mathrm{loc}}=\sum_{j} J_{i j} m_{j}+h_{i}^{\mathrm{ext}}-\beta m_{i} \sum_{j} J_{i j}^{2}\left(1-m_{j}^{2}\right) \tag{7.26}
\end{equation*}
$$

where $h_{i}^{\text {ext }}$ is the external field. The equations then read

$$
m_{i}=\tanh \left(\sum_{j(\neq i)} \beta J_{i j} m_{j}-\beta^{2} m_{i} J_{i j}^{2}\left(1-m_{j}^{2}\right)+\beta h_{i}^{\mathrm{ext}}\right) .
$$

The reason why the reaction term does not appear in the mean-field equations for ferromagnets is that it is sub-leading with respect to the first one. We now discuss why it is not in the disordered case. Let us study the order of magnitude, as powers of $N$ of each term in the r.h.s. In the first term

$$
\begin{equation*}
\sum_{j(\neq i)} J_{i j} m_{j} \simeq \sum \frac{1}{\sqrt{N}} m_{j} \simeq 1 \tag{7.27}
\end{equation*}
$$

because of the central limit theorem. In the second term

$$
\begin{equation*}
\sum_{j(\neq i)} J_{i j}^{2}\left(1-m_{j}^{2}\right) \simeq \sum \frac{1}{N}\left(1-m_{j}^{2}\right) \simeq 1 \tag{7.28}
\end{equation*}
$$

because all terms in the sum are positive definite. Thus, in disordered systems the reaction term is of the same order of the usual mean-field; a correct mean-field description has to include it. In the ferromagnetic case this term can be neglected since it is sub-leading in $N$, since $J_{i j}^{2}=J^{2} / N^{2}$ in this case.

The argument leading to the Onsager reaction term can be generalised to include the combined effect of the magnetisation of spin $i$ on a sequence of spins (or $p-1$ spins) in the sample, i.e. the effect on $i$ on $j$ and then on $k$ that comes back to $i$ in the SK model. These higher order terms are indeed negligible only if the series of all higher order effects does not diverge. The ensuing condition is $1>\beta^{2}\left(1-2 q_{\mathrm{EA}}+N^{-1} \sum_{i} m_{i}^{4}\right)$ for the SK model.

Using the fact that there is a sum over a very large number of elements, $J_{i j}^{2}$ can be replaced by its site-independent variance $\left[J_{i j}^{2}\right]=p!J^{2} /(2 N)$ in the last term in (7.27). Introducing the Edwards-Anderson parameter $q_{\mathrm{EA}}=N^{-1} \sum_{i=1} m_{i}^{2}$ the TAP equations follow:

$$
\begin{equation*}
m_{i}=\tanh \left(\beta \sum_{j \neq i} J_{i j} m_{j}-\beta^{2} J^{2} m_{i}\left(1-q_{\mathrm{EA}}\right)+\beta h_{i}\right) . \tag{7.29}
\end{equation*}
$$

The generalisation of the argument leading to the reaction term to $p$ spin interactions

$$
\begin{equation*}
H_{J}\left[\left\{s_{i}\right\}\right]=-\sum_{i_{1}<\cdots<i_{p}} J_{i_{1} \ldots i_{p}} s_{i_{1}} \ldots s_{i_{p}} \quad\left[J_{i_{1} \ldots i_{p}}^{2}\right]=\frac{J^{2} p!}{2 N^{p-1}} \tag{7.30}
\end{equation*}
$$

is not so straightforward. An alternative derivation has been given by Biroli [75]. The TAP equations for $p$-spin fully connected models read

$$
m_{i}=\tanh \left[\sum_{i_{2} \neq \cdots \neq i_{p}}\left(\frac{\beta}{(p-1)!} J_{i i_{2} \ldots i_{p}} m_{i_{2}} \ldots m_{i_{p}}-\beta^{2} m_{i} J_{i i_{2} \ldots i_{p}}^{2}\left(1-m_{i_{2}}^{2}\right) \ldots\left(1-m_{i_{p}}^{2}\right)\right)\right]
$$

where we set $h_{i}^{\text {ext }}=0$. The first contribution to the internal field is proportional to $J_{i 1_{2} \ldots i_{p}} \sim N^{-(p-1) / 2}$ and once the $p-1$ sums performed it is of order one. The reaction term instead is proportional to $J_{i i_{2} \ldots i_{p}}^{2}$ and, again, a simple power counting shows that it is $O(1)$. Using the fact that there is a sum over a very large number of elements, $J_{i_{1} \ldots i_{p}}^{2}$ can be replaced by its site-independent variance $\left[J_{i_{1} \ldots i_{p}}^{2}\right]=p!J^{2} /\left(2 N^{p-1}\right)$ in the last term in (7.31). Introducing the Edwards-Anderson parameter $q_{\text {EA }}=N^{-1} \sum_{i=1} m_{i}^{2}$ the TAP equations follow:

$$
m_{i}=\tanh \left(\frac{\beta}{(p-1)!} \sum_{i_{2} \neq \cdots \neq i_{p}} J_{i i_{2} \ldots i_{p}} m_{i_{2}} \ldots m_{i_{p}}-\frac{\beta^{2} J^{2} p}{2} m_{i}\left(1-q_{\mathrm{EA}}\right)^{p-1}+\beta h_{i}\right) .
$$

The importance of the reaction term becomes clear from the analysis of the linearised equations, expected to describe the second order critical behaviour for the SK model $(p=2)$ in the absence of an applied field. The TAP equations become

$$
\begin{equation*}
m_{i} \sim \beta \sum_{j} J_{i j} m_{j}-\beta^{2} J^{2} m_{i}+\beta h_{i} \tag{7.31}
\end{equation*}
$$

A change of basis to the one in which the $J_{i j}$ matrix is diagonal leads to $m_{\lambda} \sim \beta(\lambda-$ $\left.\beta J^{2}\right) m_{\lambda}+\beta h_{\lambda}$. The notation we use is such that $J_{\lambda}$ is an eigenvalue of the $J_{i j}$ matrix associated to the eigenvector $\vec{v}_{\lambda} . m_{\lambda}$ represents the projection of $\vec{m}$ on the eigenvector $m_{\lambda}, m_{\lambda}=\vec{v}_{\lambda} \cdot \vec{m}$, with $\vec{m}$ the $N$-vector with components $m_{i}$. The staggered susceptibility then reads

$$
\begin{equation*}
\left.\chi_{\lambda} \equiv \frac{\partial m_{\lambda}}{\partial h_{\lambda}}\right|_{h=0}=\beta\left(1-\beta J_{\lambda}+(\beta J)^{2}\right)^{-1} \tag{7.32}
\end{equation*}
$$

Random matrix theory tells us that the eigenvalues of the random matrix $J_{i j}$ are distributed with the semi-circle law. For the normalisation of the $J_{i j}$ 's that we used, the largest eigenvalue is $J_{\lambda}^{\max }=2 J$ [60]. The staggered susceptibility for the largest eigenvalue diverges at $\beta_{c} J=1$. Note that without the reaction term the divergence appears at the inexact value $T^{*}=2 T_{c}$ (see Sect. 7.4 for the replica solution of the SK model).

The TAP equations are the extremisation conditions on the TAP free-energy density:

$$
\begin{align*}
f_{J}^{\mathrm{tap}}\left(\left\{m_{i}\right\}\right)= & -\frac{1}{p!} \sum_{i_{1} \neq \cdots \neq i_{p}} J_{i_{1} \ldots i_{p}} m_{i_{1}} \ldots m_{i_{p}}-\frac{\beta}{4 p} \sum_{i_{1} \neq \cdots \neq i_{p}} J_{i_{1} \ldots i_{p}}^{2}\left(1-m_{i_{1}}^{2}\right) \ldots\left(1-m_{i_{p}}^{2}\right) \\
& -\sum_{i} h_{i} m_{i}+T \sum_{i=1}^{N}\left[\frac{1+m_{i}}{2} \ln \frac{1+m_{i}}{2}+\frac{1-m_{i}}{2} \ln \frac{1-m_{i}}{2}\right] \tag{7.33}
\end{align*}
$$

where presented for the generalised $p$-spin model. The free-energy density as a function of the local magnetizations $m_{i}$ defines what is usually called the free-energy landscape. Note that this function depends on $N \gg 1$ variables, $m_{i}$, and these are not necessarily identical in the disordered case in which the interactions between different groups of spins


Figure 7.6: Schematic representation of a rugged free-energy landscape. Application to protein folding [76].
are different. The stability properties of each extreme $\left\{m_{l}^{\alpha}\right\}$ are given by the eigenvalues of the Hessian matrix

$$
\begin{equation*}
\left.\mathcal{H}_{i j}^{J} \equiv \frac{\partial f_{J}^{\mathrm{tap}}\left(\left\{m_{k}\right\}\right)}{\partial m_{i} \partial m_{j}}\right|_{\left\{m_{l}^{\alpha}\right\}} \tag{7.34}
\end{equation*}
$$

The number of positive, negative and vanishing eigenvalues determine then the number of directions in which the extreme is a minimum, a maximum or marginal. The sets $\left\{m_{l}^{\alpha}\right\}$ for which $f_{J}^{\text {tap }}\left(\left\{m_{l}^{\alpha}\right\}\right)$ is the absolute minima yield a first definition of equilibrium or pure states.

The TAP equations apply to $\left\{m_{i}\right\}$ and not to the configurations $\left\{s_{i}\right\}$. The values of the $\left\{m_{i}^{\alpha}\right\}$ are determined as extrema of the TAP free-energy density, $f_{J}^{\mathrm{tap}}\left(\left\{m_{i}\right\}\right)$, and they not need to be the same as those of the energy, $H_{J}\left(\left\{s_{i}\right\}\right)$, a confusion sometimes encountered in the glassy literature. The coincidence of the two can only occur at $T \rightarrow 0$.

### 7.2.1 The complexity or configurational entropy

There are a number of interesting questions about the extrema of the TAP free-energy landscape, or even its simpler version in which the Onsager term is neglected, that help us understanding the static behaviour of disordered systems:

- For a given temperature, $T$, how many solutions to the mean-field equations exist? The total number of solutions can be calculated using

$$
\begin{equation*}
\mathcal{N}_{J}(T)=\prod_{i} \int_{-1}^{1} d m_{i} \delta\left(m_{i}-m_{i}^{\alpha}\right)=\prod_{i} \int_{-1}^{1} d m_{i} \delta\left(\mathrm{eq}_{i}^{J}\right)\left|\operatorname{det} \frac{\partial \mathrm{eq}_{i}^{J}}{\partial m_{j}}\right| \tag{7.35}
\end{equation*}
$$

$\left\{m_{i}^{\alpha}\right\}$ are the solutions to the TAP equations that we write as $\left\{\mathrm{eq}_{i}^{J}=0\right\}$. The last factor is the normalization of the delta function after the change of variables, it ensures that we count one each time the integration variables touch a solution to the TAP equations independently of their stability properties. We made explicit the fact the this quantity depends on temperature.
We define the complexity or configurational entropy as the logarithm of the number of solutions at temperature $T$ divided by $N$ :

$$
\begin{equation*}
\Sigma_{J}(T) \equiv N^{-1} \ln \mathcal{N}_{J}(T) \tag{7.36}
\end{equation*}
$$

The normalization with $N$ suggests that the number of solutions is actually an exponential of $N$. We shall come back to this very important point below.

- Does $\mathcal{N}_{J}(T)$ depend on $T$ and does it change abruptly at particular values of $T$ that may or may not coincide with static and dynamic phase transitions?
- One can define a free-energy level dependent complexity

$$
\begin{equation*}
\Sigma_{J}(f, T) \equiv N^{-1} \ln \mathcal{N}_{J}(f, T) \tag{7.37}
\end{equation*}
$$

where $\mathcal{N}_{J}(f, T)$ is the number solutions in the interval $[f, f+d f]$ at temperature $T$.

- From these solutions, one can identify the minima as well as all saddles of different type, i.e. with different indices $K$. These are different kinds of metastable states. Geometry constrains the number of metastable states to satisfy Morse theorem that states $\sum_{l=1}^{\mathcal{N}_{J}}(-1)^{\kappa_{l}}=1$, where $\kappa_{l}$ is the number of negative eigenvalues of the Hessian evaluated at the solution $l$, for any continuous and well-behaved function diverging at infinity in all directions. (For example, in a one-dimensional double-well function, one has $(-1)^{0}+(-1)^{1}+(-1)^{0}=1$.)
One can then count the number of solutions to the TAP equations of each index, $\mathcal{N}_{J}(K, T)$, and define the corresponding complexity

$$
\begin{equation*}
\Sigma_{J}(K, T) \equiv N^{-1} \ln \mathcal{N}_{J}(K, T) \tag{7.38}
\end{equation*}
$$

or even work at fixed free-energy density

$$
\begin{equation*}
\Sigma_{J}(K, f, T) \equiv N^{-1} \ln \mathcal{N}_{J}(K, f, T) \tag{7.39}
\end{equation*}
$$

Even more interestingly, one can analyse how are the free-energy densities of different saddles organized. For instance, one can check whether all maxima are much higher in free-energy density than saddles of a given type, etc.

- What is the barrier, $\Delta f=f_{1}-f_{0}$, between ground states and first excited states? How does this barrier scale with the actual free-energy difference, $\Delta f$ between these states? To answer this question one has to estimate the nucleation radius for the reversal of a droplet under an applied field, for instance.

The definitions of complexity given above are disorder-dependent. One might then expect that the complexity will show sample-to-sample fluctuations and be characterized by a probability distribution. The quenched complexity, $\Sigma^{\text {quenched }}$, is then the most likely value of $\Sigma_{J}$, and it is defined through

$$
\begin{equation*}
\max P\left(\Sigma_{J}\right)=P\left(\Sigma^{\text {quenched }}\right) \tag{7.40}
\end{equation*}
$$

In practice, this is very difficult to compute. Most analytic results concern the annealed complexity

$$
\begin{equation*}
\Sigma_{\mathrm{ann}} \equiv N^{-1} \ln \left[\mathcal{N}_{J}\right]=N^{-1} \ln \left[e^{N \Sigma_{J}}\right] \tag{7.41}
\end{equation*}
$$

One can show that the annealed complexity is smaller or equal than the quenched one.

### 7.2.2 Weighted averages

Having identified many solutions to the TAP equations one needs to determine now how to compute statistical averages. A natural proposal is to give a probability weight to each solution, $w_{\alpha}$, and to use it to average the value of the observable of interest:

$$
\begin{equation*}
\langle O\rangle=\sum_{\alpha} w_{\alpha}^{J} O_{\alpha} \quad \text { with } \quad O_{\alpha}=O\left(\left\{m_{i}^{\alpha}\right\}\right) \tag{7.42}
\end{equation*}
$$

where $\alpha$ labels the TAP solutions, $O_{\alpha}$ is the value that the observable $O$ takes in the TAP solution $\alpha$, and $w_{\alpha}^{J}$ are their statistical weights, satisfying the normalization condition $\sum_{\alpha} w_{\alpha}^{J}=1$. Two examples can illustrate the meaning of this average. In a spin-glass problem, if $O=s_{i}$, then $O_{\alpha}=m_{i}^{\alpha}$. In an Ising model in its ferromagnetic phase, if $O=s_{i}$, then $O_{\alpha}=m_{i}^{\alpha}= \pm m$ and $w_{\alpha}=1 / 2$. Within the TAP approach one proposes

$$
\begin{equation*}
w_{\alpha}^{J}=\frac{e^{-\beta F_{\alpha}^{J}}}{\sum_{\gamma} e^{-\beta F_{\gamma}^{J}}} \tag{7.43}
\end{equation*}
$$

with $F_{\alpha}^{J}$ the total free-energy of the $\alpha$-solution to the TAP equations. The discrete sum can be transformed into an integral over free-energy densities, introducing the degeneracy of solutions quantified by the free-energy density dependent complexity:

$$
\begin{equation*}
\langle O\rangle=\frac{1}{Z_{J}} \int d f e^{-N \beta f} \mathcal{N}_{J}(f, T) O(f)=\frac{1}{Z_{J}} \int d f e^{-N\left(\beta f-\Sigma_{J}(f, T)\right)} O(f) \tag{7.44}
\end{equation*}
$$

The normalization is the 'partition function'

$$
\begin{equation*}
Z_{J}=\int d f e^{-N \beta f} \mathcal{N}_{J}(f, T)=\int d f e^{-N\left(\beta f-\Sigma_{J}(f, T)\right)} \tag{7.45}
\end{equation*}
$$

We assumed that the labelling by $\alpha$ can be traded by a labelling by $f$ that implies that at the same free-energy density level $f$ the observable $O$ takes the same value. In the
$N \rightarrow \infty$ limit these integrals can be evaluated by saddle-point, provided the parenthesis is positive. In order to simplify the calculations, the disorder-dependent complexity is generally approximated with the annealed value introduced in eq. (7.41).

## The equilibrium free-energy

The total equilibrium free-energy density, using the saddle-point method to evaluate the partition function $Z_{J}$ in eq. (7.45), reads

$$
\begin{equation*}
-\beta f_{\mathrm{eq}}^{J}=N^{-1} \ln Z_{J}=\min _{f}\left[f-k_{B} T \Sigma_{J}(f, T)\right] \equiv \min _{f} \Phi_{J}(f, T) \tag{7.46}
\end{equation*}
$$

It is clear that $\Phi_{J}(f, T)$ is the Landau free-energy density of the problem with $f$ playing the rôle of the energy and $\Sigma_{J}$ of the entropy. If we use $f=\left(E-k_{B} T S\right) / N=e-T s$ with $E$ the actual energy and $S$ the microscopic entropy one has

$$
\begin{equation*}
\Phi_{J}(f, T)=e-k_{B} T\left(s+\Sigma_{J}(f, T)\right) \tag{7.47}
\end{equation*}
$$

Thus, $\Sigma_{J}$ is an extra contribution to the total entropy that is due to the exponentially large number of metastable states. Note that we do not distinguish here their stability.

Note that $\Sigma_{J}$ is subtracted from TAP free-energy level $f$. Thus, it is possible that in some cases states lying at a higher free-energy density $f$ but being very numerous have a lower total Landau free-energy density $\Phi$ than lower lying states that are less numerous. Collectively, higher states dominate the equilibrium measure in these cases. This phenomenon actually occurs in $p$-spin models, as explained below.

## The order parameter

Now that we know that there can be a large number of states (defined as extrema of the TAP free-energy) we have to be careful about the definition of the spin-glass order parameter.

The Edwards-Anderson parameter is understood as a property of a single state. Within the TAP formalism one then has

$$
\begin{equation*}
q_{J E A}^{\alpha}=N^{-1} \sum_{i}\left(m_{i}^{\alpha}\right)^{2} \quad \text { with } \quad m_{i}^{\alpha}=\left\langle s_{i}\right\rangle_{\alpha} \tag{7.48}
\end{equation*}
$$

being restricted to spin configurations in state $\alpha$. An average of this quantity over all extrema of the free-energy density yields $\sum_{\alpha} w_{\alpha}^{J} q_{J_{\mathrm{EA}}}^{\alpha}=\sum_{\alpha} w_{\alpha}^{J} N^{-1} \sum_{i}\left(m_{i}^{\alpha}\right)^{2}$.

Instead, the statistical equilibrium magnetisation, $m_{i}=\left\langle s_{i}\right\rangle=\sum_{\alpha} w_{\alpha}^{J} m_{i}^{\alpha}$, squared is

$$
\begin{equation*}
q_{J} \equiv\left\langle s_{i}\right\rangle^{2}=m_{i}^{2}=\left(\sum_{\alpha} w_{\alpha}^{J} m_{i}^{\alpha}\right)^{2}=\sum_{\alpha \beta} w_{\alpha}^{J} w_{\beta}^{J} m_{i}^{\alpha} m_{i}^{\beta} . \tag{7.49}
\end{equation*}
$$

If there are multiple phases, the latter sum has crossed contributions from terms with $\alpha \neq \beta$. These sums, as in a usual paramagnetic-ferromagnetic transition have to be taken
over half space-space, otherwise global up-down reversal would imply the cancellation of all cross-terms.

Clearly

$$
\begin{equation*}
q_{J E A}^{\alpha} \neq q_{J} \quad \text { and } \quad \sum_{\alpha} w_{\alpha}^{J} q_{J \mathrm{EA}}^{\alpha} \neq q_{J} \tag{7.50}
\end{equation*}
$$

### 7.3 Metastable states in two families of models

### 7.3.1 High temperatures

For all models, at high temperatures $f\left(m_{i}\right)$ is characterized by a single stable absolute minimum in which all local magnetizations vanish, as expected; this is the paramagnetic state. The $m_{i}=0$ for all $i$ minimum continues to exist at all temperatures. However, even if it is still the global absolute minimum of the TAP free-energy density, $f_{J} \mathrm{TAP}$, at low temperatures it becomes unstable thermodynamically, and it is substituted as the equilibrium state, by other non-trivial configurations with $m_{i} \neq 0$ that are the absolute minima of $\Phi$. Note the difference with the ferromagnetic problem for which the paramagnetic solution is no longer a minimum below $T_{c}$.

### 7.3.2 Low temperatures

At low temperature many equilibrium states appear (and not just two as in an Ising ferromagnetic model) and they are not related by symmetry (as spin reversal in the Ising ferromagnet or a rotational symmetry in the Heisenberg ferromagnet). These are characterized by non-zero values of the local magnetizations $m_{i}$ that are different in different states.

At low-temperatures both the naive mean-field equations and the TAP equations have an exponential in $N$ number of solutions and still an exponential in $N$ number of them correspond to absolute minima of the $m_{i}$-dependent free-energy density. This means that $\Sigma_{J}(T)$ and even $\Sigma_{J}\left(0, f_{0}, T\right)$ are quantities $O(1)$. These minima can be identified as different states that could be accessed by applying the corresponding site-dependent pinning fields.

The derivation and understanding of the structure of the TAP free-energy landscape is quite subtle and goes beyond the scope of these Lectures. Still, we shall briefly present their structure for the SK and $p$-spin models to give a flavor of their complexity.

## The SK model

The first calculation of the complexity in the SK model appeared in 1980 [?, 82]. After 35 years of research the structure of the free-energy landscape in this system is still a matter of discussion. At present, the picture that emerges is the following. The temperature-dependent annealed complexity is a decreasing function of temperature that vanishes only at $T_{c}$ but takes a very small value already at $\sim 0.6 T_{c}$. Surprisingly enough, at finite but large $N$ the TAP solutions come in pairs of minima and saddles of type one,


Figure 7.7: Left: sketch of the temperature dependent complexity, $\Sigma_{J}(T)$, of the SK. It actually vanishes only at $T_{c}$ but it takes a very small value already at $\sim 0.6 T_{c}$. Right: pairs of extrema in the SK model with $N$ large and $N \rightarrow \infty$ limit.
that is to say, extrema with only one unstable direction. These states are connected by a mode that is softer the larger the number of spins: they coalesce and become marginally stable in the limit $N \rightarrow \infty$. Numerical simulations show that starting from the saddlepoint and following the 'left' direction along the soft mode one falls into the minimum; instead, following the 'right' direction along the same mode one falls into the paramagnetic solution. See Fig. 7.7 for a sketch of these results. The free-energy difference between the minimum and saddle decreases for increasing $N$ and one finds, numerically, an averaged $\Delta f \sim N^{-1.4}$. The extensive complexity of minima and type-one saddles is identical in the large $N$ limit, $\Sigma_{J}(0, T)=\Sigma_{J}(1, T)+O\left(N^{-1}\right)[83]$ in such a way that the Morse theorem is respected. The free-energy dependent annealed complexity is a smooth function of $f$ with support on a finite interval $\left[f_{0}, f_{1}\right]$ and maximum at $f_{\text {max }}$. The Bray and Moore annealed calculation (with supersymmetry breaking) yields $f_{\max }=-0.654, \Sigma_{J}^{\max }=0.052$, $\Sigma^{\prime \prime}\left(f_{\max }\right)=8.9$. The probability of finding a solution with free-energy density $f$ can be expressed as

$$
\begin{equation*}
p_{J}(f, T)=\frac{\mathcal{N}_{J}(f, T)}{\mathcal{N}_{J}(T)}=\frac{e^{N \Sigma_{J}(f, T)}}{\mathcal{N}_{J}(T)} \sim \sqrt{\frac{N \Sigma_{J}^{\prime \prime}\left(f_{\max }\right)}{2 \pi}} e^{-\frac{N}{2}\left|\Sigma_{J}^{\prime \prime}\left(f_{\max }\right)\right|\left(f-f_{\max }\right)^{2}}, \tag{7.51}
\end{equation*}
$$

where we evaluated the total number of solutions, $\mathcal{N}_{J}(T)=\int d f e^{N \Sigma_{J}(f, T)}$, by steepest descent. The complexity, approximated by its annealed value, vanishes linearly close to $f_{0}: \Sigma_{J}(f, T) \sim \lambda\left(f-f_{0}\right)$ with $\lambda<\beta$.

Only the lowest lying TAP solutions contribute to the statistical weight. The complexity does not contribute to $\Phi$ in the large $N$ limit:

$$
\begin{align*}
\Phi & =\beta f-\Sigma_{\mathrm{ann}}(f, T) \simeq \beta f-\left(f-f_{0}\right) \lambda \\
\frac{\partial \Phi}{\partial f} & \simeq \beta-\lambda>0 \quad \text { iff } \quad \beta>\lambda \tag{7.52}
\end{align*}
$$



Figure 7.8: The complexity as a function of $f$ for the SK model.
and $\Phi_{\min } \simeq \beta f_{\text {min }}=\beta f_{0}$. See Fig. 7.8. The 'total' free-energy density in the exponential is just the free-energy density of each low-lying solution.

## The (spherical) p-spin model

The number and structure of saddle-points is particularly interesting in the $p \geq 3$ cases and it is indeed the reason why these models, with a random first order transition, have been proposed to mimic the structural glass arrest. The $p \geq 3$ model has been studied in great detail in the spherical case, that is to say, when spins are not Ising variables but satisfy the global constraint, $\sum_{i=1}^{N} s_{i}^{2}=N$.

Although in general the minima of the mean-field free energy do not coincide with the minima of the Hamiltonian, they do in the spherical $p$-spin model. Their positions in the phase space does not depend on temperature, while their self-overlap does. At $T=0$ a state (stable or metastable) is just a minimum (absolute or local) of the energy. For increasing $T$ energy minima get dressed up by thermal fluctuations, and become states but they do nor cross nor merge. Thus, the states can be labeled by their zero-temperature energy density $e^{0}$.

The complexity is given by

$$
\begin{equation*}
\Sigma(e)=\frac{1}{2}\left[-\ln \frac{p z^{2}}{2}+\frac{p-1}{2} z^{2}-\frac{2}{p^{2} z^{2}}+\frac{2-p}{p}\right] \tag{7.53}
\end{equation*}
$$

where $z$ is

$$
\begin{equation*}
z=\left[-e^{0}-\sqrt{e^{0^{2}}-e_{c}^{2}}\right] /(p-1) \tag{7.54}
\end{equation*}
$$

The complexity vanishes at

$$
\begin{equation*}
e^{0}=e_{\min }=f(p), \tag{7.55}
\end{equation*}
$$



Figure 7.9: The TAP free-energy as a function of $T$ in the spherical $p$-spin model. (1) : free energy of the paramagnetic solution for $T>T^{*}, f_{\mathrm{tot}}$ for $T<T^{*} ;(2)$ : free energy of the lowest TAP states, with zero temperature energy $e_{\min } ;(3)$ : free energy of the highest TAP states, corresponding to $e_{c} ;(4)$ : an intermediate value of $e_{0}$ leads to an intermediate value of $f$ at any temperature; (5) : $f_{\text {eq }}(T)$; the difference between curves (5) and (1) gives the complexity $T S_{c}\left(f_{\mathrm{eq}}(T), T\right)$ [93].
the ground state of the system, and it is real for zero-temperature energies $e<e_{\mathrm{th}}$ with

$$
\begin{equation*}
e_{\mathrm{th}}=e_{c}=-\sqrt{\frac{2(p-1)}{p}} \tag{7.56}
\end{equation*}
$$

$e_{\text {min }}$ is the zero- $T$ energy density that one finds with the replica calculation using a 1-step RSB Ansatz, as we shall see below. The finite- $T$ energy density of a state $\alpha$ is

$$
\begin{equation*}
e_{\alpha}=q_{\alpha}^{\frac{p}{2}} e_{\alpha}^{0}-\frac{1}{2 T}\left[(p-1) q_{\alpha}^{p}-p q_{\alpha}^{p-1}+1\right] \tag{7.57}
\end{equation*}
$$

This means that

- There can be only a finite number of states with $e<e_{0}$.
- It can be shown that below $e_{\mathrm{th}}$ minima dominate on average.
- Above $e_{\text {th }}$ one can show that there are states but these are unstable.

Each zero-temperature state is characterized by a unit $N$-vector $s_{i}^{\alpha}$ and it gives rise to a finite- $T$ state characterized by $m_{i}^{\alpha}=\sqrt{q(e, T)} s_{i}^{\alpha}$ with $q(e, T)$ given by

$$
\begin{equation*}
q^{p-2}(1-q)^{2}=T^{2} \frac{\left(e+\sqrt{e^{2}-e_{\mathrm{th}}^{2}}\right)^{2}}{(p-1)^{2}} . \tag{7.58}
\end{equation*}
$$

$(q(e, T=0)=1$ and at finite $T$ the solution with $q$ closest to 1 has to be chosen.) The self-overlap at the threshold energy, $e-e_{\mathrm{th}}$, is then

$$
\begin{equation*}
q_{\mathrm{th}}^{p-2}\left(1-q_{\mathrm{th}}\right)^{2}=T^{2} \frac{2}{p(p-1)} \tag{7.59}
\end{equation*}
$$

Another way for the $q$ equation to stop having solution, is by increasing the temperature, $T>T_{\max }\left(e^{0}\right)$, at fixed bare energy $e^{0}$. This means that, even though minima of the energy do not depend on the temperature, states, i.e. minima of the free energy do. When the temperature becomes too large, the paramagnetic states becomes the only pure ergodic states, even though the energy landscape is broken up in many basins of the energy minima. This is just one particularly evident demonstration of the fundamental different between pure states and energy minima. $T_{\max }\left(e^{0}\right)$ is obtained as the limiting temperature for which eq. (7.58) admits a solution. It is given by

$$
\begin{equation*}
T_{\max }\left(e^{0}\right)=\left(\frac{2}{p}\right)\left(\frac{p-1}{-e^{0}-\sqrt{e^{0^{2}}-e_{\mathrm{th}}^{2}}}\right)\left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} . \tag{7.60}
\end{equation*}
$$

$T_{\max }$ is a decreasing function of $e^{0}$. The last states to disappear then are the ones with minimum energy $e_{\min }$, ceasing to exist at $T_{\mathrm{TAP}} \equiv T_{\max }\left(e_{\min }\right)$.

Below a temperature $T_{d}$, an exponential (in $N$ ) number of metastable states contribute to the thermodynamics in such a non-trivial way that their combined contribution to the observables makes it the one of a paramagnet. Even if each of these states is non-trivial (the $m_{i}$ 's are different from zero) the statistical average over all of them yields results that are identical to those of a paramagnet. For example, the free-energy density is $-1 /(4 T)$ as in the $m_{i}=0$ paramagnetic solution. One finds

$$
\begin{equation*}
T_{d}=\sqrt{\frac{p(p-2)^{p-2}}{2(p-1)^{p-1}}} \tag{7.61}
\end{equation*}
$$

In the $p$-spin models there is a range of temperatures in which high lying states dominate this sum since they are sufficiently numerous so as to have a complexity that renders the combined term $\beta f-\Sigma_{J}(f, T)$ smaller (in actual calculations the disorder dependent complexity is approximated by its annealed value).

At a lower temperature $T_{s}\left(T_{s}<T_{d}\right)$ there is an entropy crisis, less than an exponential number of metastable states survive, and there is a static phase transition to a glassy state.

In short:

- Above $T_{d}$ the (unique) paramagnetic solution dominates, $q=0$ and $\Phi=f=$ $-1 /(4 T)$.
- In the interval $T \in\left[T_{s}, T_{d}\right]$ an exponentially large number of states (with $q \neq 0$ given by the solution to $\left.p q^{p-2}(1-q)=2 T^{2}\right)$ dominate the partition sum. $\Phi=-1 /(4 T)$ appearing as the continuation of the paramagnetic solution.
- At $T<T_{s}$ the lowest TAP states with $e^{0}=e_{\text {min }}$ control the partition sum. Their total free-energy $\Phi$ is different from $-1 /(4 T)$.

This picture is confirmed with other analytical studies that include the use of pinning fields adapted to the disordered situation [84], the effective potential for two coupled real replicas [85], and the dynamic approach [86], and the numerical exhaustive determination of all stationary points of the Hamiltonian in systems with size $N \leq 20$ [87]

Low temperatures, entropy crisis
The interval of definition of $\Phi_{J}(e, T)$ is the same as $\Sigma_{J}(e)$, that is $e \in\left[e_{\min }: e_{\mathrm{th}}\right]$. Assuming that at a given temperature $T$ the energy $e_{\mathrm{eq}}(T)$ minimizing $\Phi_{J}$ lies in this interval, what happens if we lower the temperature? Remember that the complexity is an increasing function of $E$, as of course is $f(e, T)$. When $T$ decreases we favor states with lower free energy and lower complexity, and therefore $e_{\mathrm{eq}}$ decreases. As a result, it must exist a temperature $T_{s}$, such that, $e_{\mathrm{eq}}\left(T_{s}\right)=e_{\text {min }}$ and thus, $\Sigma_{J}\left(e_{\mathrm{eq}}(T)\right)=\Sigma_{J}\left(e_{\min }\right)=0$. Below $T_{s}$ the bare energy $e_{\text {eq }}$ cannot decrease any further: there are no other states below the ground states $e_{\min }$. Thus, $e_{\mathrm{eq}}(T)=e_{\min }$ for each temperature $T \leq T_{s}$. As a result, if we plot the complexity of equilibrium states $\Sigma_{J}\left(e_{\mathrm{eq}}(T)\right)$ as a function of the temperature,
we find a discontinuity of the first derivative at $T_{s}$, where the complexity vanishes. A thermodynamic transition takes place at $T_{s}$ : below this temperature equilibrium is no longer dominated by metastable states, but by the lowest lying states, which have zero complexity and lowest free energy density.

We shall show that $T_{s}$ is the transition temperature found with a replica calculation. The temperature where equilibrium is given for the first time by the lowest energy states, is equal to the static transition temperature. Above $T_{s}$ the partition function is dominated by an exponentially large number of states, each with high free energy and thus low statistical weight, such that they are not captured by the overlap distribution $P(q)$. At $T_{s}$ the number of these states becomes sub-exponential and their weight nonzero, such that the $P(q)$ develops a secondary peak at $q_{s} \neq 0$.

## The threshold

The analysis of the Hessian on the threshold level reveals that these states are saddles with an extensive number of flat directions. The threshold level is then like a large flat plateau in a mountain ladscape.

## Finite dimensions

In finite-dimensional systems, only equilibrium states can break the ergodicity, i.e. states with the lowest free energy density. In other words, the system cannot remain trapped for an infinite time in a metastable state, because in finite dimension free energy barriers surrounding metastable states are always finite.

The extra free energy of a droplet of size $r$ of equilibrium phase in a background metastable phase has a positive interface contribution which grows as $r^{d-1}$, and a negative volume contribution which grows as $r^{d}$,

$$
\begin{equation*}
\Delta f=\sigma r^{d-1}-\delta f r^{d} \tag{7.62}
\end{equation*}
$$

where here $\sigma$ is the surface tension and $\delta f$ is the bulk free energy difference between the two phases. This function has always a maximum, whose finite height gives the free energy barrier to nucleation of the equilibrium phase (note that at coexistence $\delta f=0$ and the barrier is infinite). Therefore, if initially in a metastable states the system will, sooner or later, collapse in the stable state with lower free energy density. For this reason, in finite dimension we cannot decompose the Gibbs measure in metastable components. When this is done, it is always understood that the decomposition is only valid for finite times, i.e times much smaller than the time needed for the stable equilibrium state to take over. On the other hand, in mean-field systems (infinite dimension), barriers between metastable states may be infinite in the thermodynamic limit, and it is therefore possible to call pure states also metastable states, and to assign them a Gibbs weight $w_{\alpha}^{J}$. We will analyse a mean-field spin-glass model, so that we will be allowed to perform the decomposition above even for metastable states.

## Comments

There is a close relationship between the topological properties of the model and its dynamical behavior. In particular, the slowing down of the dynamics above but close to $T_{d}$ is connected to the presence of saddles, whose instability decreases with decreasing energy. In fact, we have seen that the threshold energy level $e_{t h}$ separating saddles from minima, can be associated to the temperature $T_{\mathrm{th}}=T_{d}$, marking the passage from ergodicity to ergodicity breaking. In this context the dynamical transition can be seen as a topological transition. The plateau of the dynamical correlation function, which has an interpretation in terms of cage effect in liquids, may be reinterpreted as a pseudo-thermalization inside a saddle with a very small number of unstable modes.

### 7.4 The replica method

A picture that is consistent with the one arising from the naive mean-field approximation but contradicts the initial assumption of the droplet model arises from the exact solution of fully-connected spin-glass models. These results are obtained using a method called the replica trick that we will briefly present below.

In Sect. 5.2.3 we argued that the typical properties of a disordered system can be computed from the disorder averaged free-energy

$$
\begin{equation*}
\left[F_{J}\right] \equiv \int d J P(J) F_{J} \tag{7.63}
\end{equation*}
$$

One then needs to average the logarithm of the partition funtion. In the annealed approximation one exchanges the ln with the average over disorder and, basically, considers the interactions equilibrated at the same temperature $T$ as the spins:

$$
\begin{equation*}
\left[\ln Z_{J}\right] \sim \ln \left[Z_{J}\right] \tag{7.64}
\end{equation*}
$$

This approximation turns out to be correct at high temperatures but incorrect at low ones.

The replica method allows one to compute $\left[F_{J}\right]$ for fully-connected models. It is based on the smart use of the identity

$$
\begin{equation*}
\ln Z_{J}=\lim _{n \rightarrow 0} \frac{Z_{J}^{n}-1}{n} \tag{7.65}
\end{equation*}
$$

The idea is to compute the right-hand-side for finite and integer $n=1,2, \ldots$ and then perform the analytic continuation to $n \rightarrow 0$. Care should be taken in this step: for some models the analytic continuation may not be unique. (Recall the calculation done using the Potts model with $q \rightarrow 1$ that allows one to recover results for the percolation problem.) It turns out that this is indeed the case for the emblematic Sherrington-Kirkpatrick model, as discussed by van Hemmen and Palmer [88] though it has also been recently shown that the free-energy $f(T)$ obtained by Parisi [89] with the replica trick is exact! [90, 91]

The disorder averaged free-energy is given by

$$
\begin{equation*}
-\beta\left[F_{J}\right]=-\int d J P(J) \ln Z_{J}=-\lim _{n \rightarrow 0} \frac{1}{n}\left(\int d J P(J) Z_{J}^{n}-1\right) \tag{7.66}
\end{equation*}
$$

where we have exchanged the limit $n \rightarrow 0$ with the integration over the exchanges. For integer $n$ the replicated partition function, $Z_{J}^{n}$, reads

$$
\begin{equation*}
Z_{J}^{n}=\sum_{\left\{s_{i}^{a}\right\}} e^{-\beta\left[H_{J}\left(\left\{s_{i}^{1}\right\}\right)+\cdots+H_{J}\left(\left\{s_{i}^{n}\right\}\right]\right.} \tag{7.67}
\end{equation*}
$$

Here $\sum_{\left\{s_{i}^{a}\right\}} \equiv \sum_{\left\{s_{i}^{1}= \pm 1\right\}} \cdots \sum_{\left\{s_{i}^{n}= \pm 1\right\}} . Z_{J}^{n}$ corresponds to $n$ identical copies of the original system, that is to say, all of them with the same realisation of disorder. Each copy is characterised by an ensemble of $N$ spins, $\left\{s_{i}^{a}\right\}$. We label the copies with a replica index $a=1, \ldots, n$. For $p$-spin disordered spin models $Z_{J}^{n}$ takes the form

$$
\begin{equation*}
Z_{J}^{n}=\sum_{\left\{s_{i}^{a}\right\}} e^{\frac{\beta}{p!} \sum_{a=1}^{n}\left[\sum_{i_{1} \neq \cdots \neq i_{p}} J_{\left.i_{1} \ldots i_{p} s_{i_{1}}^{a} \ldots s_{i_{p}}^{a}+\sum_{i} h_{i} s_{i}^{a}\right]} . . . . .\right.} \tag{7.68}
\end{equation*}
$$

The average over disorder amounts to computing a Gaussian integral for each set of spin indices $i_{1} \neq \cdots \neq i_{p}$, with $\left[J_{i_{1} \ldots i_{p}}^{2}\right]=J^{2} p!/\left(2 N^{p-1}\right)^{6}$. One finds

$$
\begin{equation*}
\left[Z_{J}^{n}\right]=\sum_{\left\{s_{i}^{a}\right\}} e^{\frac{\beta^{2} J^{2}}{2 N^{p-1} p!} \sum_{i_{1} \neq \cdots \neq i_{p}}\left(\sum_{a} s_{i_{1}}^{a} \ldots s_{i_{p}}^{a}\right)^{2}+\beta \sum_{a} \sum_{i} h_{i} s_{i}^{a}} \equiv \sum_{\left\{s_{i}^{a}\right\}} e^{-\beta F\left(\left\{s_{i}^{a}\right\}\right)} \tag{7.69}
\end{equation*}
$$

The function $\beta F\left(\left\{s_{i}^{a}\right\}\right)$ is not random. It depends on the spin variables only but it includes terms that couple different replica indices. Indeed,

$$
\begin{equation*}
\sum_{i_{1} \neq \cdots \neq i_{p}} \sum_{a} s_{i_{1}}^{a} \ldots s_{i_{p}}^{a} \sum_{b} s_{i_{1}}^{b} \ldots s_{i_{p}}^{b}=\sum_{a b} \sum_{i_{1} \neq \cdots \neq i_{p}}\left(s_{i_{1}}^{a} s_{i_{1}}^{b}\right) \ldots\left(s_{i_{p}}^{a} s_{i_{p}}^{b}\right) \tag{7.70}
\end{equation*}
$$

We first note that all terms are identical to one for $a=b$ since $s_{i}^{2}=1$. The sum over the spin indices (ignoring the constraint $i_{1} \neq \cdots \neq i_{p}$ that will, in any case give a subdominant contribution in the $N \rightarrow \infty$ limit) and the replica indices of such terms yields $N^{p} n$. Focusing then on the cases $a \neq b$, Ignoring the constraint $i_{1} \neq \cdots \neq i_{p}$, there are $N$ factors here

$$
\begin{equation*}
\beta F\left(\left\{s_{i}^{a}\right\}\right) \approx-\frac{N \beta^{2} J^{2}}{2 p!}\left[\sum_{a \neq b}\left(\frac{1}{N} \sum_{i} s_{i}^{a} s_{i}^{b}\right)^{p}+n\right]-\beta \sum_{a} \sum_{i} h_{i} s_{i}^{a} \tag{7.71}
\end{equation*}
$$

In writing the last expression we have dropped terms that are sub-leading in $N$. (In complete analogy with what is done for the pure $p$ spin ferromagnet. They correspond to

[^5]adding terms with self-interactions in the Hamiltonian.) The constant term $-N n \beta^{2} J^{2} / 2$ originates in the terms with $a=b$, for which $\left(s_{i}^{a}\right)^{2}=1$.

To summarise, we started with an interacting spin model. Next, we enlarged the number of variables from $N$ spins to $N \times n$ replicated spins by introducing $n$ non-interacting copies of the system. By integrating out the disorder we decoupled the sites but we payed the price of coupling the replicas. Hitherto the replica indices act as a formal tool introduced to compute the average over the bond distribution. Nothing distinguishes one replica from another and, in consequence, the 'free-energy' $F\left(\left\{s_{i}^{a}\right\}\right)$ is invariant under permutations of the replica indices.

The next step to follow is to identify the order parameters and transform the freeenergy into an order-parameter dependent expression to be rendered extremal at their equilibrium values. In a spin-glass problem we already know that the order parameter is not the global magnetisation as in a pure magnetic system but the parameter $q$ - or more generally the overlap between states. Within the replica calculation an overlap between replicas

$$
\begin{equation*}
q_{a b} \equiv N^{-1} \sum_{i} s_{i}^{a} s_{i}^{b} \tag{7.72}
\end{equation*}
$$

naturally appeared in eq. (7.71). The idea is to write the free-energy density as a function of the order parameter $q_{a b}$ and look for their extreme in complete analogy with what is done for the fully-connected ferromagnet, see B. This is, of course, a tricky business, since the order parameter is here a matrix with number of elements $n$ going to zero! A recipe for identifying the form of the order parameter (or the correct saddle-point solution) has been proposed by G. Parisi in the late 70s and early 80s [89]. This solution has been recently proven to be exact for mean-field models by two mathematical physics, F. Guerra [90] and M. Talagrand [91]. Whether the very rich physical structure that derives from this rather formal solution survives in finite dimensional systems remains a subject of debate.

Introducing the Gaussian integral (Hubbard-Stratonovich transformation)

$$
\begin{equation*}
\int d q_{a b} e^{\beta J q_{a b} \sum_{i} s_{i}^{a} s_{i}^{b}-\frac{N}{2} q_{a b}^{2}}=e^{\frac{N}{2}\left(\frac{1}{N} \beta J \sum_{i} s_{i}^{a} s_{i}^{b}\right)^{2}} \tag{7.73}
\end{equation*}
$$

for each pair of replica indices $a \neq b$, one decouples the site indices, $i$, and the averaged replicated partition function can be rewritten as

$$
\begin{equation*}
\left[Z_{J}^{n}\right]=\int \prod_{a \neq b} d q_{a b} e^{-\beta F\left(q_{a b}\right)} \tag{7.74}
\end{equation*}
$$

and

$$
\begin{align*}
\beta F\left(q_{a b}\right) & =-\frac{N \beta^{2} J^{2}}{2}\left[-\sum_{a \neq b} q_{a b}^{p}+n\right]-N \ln \zeta\left(q_{a b}\right),  \tag{7.75}\\
\zeta\left(q_{a b}\right) & =\sum_{s_{a}} e^{-\beta H\left(q_{a b}, s_{a}\right)},  \tag{7.76}\\
H\left(q_{a b}, s_{a}\right) & =-J \sum_{a b} q_{a b} s_{a} s_{b}-h \sum_{a} s_{a}, \tag{7.77}
\end{align*}
$$

where for simplicity we set $h_{i}=h$. The factor $N$ in front of $\ln \zeta$ comes from the decoupling of the site indices. Note that the transformation (7.73) serves to uncouple the sites and to obtain the very useful factor $N$ in front of the exponential. The partition function

$$
\begin{equation*}
Z\left(q_{a b}\right)=\sum_{\left\{s_{a}\right\}} e^{-\beta H\left(q_{a b}, s_{a}\right)} \tag{7.78}
\end{equation*}
$$

is the one of a fully-connected Ising model with interaction matrix $q_{a b}$. As it is posed, this problem remains unsolvable. However, important steps forward will be possible taking advantage of the $n \rightarrow 0$ limit.

## Saddle-point evaluation

Having extracted a factor $N$ in the exponential suggests to evaluate the integral over $q_{a b}$ with the saddle-point method. This, of course, involves the a priori dangerous exchange of limits $N \rightarrow \infty$ and $n \rightarrow 0$. The replica theory relies on this assumption. One then writes

$$
\begin{equation*}
\lim _{N \rightarrow \infty}-\left[f_{J}\right] \rightarrow-\lim _{n \rightarrow 0} \frac{1}{n} f\left(q_{a b}^{\mathrm{sp}}\right) \tag{7.79}
\end{equation*}
$$

and searches for the solutions to the $n(n-1) / 2$ extremization equations

$$
\begin{equation*}
\left.\frac{\delta f\left(q_{a b}\right)}{\delta q_{c d}}\right|_{q_{e f}^{\mathrm{sp}}}=0 . \tag{7.80}
\end{equation*}
$$

In usual saddle-point evaluations the saddle-point one should use is (are) the one(s) that correspond to absolute minima of the free-energy density. In the replica calculation the number of variables is $n(n-1) / 2$ that becomes negative! when $n<1$ and makes the saddle-point evaluation tricky. In order to avoid unphysical complex results one needs to focus on the saddle-points with positive (or at least semi-positive) definite Hessian

$$
\begin{equation*}
\left.\mathcal{H} \equiv \frac{\partial f\left(q_{a b}\right)}{\partial q_{c d} \partial q_{e f}}\right|_{q_{g h}^{\text {sp }}}, \tag{7.81}
\end{equation*}
$$

and these sometimes corresponds to maxima (instead of minima) of the free-energy density.

The saddle-point equations are also self-consistency equations

$$
\begin{equation*}
q_{a b}^{\mathrm{sp}}=\left\langle s_{a} s_{b}\right\rangle_{H\left(q_{a b},\left\{s_{a}\right\}\right)}=\left[\left\langle s_{a} s_{b}\right\rangle\right] \tag{7.82}
\end{equation*}
$$

where the second member means that the average is performed with the single site manyreplica Hamiltonian $H\left(q_{a b},\left\{s_{a}\right\}\right)$ and the third member is just one of the averages we would like to compute.

The partition function in eq. (7.77) cannot be computed for generic $q_{a b}$ since there is no large $n$ limit to exploit. On the contrary, $n \rightarrow 0$. Thus, one usually looks for solutions to eqs. (7.80) within a certain family of matrices $q_{a b}$. We discuss below the relevant parametrizations.

Replica symmetry ( $R S$ )
In principle, nothing distinguishes one replica from another one. This is the reason why Sherrington and Kirkpatrick looked for solutions that preserve replica symmetry:

$$
\begin{equation*}
q_{a b}=q, \quad \text { for all } a \neq b \tag{7.83}
\end{equation*}
$$

Inserting this Ansatz in (7.75) and (7.77) and taking $n \rightarrow 0$ one finds

$$
\begin{equation*}
q=\int_{-\infty}^{\infty} \frac{d z}{\sqrt{2 \pi}} e^{-z^{2} / 2} \tanh ^{2}\left(\beta \sqrt{\frac{p q^{p-1}}{2}} z+\beta h\right) \tag{7.84}
\end{equation*}
$$

This equation resembles strongly the one for the magnetisation density of the $p$-spin ferromagnet, eq. (B.3.30). The free-energy density for $p=2$ is

$$
\begin{equation*}
f=-\frac{\beta}{4}(1-q)^{2}-\int_{-\infty}^{\infty} \frac{d z}{\sqrt{2 \pi}} e^{-z^{2} / 2} \ln \left[2 \cosh \left(\beta q^{1 / 2} z+\beta h\right)\right] . \tag{7.85}
\end{equation*}
$$

Let us first discuss the case $p=2$, i.e. the SK model. In the absence of a magnetic field, one finds a second order phase transition at $T_{s}=J$ from a paramagnetic $(q=0)$ to a spin-glass phase with $q \neq 0$. In the presence of a field there is no phase transition. SK soon realized though that there is something wrong with this solution: the entropy at zero temperature is negative, $S(0)=-1 /(2 \pi)$, and this is impossible for a model with discrete spins, for which $S$ is strictly positive. de Almeida and Thouless later showed that the reason for this failure is that the replica symmetric saddle-point is not stable, since the Hessian (7.81) is not positive definite and has negative eigenvalues [92]. The eigenvalue responsible for the instability of the replica symmetric solution is called the replicon.

Comparison with the TAP equations shows that the RS Ansatz corresponds to the assumption that the local fields $h_{i}=\sum_{i_{i_{1} \ldots i_{i_{p}}}} J_{i_{1} \ldots i_{p}} m_{i_{1}} \ldots m_{i_{p}}+h$ are independent and have a Gaussian distribution with average $h$ and variance $\sigma^{2}=J^{2} q^{p-1}$. Numerical simulations clearly show that this assumption is invalid.

Interestingly enough, the numerical values for several physical quantities obtained with the replica symmetric solution do not disagree much with numerical results. For instance,


Figure 7.10: Left: a one-step replica symmetry breaking (1RSB) Ansatz. Right: a two-step replica symmetry breaking Ansatz. The elements on the main diagonal vanish identically. In the 1 RSB case the diagonal blocks have size $m \times m$. In the 2 RSB the proceudre is repeated and one has blocks of size $m_{1} \times m_{1}$ with smaller diagonal blocks of size $m_{2} \times m_{2}$.
the ground state zero-temperature energy density is $E_{\mathrm{gs}}^{\mathrm{rs}}=-0.798$ while with numerical simulations one finds $E_{\mathrm{gs}} \sim-0.76$.

For the $p>2$ model one finds that the replica symmetric solution is stable at all temperatures. However, the problem of the negative entropy remains and should be solved by another solution. The transition must then have aspects of a first-order one, with another solution appearing at low temperatures and becoming the most convenient one at the transition.

## One step replica symmetry breaking

The next challenge is to device a replica symmetry breaking Ansatz, in the form of a matrix $q_{a b}$ that is not invariant under permutations of rows or columns. There is no first principles way of doing this, instead, the structure of the Ansatz is the result of trial and error. Indeed, a kind of minimal way to break the replica symmetry is to propose a structure in blocks as the one shown in Fig. 7.10-left. The diagonal elements are set to zero as in the RS case. Square blocks of linear size $m$ close to the main diagonal are filled with a paramater $q_{1}$. The elements in the rest of the matrix take a different value $q_{0}$ and one takes $0 \leq q_{0} \leq q_{1}$. The matrix $q_{a b}$ depends on three parameters $q_{0}, q_{1}, m$ and one has to find the values such that the free-energy density is maximized! The conditions for a extreme are

$$
\begin{equation*}
\frac{\partial f\left(q_{0}, q_{1}, m\right)}{\partial q_{0}}=\frac{\partial f\left(q_{0}, q_{1}, m\right)}{\partial q_{1}}=\frac{\partial f\left(q_{0}, q_{1}, m\right)}{\partial m}=0 . \tag{7.86}
\end{equation*}
$$

In the SK model $(p=2)$ the 1RSB Ansatz yields a second order phase transition ( $q_{0}=q_{1}=0$ and $m=1$ at criticality) at a critical temperature $T_{s}=J$, that remains unchanged with respect to the one predicted by the RS Ansatz. The 1RSB solution is still unstable below $T_{s}$ and in all the low temperature phase. One notices, however, that
the zero temperature entropy, even if still negative and incorrect, takes a value that is closer to zero, $S(T=0) \approx-0.01$, the ground state energy is closer to the value obtained numerically, and the replicon eigenvalue even if still negative has an absolute value that is closer to zero. All this suggest that the 1RSB Ansatz is closer to the exact solution.

Instead, in all cases with $p \geq 3$ the 1RSB Ansatz is stable below the static critical temperature $T_{s}$ and all the way up to a new characteristic temperature $0<T_{f}<T_{s}$. Moreover, one can prove that in this range of temperatures the model is solved exactly by this Ansatz. The critical behaviour is quite peculiar: while the order parameters $q_{0}$ and $q_{1}$ jump at the transition from a vanishing value in the paramagnetic phase to a non-zero value right below $T_{s}$, all thermodynamic quantities are continuous since $m=1$ at $T_{s}$ and all $q_{0}$ and $q_{1}$ dependent terms appear multiplied by $1-m$. This is a mixed type of transition that has been baptised random first-order. Note that disorder weakens the critical behaviour in the $p \geq 3$-spin models. In the limit $p \rightarrow \infty$ the solutions become $m=T / T_{c}, q_{0}=0$ and $q=1$.

## $k$-step replica symmetry breaking

The natural way to generalize the 1RSB Ansatz is to propose a $k$-step one. In each step the off-diagonal blocks are left unchanged while the diagonal ones of size $m_{k}$ are broken as in the first step thus generating smaller square blocks of size $m_{k+1}$, close to the diagonal. At a generic $k$-step RSB scheme one has

$$
\begin{gather*}
0 \leq q_{0} \leq q_{1} \leq \cdots \leq q_{k-1} \leq q_{k} \leq 1  \tag{7.87}\\
n=m_{0} \geq m_{1} \geq \cdots \geq m_{k} \geq m_{k+1} \tag{7.88}
\end{gather*}
$$

parameters. In the $n \rightarrow 0$ limit the ordering of the parameters $m$ is reversed

$$
\begin{equation*}
0=m_{0} \leq m_{1} \leq \cdots \leq m_{k} \leq m_{k+1} \tag{7.89}
\end{equation*}
$$

In the SK model one finds that any finite $k$-step RSB Ansatz remains unstable. However, increasing the number of breaking levels the results continue to improve with, in particular, the zero temperature entropy getting closer to zero. In the $p \geq 3$ case instead one finds that the 2RSB Ansatz has, as unique solution to the saddle-point equations, one that boils down to the 1RSB case. This suggests that the 1RSB Ansatz is stable as can also be checked with the analysis of the Hessian eigenvalues: the replicon is stricly positive for all $p \geq 3$.

## Full replica symmetry breaking

In order to construct the full RSB solution the breaking procedure is iterated an infinite number of times. The full RSB Ansatz thus obtained generalizes the block structure to an infinite sequence by introducing a function

$$
\begin{equation*}
q(x)=q_{i}, \quad m_{i+1}<x<m_{i} \tag{7.90}
\end{equation*}
$$



Figure 7.11: The function $q(x)$ for a replica symmetric (left), one step replica symmetry breaking (center) and full replica symmetry breaking Ansätze.
with $0 \leq x \leq 1$. Introducing $q(x)$ sums over replicas are traded by integrals over $x$; for instance

$$
\begin{equation*}
\frac{1}{n} \sum_{a \neq b} q_{a b}^{l}=\int_{0}^{1} d x q^{l}(x) \tag{7.91}
\end{equation*}
$$

The free-energy density becomes a functional of the function $q(x)$. The extremization condition is then a hard functional equation. A Landau expansion - expected to be valid close to the assumed second order phase transition - simplifies the task of solving it. For the SK model one finds

$$
q(x)= \begin{cases}\frac{x}{2}, & 0 \leq x \leq x_{1}=2 q(1)  \tag{7.92}\\ q_{\mathrm{EA}} \equiv q_{\max }=q(1), & x_{1}=2 q(1) \leq x \leq 1\end{cases}
$$

at first order in $\left|T-T_{c}\right|$, with $q(1)=\left|T-T_{c}\right| / T_{c}$ and $x_{1}=2 q(1)$. The stability analysis yields a vanishing replicon eigenvalue signalling that the full RSB solution is marginally stable.

One can also recover the particular case of the 1RSB using a $q(x)$ with two plateaux, at $q_{0}$ and $q_{1}$ and the breaking point at $x=m$.

## Marginality condition

In the discussion above we chose the extreme that maximize the free-energy density since we were interested in studying equilibrium properties. We could, instead, use a different prescription, though a priori not justified, and select other solutions. For example, we can impose that the solution is marginally stable by requiring that the replicon eigenvalue vanishes. In the $p=2$ this leads to identical results to the ones obtained with the usual prescription since the full-RSB Ansatz is in any case marginally stable. In the $p$-spin models with $p \geq 3$ instead it turns out that the averaged properties obtained in this way correspond to the asymptotic values derived with the stochastic dynamics starting from random initial conditions. This is quite a remarkable result.

### 7.4.1 Interpretation of replica results

Let us now discuss the implications of the solution to fully-connected disordered models obtained with the, for the moment, rather abstract replica formalism.

The interpretation uses heavily the identification of pure states. Their definition is a tricky matter that we shall not discuss in detail here. We shall just assume it can be done and use the analogy with the ferromagnetic system - and its two pure states - and the TAP results at fixed disorder. As we already know, which are the pure states, its properties, number, etc. can depend on the quenched disorder realization and fluctuate from sample to sample. We shall keep this in mind in the rest of our discussion.

Let us then distinguish the averages computed within a pure state and over all configuration space. In a ferromagnet with no applied magnetic field this is simple to grasp: at high temperatures there is just one state, the paramagnet, while at low temperatures there are two, the states with positive and negative magnetization. If one computes the averaged magnetization restricted to the state of positive (negative) magnetization one finds $m_{\mathrm{eq}}>0\left(m_{\mathrm{eq}}<0\right)$; instead, summing over all configurations $m_{\mathrm{eq}}=0$ even at low temperatures. Now, if one considers systems with more than just two pure states, and one labels them with Greeks indices, averages within such states are denoted $\langle O\rangle_{\alpha}$ while averages taken with the full Gibbs measure are expressed as

$$
\begin{equation*}
\langle O\rangle=\sum_{\alpha} w_{\alpha}^{J}\langle O\rangle_{\alpha} \tag{7.93}
\end{equation*}
$$

$w_{\alpha}^{J}$ is the probability of the $\alpha$ state given by

$$
\begin{equation*}
w_{\alpha}^{J}=\frac{e^{-\beta F_{\alpha}^{J}}}{Z_{J}}, \quad \text { with } \quad Z_{J}=\sum_{\alpha} e^{-\beta F_{\alpha}^{J}} \tag{7.94}
\end{equation*}
$$

and thus satisfying the normalization condition $\sum_{\alpha} w_{\alpha}^{J}=1 . F_{\alpha}^{J}$ can be interpreted as the total free-energy of the state $\alpha$. These probabilities, as well as the state dependent averages, will show sample-to-sample fluctuations.

One can then define an overlap between states:

$$
\begin{equation*}
q_{J \alpha \beta} \equiv N^{-1} \sum_{i}\left\langle s_{i}\right\rangle_{\alpha}\left\langle s_{i}\right\rangle_{\beta}=N^{-1} \sum_{i} m_{i}^{\alpha} m_{i}^{\beta} \tag{7.95}
\end{equation*}
$$

and rename the self-overlap the Edwards-Anderson parameter

$$
\begin{equation*}
q_{J \alpha \alpha} \equiv N^{-1} \sum_{i}\left\langle s_{i}\right\rangle_{\alpha}\left\langle s_{i}\right\rangle_{\alpha} \equiv q_{J \mathrm{EA}} \tag{7.96}
\end{equation*}
$$

(assuming the result is independent of $\alpha$ ). The statistics of possible overlaps is then characterized by a probability function

$$
\begin{equation*}
P_{J}(q) \equiv \sum_{\alpha \beta} w_{\alpha}^{J} w_{\beta}^{J} \delta\left(q-q_{\alpha \beta}^{J}\right) \tag{7.97}
\end{equation*}
$$

where we included a subindex $J$ to stress the fact that this is a strongly sample-dependent quantity. Again, a ferromagnetic model serves to illustrate the meaning of $P_{J}(q)$. First, there is no disorder in this case so the $J$ label is irrelevant. Second, the high- $T$ equilibrium phase is paramagnetic, with $q=0 . P(q)$ is then a delta function with weight 1 (see the left panel in Fig. 7.12). In the low- $T$ phase there are only two pure states with identical statistical properties and $q_{\mathrm{EA}}=m^{2}$. Thus, $P(q)$ is just the sum of two delta functions with weight $1 / 2$ (central panel in Fig. 7.12).

Next, one can consider averages over quenched disorder and study

$$
\begin{equation*}
\left[P_{J}(q)\right] \equiv \int d J P(J) \sum_{\alpha \beta} w_{\alpha}^{J} w_{\beta}^{J} \delta\left(q-q_{\alpha \beta}^{J}\right) \tag{7.98}
\end{equation*}
$$

How can one access $P_{J}(q)$ or $\left[P_{J}(q)\right]$ ? It is natural to reckon that

$$
\begin{equation*}
P_{J}(q)=Z^{-2} \sum_{\sigma s} e^{-\beta H_{J}\left(\left\{\sigma_{i}\right\}\right)} e^{-\beta H_{J}\left(\left\{s_{i}\right\}\right)} \delta\left(N^{-1} \sum_{i} \sigma_{i} s_{i}-q\right) \tag{7.99}
\end{equation*}
$$

that is to say, $P_{J}(q)$ is the probability of finding an overlap $q$ between two real replicas of the system with identical disordered interactions in equilibrium at temperature $T$. This identitiy gives a way to compute $P_{J}(q)$ and its average in a numerical simulation: one just has to simulate two independent systems with identical disorder in equilibrium and calculate the overlap.

But there is also, as suggested by the notation, a way to relate the pure state structure to the replica matrix $q_{a b}$. Let us consider the simple case

$$
\begin{align*}
{\left[m_{i}\right] } & =\left[Z_{J}^{-1} \sum_{\left\{s_{i}\right\}} s_{i} e^{-\beta H_{J}\left(\left\{s_{i}\right\}\right)}\right]=\left[\frac{Z_{J}^{n-1}}{Z_{J}^{n}} \sum_{\left\{s_{i}^{1}\right\}} s_{i}^{1} e^{-\beta H_{J}\left(\left\{s_{i}^{1}\right\}\right)}\right] \\
& =\left[\frac{1}{Z_{J}^{n}} \sum_{\left\{s_{i}^{a}\right\}} s_{i}^{1} e^{-\beta \sum_{a=1}^{n} H_{J}\left(\left\{s_{i}^{a}\right\}\right)}\right] \tag{7.100}
\end{align*}
$$

where we singled out the replica index of the spin to average. This relation is valid for all $n$, in particular for $n \rightarrow 0$. In this limit the denominator approaches one and the average over disorder can be simply evaluated

$$
\begin{equation*}
\left[m_{i}\right]=\sum_{\left\{s_{i}^{a}\right\}} s_{i}^{1} e^{-\beta H_{\mathrm{eff}}\left(\left\{s_{i}^{a}\right\}\right)} \tag{7.101}
\end{equation*}
$$

and introducing back the normalization factor $Z^{n}=1=\sum_{\left\{s_{i}^{a}\right\}} e^{-\beta \sum_{a=1}^{n} H_{J}\left(\left\{s_{i}^{a}\right\}\right)}$ that becomes $Z^{n}=\left[\sum_{\left\{s_{i}^{a}\right\}} e^{-\beta \sum_{a=1}^{n} H_{J}\left(\left\{s_{i}^{a}\right\}\right)}\right]=e^{-\beta H_{\mathrm{eff}}\left(\left\{s_{i}^{a}\right\}\right)}$ we have

$$
\begin{equation*}
\left[m_{i}\right]=\left\langle s_{i}^{a}\right\rangle_{H_{\mathrm{eff}}} \tag{7.102}
\end{equation*}
$$

with $a$ any replica index. The average is taken over the Gibbs measure of a system with effective Hamiltonian $H_{\text {eff }}$. In a replica symmetric problem in which all replicas are identical this result should be independent of the label $a$. Instead, in a problem with replica symmetry breaking the averages on the right-hand-side need not be identical for all $a$. This could occur in a normal vectorial theory with dimension $n$ in which not all components take the same expected value. It is reasonable to assume that the full thermodynamic average is achieved by the sum over all these cases,

$$
\begin{equation*}
\left[m_{i}\right]=\lim _{n \rightarrow 0} \frac{1}{n} \sum_{a=1}^{n}\left\langle s_{i}^{a}\right\rangle_{H_{\mathrm{eff}}} . \tag{7.103}
\end{equation*}
$$

Let us now take a less trivial observable and study the spin-glass order parameter $q$

$$
\begin{align*}
q & \equiv\left[\left\langle s_{i}\right\rangle^{2}\right]=\left[Z_{J}^{-1} \sum_{\left\{s_{i}\right\}} s_{i} e^{-\beta H_{J}\left(\left\{s_{i}\right\}\right)} Z_{J}^{-1} \sum_{\left\{\sigma_{i}\right\}} \sigma_{i} e^{-\beta H_{J}\left(\left\{\sigma_{i}\right\}\right)}\right] \\
& =\left[\frac{Z^{n-2}}{Z^{n}} \sum_{\left\{s_{i}\right\},\left\{\sigma_{i}\right\}} s_{i} \sigma_{i} e^{-\beta H_{J}\left(\left\{s_{i}\right\}\right)-\beta H_{J}\left(\left\{\sigma_{i}\right\}\right)}\right] \\
& =\left[\frac{1}{Z_{J}^{n}} \sum_{\left\{s_{i}^{a}\right\}} s_{i}^{1} s_{i}^{2} e^{-\beta \sum_{a=1}^{n} H_{J}\left(\left\{s_{i}^{a}\right\}\right)}\right] \tag{7.104}
\end{align*}
$$

In the $n \rightarrow 0$ limit the denominator is equal to one and one can then perform the average over disorder. Introducing back the normalization one then has

$$
\begin{equation*}
q=\left\langle s_{i}^{a} s_{i}^{b}\right\rangle_{E_{\text {eff }}\left(\left\{s_{i}^{a}\right\}\right)} \tag{7.105}
\end{equation*}
$$

for any arbitrary pair of replicas $a \neq b$ (since $\left\langle s_{i}^{a} s_{i}^{a}\right\rangle=1$ for Ising spins). The average is done with an effective theory of $n$ interacting replicas characterized by $E_{\text {eff }}\left(\left\{s_{i}^{a}\right\}\right)$. Again, if there is replica symmetry breaking the actual thermal average is the sum over all possible pairs of replicas:

$$
\begin{equation*}
q=\lim _{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{a \neq b} q^{a b} . \tag{7.106}
\end{equation*}
$$

A similar argument allows one to write

$$
\begin{equation*}
q^{(k)}=\left[\left\langle s_{i_{1}} \ldots s_{i_{k}}\right\rangle^{2}\right]=\lim _{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{a \neq b} q_{a b}^{k} . \tag{7.107}
\end{equation*}
$$

One can also generalize this argument to obtain

$$
\begin{equation*}
P(q)=\left[P_{J}(q)\right]=\lim _{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{a \neq b} \delta\left(q-q^{a b}\right) \tag{7.108}
\end{equation*}
$$



Figure 7.12: $\left[P_{J}(q)\right]$ in a paramagnet (left), in a ferromagnet or a replica symmetric system (centre) and for system with full RSB (right).

Thus, the replica matrix $q_{a b}$ can be ascribed to the overlap between pure states.
Note that a small applied field, though uncorrelated with a particular pure state, is necessary to have non-zero local magnetizations and then non-zero $q$ values.

The function $P(q)$ then extends the concept of order parameter to a function. In zero field the symmetry with respect to simultaneous reversal of all spins translates into the fact that $P_{J}(q)$ must be symmetric with respect to $q=0$. [ $\left.P_{J}(q)\right]$ can be used to distinguish between the droplet picture prediction for finite dimensional spin-glasses - two pure states - that simply corresponds to

$$
\begin{equation*}
\left[P_{J}(q)\right]=\frac{1}{2} \delta\left(q-q_{\mathrm{EA}}\right)+\frac{1}{2} \delta\left(q+q_{\mathrm{EA}}\right) \tag{7.109}
\end{equation*}
$$

(see the central panel in Fig. 7.12) and a more complicated situation in which $\left[P_{J}(q)\right]$ has the two delta functions at $\pm q_{\text {EA }}$ plus non-zero values on a finite support (right panel in Fig. 7.12) as found in mean-field spin-glass models.

## The linear susceptibility

Taking into account the multiplicity of pure states, the magnetic susceptibility, eq. (7.4), and using (7.93) becomes

$$
\begin{equation*}
T \chi=T\left[\chi_{J}\right]=1-\frac{1}{N} \sum_{i}\left[\left\langle s_{i}\right\rangle^{2}\right]=1-\sum_{\alpha \beta}\left[w_{\alpha}^{J} w_{\beta}^{J}\right] q_{\alpha \beta}=\int d q(1-q) P(q) \tag{7.110}
\end{equation*}
$$

There are then several possible results for the susceptibility depending on the level of replica symmetry breaking in the system:

- In a replica symmetric problem or, equivalently, in the droplet model,

$$
\begin{equation*}
\chi=\beta\left(1-q_{\mathrm{EA}}\right) . \tag{7.111}
\end{equation*}
$$

This is also the susceptibility within a pure state of a system with a higher level of RSB.

- At the one step RSB level, this becomes

$$
\begin{equation*}
\chi=\beta\left[1-(1-m) q_{\mathrm{EA}}\right] . \tag{7.112}
\end{equation*}
$$

- For systems with full RSB one needs to know the complete $P(q)$ to compute $\chi$, as in (7.110).

Note that in systems with RSB (one step or full) the susceptibility is larger than $\beta\left(1-q_{\mathrm{EA}}\right)$.
A system with $q_{\text {EA }}=1$ in the full low-temperature phase (as the REM model or $p \rightarrow \infty$ limit of the $p$ spin model, see below) has just one configuration in each state. Systems with $q_{\text {EA }}<1$ below $T_{c}$ have states formed by a number of different configurations that is exponentially large in $N$. (Note that $q_{\text {EA }}<1$ means that the two configurations differ in a number of spins that is proportional to $N$.) The logarithm of this number is usually called the intra-state entropy.

Even if the number of pure states can be very large (exponential in $N$ ) only a fraction of them can have a non-negligible weight. This is the case if one finds, for example, $\sum_{\alpha} w_{\alpha}^{2}<+\infty$

## Symmetry and ergodicity breaking

In all $p \geq 2$ spin models there is a phase transition at a finite $T_{s}$ at which the rather abstract replica symmetry is broken. This symmetry breaking is accompanied by ergodicity breaking as in the usual case. Many pure states appear at low temperatures, each one has its reversed $s_{i} \rightarrow-s_{i}$ counterpart, but not all of them are related by real-space symmetry properties.

The one-step $R S B$ scenario
In this case the transition has first-order and second-order aspects. The order parameters $q_{0}$ and $q_{1}$ jump at the critical point as in a first-order transition but the thermodynamic quantities are continuous.

The full RSB scenario
Right below $T_{c}$ an exponential in $N$ number of equilibrium states appear. The transition is continuous, the order parameter approaches zero right below $T_{c}$. Lowering further the temperature each ergodic component breaks in many other ones. In this sense, the full spin-glass phase, $T<T_{c}$, is 'critical' and not only the single point $T_{c}$.

### 7.4.2 The pinning field

We can nevertheless choose a possible direction, given by another field $\sigma(x)$, and compute the free-energy of our system when it is weakly pinned by this external quenched field

$$
\begin{equation*}
F_{\phi}[\sigma, g, \beta]=-\frac{1}{\beta} \log \int d \phi(x) e^{-\beta H[\phi]-\frac{g}{2} \int d x(\sigma(x)-\phi(x))^{2}} \tag{7.113}
\end{equation*}
$$



Figure 7.13: The susequent phase transitions in the SK model.
where $g>0$ denotes the strength of the coupling. This free-energy (7.113) will be small when the external perturbing field $\sigma(x)$ lies in a direction corresponding to the bottom of a well of the unperturbed free-energy. Therefore, we should be able to obtain useful information about the free-energy landscape by scanning the entire space of the configurations $\sigma(x)$ to locate all the states in which the system can freeze after spontaneous ergodicity breaking $(g \rightarrow 0)$. According to this intuitive idea, we now consider the field $\sigma(x)$ as a thermalized variable with the "Hamiltonian" $F_{\phi}[\sigma, g, \beta]$. The free-energy of the field $\sigma$ at inverse temperature $\beta m$ where $m$ is a positive free parameter therefore reads

$$
\begin{equation*}
F_{\sigma}(m, \beta)=\lim _{g \rightarrow 0^{+}}-\frac{1}{\beta m} \log \int d \sigma(x) e^{-\beta m F_{\phi}[\sigma, g, \beta]} \tag{7.114}
\end{equation*}
$$

When the ratio $m$ between the two temperatures is an integer, one can easily integrate $\sigma(x)$ in Eq.(7.114) after having introduced $m$ copies $\phi^{\rho}(x)(\rho=1 \ldots m)$ of the original field to obtain the relation

$$
\begin{equation*}
F_{\sigma}(m, \beta)=\lim _{g \rightarrow 0^{+}}-\frac{1}{\beta m} \log \int \prod_{\rho=1}^{m} d \phi^{\rho}(x) e^{-\beta \sum_{\rho} H\left[\phi^{\rho}\right]+\frac{1}{2} \sum_{\rho, \lambda} g^{\rho \lambda} \int d x \phi^{\rho}(x) \phi^{\lambda}(x)} \tag{7.115}
\end{equation*}
$$

where $g^{\rho \lambda}=g\left(\frac{1}{m}-\delta^{\rho \lambda}\right)$. Let us define two more quantities related to the field $\sigma$ : its internal energy $W(m, \beta)=\frac{\partial\left(m F_{\sigma}\right)}{\partial m}$ and its entropy $S(m, \beta)=\beta m^{2} \frac{\partial F_{\sigma}}{\partial m}$. Since the case $m=1$ will be of particular interest, we shall use hereafter $F_{h s}(\beta) \equiv W(m=1, \beta)$ and $\mathcal{S}_{h s}(\beta) \equiv S(m=1, \beta)$ where $h s$ stands for "hidden states". We stress that $S(m, \beta)$ and $\beta^{2} \frac{\partial F_{\phi}}{\partial \beta}$ which are respectively the entropies of the fields $\sigma$ and $\phi$ are two distinct quantities with different physical meanings.

When the pinning field $\sigma(x)$ is thermalized at the same temperature as $\phi(x)$, that is when $m=1$, one sees from Eq.(7.115) that $F_{\phi}(\beta)=F_{\sigma}(m=1, \beta)$. The basic idea of this letter is to decompose $F_{\sigma}$ into its energetic and entropic contributions to obtain

$$
\begin{equation*}
\mathcal{S}_{h s}(\beta)=\beta\left[F_{h s}(\beta)-F_{\phi}(\beta)\right] \tag{7.116}
\end{equation*}
$$

To get some insights on the significance of the above relation, we shall now turn to the particular case of disordered mean-field systems. We shall see how it rigorously gives back some analytical results derived within the mean-field TAP and dynamical approaches. We shall then discuss the physical meaning of identity (7.116) for the general case of glassy systems.

### 7.4.3 Coupling replicas and the effective potential

Let us take a spin-configuration, $\{s\}$, in equilibrium at temperature $T^{\prime}$, that is to say, drawn from the canonical probability distribution $P[\{s\}]=\exp \left(-\beta^{\prime} H[\{s\}]\right) / Z\left(T^{\prime}\right)$. One computes the free-energy cost to keep the system at a fixed overlap $\tilde{p}=q_{s, \sigma}$ with $\{s\}$ at a temperature $T$ (in general different from $T^{\prime}$ ):

$$
\begin{align*}
V_{J}(\beta, \tilde{p},\{s\}) & =-\frac{T}{N} \ln Z_{J}(\beta, \tilde{p},\{s\})-f_{J}(T)  \tag{7.117}\\
Z_{J}(\beta, \tilde{p},\{s\}) & \equiv \sum_{\{\sigma\}} e^{-\beta H_{J}[\{\sigma\}]} \delta\left(\tilde{p}-q_{s, \sigma}\right)  \tag{7.118}\\
\beta N f_{J}(T) & =\ln Z_{J}(\beta)=\ln \sum_{\{s\}} e^{-\beta H_{J}[\{s\}]} \tag{7.119}
\end{align*}
$$

( $f_{J}(T)$ is the disorder-dependent free-energy density without constraint.) In this problem the spins $s_{i}$ are quenched variables on the same footing as the random interactions in the Hamiltonian. One then assumes that $V$ is self-averaging with respect to the quenched disorder and the probability distribution of the reference configuration $\{s\}$. One then computes the two averages:

$$
\begin{equation*}
N V\left(\beta, \beta^{\prime}, \tilde{p}\right) \equiv N\left[V_{J}(\beta, \tilde{p},\{s\})\right]_{J,\{s\}}=\left[\sum_{\{s\}} \frac{e^{-\beta^{\prime} H_{J}[\{s\}]}}{Z_{J}\left(\beta^{\prime}\right)}\left(-T \ln Z_{J}(\beta, \tilde{p},\{s\})-f_{J}(T)\right)\right]_{J} \tag{7.120}
\end{equation*}
$$

This average can be done using the replica method:

$$
\begin{equation*}
N V\left(\beta, \beta^{\prime}, \tilde{p}\right)=-T \lim _{n \rightarrow 0} \lim _{m \rightarrow 0}\left[\sum_{\{s\}} e^{-\beta^{\prime} H_{J}[\{s\}]} Z_{J}\left(\beta^{\prime}\right)^{n-1}\left(\frac{Z_{J}(\{s\} ; \tilde{p},\{s\})^{m}-1}{m}\right)\right]_{J} \tag{7.121}
\end{equation*}
$$

The analytic continuation is performed from integer $n$ and $m$. One then has

$$
\begin{equation*}
Z^{(n, m)}=\left[\sum_{\left\{s^{a}\right\}} \sum_{\left\{\sigma^{\alpha}\right\}} \exp \left[\beta^{\prime} \sum_{a=1}^{n} H\left[\left\{s^{a}\right\}\right]+\beta \sum_{\alpha=1}^{m} H\left[\left\{\sigma^{\alpha}\right\}\right]\right] \prod_{\alpha=1}^{m} \delta\left(\sum_{i} s_{i}^{1} \sigma_{i}^{\alpha}-N \tilde{p}\right)\right]_{J} \tag{7.122}
\end{equation*}
$$

After averaging over the disorder strength distribution one introduces the order parameters:

$$
\begin{equation*}
Q_{a b}=\frac{1}{N} \sum_{i} s_{i}^{a} s_{i}^{b}, \quad R_{\alpha \gamma}=\frac{1}{N} \sum_{i} \sigma_{i}^{\alpha} \sigma_{i}^{\gamma} \quad P_{a \alpha}=\frac{1}{N} \sum_{i} s_{i}^{a} \sigma_{i}^{\alpha} \tag{7.123}
\end{equation*}
$$

with $a, b=1, \ldots, n$ and $\alpha, \gamma=1, \ldots, m$. Combining the order parameters in the single $(n+m) \times(n+m)$ matrix

$$
Q=\left(\begin{array}{ll}
Q & P  \tag{7.124}\\
P^{T} & R
\end{array}\right)
$$

one finds

$$
\begin{equation*}
\left.\left.\frac{1}{N} \log Z^{(n, m)}=\frac{1}{2}\left[\beta^{\prime 2} \sum_{a=1, b=1}^{n} Q_{a b}^{p}+\beta^{2} \sum_{\alpha=1 \gamma=1}^{m} R_{\alpha, \gamma}^{p}\right)+2 \beta \beta^{\prime} \sum_{a=1}^{n} \sum_{\alpha=1}^{m} P_{a \alpha}^{p}\right)\right]+\frac{1}{2} \operatorname{Tr} \ln Q \tag{7.125}
\end{equation*}
$$

We shall not present the details of the RSB Ansatz here.
One studies different ranges of $\beta$ and $\beta^{\prime}$ and analyses the minima of $V$ with respect to $\tilde{p}$.

The effective potential for four different temperatures, $T=T^{\prime}$ for $p=3$ is shown in [85] From top to bottom, the curves represent the potential at temperature higher then $T_{d}$, equal to $T_{d}$ between $T_{d}$ and $T_{S}$, and right at $T_{S}$. We can see from the figure that for $T>T_{d}$ the potential is monotonically increasing, and the only extremum of the potential is the minimum at $\tilde{p}=0$. At the temperature $T_{d}$ where the dynamical transition happens, the potential develops for the first time a minimum with $\tilde{p} \equiv r \neq 0$. It is interesting to observe that the energy in this flex point is equal to the asymptotic value of the energy in the out-of-equilibrium dynamics. The same is true for the parameter $r$ which turns out to be equal to the dynamical Edward-Anderson parameter.

The condition for the potential of having a flex coincides with the marginality condition. Indeed the flex implies a zero eigenvalue in the longitudinal sector and at $x=1$ the replicon and the longitudinal eigenvalues are degenerate. The marginality condition is well known to give exact results for the transition temperatures in p-spin spherical models.

We have observed that in general more then one minimum can be present in the potential. In the $p$-spin model it happens that two minima develop at the same temperature $T_{d}$. The rightmost one, that we will call primary is the one with $\tilde{p}=r$, while the other, secondary, has $\tilde{p}<r$. For temperatures smaller than $T_{d}$ the minima have a finite depth, i.e. are separated by extensive barriers from the absolute minimum.

The primary minimum is easily interpreted. There the system denoted by $s$ is in the same pure state as the system $\sigma$. In the region $T_{S}<T<T_{D}$ the number of pure states is exponentially large in $N: \mathcal{N}=e^{N \Sigma(T)}$. Consequently the probability of finding two system in the same state is exponentially small and proportional to $e^{-N \Sigma(T)}$. The free energy cost to constrain two systems to be in the same state is then proportional to the logarithm of this probability, namely we have

$$
\begin{equation*}
V_{\text {primary }}=T \Sigma(T) \tag{7.126}
\end{equation*}
$$

Coherently at the statical transition temperature $T=T_{S}$ one finds $V_{\text {primary }}=0$. The quantity $\Sigma$ has been computed for the $p$-spin model as the number of solution of the TAP equation with given free energy and coincides with our calculation. The secondary minima, could aso be associated to metastable states, but at present we do not have an interpretation for them. This conclusion on the equivalence of the potential with the number of solution of the TAP equation hold also in the ROM.

The study of the potential for temperatures smaller than $T_{S}$ would require to take into account RSB effects, which would complicate a bit the analysis. However it is physically clear that the shape of the potential in that region it is not different qualitatively from the one at $T=T_{S}$. It has a minimum where $r=\tilde{p}$ are equal to the Edwards Anderson parameter and the value of potential is zero.

The study of the effective potential at different gives information about the chaotic properties of the models. We shall not develop it here.

### 7.5 The random energy model

The random energy model (REM) describes a system with $2^{N}$ independent energy levels, $E_{i}$, with a Gaussian probability density

$$
\begin{equation*}
p(E)=\left(2 \pi N J^{2}\right)^{-1 / 2} e^{-E^{2}}\left(2 N J^{2}\right) \tag{7.127}
\end{equation*}
$$

All $p$-spin Ising models do indeed have $2^{N}$ energy levels with such a Gaussian distribution. Indeed,

$$
\begin{equation*}
p(E)=\int \mathcal{D} J_{i_{i} \ldots i_{p}} P\left(J_{i_{1} \ldots i_{p}}\right) \delta\left(E-\sum_{i_{1} \ldots i_{p}} J_{i_{i} \ldots i_{p}} s_{i_{1}} \ldots s_{i_{p}}\right)=\left(2 \pi N J^{2}\right)^{-1 / 2} e^{-E^{2}}\left(2 N J^{2}\right) \tag{7.128}
\end{equation*}
$$

independently of the spin configuration. For finite $p$ different energy levels are not independent random variables but they do become independent in the limit $p \rightarrow \infty$ :

$$
\begin{equation*}
p\left(E_{1}, \ldots, E_{k}\right)=p\left(E_{1}\right) \ldots p\left(E_{k}\right) \tag{7.129}
\end{equation*}
$$

for any integer $k$.
The free-energy, $-\beta F\left(\left\{E_{i}\right\}\right)$ depends on the realisation of the $2^{N}$ energy levels. In the large $N$ limit one can prove that the free-energy density is a self-averaging quantity,


Figure 7.14: The energy-density, specific heat, susceptility of the REM.
$\lim _{N \rightarrow \infty}\left[f\left(\left\{E_{i}\right\}\right)\right]-f\left(\left\{E_{i}\right\}\right)=0$ and $\lim _{N \rightarrow \infty}\left[f^{2}\left(\left\{E_{i}\right\}\right)\right]-\left(f\left(\left\{E_{i}\right\}\right)\right)^{2}=0$. One then computes the average

$$
\begin{equation*}
-\beta[F]=\int \prod_{i} d E_{i} P\left(E_{i}\right) \ln Z\left(\left\{E_{i}\right\}\right) \tag{7.130}
\end{equation*}
$$

with

$$
\begin{equation*}
Z\left(\left\{E_{i}\right\}\right) \equiv \sum_{i=1}^{2^{N}} e^{-\beta E_{i}}=\int d E \delta\left(E-E_{i}\right) e^{-\beta E} \tag{7.131}
\end{equation*}
$$

One finds a freezing phase transition at $T_{c}=1 /(2 \ln 2)$ where the temperature-dependent energy density, $E(T) / N$, gets blocked at a constant value $-E_{0} / N=-J \sqrt{\ln 2}$ and the specific heat and entropy vanish. The magnetic susceptibility goes from the high temperature Curie law $\chi=1 / T$ to the constant value $\chi=1 / T_{c}$ at $T_{c}$.

The number of states in the interval $[E, E+d E]$ is

$$
\begin{equation*}
\mathcal{N}_{0}(E) d E=2^{N}\left(2 \pi N J^{2}\right)^{-1 / 2} e^{-E^{2}}\left(2 N J^{2}\right)=e^{N\left(\ln 2-(E / N)^{2} / 2 J^{2}\right)} \tag{7.132}
\end{equation*}
$$

If the parenthesis is negative, $\mathcal{N}_{0}(E) \rightarrow 0$ in the thermodynamic limit, if it vanishes identically $\mathcal{N}_{0}(E)$ is a constant and, otherwise, $\mathcal{N}_{0}(E)$ diverges exponentially. One then has a critical energy-density $e_{c}^{2}=2 J^{2} \ln 2$ above which there are no energy levels. Below this level there are

$$
\mathcal{N}\left(E^{\prime}<E\right)=\int_{-\infty}^{\mathcal{N}_{E}} d E^{\prime} \mathcal{N}_{0}\left(E^{\prime}\right)=\int_{-\infty}^{\mathcal{N}_{E}} d E^{\prime} e^{N\left(\ln 2-\left(E^{\prime} / N\right)^{2} / 2 J^{2}\right)} \approx e^{N\left(\ln 2-(E / N)^{2} / 2 J^{2}(7.133)\right.}
$$

since the integral is dominated by the upper limit. Il one defines the complexity of the model as

$$
\begin{equation*}
\Sigma(E)=k_{B} \ln \mathcal{N}(E)=k_{B} \tag{7.134}
\end{equation*}
$$

It is clear that the lowest energy levels, $E_{i}=E_{0}+\epsilon_{i}$, with $\epsilon_{i}$ finite dominate the partition sum (7.131) at low temperatures. One obtains the probability distribution $p\left(E_{i}\right)$ by linearizing (7.127):

$$
p(\epsilon)= \begin{cases}\beta_{c} e^{-\beta_{c}\left(\epsilon-\epsilon_{c}\right)} & \epsilon<\epsilon_{c}  \tag{7.135}\\ 0 & \epsilon>\epsilon_{c}\end{cases}
$$

with $\epsilon_{c}$ a cut-off energy. The behaviour of the low- $T$ phase is independent of $\epsilon_{c}$ and well defined in the limit $N \rightarrow \infty, \epsilon_{c} \rightarrow \infty$ with $N e^{-\beta_{c} \epsilon_{c}}$ finite. In this limit the density of levels at any $\epsilon$ becomes constant.

## A Some useful formulæ

## A. 1 Fourier transform

## Finite volume

We define the Fourier transform (FT) of a function $f(\vec{x})$ defined in a volume $V$ as

$$
\begin{equation*}
\tilde{f}(\vec{k})=\int_{V} d^{d} x f(\vec{x}) e^{-i \vec{k} \vec{x}} \tag{A.1.1}
\end{equation*}
$$

This implies

$$
\begin{equation*}
f(\vec{x})=\frac{1}{V} \sum_{\vec{k}} \tilde{f}(\vec{k}) e^{i \vec{k} \vec{x}} \tag{A.1.2}
\end{equation*}
$$

where the sum runs over all $\vec{k}$ with components $k_{i}$ satisfying $k_{i}=2 m \pi / L$ with $m$ an integer and $L$ the linear size of the volume $V$.

## Infinite volume

In the large $V$ limit these equations become

$$
\begin{align*}
\tilde{f}(\vec{k}) & =\int_{V} d^{d} x f(\vec{x}) e^{-i \vec{k} \vec{x}}  \tag{A.1.3}\\
\tilde{f}(\vec{x}) & =\int_{V} \frac{d^{d} k}{(2 \pi)^{d}} f(\vec{k}) e^{i \vec{k} \vec{x}} \tag{A.1.4}
\end{align*}
$$

On a lattice
Take now a function $f_{\vec{x}}$ defined on a lattice. Its Fourier transform is

$$
\begin{equation*}
\tilde{f}(\vec{k})=\sum_{\vec{x}} f_{\vec{x}} e^{-i \vec{k} \vec{x}} \tag{A.1.5}
\end{equation*}
$$

with the inverse

$$
\begin{equation*}
f_{\vec{x}}=\int \frac{d^{d} k}{2 \pi} f(\vec{k}) e^{i \vec{k} \vec{x}} \tag{A.1.6}
\end{equation*}
$$

and $\int d^{d} k /(2 \pi)^{d}=\prod_{i=1}^{d} \int_{-\pi}^{\pi} d k_{1} /(2 \pi) \cdots \int_{-\pi}^{\pi} d k_{d} /(2 \pi)$ with these integrals running over the first Brillouin zone in reciprocal space.

## Time domain

The convention for the Fourier transform is the time-domains is

$$
\begin{align*}
& f(\tau)=\int_{-\infty}^{\infty} \frac{d \omega}{2 \pi} e^{-i \omega \tau} f(\omega)  \tag{A.1.7}\\
& f(\omega)=\int_{-\infty}^{\infty} d \tau e^{+i \omega \tau} f(\tau) \tag{A.1.8}
\end{align*}
$$

## Properties

The Fourier transform of a real function $f(\vec{x})$ satisfies $\tilde{f}^{*}(\vec{k})=\tilde{f}(-\vec{k})$.
The Fourier transform of the theta function reads

$$
\begin{equation*}
\theta(\omega)=i \operatorname{vp} \frac{1}{\omega}+\pi \delta(\omega) \tag{A.1.9}
\end{equation*}
$$

The convolution is

$$
\begin{equation*}
[f \cdot g](\omega)=f \otimes g(\omega) \equiv \int \frac{d \omega^{\prime}}{2 \pi} f\left(\omega^{\prime}\right) g\left(\omega-\omega^{\prime}\right) \tag{A.1.10}
\end{equation*}
$$

## A. 2 Stirling

Stirling formula for the factorial of a large number reads:

$$
\begin{equation*}
\ln N!\sim N \ln N-\ln N, \quad \text { for } \quad N \gg 1 \tag{A.2.1}
\end{equation*}
$$

## A. 3 Moments

Introducing a source $h$ that couples linearly to a random variable $x$ one easily computes all moments of its distribution $p(x)$. Indeed,

$$
\begin{equation*}
\left\langle x^{k}\right\rangle=\left.\frac{\partial^{k}}{\partial h^{k}} \int d x p(x) e^{h x}\right|_{h=0} \tag{A.3.1}
\end{equation*}
$$

## A. 4 Gaussian integrals

The Gaussian integral is

$$
\begin{equation*}
I_{1} \equiv \int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{(x-\mu)^{2}}{2 \sigma^{2}}}=1 \tag{A.4.1}
\end{equation*}
$$

It is the normalization condition of the Gaussian probability density written in the normal form. One has

$$
\begin{gather*}
\int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{(x-\mu)^{2}}{2 \sigma^{2}}} x=\mu \\
\int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{(x-\mu)^{2}}{2 \sigma^{2}}} x^{2}=\sigma^{2} \tag{A.4.2}
\end{gather*}
$$

From (A.4.1) one has

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{x^{2}}{2 \sigma^{2}}+\frac{\mu x}{\sigma^{2}}}=e^{\frac{\sigma^{2} \mu^{2}}{2}} . \tag{A.4.3}
\end{equation*}
$$

The generalization to $N$ variables

$$
\begin{equation*}
I_{N} \equiv \int_{-\infty}^{\infty} \prod_{i=1}^{N} d x_{i} e^{-\frac{1}{2} \vec{x}^{t} A \vec{x}+\vec{x}^{t} \vec{\mu}} \tag{A.4.4}
\end{equation*}
$$

with

$$
\vec{x}=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\ldots \\
x_{N}
\end{array}\right), \quad \vec{\mu}=\left(\begin{array}{c}
\mu_{1} \\
\mu_{2} \\
\ldots \\
\mu_{N}
\end{array}\right), \quad A=\left(\begin{array}{ccc}
A_{11} & \ldots & A_{1 N} \\
A_{21} & \ldots & A_{2 N} \\
& \ldots & \\
A_{N 1} & \ldots & A_{N N}
\end{array}\right)
$$

and

$$
\begin{equation*}
-\frac{1}{2} \vec{x}^{t} A \vec{x}+\vec{x}^{t} \vec{\mu} \tag{A.4.5}
\end{equation*}
$$

is the most generic quadratic form. Note that $A$ plays here the role $\sigma^{-2}$ in the single variable case. One can keep the symmetric part $\left(A+A^{t}\right) / 2$ of the matrix $A$ only since the antisymmetric part $\left(A-A^{t}\right) / 2$ yields a vanishing contribution once multiplied by the vectors $\vec{x}$ and its transposed. Focusing now on a symmetric matrix, $A^{t}=A$, that we still call $A$ we can ensure that it is diagonalizable and all its eigenvalues are positive definite, $\lambda_{i}>0$. One can then define $A^{1 / 2}$ as the matrix such that $A^{1 / 2} A^{1 / 2}=A$ and its eigenvalues are the square root of the ones of $A$. Writing $\vec{x}^{t} A \vec{x}=\left(\vec{x}^{t} A^{1 / 2}\right)\left(A^{1 / 2} \vec{x}\right)=\vec{y} \vec{y}$, the integral $I_{N}$ in (A.4.4) becomes

$$
\begin{equation*}
I_{N}=\int_{-\infty}^{\infty} \prod_{i=1}^{N} d y_{i} J e^{-\frac{1}{2} \vec{y}^{t} \vec{y}+\vec{y}^{t}\left(A^{-1 / 2} \mu\right)} \tag{A.4.6}
\end{equation*}
$$

where $J=\operatorname{det}\left(A^{1 / 2}\right)^{-1}=(\operatorname{det} A)^{-1 / 2}$ is the Jacobian of the change of variables. Calling $\vec{\mu}^{\prime}$ the last factor one has the product of $N$ integrals of the type $I_{1}$; thus

$$
\begin{equation*}
I_{N}=(2 \pi)^{N / 2}(\operatorname{det} A)^{-1 / 2} e^{\frac{1}{2} \vec{\mu}^{t} A^{-1} \vec{\mu}} \tag{A.4.7}
\end{equation*}
$$

Finally, the functional Gaussian integral is the continuum limit of the $N$-dimensional Gaussian integral

$$
\begin{equation*}
\vec{x} \equiv\left(x_{1}, \ldots, x_{N}\right) \rightarrow \phi(\vec{x}) \tag{A.4.8}
\end{equation*}
$$

and

$$
\begin{equation*}
I=\int \mathcal{D} \phi e^{-\frac{1}{2} \int d^{d} x d^{d} y \phi(\vec{x}) A(\vec{x}, \vec{y}) \phi(\vec{y})+\int d^{d} x \phi(\vec{x}) \mu(\vec{x})} . \tag{A.4.9}
\end{equation*}
$$

The sum runs over all functions $\phi(\vec{x})$ with the spatial point $\vec{x}$ living in $d$ dimensions. The first and the second term in the exponential are quadratic and linear in the field, respectively. In analogy with the $I_{N}$ case the result of the path integral is

$$
\begin{equation*}
I \propto e^{\frac{1}{2} \int d^{d} x d^{d} y \mu(\vec{x}) A^{-1}(\vec{x}, \vec{y}) \mu(\vec{y})} \tag{A.4.10}
\end{equation*}
$$

where we ignore the proportionality constant. Indeed, this one depends on the definition of the path-integral measure $\mathcal{D} \phi$. Usually, the actual value of this constant is not important since it does not depend on the relevant parameters of the theory. The inverse $A^{-1}$ is defined by

$$
\begin{equation*}
\int d^{d} y A^{-1}(\vec{x}, \vec{y}) A(\vec{y}, \vec{z})=\delta(\vec{x}-\vec{z}) \tag{A.4.11}
\end{equation*}
$$

## A. 5 Wick's theorem

Take a Gaussian variable $x$ with mean $\langle x\rangle=\mu$ and variance $\sigma^{2}=\left\langle x^{2}\right\rangle-\langle x\rangle^{2}$. Its pdf is

$$
\begin{equation*}
p(x)=\left(2 \pi \sigma^{2}\right)^{-1 / 2} e^{-(x-\mu)^{2} /\left(2 \sigma^{2}\right)} \tag{A.5.1}
\end{equation*}
$$

All moments $\left\langle x^{k}\right\rangle$ can be computed with (A.3.1). One finds

$$
\begin{equation*}
\left\langle e^{h x}\right\rangle=e^{\frac{h^{2} \sigma^{2}}{2}+h \mu} \tag{A.5.2}
\end{equation*}
$$

and then

$$
\begin{equation*}
\left\langle x^{k}\right\rangle=\left.\frac{\partial^{k}}{\partial h^{k}} e^{\frac{h^{2} \sigma^{2}}{2}+\mu h}\right|_{h=0} \tag{A.5.3}
\end{equation*}
$$

from where

$$
\begin{array}{ll}
\langle x\rangle=\mu, & \left\langle x^{2}\right\rangle=\sigma^{2}+\mu^{2}, \\
\left\langle x^{3}\right\rangle=3 \sigma^{2} \mu+\mu^{3}, & \left\langle x^{4}\right\rangle=3 \sigma^{4}+6 \sigma^{2} \mu^{2}+\mu^{4}
\end{array}
$$

etc. One recognizes the structure of Wick's theorem: given $k$ factors $x$ one organises them in pairs leaving the averages $\mu$ aside. The simplest way of seeing Wick's theorem in action is by drawing examples.

The generalization to $N$ Gaussian variables is immediate. Equation (A.5.2) becomes

$$
\begin{equation*}
\left\langle e^{\vec{h} \vec{x}}\right\rangle=e^{\frac{1}{2} \vec{h} A^{-1} \vec{h}+\vec{h} \vec{\mu}} \tag{A.5.4}
\end{equation*}
$$

and the generalization of (A.5.3) leads to

$$
\begin{equation*}
\left\langle x_{i}\right\rangle=\mu_{i}, \quad\left\langle x_{i} x_{j}\right\rangle=A^{-1}{ }_{i j}+\mu_{i} \mu_{j} \tag{A.5.5}
\end{equation*}
$$

etc. In other words, whereever there is $\sigma^{2}$ in the single variable case we replace it by $A^{-1}{ }_{i j}$ with the corresponding indices.

The generalization to a field theory necessitates the introduction of functional derivatives that we describe below. For completeness we present the result for a scalar field in $d$ dimensions here

$$
\begin{equation*}
\langle\phi(\vec{x})\rangle=\mu(\vec{x}), \quad\langle\phi(\vec{x}) \phi(\vec{y})\rangle=A^{-1}(\vec{x}, \vec{y})+\mu(\vec{x}) \mu(\vec{y}), \tag{A.5.6}
\end{equation*}
$$

etc.

## A. 6 Functional analysis

A functional $F[h]$ is a function of a function $h: \vec{x} \rightarrow h(\vec{x})$. The variation of a functional $F$ when one changes the function $h$ by an infinitesimal amount allows one to define the functional derivative. More precisely, one defines $\delta F \equiv F[h+\delta h]-F[h]$ and one tries to write this as $\delta F=\int d^{d} x \alpha(\vec{x}) \delta h(\vec{x})+\frac{1}{2} \int d^{d} x d^{d} y \beta(\vec{x}, \vec{y}) \delta h(\vec{x}) \delta h(\vec{y})+\ldots$ and one defines the functional derivative of $F$ with respect to $h$ evaluated at the spatial point $\vec{x}$ as

$$
\begin{equation*}
\frac{\delta F}{\delta h(\vec{x})}=\alpha(\vec{x}), \quad \frac{\delta^{2} F}{\delta h(\vec{x}) \delta h(\vec{y})}=\beta(\vec{x}, \vec{y}) \tag{A.6.1}
\end{equation*}
$$

etc. All usual properties of partial derivatives apply.

## A. 7 The saddle-point method

Imagine one has to compute the following integral

$$
\begin{equation*}
I \equiv \int_{a}^{b} d x e^{-N f(x)} \tag{A.7.1}
\end{equation*}
$$

with $f(x)$ a positive definite function in the interval $[a, b]$, in the limit $N \rightarrow \infty$. It is clear that due to the rapid exponential decay of the integrand, the integral will be dominated by the minimum of the function $f$ in the interval. Assuming there is only one absolute minimum, $x_{0}$, one then Taylor expands $f(x)$ upto second order

$$
\begin{equation*}
f(x) \sim f\left(x_{0}\right)+\frac{1}{2} f^{\prime \prime}\left(x_{0}\right)\left(x-x_{0}\right)^{2} \tag{A.7.2}
\end{equation*}
$$

and obtains

$$
\begin{equation*}
I \sim e^{-N f\left(x_{0}\right)} \int_{a}^{b} d x e^{-N \frac{1}{2} f^{\prime \prime}\left(x_{0}\right)\left(x-x_{0}\right)^{2}}=e^{-N f\left(x_{0}\right)}\left[N f^{\prime \prime}\left(x_{0}\right)\right]^{-1 / 2} \int_{y_{a}}^{y_{b}} d y e^{-\frac{1}{2}\left(y-y_{0}\right)^{2}} \tag{A.7.3}
\end{equation*}
$$

with $y_{0} \equiv \sqrt{N f^{\prime \prime}\left(x_{0}\right)} x_{0}$ and similarly for $y_{a}$ and $y_{b}$. The Gaussian integral is just an error function that one can find in Tables.

This argument can be extended to multidimensional integrals, cases in which there is no absolute minimum within the integration interval, cases in which the function $f$ is not positive definite, etc.

## B Mean-field theory

In spite of their apparent simplicity, the statics of ferromagnetic Ising models has been solved analytically only in one and two dimensions. The mean-field approximation allows one to solve the Ising model in any spatial dimensionality. Even if the qualitative results obtained are correct, the quantitative comparison to experimental and numerical data shows that the approximation fails below an upper critical dimension $d_{u}$. It is however very instructive to see the mean-field approximation at work.

We will study a generic Ising spin Hamiltonian

$$
H_{J}\left[\left\{s_{i}\right\}\right]=-\sum_{i_{1} \ldots i_{p}} J_{i_{1} \ldots i_{p}} s_{i_{1}} \ldots s_{i_{p}}
$$

and we will scale the coupling constants $J_{i_{1} \ldots i_{p}}$ with the number of spins in the sample so as to ensure that the extensive property of the energy, $H=O(N)$. For the moment we do not restrict the sum over the spins, it will be determined by the type of interaction.

## B. 1 Ignoring correlations

The usual way of deriving the mean-field approximation is to write

$$
\begin{equation*}
s_{i}=m_{i}+\delta s_{i} \tag{B.1.4}
\end{equation*}
$$

with $\delta s_{i}=s_{i}-m_{i}$, and $m_{i}=\left\langle s_{i}\right\rangle$ the local magnetisation density. The next step is to replace this expression in each term contributing to the sum over spins in the Hamiltonian that, for concreteness, we take to have pair interactions $(p=2)$, and keep only first order terms in powers of $\delta s_{i}$

$$
\begin{equation*}
s_{i} s_{j} \simeq m_{i} m_{j}+m_{i} \delta s_{j}+m_{j} \delta s_{i}=m_{i} s_{j}+m_{j} s_{i}-m_{i} m_{j} \tag{B.1.5}
\end{equation*}
$$

The extension to $p$-spin interactions is straightforward. This leads to a model with $N$ non-interacting Ising spins coupled to $m_{i}$-dependent local fields:

$$
\begin{equation*}
\sum_{i j} J_{i j} s_{i} s_{j} \approx-\sum_{i j} J_{i j} m_{i} m_{j}+2 \sum_{i j} J_{i j} s_{i} m_{j} \tag{B.1.6}
\end{equation*}
$$

This way of presenting the approximation makes its "mean field" character transparent. The fact that correlations between different spins are neglected will become clear. This approximation cannot be accurate when the correlations are strong, i.e. close to the critical point.

Having truncated the Hamiltonian we are now able to compute the partition sum, as the model just became one of independent spins in an effective local field

$$
\begin{equation*}
h_{i}^{\mathrm{eff}}=2 \sum_{j(i)} J_{i j} m_{j} \tag{B.1.7}
\end{equation*}
$$

The second sum runs over the spins $j$ that interact with the selected spin $i$. The partition sum reads

$$
\begin{align*}
\mathcal{Z} & \approx e^{-\beta \sum_{i j} J_{i j} m_{i} m_{j}} \sum_{\left\{s_{i}= \pm 1\right\}} e^{\beta \sum_{i=1}^{N} s_{i}\left(4 \sum_{j(i)} J_{i j} m_{j}+h_{i}\right)} \\
& =e^{-\beta \sum_{i j} J_{i j} m_{i} m_{j}} \prod_{i=1}^{N} 2 \cosh \left[\beta\left(4 \sum_{j(i)} J_{i j} m_{j}+h_{i}\right)\right] \tag{B.1.8}
\end{align*}
$$

where we added an external local field $h_{i}^{\text {eff }} \mapsto h_{i}^{\text {eff }}+h_{i}$ as a pinning field or as a source to compute correlation functions (see below). ${ }^{7}$ Therefore, the free-energy density $-\beta f=$ $N^{-1} \ln \mathcal{Z}$ is

$$
\begin{equation*}
f\left(\left\{m_{i}\right\}\right) \approx \frac{1}{N} \sum_{i j} J_{i j} m_{i} m_{j}-\frac{k_{B} T}{N} \sum_{i=1}^{N} \ln \left\{2 \cosh \left[\beta\left(4 \sum_{j(i)} J_{i j} m_{j}+h_{i}\right)\right]\right\} \tag{B.1.9}
\end{equation*}
$$

With $j(i)$ we indicate the spins $j$ that interact with $i$. How many these are, and where they are situated in the sample, depend on the range of the interactions.
Exercise B. 2 Prove $\left\langle s_{i}\right\rangle=m_{i}$ for all $i$. Hint: use a "source" $H \mapsto H-\sum_{i} h_{i} s_{i}$ and express $\left\langle s_{i}\right\rangle$ as $\beta^{-1} \partial \ln \mathcal{Z} /\left.\partial h_{i}\right|_{h_{i}=0}=-\partial f / \partial h_{i}$.
Exercise B. 3 Study the scaling, with $N$, of the two terms contributing to $f$ for different kinds of interaction ranges.

## B.1.1 Uniform interactions and fields

If we now take all interactions to be equal, $J_{i j}=J$, and $h_{i}=h$, all spins see the same environment. In consequence, we can assume that all the $m_{i}$ are equal, $m_{i}=m$. Then,

$$
\begin{equation*}
f(m) \approx \frac{J}{N} \sum_{i j} m^{2}-k_{B} T \ln \left\{2 \cosh \left[\beta\left(4 J \sum_{j(i)} m+h\right)\right]\right\} \tag{B.1.10}
\end{equation*}
$$

The result of the two sums depends of the range of the interactions in the Hamiltonian. One can easily see that for the fully-connected model, the free-energy density will be order

[^6]one only if we scale $J \mapsto J / N$. Instead, for models with finite range interactions the sum over $j$ has a finite number of terms and $f$ is $\mathrm{O}(1)$ and on a lattice with coordination $z$
\[

$$
\begin{equation*}
f(m) \approx 2 J z m^{2}-k_{B} T \ln \{2 \cosh [\beta(4 J z m+h)]\} \tag{B.1.11}
\end{equation*}
$$

\]

One can see that the more spins interact with the chosen one the closer the spin sees an average field, i.e. the mean-field. The number of interacting spins increases with the range of interaction and the dimension of space in a problem with nearest neighbour interactions on a lattice.

The free-energy density is a function of the parameters $\left\{m_{i}\right\}$ that are not fixed yet. The next step is to look for the extreme values of $f\left(\left\{m_{i}\right\}\right)$ to fix them. We will carry out this analysis after presenting alternative ways to derive the mean-field free-energy density as a function of the $\left\{m_{i}\right\} \mathrm{s}$.
Exercise B. 4 Take the limit $k_{B} T \rightarrow 0$ in Eq. (B.1.11) evaluated at $h=0$ and show that the minima of the free-energy density are located at $m= \pm 1$ as they should. This kind of calculation can be used as a check to verify the factors in Eq. (B.1.11).

The connected correlation function of two spins $s_{k}$ and $s_{l}$, for the problem with $J_{i j}=J$, in the absence of an applied field is

$$
\begin{align*}
\left\langle s_{k} s_{l}\right\rangle_{c}= & \left\langle s_{k} s_{l}\right\rangle-\left\langle s_{k}\right\rangle\left\langle s_{l}\right\rangle \\
= & \frac{1}{\mathcal{Z}} \sum_{\left\{s_{n}\right\}} s_{k} s_{l} e^{-\beta J \sum_{i j}\left(m_{i} m_{j}-2 s_{i} m_{j}\right)} \\
& -\frac{1}{\mathcal{Z}} \sum_{\left\{s_{n}\right\}} s_{k} e^{-\beta J \sum_{i j}\left(m_{i} m_{j}-2 s_{i} m_{j}\right)} \frac{1}{\mathcal{Z}} \sum_{\left\{s_{n}\right\}} s_{l} e^{-\beta J \sum_{i j}\left(m_{i} m_{j}-2 s_{i} m_{j}\right)} \tag{B.1.12}
\end{align*}
$$

The first remark is that the sum over $\left\{s_{n}\right\}$ indicates $N$ sums over the states of each spin. The sums that are not the ones associated to $s_{k}$ and $s_{l}$ are identical in the numerators and denominators and therefore cancel. The factors $e^{-\beta J \sum_{i j} m_{i} m_{j}}$ also cancel since they appear in identical form in numerators and denominators. If the spins $k$ and $l$ are different, the first sum factorises

$$
\begin{align*}
\left\langle s_{k} s_{l}\right\rangle_{c}= & \frac{\sum_{s_{k}} s_{k} e^{2 \beta J \sum_{j(k)} s_{k} m_{j}} \sum_{s_{l}} s_{l} e^{2 \beta J \sum_{j(l)} s_{l} m_{j}}}{\sum_{s_{k}, s_{l}} e^{2 \beta J \sum_{j(k)} s_{k} m_{j}-2 \beta J \sum_{j(l)} s_{l} m_{l}}} \\
& -\frac{\sum_{s_{k}} s_{k} e^{2 \beta J \sum_{j(k)} s_{k} m_{j}}}{\sum_{s_{k}} e^{2 \beta J \sum_{j(k)} s_{k} m_{j}}} \frac{\sum_{s_{l}} s_{l} e^{2 \beta J \sum_{j(l)} s_{l} m_{j}}}{\sum_{s_{l}} e^{2 \beta J \sum_{j(l)} s_{l} m_{j}}} \tag{B.1.13}
\end{align*}
$$

and this result is identical to zero.

## B. 2 The naive mean-field approximation

Take an Ising model on any lattice or graph. The naive mean-field approximation consists in assuming that the probability density of the system's spin configuration can be factored in a product of independent factors [5]

$$
\begin{equation*}
P\left(\left\{s_{i}\right\}\right)=\prod_{i=1}^{N} P_{i}\left(s_{i}\right) \quad \text { with } \quad P_{i}\left(s_{i}\right)=\frac{1+m_{i}}{2} \delta_{s_{i}, 1}+\frac{1-m_{i}}{2} \delta_{s_{i},-1} \tag{B.2.14}
\end{equation*}
$$

and $m_{i}=\left\langle s_{i}\right\rangle$, where the thermal average has to be interpreted in the restricted sense, i.e. taken over one ergodic component, in a way that $m_{i} \neq 0$. Note that one introduces an order-parameter dependence in the probabilities.

Exercise B. 5 Prove that $P$ is correctly normalised and $m_{i}=\left\langle s_{i}\right\rangle$ where $\langle\ldots\rangle$ is now computed with the weight $P$. Compute $\left\langle s_{i} s_{j}\right\rangle$ and discuss the result.

## B.2.1 The free-energy density

Using this assumption one can compute the total free-energy

$$
\begin{equation*}
F=U-T S \tag{B.2.15}
\end{equation*}
$$

where the average is taken with the factorized probability distribution (B.2.14) and the entropy $S$ is given by

$$
\begin{equation*}
S=-k_{B} \sum_{\left\{s_{i}= \pm 1\right\}} P\left(\left\{s_{i}\right\}\right) \ln P\left(\left\{s_{i}\right\}\right) . \tag{B.2.16}
\end{equation*}
$$

The entropy is

$$
\begin{align*}
S & =-k_{B} \sum_{s_{i}= \pm 1} \prod_{k=1}^{N} P_{k}\left(s_{k}\right) \ln \prod_{l=1}^{N} P_{l}\left(s_{l}\right)=-k_{B} \sum_{l=1}^{N} \sum_{s_{l}= \pm 1} P_{l}\left(s_{l}\right) \ln P_{l}\left(s_{l}\right) \\
& =-k_{B} \sum_{i}\left(\frac{1+m_{i}}{2} \ln \frac{1+m_{i}}{2}+\frac{1-m_{i}}{2} \ln \frac{1-m_{i}}{2}\right) \tag{B.2.17}
\end{align*}
$$

In general one finds the internal energy

$$
\begin{equation*}
U=-\sum_{i_{1} \ldots i_{p}} J_{i_{1} \ldots i_{p}}\left\langle s_{i_{1}} \ldots s_{i_{p}}\right\rangle-h \sum_{i}\left\langle s_{i}\right\rangle=-\sum_{i_{1} \ldots i_{p}} J_{i_{1} \ldots i_{p}} m_{i_{1}} \ldots m_{i_{p}}-h \sum_{i} m_{i} \tag{B.2.18}
\end{equation*}
$$

## B.2.2 On a lattice

One can also use this approximation to treat finite dimensional models. Applied to the $d$-dimensional pure ferromagnetic Ising model with nearest-neighbour interactions on a generic lattice $J_{i j}=J$ for nearest-neighbours and zero otherwise. One finds the internal energy

$$
\begin{equation*}
U=-J \sum_{\langle i j\rangle}\left\langle s_{i} s_{j}\right\rangle-h \sum_{i}\left\langle s_{i}\right\rangle=-J \sum_{\langle i j\rangle} m_{i} m_{j}-h \sum_{i} m_{i} . \tag{B.2.19}
\end{equation*}
$$

For a uniformly applied magnetic field, all local magnetisations equal the total density one, $m_{i}=m$, and one has the "order-parameter dependent" free-energy density:

$$
\begin{equation*}
f(m)=-2 J z m^{2}-h m+k_{B} T\left(\frac{1+m}{2} \ln \frac{1+m}{2}+\frac{1-m}{2} \ln \frac{1-m}{2}\right) \tag{B.2.20}
\end{equation*}
$$

with $z$ the coordination of the lattice (we used $\sum_{\langle i j\rangle} \cdot=2 \sum_{i} \sum_{j(i)} \cdot=2 N z \cdot$ ). Although this equation looks different from eq. (B.1.11) it is just a rewriting of it and it leads to the same extreme, as we show below.

## B.2.3 The extrema

The extrema, $d f(m) / d m=0$, are given by ${ }^{8}$

$$
\begin{equation*}
m=\tanh (\beta 4 z J m+\beta h) . \tag{B.2.21}
\end{equation*}
$$

The stable states are those that also satisfy $d^{2} f / d m^{2}>0$.
Exercise B. 6 Prove that the equation that fixes the extrema of eq. (B.2.20) and (B.1.11) is indeed eq. (B.2.21).

This equation of state predicts a second order phase transition at $k_{B} T_{c}=4 z J$ when $h=0$. This transition is seen as the value of the parameters at which the equation passes from having just one solution at $m=0$ to having three solutions, one still at $m=0$, and the other two at $m \neq 0$, always at $h=0$. Since the way in which the non-vanishing solutions appear is continuous, the transition is also called continuous.

The relation $k_{B} T_{c}=4 z J$ correctly captures the fact that $T_{c}$ is proportional to $J$ and that it depends on the lattice geometry. For a cubic lattice $z=2 d$, and the dependence on $d$ is qualitatively correct in the sense that $T_{c}$ increases with increasing $d$. However, the actual value is incorrect in all finite dimensions. In particular, this treatment predicts a finite $T_{c}$ in $d=1$ which is clearly wrong. The critical behaviour is also incorrect in all finite $d$, with exponents that do not depend on dimensionality and take the mean-field values. Still, the nature of the qualitative paramagnetic-ferromagnetic transition in $d>1$ is correctly captured. Having an expression for the free-energy density as a function of the order parameter, which is determined by eq. (B.2.21), one can compute all observables and, in particular, their critical behaviour.

The Taylor expansion of the free-energy in power of $m$, close to the critical point where $m \sim 0$, yields the familiar cross over from a function with a single minima at $m$ to the double well form:

$$
\begin{equation*}
f(m) \sim \frac{1}{2}\left(k_{B} T-4 z J\right) m^{2}+\frac{k_{B} T}{12} m^{4}-h m . \tag{B.2.22}
\end{equation*}
$$

Indeed, below $k_{B} T=4 z J=T_{c}$ the sign of the quadratic term becomes negative and the function develops two minima away from $m=0$.

[^7]Exercise B. 7 Plot the full $f(m)$ and the approximate one and discuss the similarities and differences.

Exercise B. 8 Study the small $m$ behaviour of $f(m)$ in eq. (B.1.11) and compare.
Expanding the mean-field equation close to $m \simeq 0$ one deduces $m \propto\left(T_{c}-T\right)^{1 / 2}$ close to $T_{c}$, meaning that the critical exponent $\beta$ takes the value $1 / 2$. Taking the derivative of $m$ with respect to $h$ and the limit $h \rightarrow 0^{ \pm}$one easily finds that $\chi$ diverges as $\left|T-T_{c}\right|^{-1}$ and $\gamma=1$.

Exercise B. 9 Obtain these and all other mean-field exponents.

## B. 3 The fully-connected p-spin FM model

A set of models for which the mean-field approximation is exact are those defined on a the complete graph, also called fully-connected. We discuss here the generic $p$-spin model and we later specialise to the ferromagnetic class.

## B.3.1 Naive mean-field

Using the factorization of the joint probability density that defines the mean-field approximation, we have already found

$$
\begin{align*}
F\left(\left\{m_{i}\right\}\right)= & -\sum_{i_{1} \neq \cdots \neq i_{p}} J_{i_{1} \ldots i_{p}} m_{i_{1}} \ldots m_{i_{p}}-\sum_{i} h_{i} m_{i} \\
& +k_{B} T \sum_{i=1}^{N}\left(\frac{1+m_{i}}{2} \ln \frac{1+m_{i}}{2}+\frac{1-m_{i}}{2} \ln \frac{1-m_{i}}{2}\right) \tag{B.3.23}
\end{align*}
$$

where in the first sum we wrote explicitly that there are no self interactions. Recall that a Taylor expansion of the entropic contribution around $m_{i}=0$ leads to a polynomial expression that is the starting point in the Landau theory of second order phase transitions.

The local magnetizations, $m_{i}$, are then determined by requiring that they minimize the free-energy density and a positive definite Hessian,

$$
\begin{equation*}
\frac{\partial f\left(\left\{m_{j}\right\}\right)}{\partial m_{i}}=0 \quad \frac{\partial^{2} f\left(\left\{m_{j}\right\}\right)}{\partial m_{i} \partial m_{j}} \tag{B.3.24}
\end{equation*}
$$

(i.e. with all eigenvalues being positive at the extremal value). The first equation yields

$$
\begin{equation*}
m_{i}=\tanh \left(p \beta \sum_{i_{2} \neq \cdots \neq i_{p}} J_{i i_{2} \ldots i_{p}} m_{i_{2}} \ldots m_{i_{p}}+\beta h_{i}\right) \tag{B.3.25}
\end{equation*}
$$

If $J_{i_{1} \ldots i_{p}}=J /\left(p!N^{p-1}\right)$ for all $p$ uplets and the applied field is uniform, $h_{i}=h$, one can take $m_{i}=m$ for all $i$ and these expressions become (B.3.27) and (B.3.30) below, respectively. (Note that a factor $p$ ! has been added to the denominator in order to normalise
the interactions in such a way that each $p$-uplet is counted only once in the sum. This is especially useful if one wants to study the $p \rightarrow \infty$ limit.) The mean-field approximation is exact for the fully-connected pure Ising ferromagnet, as we shall show below. [Note that the fully-connected limit of the model with pair interactions $(p=2)$ is correctly attained by taking $J \rightarrow J /(2 N)$ and $z=2 d \rightarrow N$ in (B.2.21) leading to $T_{c}=J$.]

## B.3.2 Exact solution

Let us solve the ferromagnetic model exactly. The sum over spin configurations in the partition function can be traded for a sum over the variable, $x=N^{-1} \sum_{i=1}^{N} s_{i}$, that takes values $x=-1,-1+2 / N,-1+4 / N, \ldots, 1-4 / N, 1-2 / N, 1$. Neglecting subdominant terms in $N$, one then writes

$$
\begin{equation*}
\mathcal{Z}=\sum_{x} e^{-N \beta f(x)} \tag{B.3.26}
\end{equation*}
$$

with the $x$-parameter dependent 'free-energy density'

$$
\begin{equation*}
f(x)=-\frac{J}{p!} x^{p}-h x+k_{B} T\left(\frac{1+x}{2} \ln \frac{1+x}{2}+\frac{1-x}{2} \ln \frac{1-x}{2}\right) . \tag{B.3.27}
\end{equation*}
$$

The first two terms are the energetic contribution while the third one is of entropic origin since $N!/(N(1+x) / 2)!(N(1-x) / 2)$ ! spin configurations have the same magnetization density. The average of the parameter $x$ is simply the averaged magnetization density:

$$
\begin{equation*}
\langle x\rangle=\frac{1}{N} \sum_{i=1}^{N}\left\langle s_{i}\right\rangle=m \tag{B.3.28}
\end{equation*}
$$

Exercise B. 10 Prove this statement.
In the large $N$ limit, the partition function - and all averages of $x$ - can be evaluated in the saddle-point approximation (see Appendix A.7)

$$
\begin{equation*}
\mathcal{Z} \approx \sum_{\alpha} e^{-N \beta f\left(x_{\mathrm{sp}}^{\alpha}\right)} \tag{B.3.29}
\end{equation*}
$$

where $x_{\mathrm{sp}}^{\alpha}$ are the absolute minima of $f(x)$ given by the solutions to $\partial f(x) /\left.\partial x\right|_{x_{\mathrm{sp}}}=0$,

$$
\begin{equation*}
x_{\mathrm{sp}}=\tanh \left(\frac{\beta J}{(p-1)!} x_{\mathrm{sp}}^{p-1}+\beta h\right), \tag{B.3.30}
\end{equation*}
$$

together with the conditions $d^{2} f(x) /\left.d x^{2}\right|_{x_{\mathrm{sp}}^{\alpha}}>0$. Note that the contributing saddle-points should be degenerate, i.e. have the same $f\left(x_{\mathrm{sp}}^{\alpha}\right)$ for all $\alpha$, otherwise their contribution is exponentially suppressed. The sum over $\alpha$ then just provides a numerical factor of two in the case $h=0$. Now, since

$$
\begin{equation*}
x_{\mathrm{sp}}=-\partial f(x) /\left.\partial h\right|_{x_{\mathrm{sp}}}=\langle x\rangle=m, \tag{B.3.31}
\end{equation*}
$$



Figure 2.15: The free-energy density $f(m)$ of the $p=2$ (left), $p=3$ (center) and $p=4$ (right) models at three values of the temperature $T<T_{c}$ (light dashed line), $T=T_{c}$ (dark dashed line) and $T>T_{c}$ (solid line) and with no applied field. (The curves have been translated vertically.)
as we shall show in Eq. (B.3.32), the solutions to the saddle-point equations determine the order parameter. We shall next describe the phases and phase transition qualitatively and we will later justify this description analytically.

## Model in a finite field

In a finite magnetic field, eq. (B.3.30) has a unique positive - negative - solution for positive - negative - $h$ at all temperatures. The model is ferromagnetic at all temperatures and there is no phase transition in this parameter.

2nd order transition for $p=2$
In the absence of a magnetic field this model has a paramagnetic-ferromagnetic phase transition at a finite $T_{c}$. The order of the phase transition depends on the value of $p$. This can be seen from the temperature dependence of the free-energy density (B.3.27). Figure 2.15 displays $f(x)$ in the absence of a magnetic field at three values of $T$ for the $p=2$ (left), $p=3$ (center) and $p=4$ (right) models (we call the independent variable $m$ since the stationary points of $f(x)$ are located at the magnetization density of the equilibrium and metastable states, as we shall show below). At high temperature the unique minimum is $m=0$ in all cases. For $p=2$, when one reaches $T_{c}$, the $m=0$ minimum splits in two that slowly separate and move towards higher values of $|m|$ when $T$ decreases until reaching $|m|=1$ at $T=0$ (see Fig. 2.15-left). The transition occurs at $T_{c}=J$ as can be easily seen from a graphical solution to eq. (B.3.30), see Fig. 2.16left. Close but below $T_{c}$, the magnetization increases as $m \sim\left(T_{c}-T\right)^{\frac{1}{2}}$. The linear magnetic susceptibility has the usual Curie behavior at very high temperature, $\chi \approx \beta$, and it diverges as $\chi \sim\left|T-T_{c}\right|^{-1}$ on both sides of the critical point. The order parameter is continuous at $T_{c}$ and the transition is of second-order thermodynamically.

## 1st order transition for $p>2$

For $p>2$ the situation changes. For even values of $p$, at $T^{*}$ two minima (and two maxima) at $|m| \neq 0$ appear. These coexist as metastable states with the stable minimum


Figure 2.16: Graphical solution to the equation fixing the order parameter $x$ for $p=2$ (left), $p=3$ (center) and $p=4$ (right) ferromagnetic models at three values of the temperature $T<T^{*}, T=T^{*}$ and $T>T^{*}$ and with no applied field. Note that the rhs of this equation is antisymmetric with respect to $m \rightarrow-m$ for odd values of $p$ while it is symmetric under the same transformation for even values of $p$. We show the positive quadrant only to enlarge the figure. $T^{*}$ is the temperature at which a second minimum appears in the cases $p=3$ and $p=4$.
at $m=0$ until a temperature $T_{c}$ at which the three free-energy densities coincide, see Fig. 2.15-right. Below $T_{c}$ the $m=0$ minimum continues to exist but the $|m| \neq 0$ ones are favored since they have a lower free-energy density. For odd values of $p$ the free-energy density is not symmetric with respect to $m=0$. A single minimum at $m^{*}>0$ appears at $T^{*}$ and at $T_{c}$ it reaches the free-energy density of the paramagnetic one, $f\left(m^{*}\right)=f(0)$, see Fig. 2.15-center. Below $T_{c}$ the equilibrium state is the ferromagnetic minimum. For all $p>2$ the order parameter is discontinuous at $T_{c}$, it jumps from zero at $T_{c}^{+}$to a finite value at $T_{c}^{-}$. The linear magnetic susceptibility also jumps at $T_{c}$. While it equals $\beta$ on the paramagnetic side, it takes a finite value given by eqn. (B.3.33) evaluated at $m^{*}$ on the ferromagnetic one. In consequence, the transition is of first-order.

## Pinning field, broken ergodicity and spontaneous broken symmetry

The saddle-point equation (B.3.30) for $p=2$ [or the mean-field equation (B.2.21)] admits two equivalent solutions in no field. What do they correspond to? They are the magnetisation density of the equilibrium ferromagnetic states with positive and negative value. At $T<T_{c}$ if one computes $m=N^{-1} \sum_{i=1}^{N}\left\langle s_{i}\right\rangle=\sum_{x} e^{-\beta N f(x)} x$ summing over the two minima of the free-energy density one finds $m=0$ as expected by symmetry. Instead, if one computes the averaged magnetisation density with the partition sum restricted to the configurations with positive (or negative) $x$ one finds $m=\left|m_{\mathrm{sp}}\right|$ (or $m=-\left|m_{\mathrm{sp}}\right|$ ).

In practice, the restricted sum is performed by applying a small magnetic field, computing the statistical properties in the $N \rightarrow \infty$ limit, and then setting the field to zero. In other words,

$$
\begin{equation*}
m_{ \pm} \equiv \frac{1}{N} \sum_{i=1}^{N}\left\langle s_{i}\right\rangle_{ \pm}=\left.\left(\frac{1}{\beta N} \frac{\partial \ln Z}{\partial h}\right)\right|_{h \rightarrow 0^{ \pm}}=-\left.\frac{\partial f\left(x_{\mathrm{sp}}\right)}{\partial h}\right|_{h \rightarrow 0^{ \pm}}= \pm\left|x_{\mathrm{sp}}\right| \tag{B.3.32}
\end{equation*}
$$

By taking the $N \rightarrow \infty$ limit in a field one selects the positive (or negatively) magnetised
states.
For all odd values of $p$ the phase transition is not associated to symmetry breaking, since there is only one non-degenerate minimum of the free-energy density that corresponds to the equilibrium state at low temperature. The application of a pinning field is then superfluous.

For any even value of $p$ and at all temperatures the free-energy density in the absence of the field is symmetric with respect to $m \rightarrow-m$, see the left and right panels in Fig. 2.15. The phase transition corresponds to a spontaneous symmetry breaking between the states of positive and negative magnetisation. One can determine the one that is chosen when going through $T_{c}$ either by applying a small pinning field that is taken to zero only after the thermodynamic limit, or by imposing adequate boundary conditions. Once a system sets into one of the equilibrium states this is completely stable in the $N \rightarrow \infty$ limit. In pure static terms this means that one can separate the sum over all spin configurations into independent sums over different sectors of phase space that correspond to each equilibrium state. In dynamic terms it means that temporal and statistical averages (taken over all configurations) in an infinite system do not coincide.

The magnetic linear susceptibility for generic $p$ is given by

$$
\begin{equation*}
\left.\chi \equiv \frac{\partial m}{\partial h}\right|_{h \rightarrow 0^{ \pm}}=\left.\frac{\partial x_{\mathrm{sp}}}{\partial h}\right|_{h \rightarrow 0^{ \pm}}=\frac{\beta}{\cosh ^{2}\left(\frac{\beta J}{(p-1)!} x_{\mathrm{sp}}^{p-1}\right)-\frac{\beta J}{(p-2)!} x_{\mathrm{sp}}^{p-2}} \tag{B.3.33}
\end{equation*}
$$

For $p=2$, at $T>T_{c}, x_{\mathrm{sp}}=0$ the susceptibility is given by $(T-J)^{-1}$ predicting the second order phase transition with a divergent susceptibility at $T_{c}=J$. Approaching $T_{c}$ from below the two magnetized states have the same divergent susceptibility, $\chi \sim\left(T_{c}-T\right)^{-1}$.

For $p>2$, at $T>T_{c}, x_{\mathrm{sp}}=0$ and the susceptibility takes the Curie form $\chi=\beta$. The Curie law, $\chi=\beta$, jumps to a different value at the critical temperature due to the fact that $x_{\text {sp }}$ jumps.

## B. 4 Related problems

The mean-field solution of the random-field Ising model is described in https://inordinatum.wordpress.com/2013/01/20/mean-field-solution-of-the-random-field-ising-model/
see also
T. Schneider and E. Pytte, Random-field instability of the ferromagnetic state, Phys. Rev. B 15, 1519 (1977).

## C Classical results in statistical physics

## C. 1 High temperature expansion

The partition function of the Ising ferromagnet reads

$$
\begin{equation*}
Z=\sum_{s_{i}= \pm 1} e^{\beta J \sum_{\langle i j\rangle} s_{i} s_{j}}=\sum_{s_{i}= \pm 1} \prod_{\langle i j\rangle} e^{\beta J s_{i} s_{j}} \tag{C.1.34}
\end{equation*}
$$

Using the identity $e^{\beta J s_{i} s_{j}}=a\left(1+b s_{i} s_{j}\right)$ with $a=\cosh (\beta J)$ and $b=\tanh (\beta J)$ and the fact that $b$ is order $\beta$, an expansion if powers of $b$ can be established. The average of products of the spins $s$ 's that remains can be non-zero only if each spin appears an even number of times $s$. The expansion can then be represented as graphs on the lattice, a representation that makes the enumeration of terms easier.

## C. 2 Lee-Yang theorem

The LeeĐYang theorem states that if partition functions of models with ferromagnetic interactions are considered as functions of an external field, then all zeros are purely imaginary (or on the unit circle after a change of variable) [94].

## C. 3 Critical behaviour

Second order phase transitions are characterised by the diverge of the correlation length. In normal conditions, far from the critical point, the correlation function of the fluctuations of an observable decay as an exponential of the distance between the measuring points:

$$
\begin{equation*}
C(\vec{r}) \equiv\left\langle\left[O\left(\vec{r}+\vec{r}^{\prime}\right)-\left\langle\left(O\left(\vec{r}+\vec{r}^{\prime}\right)\right\rangle\right]\left[O\left(\vec{r}^{\prime}\right)-\left\langle\left(O\left(\vec{r}^{\prime}\right)\right\rangle\right)\right\rangle\right]\right\rangle \simeq e^{-r / \xi} . \tag{C.3.35}
\end{equation*}
$$

$\xi$ is the correlation length that diverges at the critical point as

$$
\begin{equation*}
\xi \simeq\left|T-T_{c}\right|^{-\nu} \tag{C.3.36}
\end{equation*}
$$

with $\nu$ a critical exponent. A power-law singularities in the length scales leads to powerlaw singularities in observable quantities. We summarise in Table C. 4 all the critical exponents associated to various quantities in a second order phase transition. The values of the critical exponents generally do not depend on the microscopic details but only on the space dimensionality and the symmetries of the system under consideration.

The collection of all these power laws characterizes the critical point and is usually called the critical behavior.

|  | exponent | definition | conditions |
| :--- | :---: | :---: | :--- |
| Specific heat | $\alpha$ | $c \propto\|u\|^{-\alpha}$ | $u \rightarrow 0, h=0$ |
| Order parameter | $\beta$ | $m \propto(-u)^{\beta}$ | $u \rightarrow 0-, h=0$ |
| Susceptibility | $\gamma$ | $\chi \propto\|u\|^{-\gamma}$ | $u \rightarrow 0, h=0$ |
| Critical isotherm | $\delta$ | $h \propto\|m\|^{\delta} \operatorname{sign}(m)$ | $h \rightarrow 0, u=0$ |
| Correlation length | $\nu$ | $\xi \propto\|r\|^{-\nu}$ | $r \rightarrow 0, h=0$ |
| Correlation function | $\eta$ | $G(\vec{r}) \propto\|\vec{r}\|^{-d+2-\eta}$ | $r=0, h=0$ |

Table C.4: Definitions of the commonly used critical exponents. $m$ is the order parameter, e.g. the magnetization, $h$ is an external conjugate field, e.g. a magnetic field, $u$ denotes the distance from the critical point, e.g. $\left|T-T_{c}\right|$, and $d$ is the space dimensionality.

Whether fluctuations influence the critical behavior depends on the space dimensionality $d$. In general, fluctuations become less important with increasing dimensionality.

In sufficiently low dimensions, i.e. below the lower critical dimension $d_{l}$, fluctuations are so strong that they completely destroy the ordered phase at all (nonzero) temperatures and there is no phase transition. Between $d_{l}$ and the upper critical dimension $d_{u}$, order at low temperatures is possible, there is a phase transition, and the critical exponents are influenced by fluctuations (and depend on $d$ ). Finally, for $d>d_{u}$, fluctuations are unimportant for the critical behavior, and this is well described by mean-field theory. The exponents become independent of d and take their mean-field values. For example, for Ising ferromagnets, $d_{l}=1$ and $d_{u}=4$, for Heisenberg ferromagnets $d_{l}=2$ and $d_{u}=4$.

## C. 4 Finite size scaling

Finite-size scaling controls the finite-size effects close to second order phase transitions. Say that the critical point, $T_{c}$, is approached by tuning the control parameter $T$, and that a correlation length, $\xi$, diverges as $\xi \simeq\left|T-T_{c}\right|^{-\nu}$ close to $T_{c}$. A generic observable $X$ is expected to scale as

$$
\begin{equation*}
X_{L}(T) \simeq\left(T-T_{c}\right)^{-\chi_{\times}} g\left(\left(p-p_{c}\right) L^{1 / \nu}\right) \propto \xi^{\chi_{\times} / \nu} g\left((L / \xi)^{1 / \nu}\right) \tag{C.4.37}
\end{equation*}
$$

close to $T_{c}$, with

$$
g(y) \simeq \begin{cases}\text { const } & \text { for } \quad y \rightarrow 0  \tag{C.4.38}\\ y^{\chi_{\times} / \nu} & \text { for } y \rightarrow \infty\end{cases}
$$

thus ensuring

$$
X_{L}(T) \simeq \begin{cases}\xi^{\chi_{x} / \nu} & \text { for } \quad L \gg \xi  \tag{C.4.39}\\ L^{\chi_{x} / \nu} & \text { for } \quad L \ll \xi\end{cases}
$$

## C.4.1 One dimensional case

Take the disordered Ising chain

$$
\begin{equation*}
H=-\sum_{i=1}^{N}\left(J_{i} s_{i} s_{i+1}+h_{i} s_{i}\right) \tag{C.4.40}
\end{equation*}
$$

with $J_{i}$ and $h_{i}$ random exchanges and random fields taken from probability distribution. Impose periodic boundary conditions such that $s_{N+1}=s_{1}$. The partition function can be evaluated with the transfer matrix method introduced by Kramers and Wannier [30] and Onsager [31]. Indeed,

$$
\begin{equation*}
Z_{N}=\sum_{\left\{s_{i}= \pm 1\right\}} T_{1 s_{1} s_{2}} T_{2 s_{2} s_{3}} \ldots T_{N s_{N} s_{1}}=\operatorname{Tr} \prod_{i=1}^{N} T_{i} \tag{C.4.41}
\end{equation*}
$$

where $T_{i}$ are $2 \times 2$ matrices in which one takes the two row and column indices to take the values $\pm 1$. Then

$$
T_{i}=\left(\begin{array}{cc}
e^{\beta\left(J_{i}+h_{i}\right)} & e^{\beta\left(-J_{i}+h_{i}\right)}  \tag{C.4.42}\\
e^{\beta\left(-J_{i}-h_{i}\right)} & e^{\beta\left(J_{i}-h_{i}\right)}
\end{array}\right)
$$

Note that, for random exchanges and/or fields Eq. (C.4.41) is a product of random matrices. Methods from random matrix theory can then be used to study disordered spin chains [71].

The free-energy per spin is given by

$$
\begin{equation*}
-\beta f_{N}=-\frac{1}{N} \ln Z_{N}=-\frac{1}{N} \ln \operatorname{Tr} \prod_{i=1}^{N} T_{i} \tag{C.4.43}
\end{equation*}
$$

The thermodynamic quantities energy per spin, magnetic susceptibility and other can be computed from this expression. The local quantities, such as local averaged magnetization or correlation functions are evaluated with the help of the spin operator

$$
\Sigma=\left(\begin{array}{cc}
1 & 0  \tag{C.4.44}\\
0 & -1
\end{array}\right)
$$

as

$$
\begin{align*}
\left\langle s_{i}\right\rangle & =\frac{1}{Z_{N}} \operatorname{Tr} T_{1} T_{2} \ldots T_{i-1} \Sigma T_{i} \ldots T_{N}  \tag{C.4.45}\\
\left\langle s_{i} s_{j}\right\rangle & =\frac{1}{Z_{N}} \operatorname{Tr} T_{1} \ldots T_{i-1} \Sigma T_{i} \ldots T_{j-1} \Sigma T_{j} \ldots T_{N} \tag{C.4.46}
\end{align*}
$$

This problem was solved above with the change of variables $\sigma_{i}=s_{i} s_{i+1}$. Let us see now how one can solve it with the transfer matrix method. For $h_{i}=0$, all matrices can be diagonalized with the change of basis, $P U_{i} P^{-1}=T_{i}$ with

$$
P=\left(\begin{array}{cc}
1 & 1  \tag{C.4.47}\\
1 & -1
\end{array}\right) \quad U_{i}=\left(\begin{array}{cc}
2 \cosh \beta J_{i} & 0 \\
0 & 2 \sinh \beta J_{i}
\end{array}\right)
$$

and $P^{-1}=P / 2$. One then has

$$
\begin{equation*}
Z_{N}=\left(\prod_{i=1}^{N}\left(2 \cosh \beta J_{i}\right)\right)\left(\prod_{i=1}^{N}\left(1+\tanh \beta J_{i}\right)\right) \approx \prod_{i=1}^{N}\left(2 \cosh \beta J_{i}\right) \tag{C.4.48}
\end{equation*}
$$

More details on this kind of chains can be found in [71].

## D The random field Ising model on the complete graph

Consider the fully connected random field Ising model

$$
\begin{equation*}
H\left[\left\{s_{i}\right\}\right]=-\frac{J}{N} \sum_{i \neq j} s_{i} s_{j}-\sum_{i} h_{i} s_{i} \tag{D.0.1}
\end{equation*}
$$

with Gaussian distributed random fields such that

$$
\begin{equation*}
\left[e^{-s h}\right]_{h}=\int_{-\infty}^{\infty} \frac{d h}{\sqrt{2 \pi \sigma_{h}^{2}}} e^{-\frac{h^{2}}{2 \sigma_{h}^{2}}} e^{-s h}=e^{\frac{\sigma_{h}^{2} s^{2}}{2}} \tag{D.0.2}
\end{equation*}
$$

The disorder averaged free energy density can be computed with the replica trick [95]

$$
\begin{equation*}
-\beta[f]_{h}=\lim _{N \rightarrow \infty} \lim _{n \rightarrow 0} \frac{\left[Z^{n}\right]_{h}-1}{n N} \tag{D.0.3}
\end{equation*}
$$

The average of the replicated partition sum reads

$$
\begin{align*}
{\left[Z^{n}\right]_{h} } & =\sum_{s_{i}^{a}= \pm 1} e^{\frac{\beta J}{N} \sum_{a} \sum_{i \neq j} s_{i}^{a} s_{j}^{a}}\left[e^{\beta \sum_{i} \sum_{a} s_{i}^{a} h_{i}}\right]_{h} \\
& =\sum_{s_{i}^{a}= \pm 1} e^{\frac{\beta J}{N} \sum_{a} \sum_{i \neq j} s_{i}^{a} s_{j}^{a}} \prod_{i} e^{\frac{\beta^{2} \sigma_{h}^{2}}{2} \sum_{a} \sum_{b} s_{i}^{a} s_{i}^{b}} \tag{D.0.4}
\end{align*}
$$

We now decouple the term in the first exponential by using a Hubbard-Stratonovich transformation. First we rewrite the sum as

$$
\begin{equation*}
\sum_{i \neq j} s_{i}^{a} s_{j}^{a}=\sum_{i j} s_{i}^{a} s_{j}^{a}-\sum_{i}\left(s_{i}^{a}\right)^{2}=\left(\sum_{i} s_{i}^{a}\right)^{2}-N \tag{D.0.5}
\end{equation*}
$$

and we focus on the first term only, that we decouple using

$$
\begin{equation*}
e^{\frac{\beta J}{N} \sum_{a}\left(\sum_{i} s_{i}^{a}\right)^{2}}=\frac{1}{(2 \pi)^{n / 2}} \int d x_{a} e^{x_{a} \sqrt{2 \beta J} N \sum_{i} s_{i}^{a}-\frac{1}{2} x_{a}^{2}} . \tag{D.0.6}
\end{equation*}
$$

The averaged replicated partition function is

$$
\begin{equation*}
\left[Z^{n}\right]_{h}=\left(\frac{N}{2 \pi}\right)^{n / 2} \sum_{s_{i}^{a}= \pm 1} \int d \tilde{x}_{a} e^{-\frac{N}{2} \sum_{a} \tilde{x}_{a}^{2}+\sqrt{2 \beta J} \sum_{i} \sum_{a} \tilde{x}_{a} s_{i}^{a}+\frac{\beta^{2} \sigma_{h}^{2}}{2} \sum_{i}\left(\sum_{a} s_{i}^{a}\right)^{2}} \tag{D.0.7}
\end{equation*}
$$

where we rescaled $x_{a} \mapsto \sqrt{N} \tilde{x}_{a}$. It became the one of independent spins $s_{i}^{a}$ with partition sum

$$
\begin{equation*}
Z_{1}\left(\tilde{x}_{a}\right)=\sum_{s= \pm 1} e^{\sqrt{2 \beta J} \sum_{a} \tilde{x}_{a} s_{a}+\frac{\beta^{2} J^{2}}{2}\left(\sum_{a} s_{a}\right)^{2}} \tag{D.0.8}
\end{equation*}
$$

The averaged replicated partition sum is now

$$
\begin{equation*}
\left[Z^{n}\right]_{h}=e^{-\beta J}\left(\frac{N}{2 \pi}\right)^{n / 2} \int d \tilde{x}_{a} e^{-N\left[\frac{1}{2} \sum_{a} \tilde{x}_{a}^{2}+\ln Z_{1}\left(\tilde{x}_{a}\right)\right]} \tag{D.0.9}
\end{equation*}
$$

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[^0]:    ${ }^{1}$ Note that Ising model does not have a natural dynamics associated to it. We shall see in Section ?? how a dynamic rule is attributed to the evolution of the spins.

[^1]:    ${ }^{2}$ The calculation goes as follow: $\left\langle F_{i}\right\rangle=\sum_{j} J_{i j}\left\langle m_{j}\right\rangle=0$ and $\left\langle F_{i}^{2}\right\rangle=\sum_{j k} J_{i j} J_{i k}\left\langle m_{j} m_{k}\right\rangle=\sum_{j} J_{i j}^{2}$

[^2]:    ${ }^{3}$ The spectrum of a large symmetric random matrix can be evaluated with several methods, including the replica trick, as explained in [61].

[^3]:    ${ }^{4}$ the notion of a pure state will be made more precise below. A mathematical definition can be given by it lies beyond the scope of these lectures.

[^4]:    ${ }^{5}$ Note that $\chi_{i j}$ can take negative values. Moreover, the sum over $i \neq j$ has $\mathrm{O}\left(N^{2}\right)$ terms of different sign and then central limit theorem implies that, if they are uncorrelated, the result is $\mathrm{O}(N)$ that once normalised by $N$ yields a value $\mathrm{O}(1)$. The further average over the $J_{i j}$ yields the vanishing result.

[^5]:    ${ }^{6}$ We use $\int d J_{\alpha} e^{-J_{\alpha}^{2} /\left(2 \sigma^{2}\right)-J_{\alpha} x} \propto e^{x^{2} \sigma^{2} / 2}$

[^6]:    ${ }^{7}$ The reason for the additional factor 2 in front of the sum $\sum_{j(i)}$ is that $\sum_{i j}=2 \sum_{i} \sum_{j(i)}$.

[^7]:    ${ }^{8}$ Note that this approximation amounts to replacing the exact equation $m_{i}=\left\langle\tanh \beta\left(h+\sum_{j} J_{i j} s_{j}\right)\right\rangle$ by $m_{i}=\tanh \beta\left(h+\sum_{j} J_{i j} m_{j}\right)$.

