MASTER COMPLEX SYSTEMS

QUANTUM PHYSICS OUT OF EQUIL.

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ENSEIGNEMENT

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QUANTUM

PURE STATE VECTOR $\in \mathcal{H}$ $\quad k \leq \dim \mathcal{H}$

$$|\psi\rangle = \sum_{k=1}^{K} c_k |\psi_k\rangle$$ KET

LINEAR SUPERP. PRINCIPLE

$c_k \in \mathbb{C}$ $\quad \{|\psi_k\rangle\}$ ORTHON. BASIS OF $\mathcal{H}$

k

$$\sum_{k=1} |c_k|^2 = 1$$

$$\langle\psi| = \sum_{k=1}^{K} c_k^* \langle\psi_k|$$ DUAL BRA

INNER PRODUCT $\quad \langle\psi|\psi\rangle = 1$ NORM

$$\sum_{k=1}^{K} |c_k|^2 = 1$$

eg. SPIN

CONTINUOUS INFO THROUGH $\{c_k\}$

OPERATORS

ACTS ON A VECTOR AND PRODUCES A VECTOR

$$\hat{A}: \quad |\psi\rangle \in \mathcal{H} \longrightarrow \hat{A}|\psi\rangle \in \mathcal{H}$$

RANK 1

$$\hat{\theta} = |\phi\rangle\langle\psi|$$

CONJUGATE $\quad \hat{\theta}^\dagger = |\psi\rangle\langle\phi|$

HERMITIAN $\quad \hat{\theta}^\dagger = \hat{\theta}$

eg. PROJECTOR $\quad \hat{P} = |\psi\rangle\langle\psi|$

$\langle\psi|\psi\rangle = 1$ $\qquad \hat{P}^2 = \hat{P} \qquad \hat{P}^\dagger = \hat{P}$

ex. SPIN PROBLEM CONSTRUCT NON-HERM & HERM. OP.

CLOSURE RELATION $\sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{1}$

ex QU-BIT

GENERAL RANK

MEASURE OF HOW MANY INDEP DIRECTIONS THE MAP PRODUCES

eg $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ TWO INDEP COLUMNS

if $\alpha \vec{v}_1 + \beta \vec{v}_2 = 0$ $\alpha \neq 0$ $\beta \neq 0$ $\Rightarrow$ DEP.

OTHERWISE INDEP.

IT CANNOT BE WRITTEN AS $|\phi\rangle\langle\psi|$

BUT IT CAN BE WRITTEN AS

$\hat{A} = \sum_{ij} A_{ij} |\psi_i\rangle\langle\psi_j|$ $\hat{A}$ DIAGONALIZ.

WITH $A_{ij} = \lambda_i \delta_{ij}$ $|\psi_i\rangle$ EIGENVECTOR W/ EIGENVALUE λ_i

MEASUREMENT

A MEASURABLE OBSERV $\hat{A} = \hat{A}^+$

HOW TO MEASURE ? IDENTIF

$$\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$$

WRITE $|\psi\rangle = \sum_n c_n\,|a_n\rangle$

$$c_n = \langle a_n|\psi\rangle$$

RESULT of MEASUREMENT : ONE a_n

WITH PROBABILITY

BORN RULE $P_n = |c_n|^2 = |\langle\psi|a_n\rangle|^2$

MEASURE $\hat{A}$ STATE of SYST

COLLAPSES on $|a_n\rangle$ WHEN GETTING a_n

VON NEUMANN POSTULATE

(EXTENSION for DEG. $P_n = \sum_i |\langle a_n^i|\psi\rangle|^2$)

EXPECTATION VALUES

THEOR CONSTRUCT

WHAT ONE SHOULD MEASURE ON AVER

OVER MANY TRIALS

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_n |c_n|^2 a_n$$

ex $\langle \hat{\sigma}_z \rangle$ on $|\psi\rangle = |0\rangle$ SPIN UP

or $a|0\rangle + b|1\rangle = |\psi\rangle$

$$\langle \hat{\sigma}_z \rangle = (b^* \langle 1| + a^* \langle 0|)\, \hat{\sigma}_z\, (a|0\rangle + b|1\rangle) =$$
$$= (b^* \langle 1| + a^* \langle 0|)(a|0\rangle - b|1\rangle)$$
$$= |a|^2 - |b|^2$$

UNITARY OP

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \mathbb{1}$$

CONS. NORM CONSERVED

$$|\psi'\rangle = \hat{U}|\psi\rangle \qquad \langle \psi' | \psi' \rangle = \langle \psi | \psi \rangle$$

DENSITY OPERATOR

DESCRIBES THE STATE OF THE SYSTEM, BEYOND WHEN IT'S IN A PURE STATE. ALSO FOR MIXED STATES (STAT. SENSE).

- PURE STATES

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

Basis of $\mathcal{H}$

$$= \sum_i c_i |\psi_i\rangle \sum_j c_j^* \langle\psi_j|$$

$$= \sum_{i,j} c_i^* c_j \, |\psi_j\rangle\langle\psi_i|$$

ELEMENTS $\langle\psi_n|\hat{\rho}|\psi_m\rangle$ DENS MATRIX

$$\langle\psi_n|\hat{\rho}|\psi_m\rangle = \sum_{i,j} c_i^* c_j \underbrace{\langle\psi_n|\psi_j\rangle}_{\delta_{jn}} \underbrace{\langle\psi_i|\psi_m\rangle}_{\delta_{im}}$$

$$= c_m^* c_n$$

ρ_{nn} DIAG $\Rightarrow$ POPULATIONS

GIVE PROB of RESULTS of

MEASUREMENTS $|c_n|^2$

OF THE OP. WHICH GIVES RISE TO

THE BASIS $\{|\psi_i\rangle\}$ USED.

$$\sum_n \rho_{nn} = \sum_n |c_n|^2 = 1$$ By norm.

$$\Rightarrow \quad Tr\,\hat{\rho} = 1$$

$\rho_{n\neq m}$ off DIAG $\Rightarrow$ COHERENCES

THE ELEM ρ_{nm} DEP ON THE BASIS CHOSEN

- IMPORTANT ρ CAN BE DIAG IN ONE BASIS & NON-DIAG IN ANOTHER
- $\hat{\rho} = \hat{\rho}^\dagger$ HERMITIAN (in BASIS)
- ALSO POSITIVE SEMI- DEF

ex. HERE

- MIXED STATES

IF WE DO NOT KNOW WELL THE STATE OF THE SYST → IGNORANCE

$$\hat{\rho} = \sum_{\alpha} p_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$\sum_{\alpha=1}^{N} p_\alpha = 1 \qquad p_\alpha \in \mathbb{R} \qquad p_\alpha \geq 0$$

THE $|\psi_\alpha\rangle$ NORM. BUT NOT NEC. ORTHOG.

STATISTICAL ("NOT QUANTUM")

ex SPIN MEASURED W/ $\hat{S}^2$ BUT WE DO NOT KNOW OUTCOME

HERMITIAN

$$\hat{\rho}^\dagger = \left(\sum_\alpha p_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha| \right)^\dagger = \sum_\alpha p_\alpha^* \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$= \sum_\alpha p_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

ex PROVE $\quad Tr\, \hat{\rho} = 1$

$Tr\, \hat{\rho}^2 < 1$ if MIXED

$Tr\, \hat{\rho}^2 = Tr\, \hat{\rho}$ if PURE

$\langle\psi|\hat{\rho}|\psi\rangle \geq 0 \quad \forall\, |\psi\rangle$

$\hat{\rho} = \hat{\rho}^\dagger$ AND $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$

$\Rightarrow$ EIGEN. $\lambda_k \in R^+ \quad \lambda_k \geq 0$

EXP. VALUE OF AN OBSERV. $\hat{\theta}$

$$\langle\hat{\theta}\rangle = Tr\left(\hat{\rho}\,\hat{\theta}\right)$$

FOR A PURE STATE $\quad \langle\hat{\theta}\rangle = \langle\psi|\hat{\theta}|\psi\rangle$

AS DEFINED ABOVE $\quad$ ex. SHOW IT

FOR A MIXED STATE $\hat{\rho}$

$$\langle \hat{\Theta} \rangle = \sum_{\alpha} p_\alpha \sum_m a_m |c_\alpha^m|^2$$

ex. SHOW IT

STAT-PHYSICS IN EQUIL

MICROCANONICAL

$[E, E+\delta E]$ ENERGY SHELL

$$\hat{\rho} = \frac{1}{\Omega} \sum_{n \in \text{SHELL}} |n\rangle\langle n|$$

$$\hat{H}|n\rangle = E_n |n\rangle$$

Ω # STATES WITH $E_n \in [E, E+\delta E]$

$$S = k_B \ln \Omega$$

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

$$\langle \hat{\theta} \rangle = Tr\left(\hat{\rho}\hat{\theta}\right)$$

$$= \frac{1}{\Omega} \sum_{n \in \text{SHELL}} \langle n|\hat{\theta}|n\rangle$$

ex. TWO LEVEL SYST — MORE BY END OF LECTURES

CANONICAL

$$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z}$$

$$Z = Tr\, e^{-\beta\hat{H}} = \sum_n \langle n|e^{-\beta\hat{H}}|n\rangle$$

$$= \sum_n e^{-\beta E_n}$$

$$p(|n\rangle) = \frac{e^{-\beta E_n}}{Z}$$

PROB of STATE $|n\rangle$

FOR PROB of E_n:

$$p(E_n) = g(E_n)\frac{e^{-\beta E_n}}{Z}$$

$$-\beta F = \ln Z$$

Disordered Systems (Many Body)

$$\hat{H} = -\frac{1}{2} \sum_{ij} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Gamma \sum_i \hat{\sigma}_i^x$$

NON COMMUTING TERMS

$$Z = Tr\, e^{-\beta \hat{H}}$$

$$-\beta F = \ln Z$$

HAVE TO AVERAGE OVER $P(\{J_{ij}\})$

THE $\ln Z$ - QUENCHED AVER.

$\Rightarrow$ REPLICAS.

BUT, HOW TO DEAL W/ OPERATORS ?

PATH INT. FORMULATION

$\Rightarrow$ LATER

MANY BODY SYSTEMS

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \qquad \dim \mathcal{H} = k^N$$

$$= \bigotimes_{i=1}^{N} \mathcal{H}_i$$

TENSOR PRODUCT OF HILBERT SP.

$$|\psi\rangle = \sum_{k_1=1}^{k} \cdots \sum_{k_N=1}^{k} c_{k_1 \cdots k_N} \, |\psi_1^{k_1}\rangle \otimes \cdots \otimes |\psi_N^{k_N}\rangle$$

SYST 1 STATE k_1 etc.

$$\sum_{k_1=1}^{k} \cdots \sum_{k_N=1}^{k} |c_{k_1 \cdots k_N}|^2 = 1$$

ex 2 QU-BITS

REDUCED DENSITY OPERATOR

$$\hat{\rho}_A^{RED} = \mathrm{Tr}_B \, \hat{\rho}_{AB}$$

- EX. SEPARABLE PURE STATE:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\hat{\rho}_A^{RED} = \mathrm{Tr}_B \, |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$= \hat{\rho}_A = |\psi_A\rangle\langle\psi_A|$$

OTHERWISE $\hat{\rho}_A^{RED} \neq \hat{\rho}_A$ ENTANGLED PURE $|\psi_{AB}\rangle$

- MIXED STATE FROM RED OF NON-SEP ST.

$$|\psi_{AB}\rangle = \sum_{nm} C_{nm} \, |a_n\rangle \otimes |b_m\rangle$$

FOLLOW NOTES · p. 39

- PURIFICATION FOLLOW NOTES. p.40
- EXPECTED VALUES IN REDUCED STATES.

EXPECTED VALUES

$$\langle \hat{\theta}_A \rangle = Tr_{AB} \; \hat{\theta}_A \; \hat{\rho}_{AB}$$

$$= Tr_A \; \hat{\theta}_A \; Tr_B \; \hat{\rho}_{AB}$$

$$= Tr_A \; \hat{\theta}_A \; \hat{\rho}_A^{RED}$$

MENTION EX. IN NOTES WORKED OUT IN FULL DETAIL 2.10 AND 2.11

ENTANGLEMENT

THE QUANTUM STATES of TWO SUBSYST ARE DEPENDENT ON EACH OTHER

SCHRÖDINGER 1935

PERFECT CORRELATION.

eg. ONE PART $\vec{S}=0$ DECAYS IN TWO

PART $\Rightarrow$ SUM OF SPINS $=0$

MEASURE $\vec{S}$ IN ONE

$\Rightarrow$ THE OTHER ONE MUST BE KNOWN TOO

INDEP. of DIST ! PERFECT CORR.

Def. WORKING W/ STATES

$|\psi\rangle_{AB}$ is ENTANGLED if IT CANNOT BE WRITTEN AS A SEPARABLE STATE

$$|\psi\rangle_{AB} = \sum_{ab} c_{ab} \, |\psi_a\rangle_A \otimes |\psi_b\rangle_B$$

IS GENERIC CASE, AND $\neq$ FROM

$$|\psi\rangle_{AB} \overset{SEP}{=} |\psi\rangle_A \otimes |\psi_B\rangle$$

$$= \sum_a c_a^A \, |\psi_a\rangle \otimes \sum_b c_b^B \, |\psi_b\rangle$$

IN AN ENTANGLED STATE,
A MEASUREMENT DONE ON B COLLAPSES WHAT
A CAN BE. INSTANTANEOUSLY & INDEP OF
DIST BTW A AND B.

WORKING WITH DENSITY OPERATORS

$$\hat{\rho}_{AB} \overset{PROD}{=} \hat{\rho}_A \otimes \hat{\rho}_B \qquad \text{PRODUCT STATE} \qquad (1)$$

$$\hat{\rho}_{AB} \overset{SEP}{=} \sum_\alpha p_\alpha \, \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha \qquad (2)$$

STATISTICAL MIXTURE OF PRODUCT STATES

BOTH ARE SEPARABLE.

(1) NO QUANTUM NOR CLASSICAL CORR

(2) ONLY CLASSICAL CORR

SEPARABLE DENS. OP.

$$\operatorname{Tr}_B \rho_{AB} \overset{\text{PROD.}}{=} \operatorname{Tr}_B \hat{\rho}_A \otimes \hat{\rho}_B$$

$$= \hat{\rho}_A$$

$$\operatorname{Tr}_B \rho_{AB} \overset{\text{SEP}}{=} \operatorname{Tr}_B \sum_\alpha p_\alpha \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha \otimes \operatorname{Tr}_B \hat{\rho}_B^\alpha$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha$$

A MIXTURE OF THE DENS OP OF A.

ENTANGLED DENSITY OPERATOR

$$\hat{\rho}_{AB}^{ENT.} \neq \hat{\rho}_{AB}^{SEP}$$

IT IS HARD TO KNOW WHETHER A GENERIC $\hat{\rho}_{AB}$ IS ENTANGLED OR NOT

NP-HARD

LOCAL PROBS OF ENT STATES CANNOT BE TREATED INDEP.

SEPARAB. IS INDEP OF BASIS

PROOF IT IS INVAR UNDER LOCAL UNITARIES $\hat{U}_A \otimes \hat{U}_B$

$|\psi\rangle$ SEPAR. $\Rightarrow$

$$|\psi\rangle = |\psi_A\rangle \otimes |\phi_B\rangle$$

FOR $\hat{U} = \hat{U}_A \otimes \hat{U}_B$

$$\hat{U}|\psi\rangle = \hat{U}_A|\psi_A\rangle \otimes \hat{U}_B|\psi_B\rangle$$

SEP. AS WELL

$\hat{U}|\psi\rangle$ IN ANOTHER BASIS

ENTANG. ALSO INDEP BASIS.

BELL STATES

Z- BASIS USED

ENTANGLED STATES OF TWO SPIN - ½

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle\right)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle\right)$$

MEASURES ONE ⇒ KNOWS THE OTHER !

MAXIMALLY ENTANGLED WITH CERTAINTY

PROPERTIES

- NORMALIZED TO 1
- PURE STATES $\hat{\rho}^2 = \hat{\rho}$

- ENTANGLED

- SYMM OR ANTI SYMM $A \leftrightarrow B$

  $|\Phi^{\pm}\rangle$ $|\psi^{+}\rangle$ $|\psi^{-}\rangle$

- WRITE THEM IN TERMS OF $|+\rangle, |-\rangle$

  AND CHECK SYMM. PROP. ABOVE

  $|+\rangle, |-\rangle$ EIGENSTATES OF $\hat{\sigma}^x$

- CALCULATE THE EIGENVALUES OF

  THE TOTAL SPIN OPERATOR

$$\hat{S}^2 = \left(\hat{\vec{S}}_A + \hat{\vec{S}}_B\right)^2$$

$S(S+1)$ $\begin{cases} S=1 & |\Phi^{\pm}\rangle \quad |\psi^{+}\rangle \\ S=0 & |\psi^{-}\rangle \end{cases}$

STATES IN TRIPLET $S=1$

STATE IN SINGLET $S=0$

- EVALUATE THE TOTAL SPIN IN EACH OF THE DIRECTIONS

  x, y, z USING THE Z BASIS

  IN THE STATE

  $|\Phi^{\pm}\rangle \Rightarrow$ EQUAL

  $|\psi^{\pm}\rangle \Rightarrow$ OPPOSITE

CALCULATE

$$P_{00} = \left| \langle 0| \otimes \langle 0| \, |\Phi^{\pm}\rangle \right|^2$$

P_{m}, P_{01}, P_{10} FOR Z- DIRECTION

$$P_{++} = \left| \langle +| \otimes \langle +| \, |\Phi^{\pm}\rangle \right|^2$$

P_{--}, P_{+-}, P_{-+} FOR X DIRECTION etc.

FOR $|\Phi^{\pm}\rangle$ AND SIMILARLY FOR $|\psi^{\pm}\rangle$

IMPORTANT

$$\langle \psi^-| \, \hat{\sigma}^c \otimes \mathbb{I} + \mathbb{I} \otimes \hat{\sigma}^c \, |\psi^-\rangle = 0$$

- $\underset{B}{Tr}\ |\phi^{\pm}\rangle\langle\phi^{\pm}| = \frac{\mathbb{I}}{2} = \underset{B}{Tr}\ |\psi^{\pm}\rangle\langle\psi^{\pm}|$

  $\underset{A}{Tr}\ |\Phi^{\pm}\rangle\langle\Phi^{\pm}| = \frac{\mathbb{I}}{2} = \underset{A}{Tr}\ |\psi^{\pm}\rangle\langle\psi^{\pm}|$

- SHOW THAT THE FULL KNOWLEDGE OF NOT MEASURED QU-BIT ONCE THE OTHER ONE IS MEASURED APPLIED TO BELL STATES SINCE $|0\rangle$ AND $|1\rangle$ ARE ORTHOGONAL.

  eg $|\phi\rangle$ 2nd $|\psi_1\rangle$ or $|\psi_2\rangle$ FIRST

$$|\psi\rangle = \left(|\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle\right)\frac{1}{\sqrt{2}}$$

MEASURE 2nd $\Rightarrow$

$$|\psi\rangle_{\text{AFTER}} = \left(|\psi_1\rangle + |\psi_2\rangle\right)\frac{1}{\sqrt{2}}$$

EPR PARADOX

EXPLAIN

BELL INEQUALITY CLASSICAL

$$P^{xy}_{SAME} + P^{xz}_{SAME} + P^{yz}_{SAME} \geq 1$$

QUANTUM

WE ESTIMATE THE lhs AND FIND < 1
FOR THREE COPLANAR AXES x, y, z AT 120^0

FOR $|\Phi^+\rangle$

$$P^{xy}_{SAME} = P^{xz}_{SAME} = P^{yz}_{SAME} = 1/4 = \cos^2(\theta/2)$$

$\Rightarrow$ SUM $3 \times \frac{1}{4} = \frac{3}{4} < 1$ CONTRAD.

VON NEUMANN & RENYI ENTROPIES

$$S_{VN} = -k_B \, Tr \, \hat{\rho}_A \ln \hat{\rho}_A$$

$$S_R = \frac{k_B}{1-q} \ln Tr \, \hat{\rho}_A^{\,q}$$

REPRESENT IN TERMS OF EIGENVALUES OF $\hat{\rho}$:

$$S_{VN} = -k_B \sum_{i=1}^{D} \lambda_i \ln \lambda_i$$

$$S_R = k_B \frac{1}{1-q} \ln \sum_i \lambda_i^{\,q}$$

λ_i EIGENVALUES OF $\hat{\rho}$

FROM RÉNYI TO VON NEUMANN ENTROPY

$$\lim_{q\to 1} S_R = k_B \lim_{q\to 1^+} \ln \frac{\mathrm{Tr}\ \hat{\rho}_A^q}{1-q}$$

$\frac{0}{0}$ SINCE $\sum_i \lambda_i = 1$

$$= k_B \lim_{q\to 1} \frac{\ln \sum_i \lambda_i^q}{1-q}$$

EXPAND AROUND $q=1$

$$\simeq k_B \left.\frac{\frac{d}{dq}\ln \sum_i \lambda_i^q \,(q-1)}{1-q}\right|_{q=1} \qquad (-1)$$

$$= -k_B \left.\frac{\frac{d}{dq}\left(\sum_j \lambda_j^q\right)}{\sum_i \lambda_i}\right|_{q=1}, \qquad \sum_i \lambda_i = 1$$

$$= -k_B \sum_j \frac{d}{dq}\lambda_j^q$$

$$\frac{d}{dq}\,\lambda_j^q = ? \qquad y = \lambda_j^q \Rightarrow \ln y = q \ln \lambda_j$$

$$\frac{d \ln y}{dq} = \ln \lambda_j = \frac{1}{y}\frac{dy}{dq}$$

$$\Rightarrow \frac{dy}{dq} = y \ln \lambda_j = \lambda_j^q \ln \lambda_j$$

$$\frac{d}{dq}\,\lambda_j^q = \lambda_j^q \ln \lambda_j$$

THEN

$$\lim_{q \to 1} S_R = -k_B \sum_j \lambda_j^q \ln \lambda_j \Big|_{q=1}$$

$$= -k_B \sum_j \lambda_j \ln \lambda_j$$

$$= S_{vN}$$

REPLICA VERSION

$$S_{vN} = -k_B \lim_{q \to 1} \frac{\partial}{\partial q} \ln \operatorname{Tr} \hat{\rho}^q$$

METHODS TO CALC RENYI ENTROPY: REPLICAS (POWER HELPS)

- CALCULATE THE POWER FOR $q \geq 2$ $q \in \mathbb{N}$
- CONTINUE TO REAL q
- TAKE $q \to 1^+$

PURE STATE $S_{VN} = 0$

$\lambda_k = 1$ $\lambda_{j(\neq k)} = 0$

REDUCED DENSITY OP. OF MAX ENTANGLED PURE STATE

$\rho = \mathbb{I}/D$ $\lambda_j = 1/D$ $\forall j$

$S_{VN} = k_B \ln D$

ENTANGLEMENT ENTROPY

$$S_{VN}^{ENT} = -k_B \, Tr \; \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$S_R^{ENT} = \frac{k_B}{1-q} \ln Tr \left(\hat{\rho}_A^{RED} \right)^q$$

ENTANGLEMENT SPECTRUM

$$\hat{\rho} = e^{-\hat{H}_E} \quad \Rightarrow -\ln \hat{\rho} = \hat{H}_E$$

$\hat{H}_E$ IS NOT A PROPER HAMILTONIAN.
IT'S ADIM. BECAUSE TO DEF $\{\lambda_i\}$

$$\hat{H}_E |v_i\rangle = \bar{\lambda}_i |v_i\rangle$$

$\{\bar{\lambda}_i\}$ SPECTRUM

NB S^{ENT} CAN BE EQUAL WHILE

$\{\lambda_i\}$ Diff $\Rightarrow$ DISTINGUISH SYSTEMS.

FOR PURE STATES $\{\lambda_i\}$ SCHMIDT DECOMP.

ENTANGLEM. SPECTRUM $\exists$ FOR MIXED STATES

BUT CLASS. CONTRIB TOO. (FROM A_E)

PURE STATES $S^{ENT}(A) = S^{ENT}(B)$

PROOF :

$$\hat{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i}\ |u_i\rangle_A \otimes |v_i\rangle_B$$

$$S_A^{ENT} = -k_B\, Tr\ \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

$$S_B^{ENT} = -k_B \, Tr \; \hat{\rho}_B^{RED} \ln \hat{\rho}_B^{RED}$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

SINCE $\hat{\rho}_A^{RED} = \underset{B}{Tr} \; \tilde{\rho}_{AB}$

$$= \sum_i \lambda_i \, |u_i\rangle_A \, {}_A\langle u_i|$$

$\hat{\rho}_B^{RED} = \underset{A}{Tr} \; \hat{\rho}_{AB}$

SAME EIGENVALUES $\{\lambda_i\}$

$$= \sum_i \lambda_i \, |v_i\rangle_B \, {}_B\langle v_i|$$

SEPARABLE PURE STATE $\rightarrow S^{ENT} = 0$

$|\psi_{AB}\rangle = |\psi\rangle_A \otimes |\psi_B\rangle$ SEP.

$$\hat{\rho}_{AB} = \left(|\psi\rangle_A \otimes |\psi_B\rangle\right)\left({}_A\langle\psi| \otimes {}_B\langle\psi|\right)$$

$$= \left(|\psi\rangle_A \, {}_A\langle\psi|\right) \otimes |\psi\rangle_B \, {}_B\langle\psi|$$

$$= \hat{\rho}_A \otimes \hat{\rho}_B$$

$$\hat{\rho}_A^{RED} = \underset{B}{Tr}\ \hat{\rho}_{AB} = \hat{\rho}_A$$

$$S_A^{ENT} = -k_B\, Tr\ \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B\, Tr\ \hat{\rho}_A \ln \hat{\rho}_A$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

BUT $\lambda_i = 1$ FOR SOME i 0 OTHERWISE

$$\Rightarrow S_A^{ENT} = 0$$

FOR A PURE SEP ST.

FOR A MIXED SEP STATE :

Proof

$$\hat{\rho}_{AB}^{SEP} = \sum_\alpha p_\alpha \, \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha$$

$$0 \le p_\alpha \le 1 \qquad \sum_\alpha p_\alpha = 1$$

$$\hat{\rho}_A^{RED} = \mathrm{Tr}_B \, \hat{\rho}_{AB}^{SEP}$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha$$

CLASS PROB MIX OF PURE $\hat{\rho}_A^\alpha$ STATES

I ORTHOG.

$$S^{ENT} = -k_B \, \mathrm{Tr} \, \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B \sum_\alpha p_\alpha \ln p_\alpha$$

CLASSICAL SHANNON ENTROPY

FOR SEP. STATE $C_{ab} = C_a \cdot C_b'$ IN ONE EX.

$$\det \hat{\rho}_A^{RED} = |ad - bc|^2$$

$$|b|^2|c|^2 + |a|^2|d|^2 - ab^* c^* d - a^* b c d^*$$

$a = C_{00} = C_0 C_0'$ $\qquad$ $c = C_{10} = C_1 C_0'$

$b = C_{01} = C_0 C_1'$ $\qquad$ $d = C_{11} = C_1 C_1'$

$$\det \hat{\rho}_A^{RED} =$$

NB: $ad - bc$ CONCURRENCE IS ZERO IN THIS CASE $\Rightarrow$ SEP.

$$= |C_0 C_1'|^2 |C_1 C_0'|^2 +$$

$$\| \quad \text{(same)}$$

$$+ |C_0 C_0'|^2 |C_1 C_1'|^2 -$$

$$- C_0 C_0' C_0^* C_1'^* C_1^* C_0'^* C_1 C_1' \Big] =$$

$$- C_0^* C_0'^* C_0 C_1' C_1 C_0' C_1^* C_1'^*$$

$$= 2|C_0 C_1' C_1 C_0'|^2 - 2|C_0 C_0' C_1 C_0'|^2$$

$$= 0$$

PROPERTIES

$$|\psi\rangle_{AB} = \sum_{\alpha}^{D} \sqrt{\lambda_\alpha}\, |\phi_\alpha\rangle_A \otimes |\psi_\alpha\rangle_B$$ SCHMIDT DECOMP

FOR LOW ENTANGLED STATES

THE SUM CAN BE CUT AT AN $n << D = \min(D_A, D_B)$

BASIS OF MATRIX PRODUCT STATES & NUM. METHODS.

SCALING WITH SIZE

A TYPICAL QUANTUM STATE WILL SATISFY:

$$S(A) \propto |A| \sim L^d$$ VOLUME OF A

eg. THERMAL STATES

BUT GROUND STATES OF GAPPED SYST.

$$S(A) \propto |\partial A| \sim L^{d-1}$$ AREA

LOWER ENTANG ENTROPY $\Rightarrow$ NUMERICS OK.

AT CRITICAL GROUND STATES 1d SYST

$$S(A) \sim \frac{c}{3} \ln(L/a)$$

L INTERVAL LENGTH IN AN ∞ CHAIN

CRITICAL STATES IN $d > 1$

$$S(A) \sim L^{d-1} \ln L \quad \text{or} \quad L^{d-1}$$

TIME EVOLUTION

SCHRÖDINGER

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \hat{H} |\psi, t\rangle$$

$$|\psi, t\rangle = \hat{U}(t, t') |\psi, t'\rangle$$

$$\hat{U}^{-1}(t, t_0)\, \hat{U}(t, t_0) = \hat{\mathbb{I}}$$ INVERSE

$$\hat{U}^{\dagger}(t, t_0)\, \hat{U}(t, t_0) = \hat{\mathbb{I}}$$ ENSURES CONS. PROB.

$$\langle \psi, t | \psi, t \rangle = 1$$

COMPOSITION $\hat{U}(t, t_1)\, \hat{U}(t_1, t_0) = \hat{U}(t, t_0)$

CONTINUITY $\lim_{dt \to 0} \hat{U}(t + dt, t) = \hat{U}(t, t) = \hat{\mathbb{I}}$

$$i\hbar \frac{d\hat{U}(t, t_0)}{dt} = \hat{H}(t)\, \hat{U}(t, t_0)$$

SOLUTION FOR $\hat{H}(t) = \hat{H}$ NO TIME DEP.

$$\hat{U}(t,t_0) = e^{-i/\hbar\, \hat{H}(t-t_0)}$$

HEISENBERG

$$\hat{A}_H(t,t_0) = \hat{U}^\dagger(t,t_0)\, \hat{A}_S\, \hat{U}(t,t_0)$$

$$\hat{A}_S = \hat{A}_H(t_0,t_0)$$

EXPECT. VALUE

$$\langle \hat{A}_S \rangle(t) = \langle \psi,t | \hat{A}_S | \psi,t \rangle$$

$$= \langle \psi,t_0 | \hat{U}^\dagger(t,t_0)\, \hat{A}_S\, \hat{U}(t,t_0) | \psi,t_0 \rangle$$

$$= \langle \psi,t_0 | \hat{A}_H(t,t_0) | \psi,t_0 \rangle$$

$$= \langle \hat{A}_H \rangle(t)$$

HEISENBERG'S EQN. OF OPERATORS

$$\frac{d\vec{A}_H(t,t_0)}{dt} = \frac{d\vec{U}^\dagger(t,t_0)}{dt}\,\vec{A}_S\,\vec{U}(t,t_0) +$$

$$+\,\vec{U}^\dagger(t,t_0)\,\vec{A}_S\,\frac{d\vec{U}(t,t_0)}{dt}$$

USING EVOL EQ FOR $\hat{U}^\dagger$ AND $\hat{U}$:

$$= \frac{-i}{\hbar}\left[\vec{A}_H(t,t_0),\,\vec{H}\right]$$

IF $\hat{A}_S(t) \Rightarrow \left(\partial_t \hat{A}_H\right)_H$

INTER. PICTURE IF $\hat{H}(t) \Rightarrow$ REPLACE ABOVE

$$\hat{H} = \hat{H}_0 + \hat{H}_{INT}$$

$$|\psi_I,t\rangle \equiv e^{i/\hbar\,\hat{H}_0 t}\,|\psi_S,t\rangle$$

$$\hat{\Theta}_I(t) \equiv e^{i/\hbar\,\hat{H}_0 t}\,\hat{\Theta}_S\,e^{-i/\hbar\,\hat{H}_0 t}$$

DEFS OF STATES & OP. FROM SCHRÖDINGER'S

EVOLVED OPERATORS IN INT PICTURE

$$i\hbar \frac{d}{dt}|\psi_I,t\rangle = i\hbar \frac{d}{dt}\, e^{i/\hbar\, \hat{H}_0 t}\, |\psi_S,t\rangle$$

$$= i\hbar \left[\frac{i}{\hbar} \hat{H}_0\, e^{\frac{i}{\hbar}\hat{H}_0 t}\, |\psi_S,t\rangle + e^{i/\hbar\, \hat{H}_0 t}\, \frac{d|\psi_S,t\rangle}{dt} \right]$$

$$= -\hat{H}_0\, e^{\frac{i}{\hbar}\hat{H}_0 t}\, |\psi_S,t\rangle + e^{i/\hbar\, \hat{H}_0 t}\, \hat{H}\, |\psi_S,t\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0 t} \left[-\hat{H}_0 + \hat{H}_0 + \hat{H}_{INT} \right] |\psi_S,t\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0 t}\, \hat{H}_{INT}\, |\psi_S,t\rangle$$

$$i\hbar \frac{d|\psi_I,t\rangle}{dt} = \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t}\, \hat{H}_{INT}\, e^{-\frac{i}{\hbar}\hat{H}_0 t}}\, \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t}\, |\psi_S,t\rangle}$$

$$= \hat{H}^I_{INT}(t)\, |\psi_I,t\rangle$$

$$i\hbar \frac{d|\psi_{I,t}\rangle}{dt} = \hat{H}^{I}_{INT}(t) \; |\psi_{I,t}\rangle$$

EVOL. OF OPERATORS IN INT. PICTURE

$$\hat{A}_I(t) = e^{i/\hbar \hat{H}_0 t} \hat{A}_S \, e^{-i/\hbar \hat{H}_0 t}$$

$$\frac{d\hat{A}_I(t)}{dt} = \frac{i}{\hbar} \hat{H}_0 \, e^{i/\hbar \hat{H}_0 t} \hat{A}_S \, e^{-i/\hbar \hat{H}_0 t}$$

$$+ \; e^{i/\hbar \hat{H}_0 t} \hat{A}_S \left(\frac{-i}{\hbar}\right) e^{-i/\hbar \hat{H}_0 t} \hat{H}_0$$

$$= \frac{i}{\hbar} \hat{H}_0 \hat{A}_I(t) - \frac{i}{\hbar} \hat{A}_I(t) \hat{H}_0$$

$$= \frac{i}{\hbar} [\hat{H}_0, \hat{A}_I(t)]$$

EVOLUTION OF DENSITY OPERATOR

IT'S AN OP THAT REPRESENTS THE STATE OF SYST => IT EVOLVES IN SCHRÖDINGER PICT

$$\hat{\rho}(t) = \sum_\alpha p_\alpha \, |\psi_\alpha, t\rangle\langle\psi_\alpha, t|$$

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = i\hbar \sum_\alpha p_\alpha \left(\frac{d|\psi_\alpha(t)\rangle}{dt} \langle\psi_\alpha, t| + |\psi_\alpha, t\rangle \frac{d\langle\psi_\alpha, t|}{dt} \right)$$

USING

$$i\hbar \frac{d|\psi_\alpha, t\rangle}{dt} = \hat{H}_s |\psi_\alpha, t\rangle$$

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \cancel{i\hbar} \sum_\alpha p_\alpha \left(\frac{1}{\cancel{i\hbar}} \hat{H}_s |\psi_\alpha, t\rangle\langle\psi_\alpha, t| - \frac{1}{\cancel{i\hbar}} |\psi_\alpha, t\rangle\langle\psi_\alpha, t| \hat{H}_s^\dagger \right)$$

$$= \sum_\alpha p_\alpha \left(\hat{H}_s |\psi_\alpha, t\rangle\langle\psi_\alpha, t| - |\psi_\alpha, t\rangle\langle\psi_\alpha, t| H_s \right)$$

$$= \hat{H}_s \hat{\rho}(t) - \hat{\rho}(t) \hat{H}_s$$

$$= [\hat{H}_s, \hat{\rho}(t)] = i\hbar \frac{d\hat{\rho}(t)}{dt}$$

THE QUANTUM DYN DOESN'T CHANGE THE PURITY OF A SYST AS QUANT. BY

$$\mathrm{Tr}\,\hat{\rho}^2$$

$$\frac{d}{dt}\mathrm{Tr}\,\hat{\rho}^2 = \mathrm{Tr}\; 2\hat{\rho}\frac{d\hat{\rho}}{dt}$$

$$= 2\,\mathrm{Tr}\;\hat{\rho}\;\frac{1}{i\hbar}\left[\hat{H}_s,\ \hat{\rho}(H)\right]$$

$$= \frac{2}{i\hbar}\,\mathrm{Tr}\;\hat{\rho}\left(\hat{H}_s\hat{\rho} - \hat{\rho}\,\hat{H}_s\right)$$

CYCLIC PROP Tr

$$= \frac{2}{i\hbar}\,\mathrm{Tr}\left(\rho\hat{H}_s\hat{\rho} - \hat{\rho}\hat{H}_s\hat{\rho}\right) = 0$$

ENTROPY CONSERVED

$$S_{VN} = -k_B\,\mathrm{Tr}\;\hat{\rho}\ln\hat{\rho}$$

$$\frac{dS_{VN}}{dt} = -k_B\,\mathrm{Tr}\left[\frac{d\hat{\rho}}{dt}\ln\hat{\rho} + \hat{\rho}\,\hat{\rho}^{-1}\frac{d\hat{\rho}}{dt}\right]$$

$$= -k_B \mathrm{Tr}\left\{ \frac{1}{i\hbar} [\hat{H}_S, \hat{\rho}] \ln \hat{\rho} + \frac{1}{i\hbar} [\hat{H}_S, \hat{\rho}] \right\}$$

$$= -\frac{k_B}{i\hbar} \mathrm{Tr}\left\{ (\hat{H}_S \hat{\rho} - \hat{\rho} \hat{H}_S) \ln \hat{\rho} + (\hat{H}_S \hat{\rho} - \hat{\rho} \hat{H}_S) \right\}$$

$\to 0$ UNDER Tr

$$= -\frac{k_B}{i\hbar} \mathrm{Tr}\left(\hat{H}_S \hat{\rho} \ln \hat{\rho} - \hat{H}_S \ln \hat{\rho}\, \hat{\rho} \right) = 0$$

$\to 0$ UNDER Tr

JUSTIFYING BOLTZMANN $\hat{\rho} \propto e^{-\beta \hat{H}_S}$

IT SOLVES THE EVOL EQ FOR $\hat{\rho}$.

ACTUALLY ANY FUNCTION OF $\hat{H}_S$ SOLVES THE HEISENBERG EQ. $\Longrightarrow$ ENSEMBLES.

NO CLONING THEOREM.

IT'S IMPOSSIBLE TO COPY A QUANTUM STATE

TAKE $|\psi\rangle = |0\rangle_A$ AND TRY TO COPY IT

ASSUME $\hat{U}(|0\rangle_A \otimes |0\rangle_B) = |0\rangle_A \otimes |0\rangle_B$

$\hat{U}(|1\rangle_A \otimes |0\rangle_B) = |1\rangle_A \otimes |1\rangle_B$

SUCH A $\hat{U}$ EXIST. THAT'S THE ASSUMPTION

THAT IS WE WANT TO COPY STATE A ON B.

B WAS INITIALIZED IN $|0\rangle$.

B BECOMES = A AND A DOESN'T CHANGE

NOW, WHAT HAPPENS WITH $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|1\rangle_A + |0\rangle_A)$

CAN WE ALSO COPY IT WITH

$\hat{U}$?

$$\hat{U}\left(|\psi_A\rangle \otimes |0\rangle_B\right) =$$

$$\frac{1}{\sqrt{2}}\ \hat{U}\left[\left(|1\rangle_A + |0\rangle_A\right) \otimes |0\rangle_B\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\hat{U}|1\rangle_A \otimes |0\rangle_B + \hat{U}|0\rangle_A \otimes |0\rangle_B\right] =$$

$$= \frac{1}{\sqrt{2}}\left[|1\rangle_A \otimes |1\rangle_B + |0\rangle_A \otimes |0\rangle_B\right]$$

$\neq |\psi_A\rangle \otimes |\psi_A\rangle$ NOT POSSIBLE TO CLONE GEN. STATES WHICH ARE NOT JUST BASIS ONES

DEVELOPM of ENTANGLEMENT

SAY $\hat{\rho}_{AB}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_B(0)$

AND $\hat{\rho}_A(0)$, $\hat{\rho}_B(0)$ THE DENS of of TWO PURE STATES

eg $|\psi,0\rangle = |0\rangle \otimes |1\rangle$

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{\sigma}_x^A \hat{\sigma}_x^B + \hat{\sigma}_y^A \hat{\sigma}_y^B\right)$$

$$|\psi,t\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi,0\rangle$$

$$= \cos\omega t\ |0\rangle\otimes|1\rangle - i\sin\omega t\ |1\rangle\otimes|0\rangle$$

$$\hat{\rho}_A^{RED}(t) = \mathrm{Tr}_B\ |\psi,t\rangle\langle\psi,t|$$

$$= \sum_{i=0,1} {}_B\langle i|\psi,t\rangle\langle\psi,t|i\rangle_B$$

$$\hat{\rho}_A^{RED}(t) = \begin{pmatrix} \cos^2\omega t & 0 \\ 0 & \sin^2\omega t \end{pmatrix}$$

$$\mathrm{Tr}\left(\hat{\rho}_A^{RED}(t)\right)^2 \neq \mathrm{Tr}\,\hat{\rho}_A^{RED}(t)$$

THEY ARE = ONLY AT $\omega t = \pi/4$

QUANTUM QUENCHES EXPLAIN

QUANTUM CHAOS

CLASSICALLY TRAJECTORIES IN PHASE SPACE, SO

HERE $\{x,p\} \to \{x\}$

$x(0)$ $x(0)+\delta x(0)$ (NOTATION)

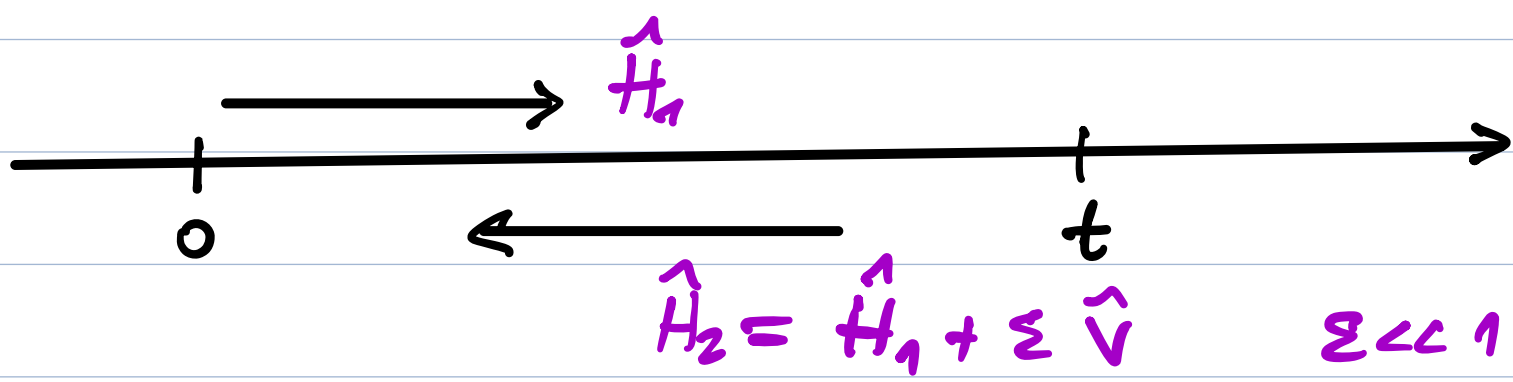
HOW DO THEY SEPARATE ?

$\delta x(t) \sim e^{\lambda t}$ CHAOTIC ONES

$\delta x(t) \sim$ NO.

LOSCHMIDT ECHO

$$M(t) = \left| \langle \psi, 0 | \, e^{\frac{i}{\hbar} H_2 t} \, e^{-\frac{i}{\hbar} \hat{H}_1 t} \, | \psi, 0 \rangle \right|^2$$



SIMILAR, OTOCs

$$C(t) = \langle [\hat{W}(t), \hat{V}(0)]^\dagger [\hat{W}(t), \hat{V}(0)] \rangle$$

$$= \mathrm{Tr}\left[[\hat{W}(t), \hat{V}(0)]^\dagger [\hat{W}(t), \hat{V}(0)] \hat{\rho} \right]$$

$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z}$ TYPICALLY USED

IF HERMITIAN $\hat{W}, \hat{V}$

$$= -\mathrm{Tr}\left([\hat{W}(t), \hat{V}(0)]^2 \hat{\rho} \right)$$

$$F(t) \equiv \langle W^\dagger(t) \hat{V}^\dagger \hat{W}(t) \hat{V} \rangle$$

$$C(t) = -\langle \left(\hat{W}(t) \hat{V}(0) - \hat{V}(0) \hat{W}(t) \right)$$

$$\left(\hat{W}(t) \hat{V}(0) - \hat{V}(0) \hat{W}(t) \right) \rangle$$

$= -\langle \hat{W}(t) \hat{V}(0) \hat{W}(t) \hat{V}(0) \rangle$ ~ $F(t)$

$+ \langle \hat{W}(t) \hat{V}(0) \hat{V}(0) \hat{W}(t) \rangle$

$+ \langle \hat{V}(0) \hat{W}(t) \hat{W}(t) \hat{V}(0) \rangle$) ASSUME ☺

$- \langle \hat{V}(0) \hat{W}(t) \hat{V}(0) \hat{W}(t) \rangle$ ALSO ~ $F(t) \Rightarrow 2$

HAVE TO TRANSF THE OTHER TERMS USING

$$\hat{W}(t) = e^{i/\hbar \hat{H} t} \; \hat{W} \; e^{-i/\hbar \hat{H} t}$$

$$\langle \hat{V}(0) \; \hat{W}(t) \; \hat{V}(0) \; \hat{W}(t) \rangle =$$

$$= \langle W(t) \; \hat{V}(0) \; \hat{W}(t) \; \hat{V}(0) \rangle = F(t)$$

THEN,

$$C(t) = 2 \left(1 - F(t)\right)$$

OTHERWISE

$$C(t) = 2 \left(1 - \mathrm{Re}\, F(t)\right)$$

SEMICLASS LIMIT

$$[\hat{x}(t), \hat{p}(0)] \longrightarrow i\hbar \{x(t), p(0)\} = i\hbar \frac{\partial x(t)}{\partial x(0)}$$

Interesting that one doesn't have more MCH

Chaotic that $e^{\lambda t}$

COUPLING TO BATHS

- CLASSICAL $\mathcal{Z} = \sum_{\text{conf}} e^{-\beta H_{TOT}}$
- QUANTUM $\mathcal{Z}$
- CLASSICAL DYNAMICS $\longrightarrow$ LANGEVIN

MENTION $\check{Z}$dyn

- SCHWINGER KELDYSH

LINDBLAD EQ

$$\frac{d\hat{\rho}^{RED}}{dt} = \frac{-i}{\hbar}\left[\hat{H}, \hat{\rho}^{RED}\right] + \sum_{i=1}^{N^2-1} \gamma_i \left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^{\dagger} - \frac{1}{2}\left\{ \hat{L}_i^{\dagger}\hat{L}_i, \hat{\rho}^{RED} \right\} \right)$$

$$\hat{H} = \hat{H}_{SYST} + \hat{\Lambda}(\{\gamma_i\})$$

↖ PARAMETERS OF COUPLING TO ENV.

$\hat{L}_i$ JUMP OPERATORS
DERIVED OR GUESSED
USUALLY, A FEW

γ_i DAMPING RATES $\in \mathbb{R}$ AND ≥ 0

$\gamma_i = 0$ VON NEUMANN EQ.

UNITARY DYNAMICS

PROPERTIES of LINDBLAD EQ.

c) CONSERVES THE POSITIVITY of $\hat{\rho}^{RED}$

$$\langle \psi | \hat{\rho}^{RED} | \psi \rangle \geq 0 \quad \forall \, |\psi\rangle$$

b) TRACE PRESERVATION

a) CONSERVES $\hat{\rho}^{RED} = \hat{\rho}^{RED\,\dagger}$

$\Rightarrow$ CONST ON $\{\gamma_i\}$

Proof

a) $\hat{\rho}^{RED} = \hat{\rho}^{RED\dagger}$?

TAKE $\dagger$ OF LINDBLAD EQ.

$$\frac{d}{dt}\hat{\rho}^{RED\dagger} = \frac{(-i)^*}{\hbar}\left[\hat{H}, \hat{\rho}^{RED}\right]^\dagger$$

$$+ \sum_{i=1}^{N^2-1} \gamma_i^* \left(\left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^\dagger \right)^\dagger - \frac{1}{2} \left\{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED} \right\}^\dagger \right)$$

USING:

$$\left(\hat{L}_i^\dagger \hat{L}_i \hat{\rho}^{RED} + \hat{\rho}^{RED} \hat{L}_i^\dagger \hat{L}_i \right)^\dagger =$$

$$= \hat{\rho}^{RED\dagger} \hat{L}_i^\dagger \hat{L}_i + \hat{L}_i^\dagger \hat{L}_i \hat{\rho}_{RED}^\dagger$$

$$= \left\{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED\dagger} \right\} \quad \text{LAST TERM}$$

$$\left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^\dagger\right)^\dagger = \hat{L}_i \hat{\rho}^{RED\dagger} \hat{L}_i^\dagger$$ NEXT-TO LAST

$$[\hat{H}, \hat{\rho}^{RED}]^\dagger = \left(\hat{H}\hat{\rho}^{RED} - \hat{\rho}^{RED}\hat{H}\right)^\dagger$$

$$= \hat{\rho}^{RED\dagger}\hat{H}^\dagger - \hat{H}^\dagger \hat{\rho}^{RED\dagger}$$

$$= \hat{\rho}^{RED\dagger}\hat{H} - \hat{H}\hat{\rho}^{RED\dagger}$$ 1st TERM IN RHS

REPLACING

$$\frac{d}{dt}\hat{\rho}^{RED\dagger} = \overbrace{(-i)^x}^{(-i)} \frac{(-1)}{\hbar} [\hat{H}, \hat{\rho}^{RED\dagger}]$$

$$+ \sum_{i=1}^{N^2-1} \gamma_i^* \left(\hat{L}_i \hat{\rho}^{RED\dagger} \hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED\dagger}\} \right)$$

γ_i

$\hat{\rho}^{RED\dagger}$ SATISFIES SAME EQ.

b) $\frac{d}{dt}\left(\mathrm{Tr}\ \hat{\rho}^{RED}\right) \overset{?}{=} 0$

CONST

rhs: $\mathrm{Tr}\ [\hat{H}, \hat{\rho}^{RED}] =$

$= \mathrm{Tr}\left(\hat{H}\hat{\rho}^{RED} - \hat{\rho}^{RED}\hat{H}\right) = 0$

$\mathrm{Tr}\left(\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i^\dagger \hat{L}_i \hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_i^\dagger \hat{L}_i\right)$

$= \mathrm{Tr}\left(\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{\rho} \hat{L}_i^\dagger\right)$

$= 0$ ok.

c) POSITIVITY

LONG PROOF.

Ex.

$$|\psi_i t+dt\rangle = \begin{cases} e^{-i\hat{G}}\,|\psi_i t\rangle & p=\lambda dt \\ |\psi_i t\rangle & p=1-\lambda dt \end{cases}$$

$\hat{G}=\hat{G}^{\dagger}$ $\qquad \hat{U}=e^{-i\hat{G}}$ $\qquad p=\lambda dt$

$$\hat{\rho}(t+dt) = |\psi_i t+dt\rangle\langle\psi_i t+dt|$$

$$= (1-\lambda dt)\,\hat{\rho}(t) + \lambda dt\; e^{-i\vec{G}}\,\hat{\rho}(t)\,e^{i\vec{G}}$$

$$\frac{d\hat{\rho}(t)}{dt} = -\lambda\left(\hat{\rho}(t) - e^{-i\vec{G}}\,\hat{\rho}(t)\,e^{i\vec{G}}\right)$$

rhs is of LINDBLAD FORM:

$$\text{rhs} = \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}$$

$$= \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\left(\hat{L}^\dagger\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L}\right)$$

WITH $\hat{L} = \sqrt{\lambda}\, e^{-i\hat{G}}$ AND $\gamma = 1$

$$= \lambda\, e^{-i\hat{G}}\hat{\rho}\, e^{i\hat{G}} - \frac{1}{2}2\lambda\hat{\rho}$$

$$= -\lambda\hat{\rho} + \lambda\, e^{-i\hat{G}}\hat{\rho}\, e^{i\hat{G}}$$

OK

HOW DOES THE EQ LOOK LIKE IN MATHICAL TERM ?

LAST TERM in rhs:

$$\begin{pmatrix} e^{-i\varphi_1} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-i\varphi_1} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} \rho_{11} e^{i\varphi_1} & \rho_{12} e^{i\varphi_2} \\ \rho_{21} e^{i\varphi_1} & \rho_{22} e^{i\varphi_2} \end{pmatrix} =$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} e^{-i(\varphi_1-\varphi_2)} \\ \rho_{21} e^{i(\varphi_1-\varphi_2)} & \rho_{22} \end{pmatrix}$$

1st TERM in rhs $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

ADDING w/ sgn:

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & \rho_{12}(1-e^{-i(\varphi_1-\varphi_2)}) \\ \rho_{21}(1-e^{i(\varphi_1-\varphi_2)}) & 0 \end{pmatrix}$$

$$\dot{\rho}_{11} = \dot{\rho}_{22} = 0 \quad \Rightarrow \quad \rho_{11} = \text{cons} \qquad \rho_{22} = \text{const}$$

$$\dot{\rho}_{12} = -\lambda \rho_{12} \left(1 - \cos(g_1 - g_2) + i \sin(g_1 - g_2)\right)$$

$$\dot{\rho}_{21} = -\lambda \rho_{21} \left(1 - \cos(g_1 - g_2) - i \sin(g_1 - g_2)\right)$$

$$\frac{d \ln \rho_{12}}{dt} = -\lambda \left(1 - \cos(g_1 - g_2) + i \sin(g_1 - g_2)\right)$$

$$\rho_{12}(t) = \rho_{12}(0)\, e^{\underbrace{-\lambda \left(1 - \cos(g_1 - g_2)\right) t}_{\to 0}} \underbrace{e^{-i\lambda \sin(g_1 - g_2) t}}_{\text{etc. BOUNDED}}$$

EXP. RELAX. SAME FOR $\rho_{21}(t)$.

EXP DECAY of COHERENCES ρ_{12}, ρ_{21} AND FREQ SHIFT $\lambda \sin(g_1 - g_2)$

TO BE CHECKED.

EX 2

$$|\psi, t+dt\rangle = \begin{cases} |\psi,t\rangle & 1-\lambda dt \\ \hat{\sigma}^y |\psi,t\rangle & \lambda dt \end{cases}$$

$$\hat{\rho}(t+dt) = (1-\lambda dt)\ \hat{\rho}(t) + \lambda dt\ \hat{\sigma}^y\ \hat{\rho}(t)\ \hat{\sigma}^y$$

$$\frac{d\hat{\rho}(t)}{dt} = -\lambda \left[\hat{\rho}(t) - \hat{\sigma}^y\ \hat{\rho}(t)\ \hat{\sigma}^y \right]$$

$$= \gamma \left[\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\left(\hat{L}^\dagger \hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L} \right) \right]$$

THE ORANGE EQUALITIES if

$\gamma = 1 \qquad \hat{L} = \sqrt{\lambda}\ \hat{\sigma}^y$

SINCE 2nd LINE EQUALS:

$$= \lambda \left[\hat{\sigma}^y \hat{\rho}\ \hat{\sigma}^{y\dagger} - \frac{1}{2}\left(\hat{\sigma}^{y\dagger}\hat{\sigma}^y\hat{\rho} + \hat{\rho}\,\hat{\sigma}^{y\dagger}\hat{\sigma}^y \right) \right]$$

$$= \lambda \left[\hat{\sigma}^y \hat{\rho} \hat{\sigma}^y - \frac{1}{2} \cdot 2 \hat{\rho} \right]$$

$$= \lambda \left(\hat{\sigma}^y \hat{\rho} \hat{\sigma}^y - \hat{\rho} \right)$$

$$= -\lambda \left(\hat{\rho} - \hat{\sigma}^y \hat{\rho} \hat{\sigma}^y \right)$$ AND THIS IS THE 1ST LINE

THE DENSITY MATRIX FOLLOW:

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$+ \lambda \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} +$$

$$+\lambda \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i\rho_{12} & -i\rho_{11} \\ i\rho_{22} & -i\rho_{21} \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} + \lambda \begin{pmatrix} \rho_{22} & -\rho_{21} \\ -\rho_{12} & \rho_{11} \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11}-\rho_{22} & \rho_{12}+\rho_{21} \\ \rho_{21}+\rho_{12} & \rho_{22}-\rho_{11} \end{pmatrix}$$

SO THESE IMPLY

$$\dot{\rho}_{11} = -\lambda(\rho_{11}-\rho_{22}) = -\dot{\rho}_{22}$$

$$\dot{\rho}_{12} = -\lambda(\rho_{12}+\rho_{21}) = \dot{\rho}_{21}$$

$$\dot{\rho}_{21} = -\lambda(\rho_{21}+\rho_{12}) = \dot{\rho}_{12}$$

$$\dot{\rho}_{22} = -\lambda(\rho_{22}-\rho_{11}) = -\dot{\rho}_{11}$$

$$\frac{d\rho_{11}}{dt} = -\lambda\left(\rho_{11} - \rho_{22}\right)$$

WORKING WITH THE EQS. FOR ρ_{11}, ρ_{22}

$$\frac{d\rho_{22}}{dt} = -\lambda\left(\rho_{22} - \rho_{11}\right)$$

ADD $\quad \frac{d}{dt}\left(\rho_{11} + \rho_{22}\right) = 0 \Rightarrow$

$$\rho_{11}(t) + \rho_{22}(t) = \text{CTE}$$

$$\rho_{22}(t) = \text{CTE} - \rho_{11}(t)$$

SUBTRACT $\quad \frac{d}{dt}\left(\rho_{11} - \rho_{22}\right) = -2\lambda\left(\rho_{11} - \rho_{22}\right)$

$$\rho_{11}(t) - \rho_{22}(t) = \left(\rho_{11}(0) - \rho_{22}(0)\right) e^{-2\lambda t}$$

$$2\rho_{11}(t) - \text{CTE} = \left(2\rho_{11}(0) - \text{CTE}\right) e^{-2\lambda t}$$

$$\rho_{11}(t) = \frac{CTE}{2} + \left(\rho_{11}(0) - \frac{CTE}{2}\right) e^{-2\lambda t}$$

$$\xrightarrow[t \to \infty]{} \frac{CTE}{2}$$

$$\rho_{22}(t) = CTE - \rho_{11}(t) \longrightarrow \frac{CTE}{2}$$

WE WORK NOW WITH EQS FOR ρ_{12}, ρ_{21}

$$\dot{\rho}_{21} = -\lambda\left(\rho_{21} + \rho_{12}\right) = \dot{\rho}_{12}$$

$$\dot{\rho}_{12} = -\lambda\left(\rho_{12} + \rho_{21}\right)$$

ADD & SUBTRACT THEM

$$\frac{d}{dt}\left(\rho_{21}(t) + \rho_{12}(t)\right) = -\lambda 2 \left(\rho_{21}(t) + \rho_{12}(t)\right)$$

$$\rho_{12}(t) + \rho_{21}(t) = \left(\rho_{12}(0) + \rho_{21}(0)\right) e^{-2\lambda t}$$

EXP. DECAY

$$\frac{d}{dt}\left(\rho_{12}(t)-\rho_{21}(t)\right)=0 \Rightarrow 2i\,\mathrm{Im}\,\rho_{12}(t)=\mathrm{cte.}$$

$$\rho_{12}(t) \longrightarrow i\,\mathrm{Im}\,\rho_{12}(0)$$

$$\rho_{11}(t) \longrightarrow 1/2 \longleftarrow \rho_{22}(t)$$

EX. DEPHASING

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}^z$$

$$\hat{L} = \sqrt{\gamma}\,\hat{\sigma}^z$$

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\,[\hat{H},\hat{\rho}]$$

$$+\gamma\left(\hat{\sigma}^z\hat{\rho}\hat{\sigma}^z - \hat{\rho}\right)$$

$$[\hat{H},\hat{\rho}] = \frac{\hbar\omega}{2}\,[\hat{\sigma}^z,\hat{\rho}]$$

$$= \frac{\hbar\omega}{2}\left(\hat{\sigma}^z\hat{\rho} - \hat{\rho}\hat{\sigma}^z\right)$$

"DETERMINISTIC TERM"

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} \\ -\rho_{21} & -\rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & -\rho_{12} \\ \rho_{21} & -\rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

"JUMP OP."

$$\hat{\sigma}^z \hat{\rho}^1 \hat{\sigma}^z - \hat{\rho}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & -\rho_{12} \\ \rho_{21} & -\rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

THE EQ IS

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = \frac{-i}{\hbar}\frac{\hbar\omega}{2}\begin{pmatrix} 0 & 2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

$$+\gamma\begin{pmatrix} 0 & -2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

$$\dot{\rho}_{11} = 0 \quad \Longrightarrow \quad \rho_{11} = \text{CTE}$$

$$\dot{\rho}_{12} = -i\omega\,\rho_{12} - 2\gamma\,\rho_{12}$$

$$\dot{\rho}_{21} = +i\omega\,\rho_{21} - 2\gamma\,\rho_{21}$$

$$\dot{\rho}_{22} = 0 \quad \Longrightarrow \quad \rho_{22} = \text{CTE}$$

$$\rho_{12}(t) = \rho_{12}(0)\, e^{(-2\gamma - i\omega)t}$$

$$\rho_{21}(t) = \rho_{21}(0)\, e^{(-2\gamma + i\omega)t}$$

$$\rho_{12}(t) = \rho_{12}(0)\, e^{-2\gamma t}(\cos\omega t - i\sin\omega t)$$

$$\rho_{21}(t) = \rho_{21}(0)\, e^{-2\gamma t}(\cos\omega t + i\sin\omega t)$$

$$(\rho_{11} = \alpha_{TE}) + (\rho_{22} = \alpha_{TE'}) = 1$$

$$\rho_{11} = 1 - \rho_{22}$$

DERIVATION OF LINDBLAD'S

$$\hat{H}_T = \hat{H}_S + \hat{H}_E + \hat{H}_{int}$$

$\hat{H}_S$ IN $\mathcal{H}_S$

$\hat{H}_E$ IN $\mathcal{H}_E$

$\hat{H}_{int}$ IN $\mathcal{H}_T$ $\qquad \hat{H}_{int} = \alpha \sum \gamma_i \hat{S}_i \otimes \hat{E}_i$

$\alpha \in \mathbb{R}^+ \qquad \gamma_i \in \mathbb{R}^+ \qquad \hat{S}_i^\dagger = \hat{S}_i \qquad \hat{E}_i^\dagger = \hat{E}_i$

EVOLUTION OF $\hat{\rho}_T(t)$

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}_T, \hat{\rho}(t)]$$

WE WANT TO DERIVE AN EQ FOR $\hat{\rho}_S(t)$ RED

WE WILL USE SOME ASSUMPTIONS:

- WEAK COUPLING S-E LIMIT BORN

- CORR S-E DEVELOP SLOWLY

- E is NOT MODIFIED BY S:
  EXC. OF E DECAY QUICKLY

- FAST OR. TERM WRT SYSTEM'S
  EVOL. TIME-SCALE ARE NEGLIGIBLE
  MARKOV DYN.

- SECULAR / ROTATING WAVE (FOR SYST)
  DROP FAST OR. TERMS

IT'S AN APPROX !

WORK IN INTERACTION PICTURE

$$\hat{\theta}_I(t) = e^{\frac{i}{\hbar}(\hat{H}_S + \hat{H}_E)t}\ \hat{\theta}\ e^{-\frac{i}{\hbar}(\hat{H}_S + \hat{H}_E)t}$$

ONE PROVES THAT $\hat{\rho}_I(t)$ EVOLVES AS

IT WAS AN EX. LAST WEEK

$$\frac{d\hat{\rho}_T^I(t)}{dt} = -\frac{i}{\hbar}\alpha\left[\hat{H}_{int}^I(t),\ \hat{\rho}_T^I(t)\right] \qquad (*)$$

FORMAL INTEGRATION

$$\hat{\rho}_T^I(t) = \hat{\rho}_T^I(0) - \frac{i}{\hbar}\alpha\int_0^t ds\ \left[\hat{H}_{int}^I(s),\ \hat{\rho}_T^I(s)\right]$$

WE COULD TRY INT OPER DATA HERE

BUT ⟶ IMPOSSIBLE

& NON-MARKOVIAN

ITERATE THIS SOLUTION BY INSERTING IT IN EQ. (*)

$$\frac{d\hat{\rho}_T^I(t)}{dt} = \frac{-i}{\hbar}\alpha\left[\hat{H}_{int}^I(t),\ \hat{\rho}_T^I(0)\right]$$

$$+\underbrace{\frac{-i}{\hbar}\alpha\left(\frac{-i\alpha}{\hbar}\right)}_{-(\alpha/\hbar)^2}\left[\hat{H}_{int}^I(t),\int_0^t ds\ \left[\hat{H}_{int}^I(s),\hat{\rho}_T^I(s)\right]\right]$$

$$= -\int_0^t ds\ \left[\hat{H}_{int}^I(t),\left[\hat{H}_{int}^I(s),\hat{\rho}_T^I(s)\right]\right]$$

THIS ONE IS EXACT

ONCE MORE ⟶ $\hat{\rho}_T^I(0)$

$$+\ \mathcal{O}\left((\alpha/\hbar)^3\right) \longrightarrow 0$$

IF WE DROP IT :

$$\frac{d\rho_T^I(t)}{dt} = \frac{-i}{\hbar}\alpha \left[H_{int}^I(t),\ \rho_T^I(0) \right]$$

$$- \left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \left[H_{int}^I(t),\ \left[H_{int}^I(s),\ \hat{\rho}_T^I(0) \right] \right]$$

$$+ \Theta\left((\alpha/\hbar)^3\right)$$

AN APPROX EQ.

BACK TO THE EXACT EQ ABOVE AND INT. OVER E :

$$\frac{d\rho_S^{red.I}(t)}{dt} = \frac{-i}{\hbar}\alpha \operatorname*{Tr}_E \left[H_{int}^I(t),\ \hat{\rho}_T^I(0) \right]$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \operatorname*{Tr}_E \left[H_{int}^I(t),\ \left[H_{int}^I(s),\ \hat{\rho}_T^I(s) \right] \right]$$

(**)

PROBLEM $\rho_T^I(s)$ HERE

ASSUMPTIONS

$t=0 \qquad \hat{\rho}_T(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$

ALL PICTURES THE SAME $\quad 0 = [\hat{H}_E, \hat{\rho}_E(0)]$

$t=0 \qquad \hat{\rho}_S^{RED}(0) = \hat{\rho}_S(0)$ STATIONARY BATH

WE CAN NOW TRACE OVER E THE 1st TERM IN rhs OF (**)
AND SHOW IT VANISHES IF

$$Tr\left(\hat{E}_i(t)\,\hat{\rho}_E(0)\right) = 0$$

WHICH WE ASSUME - (TRANSL. OF $\hat{H}_T$)

THEN,

$$\frac{d\hat{\rho}_s^{red.I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_T^I(s) \right] \right]$$

BORN APPROX WEAK COUPLING AND LARGE BATH

$$\tau_{SE}, \tau_E \ll \tau_S$$

$$\hat{\rho}_T(t) = \hat{\rho}_s^{RED}(t) \otimes \hat{\rho}_E(0)$$

SYSTEM'S STATE MODIFIED BY ENV.
BUT ENV IS NOT BY SYST

$$\frac{d\hat{\rho}_s^{red.I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_s^{RED,I}(s) \otimes \hat{\rho}_E(0) \right] \right]$$

STILL NON-MARKOV

TIME-LOCAL ASSUMPTION

$$\rho_S^{red,I}(s) \longrightarrow \rho_S^{red,I}(t)$$

IN INTEGRAL

REDFIELD

ALREADY TIME-LOCAL

BUT STILL NOT

LINDBLAD

$$\frac{d\rho_S^{red,I}(t)}{dt} = -\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[H_{int}^I(t), \left[H_{int}^I(s), \hat{\rho}_S^{RED,I}(t) \otimes \hat{\rho}_E(0) \right] \right]$$

$s \to t$

BORN-REDFIELD EQ.

ALREADY $\left(\frac{\alpha}{\hbar}\right)^2 \Longrightarrow \hat{\rho}_S^{RED,I}(s)$ DOESN'T CHANGE MUCH.

CHANGE INTEGRATION VARIABLES

$$s' = t - s \qquad s \in [0,t]$$

$$ds' = -ds \qquad s' \in [0,t]$$

$$\frac{d\rho_S^{red,I}(t)}{dt} = -\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds' \, \mathrm{Tr}_E \left[\overset{I}{H}_{int}(t), \left[\underbrace{\overset{I}{H}_{int}(t-s)}_{\uparrow \text{MEMORY EFFECTS}}, \hat{\rho}_S^{RED,I}(t) \otimes \hat{\rho}_E(0) \right] \right]$$

BUT DON'T DO IT HERE. GO ON WITH PREVIOUS EXPRESSION.

$$-\left(\frac{\hbar}{\alpha}\right)^2 \frac{d\rho_S^{red,I}(t)}{dt} = \int_0^t ds \, \mathrm{Tr}_E \left[\overset{I}{H}_{int}(t), \left[\overset{I}{H}_{int}(s), \hat{\rho}_S^{RED,I}(t) \otimes \hat{\rho}_E(0) \right] \right]$$

NOW, REPLACE $H_{int} = \sum_i^I \gamma_i \hat{S}_i \otimes \hat{E}_i$

OPEN THE COMMUTATORS, REARRANGE ALL ARISING TERMS

CORRELATIONS LIKE

$$\mathrm{Tr}_E \; \hat{E}_i(s) \, \hat{E}_j^{\dagger}(t) \, \vec{\rho}_E(0)$$

USUALLY HERMITIAN $E_i^{\dagger} = E_i$

APPEAR

CALL THEM $\Gamma_{ij}(s,t) = \Gamma_{ij}(s-t)$

ASSUMING STATIONARITY OF BATH

IT GETS LONG AND NOT SO INTERESTING

FURTHER ASSUMPTION **WHITE BATH**

$$\Gamma_{ij}(t,s) = h_{ij}\,\delta(t-s)$$

$h_{ij} \in \mathbb{R}$, $\neq 0$

VERY QUICK RELAX OF BATH AND MEMORY

$$-\left(\frac{\hbar^2}{\alpha^2}\right)\frac{d\rho_s^{RED,I}(t)}{dt}$$

$$\sum_{ij} h_{ij}\left[-\hat{S}_j(t)\,\hat{\rho}_s^{RED,I}(t)\,\hat{S}_i^{\dagger}(t) + \frac{1}{2}\left\{\hat{S}_i^{\dagger}(t)\,\hat{S}_j(t),\ \hat{\rho}_s^{RED,I}(t)\right\}\right]$$

GET RID OF $\sum_{ij}$ WITH A DIAGONALIZATION

$$\hat{L}_i(t) = \sum_k d_{ik}\,\hat{S}_k(t)$$

$$\Gamma_{ij} = \sum_{k\ell} d_{ik}\,h_{k\ell}\,d^{*}_{\ell j} = \Gamma_i\,\delta_{ij}$$

$\Rightarrow$ CUTS THE DOUBLE SUM INTO A SINGLE ONE OVER i

MOVING BACK TO SCHRÖDINGER'S PICT

$$\hat{\rho}_S^{RED,S}(t) = e^{-\frac{i}{\hbar}\hat{H}_S t}\, \hat{\rho}_S^{RED,I}(t)\, e^{+\frac{i}{\hbar}\hat{H}_S t}$$

$\hat{H}_S$ SCHRÖD. PICT SYST'S HAMILT

$\Longrightarrow$

$$\frac{d\hat{\rho}_S^{RED}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{H}_S, \hat{\rho}_S^{RED}(t)\right]$$

$$+ \frac{\alpha^2}{\hbar^2} \sum_i \Gamma_i \left(\hat{L}_i \hat{\rho}_S^{RED}(t) \hat{L}_i^\dagger - \frac{1}{2}\left\{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho}_S^{RED}(t) \right\} \right)$$

NO SHIFT OF $\hat{H}_S$

THIS is A QUITE BRUTAL APPROX FOR BATH
↳ WHITE

BETTER ONES:

ROTATING WAVE APPROX

$$S = \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} |\epsilon\rangle\langle\epsilon'|$$

$$\hat{H}_s |\epsilon\rangle = \epsilon |\epsilon\rangle$$

$$\hat{S}_I(t) = e^{\frac{i}{\hbar}\hat{H}_s t} \hat{S} e^{-\frac{i}{\hbar}\hat{H}_s t}$$

$$= e^{\frac{i}{\hbar}\hat{H}_s t} \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} |\epsilon\rangle\langle\epsilon'| e^{-\frac{i}{\hbar}\hat{H}_s t}$$

$$= \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} e^{\frac{i}{\hbar}\epsilon t} e^{-\frac{i}{\hbar}\epsilon' t} |\epsilon\rangle\langle\epsilon'|$$

$$= \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} e^{\frac{i}{\hbar}(\epsilon-\epsilon')t} |\epsilon\rangle\langle\epsilon'|$$

$$= \sum_{\omega} \hat{S}(\omega) e^{-i\omega t}$$

$$\hat{S}(\omega) \equiv S_{\epsilon\epsilon'} |\epsilon\rangle\langle\epsilon'| \Big|_{\epsilon-\epsilon' = -\omega}$$

$$\hat{S}(\omega)^{\dagger} = \hat{S}(-\omega)$$

THE INT. HAM. IN INT PICT

$$\hat{H}^{I}_{int}(t) = \sum_i \gamma_i \sum_\omega e^{-i\omega t} \hat{S}_i(\omega) \otimes \hat{E}_i(t)$$

GOING BACK TO THE EQ. FOR $\hat{\rho}_S^{red,I}(t)$

$$-\frac{\hbar^2}{\alpha^2} \frac{d\hat{\rho}_S^{red,I}(t)}{dt} =$$

$$\sum_{ij} \sum_{\omega\omega'} \Gamma_{ij}(\omega)\, e^{i(\omega-\omega')t}$$

$$\Big(\hat{S}_i^{\dagger}(\omega)\, S_j(\omega')\, \hat{\rho}_S^{red,I}(t) - \hat{S}_i(\omega)\, \hat{\rho}_S^{red,I}(t)\, \hat{S}_j^{\dagger}(\omega') + \hat{\rho}_S^{red,I}(t)\, \hat{S}_i^{\dagger}(\omega)\, \hat{S}_j(\omega') - \hat{S}_i(\omega')\, \hat{\rho}_S(t)\, \hat{S}_j^{\dagger}(\omega) \Big)$$

ASSUMPTION $\longrightarrow$ $\omega = \omega'$ ONLY.

VALID IF $|\omega - \omega'| \gg$ REL. TIME

WE HAVE TAKEN $t \to \infty$ IN LIMIT OF INT

RE-ARRANGING

$$\frac{d\rho_S^{RED.I}(t)}{dt} = \frac{-i}{\hbar} \sum_\omega \sum_{ij} \Gamma_{ij}(\omega) \left[\underbrace{\hat{S}_i^\dagger(\omega)\hat{S}_i(\omega)}_{\text{SHIFT}}, \hat{\rho}_S^{RED.I}(t) \right]$$

$$+ \sum_\omega \sum_{ij} \gamma_{ij}(\omega) \left(\hat{S}_j(\omega) \hat{\rho}_S^{RED.I}(t) \hat{S}_i^\dagger(\omega) \right.$$

$$\left. - \frac{1}{2} \left\{ \hat{S}_i^\dagger(\omega) \hat{S}_j(\omega), \hat{\rho}_S^{RED.I}(t) \right\} \right)$$

BACK TO SCHRÖD. PICTURE 1ST TERM

ADD $\hat{H}_S$ TO COMM.

$\alpha^2/\hbar^2$ HAS BEEN ABSORBED

$\gamma_{ij}(\omega) \geq 0$