MASTER COMPLEX SYSTEMS

QUANTUM PHYSICS OUT OF EQUIL.

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ENSEIGNEMENT

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QUANTUM

PURE STATE

VECTOR $\in \mathcal{H}$

$$|\psi\rangle = \sum_{k=1}^{K} c_k |\psi_k\rangle$$

KET

LINEAR SUPERP. PRINCIPLE

$c_k \in \mathbb{C}$ $\{|\psi_k\rangle\}$ ORTHON. BASIS of $\mathcal{H}$

$$\sum_{k=1}^{K} |c_k|^2 = 1$$

$$\langle\psi| = \sum_{k=1}^{K} c_k^* \langle\psi_k|$$

DUAL BRA

INNER PRODUCT $\langle\psi|\psi\rangle = 1$ NORM

$$\sum_{k=1}^{K} |c_k|^2 = 1$$

eg. SPIN

CONTINUOUS INFO THROUGH $\{c_k\}$

OPERATORS

ACTS ON A VECTOR AND PRODUCES A VECTOR

$$\hat{A}: \quad |\psi\rangle \in \mathcal{H} \longrightarrow \hat{A}|\psi\rangle \in \mathcal{H}$$

RANK 1

$$\hat{\theta} = |\phi\rangle\langle\psi|$$

CONJUGATE $\quad \hat{\theta}^\dagger = |\psi\rangle\langle\phi|$

HERMITIAN $\quad \hat{\theta}^\dagger = \hat{\theta}$

eg. PROJECTOR $\quad \hat{P} = |\psi\rangle\langle\psi|$

$$\hat{P}^2 = \hat{P} \qquad \hat{P}^\dagger = \hat{P}$$

ex. SPIN PROBLEM CONSTRUCT NON-HERM & HERM. OP.

CLOSURE RELATION $\sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{1}$

ex Qu-Bit

GENERAL RANK

MEASURE OF HOW MANY INDEP DIRECTIONS THE MAP PRODUCES

eg $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ TWO INDEP COLUMNS

if $\alpha \vec{v}_1 + \beta \vec{v}_2 = 0 \Rightarrow$ DEP.

OTHERWISE INDEP.

IT CANNOT BE WRITTEN AS $|\phi\rangle\langle\psi|$

BUT IT CAN BE WRITTEN AS

$$\hat{A} = \sum_{ij} A_{ij} |\psi_i\rangle\langle\psi_j|$$

WITH $A_{ij} = \lambda_i \delta_{ij}$ $|\psi_i\rangle$ EIGENVECTOR W/ EIGENVALUE λ_i

MEASUREMENT

A MEASURABLE OBSERV $\hat{A} = \hat{A}^+$

HOW TO MEASURE ? IDENTIF

$\hat{A} |a_n\rangle = a_n |a_n\rangle$

WRITE $|\psi\rangle = \sum_n c_n |a_n\rangle$

$c_n = \langle a_n | \psi \rangle$

RESULT of MEASUREMENT : ONE a_n

WITH PROBABILITY $P_n = |c_n|^2 = |\langle \psi | a_n \rangle|^2$

BORN RULE MEASURE $\hat{A}$ STATE of SYST

COLLAPSES on $|a_n\rangle$ WHEN GETTING a_n

EXPECTATION VALUES

THEOR CONSTRUCT

WHAT ONE SHOULD MEASURE ON AVER

OVER MANY TRIALS

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_n |c_n|^2 a_n$$

ex $\langle \hat{\sigma}_z \rangle$ on $|\psi\rangle = |0\rangle$ SPIN UP

or $a|0\rangle + b|1\rangle = |\psi\rangle$

UNITARY OP

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \mathbb{1}$$

CONS. NORM CONSERVED

$$|\psi'\rangle = \hat{U}|\psi\rangle \qquad \langle \psi' | \psi' \rangle = \langle \psi | \psi \rangle$$

DENSITY OPERATOR

DESCRIBES THE STATE OF THE SYSTEM, BEYOND WHEN IT'S IN A PURE STATE. ALSO FOR MIXED STATES (STAT. SENSE).

- PURE STATES

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

Basis of $\mathcal{H}$

$$= \sum_j c_j |\psi_j\rangle \sum_i c_i^* \langle\psi_i|$$

$$= \sum_{i,j} c_i^* c_j |\psi_j\rangle\langle\psi_i|$$

ELEMENTS $\langle\psi_n|\hat{\rho}|\psi_m\rangle$ DENS MATRIX

$$\langle\psi_n|\hat{\rho}|\psi_m\rangle = \sum_{i,j} c_i^* c_j \underbrace{\langle\psi_n|\psi_j\rangle}_{\delta_{jn}} \underbrace{\langle\psi_i|\psi_m\rangle}_{\delta_{im}}$$

$$= c_m^* c_n$$

ρ_{nn} DIAG $\Rightarrow$ POPULATIONS

GIVE PROB of RESULTS of MEASUREMENTS $|c_n|^2$

OF THE OP. WHICH GIVES RISE TO THE BASIS $\{|\psi_i\rangle\}$ USED.

$$\sum_n \rho_{nn} = \sum_n |c_n|^2 = 1 \quad \text{BY NORM.}$$

$$\Rightarrow \quad \mathrm{Tr}\,\hat{\rho} = 1$$

$\rho_{n \neq m}$ OFF DIAG $\Rightarrow$ COHERENCES

THE ELEM ρ_{nm} DEP ON THE BASIS CHOSEN

- IMPORTANT ρ CAN BE DIAG IN ONE BASIS & NON-DIAG IN ANOTHER

- $\hat{\rho} = \hat{\rho}^\dagger$ HERMITIAN (IN BASIS)

EX. HERE

- MIXED STATES

if WE DO NOT KNOW WELL THE STATE OF THE SYST → IGNORANCE

$$\hat{S} = \sum_{\alpha} p_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$\sum_{\alpha=1}^{N} p_\alpha = 1 \qquad p_\alpha \in \mathbb{R}$$

STATISTICAL ("NOT QUANTUM")

ex SPIN MEASURED w/ $\hat{S}^2$ BUT WE DO NOT KNOW OUTCOME

HERMITIAN

$$\hat{S}^\dagger = \left(\sum_{\alpha} p_\alpha \, |\psi_\alpha\rangle\langle\psi_\alpha| \right)^\dagger$$

$$= \sum_{\alpha} p_\alpha^* \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$= \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

ex PROVE $\mathrm{Tr}\,\hat{\rho} = 1$

$\mathrm{Tr}\,\hat{\rho}^2 < 1$ if MIXED

$\mathrm{Tr}\,\hat{\rho}^2 = \mathrm{Tr}\,\hat{\rho}$ if PURE

$\langle\psi|\hat{\rho}|\psi\rangle \geq 0 \quad \forall\, |\psi\rangle$

$\hat{\rho} = \hat{\rho}^\dagger$ AND $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$

$\Rightarrow$ EIGEN. $\lambda_k \in R^+$

EXP. VALUE of AN OBSERV. $\hat{\theta}$

$$\langle\hat{\theta}\rangle = \mathrm{Tr}\left(\hat{\rho}\,\hat{\theta}\right)$$

FOR A PURE STATE $\langle\hat{\theta}\rangle = \langle\psi|\hat{\theta}|\psi\rangle$

AS DEFINED ABOVE ex. SHOW IT

FOR A MIXED STATE $\hat{\rho}$

$$\langle \hat{\theta} \rangle = \sum_{\alpha} p_{\alpha} \sum_{m} a_m |c_{\alpha}^{m}|^2$$

ex. SHOW IT

STAT. PHYSICS IN EQUIL

MICROCANONICAL

$[E, E+\delta E]$ ENERGY SHELL

$$\hat{\rho} = \frac{1}{\Omega} \sum_{n \in \text{SHELL}} |n\rangle\langle n|$$

$$\hat{H}|n\rangle = E_n |n\rangle$$

Ω # STATES WITH $E_n \in [E, E+\delta E]$

$$S = k_B \ln \Omega$$

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

$$\langle \hat{\theta} \rangle = Tr\left(\hat{\rho}\hat{\theta}\right)$$

$$= \frac{1}{\Omega} \sum_{n \in \text{SHELL}} \langle n|\hat{\theta}|n\rangle$$

ex. TWO LEVEL SYST — MORE BY END OF LECTURES

CANONICAL

$$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z}$$

$$Z = Tr\, e^{-\beta\hat{H}} = \sum_n \langle n|e^{-\beta\hat{H}}|n\rangle$$

$$= \sum_n e^{-\beta E_n}$$

$$p(E_n) = \frac{e^{-\beta E_n}}{Z}$$

$$-\beta F = \ln Z$$

Disordered Systems (Many Body)

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\,\hat{\sigma}_i^z\,\hat{\sigma}_j^z + \Gamma\sum_i \hat{\sigma}_i^x$$

NON COMMUTING TERMS

$$Z = \mathrm{Tr}\; e^{-\beta\hat{H}}$$

$$-\beta F = \ln Z$$

HAVE TO AVERAGE OVER $P(\{J_{ij}\})$ THE $\ln Z$

$\Rightarrow$ REPLICAS.

BUT, HOW TO DEAL W/ OPERATORS ?

PATH INT. FORMULATION

$\Rightarrow$ LATER

MANY BODY SYSTEMS

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$$

$$= \bigotimes_{i=1}^{N} \mathcal{H}_i$$

UNION OF HILBERT SP.

$$|\psi\rangle = \sum_{k_1=1}^{k} \cdots \sum_{k_N=1}^{k} c_{k_1 \cdots k_N} |\psi_1^{k_1}\rangle \otimes \cdots \otimes |\psi_N^{k_N}\rangle$$

SYST 1 STATE k_1 etc.

$$\sum_{k_1=1}^{k} \cdots \sum_{k_N=1}^{k} |c_{k_1 \cdots k_N}|^2$$

ex 2 QU-BITS

REDUCED DENSITY OPERATOR

$$\hat{\rho}_A^{RED} = \underset{B}{Tr}\ \hat{\rho}_{AB}$$

- EX. SEPARABLE PURE STATES

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\hat{\rho}_A^{RED} = \underset{B}{Tr}\ |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$= \hat{\rho}_A = |\psi_A\rangle\langle\psi_A|$$

OTHERWISE $\hat{\rho}_A^{RED} \neq \hat{\rho}_A$

- MIXED STATES FROM RED OF NON-SEPST.

$$|\psi_{AB}\rangle = \sum_{nm} c_{nm}\, |a_n\rangle \otimes |b_m\rangle$$

FOLLOW NOTES · p. 39

- PURIFICATION FOLLOW NOTES. p.40
- EXPECTED VALUES IN REDUCED STATES.

EXPECTED VALUES 27/11

$$\langle \hat{\theta}_A \rangle = Tr_{AB}\, \hat{\theta}_A \hat{\rho}_{AB}$$

$$= Tr_A\, \hat{\theta}_A\, Tr_B\, \hat{\rho}_{AB}$$

$$= Tr_A\, \hat{\theta}_A\, \hat{\rho}_A^{RED}$$

MENTION EX. IN NOTES WORKED OUT IN FULL DETAIL 2.10 AND 2.11

ENTANGLEMENT

THE QUANTUM STATES of TWO SUBSYST ARE DEPENDENT ON EACH OTHER

SCH. 1935

PERFECT CORRELATION.

eg. ONE PART $\vec{S}=0$ DECAYS IN TWO PART $\Rightarrow$ SUM OF SPINS $=0$

MEASURE $\hat{S}^2$ IN ONE $\rightarrow$ RESULT UP

$\Rightarrow$ THE OTHER ONE MUST BE DOWN

INDEP. of DIST ! PERFECT CORR.

Def. WORKING W/ STATES

$|\psi\rangle_{AB}$ is ENTANGLED IF IT CANNOT BE WRITTEN AS A SEPARABLE STATE

$$|\psi\rangle_{AB} = \sum_{ab} c_{ab} \, |\psi_a\rangle_A \otimes |\psi_b\rangle_B$$

IS GENERIC CASE, AND $\neq$ FROM

$$|\psi\rangle_{AB}^{SEP} = |\psi\rangle_A \otimes |\psi_B\rangle$$

$$= \sum_a c_a^A \, |\psi_a\rangle \otimes \sum_b c_b^B \, |\psi_b\rangle$$

IN AN ENTANGLED STATE, A MEASUREMENT DONE ON B COLLAPSES WHAT A CAN BE. INSTANTANEOUSLY & INDEP OF DIST BTW A AND B.

WORKING WITH DENSITY OPERATORS

$$\hat{\rho}_{AB}^{PROD} = \hat{\rho}_A \otimes \hat{\rho}_B \quad \text{PRODUCT STATE} \quad (1)$$

$$\hat{\rho}_{AB}^{SEP} = \sum_\alpha p_\alpha \, \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha \quad (2)$$

STATISTICAL MIXTURE OF PRODUCT STATES

BOTH ARE SEPARABLE.

(1) NO QUANTUM NOR CLASSICAL CORR

(2) ONLY CLASSICAL CORR

SEPARABLE DENS. OP.

$$\operatorname{Tr}_B \rho_{AB} \overset{PROP}{=} \operatorname{Tr}_B \hat{\rho}_A \otimes \hat{\rho}_B$$

$$= \hat{\rho}_A$$

$$\operatorname{Tr}_B \rho_{AB} \overset{SEP}{=} \operatorname{Tr}_B \sum_\alpha p_\alpha \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha \otimes \operatorname{Tr}_B \hat{\rho}_B^\alpha$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha$$

A MIXTURE OF THE DENS OP OF A.

ENTANGLED DENSITY OPERATOR

$$\hat{\rho}_{AB}^{ENT.} \neq \hat{\rho}_{AB}^{SEP}$$

IT IS HARD TO KNOW WHETHER A GENERIC $\hat{\rho}_{AB}$ IS ENTANGLED OR NOT

NP-HARD

LOCAL PROBS OF ENT STATES CANNOT BE TREATED INDEP.

SEPARAB. IS INDEP OF BASIS

PROOF

$|\psi\rangle$ SEPAR. $\Rightarrow$

$$|\psi\rangle = |\psi_A\rangle \otimes |\phi_B\rangle$$

$$\hat{U}|\psi\rangle = \hat{U}_A|\psi_A\rangle \otimes \hat{U}_B|\psi_B\rangle$$

SEP. AS WELL

$\hat{U}|\psi\rangle$ IN ANOTHER BASIS

ENTANG. ALSO INDEP BASIS.

$5' + 15' = 20'$

BELL STATES

ENTANGLED STATES of TWO SU(2) SPINS

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle\right)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle\right)$$

MEASURES ONE $\Rightarrow$ KNOWS THE OTHER !

MAXIMALLY ENTANGLED WITH CERTAINTY

PROPERTIES

- NORMALIZED TO 1
- PURE STATES $\hat{\rho}^2 = \hat{\rho}$

- ENTANGLED

- SYMM OR ANTI SYMM $A \leftrightarrow B$

  $|\Phi^{\pm}\rangle$ $|\psi^{+}\rangle$ $|\psi^{-}\rangle$

- WRITE THEM IN TERMS OF $|+\rangle, |-\rangle$ AND CHECK SYMM. PROP. ABOVE

  $|+\rangle, |-\rangle$ EIGENSTATES OF $\hat{\sigma}^x$

- CALCULATE THE EIGENVALUES OF THE TOTAL SPIN OPERATOR

$$\hat{S}^2 = \left(\hat{\vec{S}}_A + \hat{\vec{S}}_B\right)^2$$

$S(S+1)$ $\begin{cases} S=1 & |\Phi^{\pm}\rangle \quad |\psi^{+}\rangle \\ S=0 & |\psi^{-}\rangle \end{cases}$

STATES IN TRIPLET $\longrightarrow$ BOSONS

STATE IN SINGLET $\longrightarrow$ FERMIONS

- EVALUATE THE TOTAL SPIN IN EACH OF THE DIRECTIONS

  x, y, z

  IN THE STATE

  $|\Phi^{\pm}\rangle \Rightarrow$ EQUAL

  $|\psi^{\pm}\rangle \Rightarrow$ OPPOSITE

CALCULATE

$$P_{00} = \left| \langle 0| \otimes \langle 0| \, |\Phi^{\pm}\rangle \right|^2$$

P_{11}, P_{01}, P_{10} FOR z- DIRECTION

$$P_{++} = \left| \langle +| \otimes \langle +| \, |\Phi^{\pm}\rangle \right|^2$$

P_{--}, P_{+-}, P_{-+} FOR x DIRECTION etc.

FOR $|\Phi^{\pm}\rangle$ AND SIMILARLY FOR $|\psi^{\pm}\rangle$

IMPORTANT

$$\langle \psi^- | \, \hat{\sigma}^c \otimes \mathbb{I} + \mathbb{I} \otimes \hat{\sigma}^c \, |\psi^-\rangle = 0$$

- $\underset{B}{Tr} |\phi^{\pm}\rangle = \frac{\mathbb{I}}{2} = \underset{B}{Tr} |\psi^{\pm}\rangle$

  $\underset{A}{Tr} |\Phi^{\pm}\rangle = \frac{\mathbb{I}}{2} = \underset{A}{Tr} |\psi^{\pm}\rangle$

- SHOW THAT THE FULL KNOWLEDGE OF NOT MEASURED QU-BIT ONCE THE OTHER ONE IS MEASURED APPLIES TO BELL STATES SINCE $|0\rangle$ AND $|1\rangle$ ARE ORTHOGONAL.

eg $|\phi\rangle$ 2nd $|\psi_1\rangle$ or $|\psi_2\rangle$ FIRST

$$|\psi\rangle = \left(|\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle\right) \frac{1}{\sqrt{2}}$$

MEASURE 2nd $\Rightarrow$

$$|\psi\rangle_{\text{AFTER}} = \left(|\psi_1\rangle + |\psi_2\rangle\right) \frac{1}{\sqrt{2}}$$

EPR PARADOX AFTER BREAK.

EXPLAIN 15'

BELL INEQUALITY CLASSICAL 15'

$$P_{SAME}^{xy} + P_{SAME}^{xz} + P_{SAME}^{yz} \geq 1$$

QUANTUM

WE ESTIMATE THE lhs AND FIND < 1

VON NEUMANN & RENYI ENTROPIES

$$S_{VN} = - k_B \, Tr \; \hat{\rho} \ln \hat{\rho}$$

$$S_R = \frac{k_B}{1-q} \ln Tr \; \hat{\rho}_A^{\,q}$$

REPRESENT IN TERMS OF EIGENVALUES OF $\hat{\rho}$:

$$S_{VN} = - k_B \sum_{i=1}^{D} \lambda_i \ln \lambda_i$$

$$S_R = k_B \frac{1}{1-q} \ln \sum_i \lambda_i^{\,q}$$

λ_i EIGENVALUES OF $\hat{\rho}$

FROM RÉNYI TO VON NEUMANN ENTROPY

$$\lim_{q\to 1} S_R = k_B \lim_{q\to 1^+} \ln \frac{\mathrm{Tr}\,\hat{\rho}_A^q}{1-q}$$

$\frac{0}{0}$ SINCE $\sum_i \lambda_i = 1$

$$= k_B \lim_{q\to 1} \frac{\ln \sum_i \lambda_i^q}{1-q}$$

EXPAND AROUND $q=1$

$$\simeq k_B \left.\frac{\frac{d}{dq}\ln \sum_i \lambda_i^q}{\cancel{1-q}}\,\cancel{(q-1)}\,\right|_{q=1} \qquad (-1)$$

$$= -k_B \left.\frac{\frac{d}{dq}\left(\sum_j \lambda_j^q\right)}{\underbrace{\sum_i \lambda_i}_{=1}}\right|_{q=1}$$

$$= -k_B \sum_j \frac{d}{dq}\lambda_j^q$$

$$\frac{d}{dq}\lambda_j^q = ? \qquad y = \lambda_j^q \Rightarrow \ln y = q \ln \lambda_j$$

$$\frac{d\ln y}{dq} = \ln \lambda_j = \frac{1}{y}\frac{dy}{dq}$$

$$\Rightarrow \frac{dy}{dq} = y \ln \lambda_j = \lambda_j^q \ln \lambda_j$$

$$\frac{d}{dq}\lambda_j^q = \lambda_j^q \ln \lambda_j$$

THEN $$\lim_{q\to 1} S_R = -k_B \sum_j \lambda_j^q \ln \lambda_j \Big|_{q=1}$$

$$= -k_B \sum_j \lambda_j \ln \lambda_j$$

$$= S_{vN}$$

REPLICA VERSION

$$S_{vN} = -k_B \lim_{q\to 1} \frac{\partial}{\partial q} \ln \mathrm{Tr}\, \hat{\rho}^q$$

METHODS TO CALC RENYI ENTROPY: REPLICAS
(POWER HELPS)

- CALCULATE THE POWER FOR $q \geq 2$
- CONTINUE TO REAL q
- TAKE $q \to 1^+$

PURE STATE $S_{VN} = 0$

$\lambda_k = 1 \quad \lambda_{j(\neq k)} = 0$

MAX ENTANGLED STATE

$\hat{\rho} = \mathbb{I}/D \qquad \lambda_j = 1/D \quad \forall j$

$S_{VN} = k_B \ln D$

ENTANGLEMENT ENTROPY

$$S_{VN}^{ENT} = -k_B \, Tr \, \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$S_R^{ENT} = \frac{k_B}{1-q} \ln Tr \left(\hat{\rho}_A^{RED} \right)^q$$

ENTANGLEMENT SPECTRUM

$$\hat{\rho} = e^{-\hat{H}_E} \quad \Rightarrow -\ln \hat{\rho} = \hat{H}_E$$

$\hat{H}_E$ IS NOT A PROPER HAMILTONIAN.
IT'S ADIM. SERVES TO DEF $\{\lambda_i\}$

$$\hat{H}_E |v_i\rangle = \lambda_i |v_i\rangle$$

$\{\lambda_i\}$ SPECTRUM

NB S^{ENT} CAN BE EQUAL WHILE

$\{\lambda_i\}$ DIFF $\Rightarrow$ DISTINGUISH SYSTEMS.

FOR PURE STATES $\{\lambda_i\}$ SCHMIDT DECOMP.

ENTANGLEM. SPECTRUM $\exists$ FOR MIXED STATES BUT CLASS. CONTRIB TOO. (FROM H_E)

PURE STATES $\quad S^{ENT}(A) = S^{ENT}(B)$

PROOF :

$$\hat{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i}\ |u_i\rangle_A \otimes |v_i\rangle_B$$

$$S_A^{ENT} = -k_B\, Tr\ \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

$$S_B^{ENT} = -k_B \, Tr \, \hat{\rho}_B^{RED} \ln \hat{\rho}_B^{RED}$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

SINCE $\hat{\rho}_A^{RED} = \underset{B}{Tr} \, \tilde{\rho}_{AB}$

$$= \sum_i \lambda_i \, |u_i\rangle_A$$

$\hat{\rho}_B^{RED} = \underset{A}{Tr} \, \hat{\rho}_{AB}$

SAME EIGENVALUES $\{\lambda_i\}$

$$= \sum_i \lambda_i \, |u_i\rangle_B$$

SEPARABLE PURE STATE $\rightarrow$ $S^{ENT} = 0$

$|\psi_{AB}\rangle = |\psi\rangle_A \otimes |\psi_B\rangle$ SEP.

$$\hat{\rho}_{AB} = \left(|\psi\rangle_A \otimes |\psi_B\rangle\right)\left(\underset{A}{\langle\psi|} \otimes \underset{B}{\langle\psi|}\right)$$

$$= \left(|\psi\rangle_A \underset{A}{\langle\psi|}\right) \otimes |\psi\rangle_B \underset{B}{\langle\psi|}$$

$$= \hat{\rho}_A \otimes \hat{\rho}_B$$

$$\hat{\rho}_A^{RED} = \underset{3}{Tr}\ \hat{\rho}_{AB} = \hat{\rho}_A$$

$$S_A^{ENT} = -k_B\, Tr\ \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B\, Tr\ \hat{\rho}_A \ln \hat{\rho}_A$$

$$= -k_B \sum_i \lambda_i \ln \lambda_i$$

BUT $\lambda_i = 1$ FOR SOME i 0 OTHERWISE

$$\Rightarrow S_A^{ENT} = 0$$

FOR A PURE SEP ST.

FOR A MIXED SEP STATE :

PROOF

$$\hat{\rho}_{AB}^{SEP} = \sum_\alpha p_\alpha \, \hat{\rho}_A^\alpha \otimes \hat{\rho}_B^\alpha$$

$$0 \le p_\alpha \le 1 \qquad \sum_\alpha p_\alpha = 1$$

$$\hat{\rho}_A^{RED} = \mathrm{Tr}_B \, \hat{\rho}_{AB}^{SEP}$$

$$= \sum_\alpha p_\alpha \hat{\rho}_A^\alpha$$

CLASS PROB MIX OF $\hat{\rho}_A^\alpha$ STATES

$$S^{ENT} = -k_B \, \mathrm{Tr} \, \hat{\rho}_A^{RED} \ln \hat{\rho}_A^{RED}$$

$$= -k_B \sum_\alpha p_\alpha \ln p_\alpha$$

CLASSICAL SHANNON ENTROPY

30' APPROX.

EXERCISES HERE

FOR SEP. STATE $C_{ab} = c_a \cdot c'_b$ IN ONE EX.

$$\det \hat{\rho}_A^{RED} =$$

$$|b|^2|c|^2 + |a|^2|d|^2 - ab^* c^* d - a^* b c d^*$$

$$a = C_{00} = c_0 c_0' \qquad c = C_{10} = c_1 c_0'$$

$$b = C_{01} = c_0 c_1' \qquad d = C_{11} = c_1 c_1'$$

$$\det \hat{\rho}_A^{RED} =$$

$$= |c_0 c_1'|^2 |c_1 c_0'|^2 +$$

$$+ |c_0 c_0'|^2 |c_1 c_1'|^2 -$$

$$- c_0 c_0' \; c_0^* \; c_1'^* \; c_1^* \; c_0'^* \; c_1 c_1'$$

$$- c_0^* c_0'^* \; c_0 c_1' \; c_1 c_0' \; c_1^* c_1'^* \Big] =$$

$$= 2|C_0 C_1' C_1 C_0'|^2 - 2|C_0 C_1' C_1 C_0'|^2$$

$$= 0$$

PROPERTIES

$$|\psi\rangle_{AB} = \sum_{\alpha}^{D} \sqrt{\lambda_\alpha}\, |\phi_\alpha\rangle\langle\psi_\alpha|$$ SCHMIDT DECOMP

FOR LOW ENTANGLED STATES

THE SUM CAN BE CUT AT AN $n << D$.

BASIS OF MATRIX PRODUCT STATES & NUM. METHODS.

SCALING WITH SIZE

A TYPICAL QUANTUM STATE WILL SATISFY:

$S(A) \propto |A| \sim L^d$ VOLUME OF A

eg. THERMAL STATES

BUT GROUND STATES OF GAPPED SYST.

$$S(A) \propto |\partial A| \sim L^{d-1}$$ AREA

LOWER ENTANG ENTROPY $\Rightarrow$ NUMERICS OK.

AT CRITICAL GROUND STATES 1d SYST

$$S(A) \sim \frac{c}{3} \ln(L/a)$$

CRITICAL STATES IN $d > 1$

$$S(A) \sim L^{d-1} \ln L \quad \text{or} \quad L^{d-1}$$

TIME EVOLUTION

SCHRÖDINGER

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \hat{H} |\psi, t\rangle$$

$$|\psi, t\rangle = \hat{U}(t, t') |\psi, t'\rangle$$

$$\hat{U}^{-1}(t, t_0)\, \hat{U}(t, t_0) = \hat{\mathbb{I}} \qquad \text{INVERSE}$$

$$\hat{U}^{\dagger}(t, t_0)\, \hat{U}(t, t_0) = \hat{\mathbb{I}} \qquad \text{ENSURES CONS. PROB.}$$

$$|\langle \psi, t | \psi, t \rangle|^2 = 1$$

COMPOSITION $\quad \hat{U}(t, t_1)\, \hat{U}(t_1, t_0) = \hat{U}(t, t_0)$

CONTINUITY $\quad \lim_{dt \to 0} \hat{U}(t + dt, t) = \hat{U}(t, t) = \hat{\mathbb{I}}$

$$i\hbar \frac{d\hat{U}(t, t_0)}{dt} = \hat{H}(t)\, \hat{U}(t, t_0)$$

SOLUTION FOR $\hat{H}(t) = \hat{H}$ NO TIME DEP.

$$\hat{U}(t,t_0) = e^{-i/\hbar\, \hat{H}(t-t_0)}$$

HEISENBERG

$$\hat{A}_H(t,t_0) = \hat{U}^\dagger(t,t_0)\, \hat{A}_S\, \hat{U}(t,t_0)$$

$$\hat{A}_S = \hat{A}_H(t_0,t_0)$$

EXPECT. VALUE

$$\langle \hat{A}_S \rangle(t) = \langle \psi,t | \hat{A}_S | \psi,t \rangle$$

$$= \langle \psi,t_0 | \hat{U}^\dagger(t,t_0)\, \hat{A}_S\, \hat{U}(t,t_0) | \psi,t_0 \rangle$$

$$= \langle \psi,t_0 | \hat{A}_H(t,t_0) | \psi,t_0 \rangle$$

$$= \langle \hat{A}_H \rangle(t)$$

HEISENBERG'S EQN. OF OPERATORS

$$\frac{d\hat{A}_H(t,t_0)}{dt} = \frac{d\hat{U}^\dagger(t,t_0)}{dt}\hat{A}_S\hat{U}(t,t_0) +$$

$$+ \hat{U}^\dagger(t,t_0)\hat{A}_S\frac{d\hat{U}(t,t_0)}{dt}$$

USING EVOL EQ FOR $\hat{U}^\dagger$ AND $\hat{U}$:

$$= \frac{-i}{\hbar}\left[\hat{A}_H(t,t_0), \hat{H}\right]$$

INTER. PICTURE

$$\hat{H} = \hat{H}_0 + \hat{H}_{INT}$$

$$|\psi_I,t\rangle \equiv e^{i/\hbar\,\hat{H}_0 t}\,|\psi_S,t\rangle$$

$$\hat{\Theta}_I(t) \equiv e^{i/\hbar\,\hat{H}_0 t}\,\hat{\Theta}_S\,e^{-i/\hbar\,\hat{H}_0 t}$$

DEFS OF STATES & OP. FROM SCHRÖDINGER'S

EVOL EQ FOR STATES IN INT PICTURE

$$i\hbar \frac{d}{dt} |\psi_I, t\rangle = i\hbar \frac{d}{dt} e^{i/\hbar \hat{H}_0 t} |\psi_S, t\rangle$$

$$= i\hbar \left[\frac{i}{\hbar} \hat{H}_0 e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi_S, t\rangle + e^{i/\hbar \hat{H}_0 t} \frac{d|\psi_S, t\rangle}{dt} \right]$$

$$= -\hat{H}_0 e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi_S, t\rangle + e^{i/\hbar \hat{H}_0 t} \hat{H} |\psi_S, t\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0 t} \left[-\hat{H}_0 + \hat{H}_0 + \hat{H}_{INT} \right] |\psi_S, t\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{H}_{INT} |\psi_S, t\rangle$$

$$i\hbar \frac{d|\psi_I, t\rangle}{dt} = \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{H}_{INT} e^{-\frac{i}{\hbar}\hat{H}_0 t}}_{\hat{H}_{INT}^I(t)} \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi_S, t\rangle}_{|\psi_I, t\rangle}$$

$$= \hat{H}_{INT}^I(t) |\psi_I, t\rangle$$

$$i\hbar \frac{d|\psi_{I,t}\rangle}{dt} = \hat{H}^{I}_{INT}(t) \; |\psi_{I,t}\rangle$$

EVOL. OF OPERATORS IN INT. PICTURE

$$\hat{A}_I(t) = e^{i/\hbar \hat{H}_0 t} \; \hat{A}_S \; e^{-i/\hbar \hat{H}_0 t}$$

$$\frac{d\hat{A}_I(t)}{dt} = \frac{i}{\hbar} \hat{H}_0 \, e^{i/\hbar \hat{H}_0 t} \; \hat{A}_S \; e^{-i/\hbar \hat{H}_0 t} + e^{i/\hbar \hat{H}_0 t} \; \hat{A}_S \left(-\frac{i}{\hbar}\right) e^{-i/\hbar \hat{H}_0 t} \; \hat{H}_0$$

$$= \frac{i}{\hbar} \hat{H}_0 \; \hat{A}_I(t) - \frac{i}{\hbar} \hat{A}_I(t) \; \hat{H}_0$$

$$= \frac{i}{\hbar} \left[\hat{H}_0, \hat{A}_I(t)\right]$$

EVOLUTION OF DENSITY OPERATOR

IT'S AN OP THAT REPRESENTS THE STATE OF SYST => IT EVOLVES IN SCHRÖDINGER PICT

$$\hat{\rho}(t) = \sum_\alpha p_\alpha \, |\psi_\alpha, t\rangle\langle\psi_\alpha, t|$$

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = i\hbar \sum_\alpha p_\alpha \left(\frac{d|\psi_\alpha(t)\rangle}{dt} \langle\psi_\alpha, t| + |\psi_\alpha, t\rangle \frac{d\langle\psi_\alpha, t|}{dt} \right)$$

USING

$$i\hbar \frac{d|\psi_\alpha, t\rangle}{dt} = \hat{H}_s |\psi_\alpha, t\rangle$$

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = \cancel{i\hbar} \sum_\alpha p_\alpha \left(\frac{1}{\cancel{i\hbar}} \hat{H}_s |\psi_\alpha, t\rangle\langle\psi_\alpha, t| - \frac{1}{\cancel{i\hbar}} |\psi_\alpha, t\rangle\langle\psi_\alpha, t| \hat{H}_s^\dagger \right)$$

$$= \sum_\alpha p_\alpha \left(\hat{H}_s |\psi_\alpha, t\rangle\langle\psi_\alpha, t| - |\psi_\alpha, t\rangle\langle\psi_\alpha, t| H_s \right)$$

$$= \hat{H}_s \hat{\rho}(t) - \hat{\rho}(t) \hat{H}_s$$

$$= [\hat{H}_s, \hat{\rho}(t)] = i\hbar \frac{d\hat{\rho}(t)}{dt}$$

THE QUANTUM DYN DOESN'T CHANGE THE PURITY OF A SYST AS QUANT. BY $\mathrm{Tr}\,\hat{\rho}^2$

$$\frac{d}{dt}\mathrm{Tr}\,\hat{\rho}^2 = \mathrm{Tr}\; 2\hat{\rho}\frac{d\hat{\rho}}{dt}$$

$$= 2\,\mathrm{Tr}\;\hat{\rho}\;\frac{1}{i\hbar}\left[\hat{H}_S, \hat{\rho}(H)\right]$$

$$= \frac{2}{i\hbar}\,\mathrm{Tr}\;\hat{\rho}\left(\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\right)$$

cyclic prop Tr

$$= \frac{2}{i\hbar}\,\mathrm{Tr}\left(\rho\hat{H}\hat{\rho} - \hat{\rho}\hat{H}\rho\right) = 0$$

ENTROPY CONSERVED

$$S_{VN} = -k_B\,\mathrm{Tr}\;\hat{\rho}\ln\hat{\rho}$$

$$\frac{dS_{VN}}{dt} = -k_B\,\mathrm{Tr}\left[\frac{d\hat{\rho}}{dt}\ln\hat{\rho} + \hat{\rho}\hat{\rho}^{-1}\frac{d\hat{\rho}}{dt}\right]$$

$$= -k_B \, \mathrm{Tr} \left\{ \frac{1}{i\hbar} \left[\hat{H}_s, \hat{\rho} \right] \ln \hat{\rho} + \frac{1}{i\hbar} \left[\hat{H}_s, \hat{\rho} \right] \right\}$$

$$= -\frac{k_B}{i\hbar} \, \mathrm{Tr} \left\{ \left(\hat{H}_s \hat{\rho} - \hat{\rho} \hat{H}_s \right) \ln \hat{\rho} + \left(\hat{H}_s \hat{\rho} - \hat{\rho} \hat{H}_s \right) \right\}$$

$\leftarrow$ 0 UNDER Tr

$$= -\frac{k_B}{i\hbar} \, \mathrm{Tr} \left(\hat{H}_s \hat{\rho} \ln \hat{\rho} - \hat{H}_s \ln \hat{\rho} \, \hat{\rho} \right) = 0$$

0 UNDER Tr

JUSTIFYING BOLTZMANN $\hat{\rho} \propto e^{-\beta \hat{H}}$

IT SOLVES THE EVOL EQ FOR $\hat{\rho}$.

NO CLONING THEOREM.

IT'S IMPOSSIBLE TO COPY A QUANTUM STATE

TAKE $|\psi\rangle = |0\rangle_A$ AND TRY TO COPY IT

ASSUME $\hat{U}\,(|0\rangle_A \otimes |0\rangle_B) = |0\rangle_A \otimes |0\rangle_B$

$\hat{U}\,(|1\rangle_A \otimes |0\rangle_B) = |1\rangle_A \otimes |1\rangle_B$

SUCH A $\hat{U}$ EXIST. THAT'S THE ASSUMPTION

THAT IS WE WANT TO COPY STATE A ON B.
B WAS INITIALIZED IN $|0\rangle$.
B BECOMES = A AND A DOESN'T CHANGE

NOW, WHAT HAPPENS WITH $|\psi\rangle_A = \frac{1}{\sqrt{2}}\left(|1\rangle_A + |0\rangle_A\right)$
CAN WE ALSO COPY IT WITH
$\hat{U}$?

$$\hat{U}\left(|\psi_A\rangle \otimes |0\rangle_B\right) =$$

$$\frac{1}{\sqrt{2}}\,\hat{U}\left[\left(|1\rangle_A + |0\rangle_A\right) \otimes |0\rangle_B\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\hat{U}|1\rangle_A \otimes |0\rangle_B + \hat{U}|0\rangle_A \otimes |0\rangle_B\right] =$$

$$= \frac{1}{\sqrt{2}}\left[|1\rangle_A \otimes |1\rangle_B + |0\rangle_A \otimes |0\rangle_B\right]$$

$$\neq |\psi_A\rangle \otimes |\psi_A\rangle$$

NOT POSSIBLE TO CLONE GEN. STATES WHICH ARE NOT JUST BASIS ONES

DEVELOPM of ENTANGLEMENT

SAY $\hat{\rho}_{AB}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_B(0)$

AND $\hat{\rho}_A(0)$, $\hat{\rho}_B(0)$ THE DENS OF OF TWO PURE STATES

eg $|\psi,0\rangle = |0\rangle \otimes |1\rangle$

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{\sigma}_x^A \hat{\sigma}_x^B + \hat{\sigma}_y^A \hat{\sigma}_y^B\right)$$

$$|\psi,t\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi,0\rangle$$

$$= \cos\omega t\ |0\rangle\otimes|1\rangle - i\sin\omega t\ |1\rangle\otimes|0\rangle$$

$$\hat{\rho}_A(t) = \operatorname*{Tr}_B\ |\psi,t\rangle\langle\psi,t|$$

$$= \sum_{i=0,1} {}_B\langle i|\psi,t\rangle\langle\psi,t|i\rangle_B$$

$$\rho_A(t) = \begin{pmatrix} \sin^2\omega t & 0 \\ 0 & \cos^2\omega t \end{pmatrix}$$

$$\operatorname{Tr}\hat{\rho}_A^2(t) = \operatorname{Tr}\hat{\rho}_A(t)$$

QUANTUM QUENCHES EXPLAIN

QUANTUM CHAOS

CLASSICALLY TRAJECTORIES

$x(0)$ $x(0)+\delta x(0)$

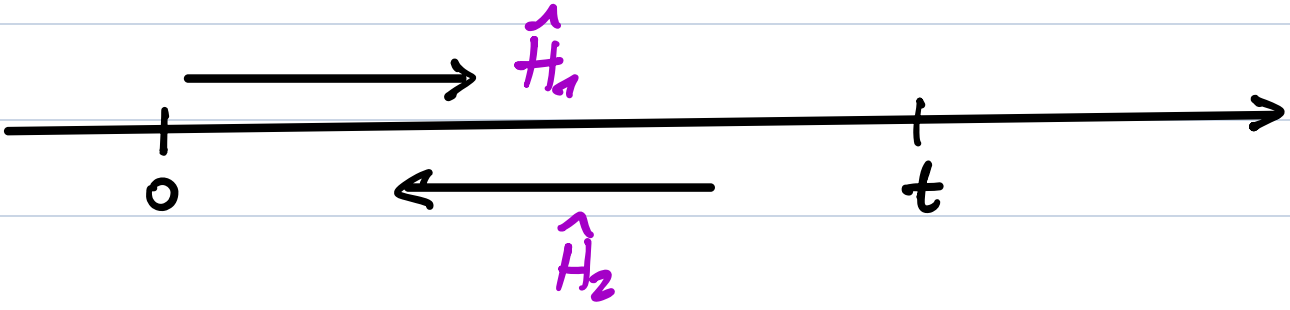
HOW DO THEY SEPARATE ?

$\delta x(t) \sim e^{\lambda t}$ CHAOTIC ONES

$\delta x(t) \sim$ NO.

LOSCHMIDT ECHO

$$M(t) = \left| \langle \psi_1 | \, e^{\frac{i}{\hbar} \hat{H}_2 t} \, e^{-\frac{i}{\hbar} \hat{H}_1 t} \, |\psi(0)\rangle \right|^2$$



SIMILAR, OTOCs

$$C(t) = \langle [\hat{W}(t), \hat{V}(0)]^\dagger [\hat{W}(t), \hat{V}(0)] \rangle$$

$$= Tr \left[[\hat{W}(t), \hat{V}(0)]^\dagger [\hat{W}(t), \hat{V}(0)] \hat{\rho} \right]$$

$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z}$ TYPICALLY USED

IF HERMITIAN

$$= -Tr\left([\hat{W}(t), \hat{V}(0)]^2 \hat{\rho} \right)$$

$$F(t) \equiv \langle W^\dagger(t) \hat{V}^\dagger \hat{W}(t) \hat{V} \rangle$$

$$C(t) = \langle \left(\hat{W}(t) \hat{V}(0) - \hat{V}(0) \hat{W}(t) \right)$$

$$\left(\hat{W}(t) \hat{V}(0) - \hat{V}(0) \hat{W}(t) \right) \rangle$$

$$= -\langle \hat{W}(t) \hat{V}(0) \hat{W}(t) \hat{V}(0) \rangle \quad \text{IN } F(t)$$

$$+ \langle \hat{W}(t) \hat{V}(0) \hat{V}(0) \hat{W}(t) \rangle$$

$$+ \langle \hat{V}(0) \hat{W}(t) \hat{W}(t) \hat{V}(0) \rangle \quad \} \text{ ASSUME } \equiv$$

$$- \langle \hat{V}(0) \hat{W}(t) \hat{V}(0) \hat{W}(t) \rangle \quad \text{ALSO IN } F(t) \Rightarrow 2$$

HAVE TO TRANSF THE OTHER TERMS USING

$$\hat{W}(t) = e^{i/\hbar \hat{H} t} \; \vec{W} \; e^{-i/\hbar \hat{H} t}$$

$$\langle \hat{V}(0) \; \hat{W}(t) \hat{V}(0) \; \vec{W}(t) \rangle =$$

$$= \langle W(t) \; \vec{V}(0) \; \vec{W}(t) \; \vec{V}(0) \rangle = F(t)$$

THEN,

$$C(t) = 2 \left(1 - F(t)\right)$$

OTHERWISE

$$C(t) = 2 \left(1 - \mathrm{Re}\, F(t)\right)$$

SEMICLASS. LIMIT $\quad [\hat{x}(t), \hat{p}(0)] \longrightarrow i\hbar \dfrac{\delta x(t)}{\delta x(0)}$

INTEGRABLY SYST OTOC DOESN'T EVOLVE MUCH

CHAOTIC SYST $\sim e^{\lambda t}$

COUPLING TO BATHS

- CLASSICAL $\mathcal{Z} = \sum_{conf} e^{-\beta H_{TOT}}$ 10'
- QUANTUM Z
- CLASSICAL DYNAMICS $\longrightarrow$ LANGEVIN 30'

  MENTION $\dot{Z}_{dyn}$
- SCHWINGER KELDYSH 30'

LINDBLAD EQ

$$\frac{d\hat{\rho}^{RED}}{dt} = \frac{-i}{\hbar}\left[\hat{H}, \hat{\rho}^{RED}\right] + \sum_{i=1}^{N^2-1} \gamma_i \left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^{\dagger} - \frac{1}{2}\left\{ \hat{L}_i^{\dagger}\hat{L}_i, \hat{\rho}^{RED} \right\} \right)$$

$$\hat{H} = \hat{H}_{SYST} + \hat{\Lambda}(\{\gamma_i\})$$

↖ COUPLING TO ENV.

$\hat{L}_i$ JUMP OPERATORS
DERIVED OR GUESSED
USUALLY, A FEW

γ_i DAMPING RATES ≥ 0

$\gamma_i = 0$ VON NEUMANN EQ.

UNITARY DYNAMICS

PROPERTIES of LINDBLAD EQ.

c) CONSERVES THE POSITIVITY of $\hat{\rho}^{RED}$

$$\langle \psi | \hat{\rho}^{RED} | \psi \rangle \geq 0 \quad \forall | \psi \rangle$$

b) TRACE PRESERVATION

a) CONSERVES $\hat{\rho}^{RED} = \hat{\rho}^{RED\dagger}$

$\Rightarrow$ CONST ON $\{\gamma_i\}$

Proof

a) $\hat{\rho}^{RED} = \hat{\rho}^{RED\dagger}$?

TAKE $\dagger$ OF LINDBLAD EQ.

$$\frac{d}{dt}\hat{\rho}^{RED\dagger} = \frac{(-i)^*}{\hbar}\left[\hat{H}, \hat{\rho}^{RED}\right]^\dagger$$

$$+ \sum_{i=1}^{N^2-1} \gamma_i^* \left(\left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^\dagger\right)^\dagger - \frac{1}{2}\left\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED}\right\}^\dagger \right)$$

USING:

$$\left(\hat{L}_i^\dagger \hat{L}_i \hat{\rho}^{RED} + \hat{\rho}^{RED} \hat{L}_i^\dagger \hat{L}_i\right)^\dagger =$$

$$= \hat{\rho}^{RED\dagger} \hat{L}_i^\dagger \hat{L}_i + \hat{L}_i^\dagger \hat{L}_i \hat{\rho}_{RED}^\dagger$$

$$= \left\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED\dagger}\right\} \quad \text{LAST TERM}$$

$$\left(\hat{L}_i \hat{\rho}^{RED} \hat{L}_i^\dagger\right)^\dagger = \hat{L}_i^\dagger \hat{\rho}^{RED\dagger} \hat{L}_i$$ NEXT-TO LAST

$$[\hat{H}, \hat{\rho}^{RED}]^\dagger = \left(\hat{H}\hat{\rho}^{RED} - \hat{\rho}^{RED}\hat{H}\right)^\dagger$$

$$= \hat{\rho}^{RED\dagger}\hat{H}^\dagger - \hat{H}^\dagger \hat{\rho}^{RED\dagger}$$

$$= \hat{\rho}^{RED\dagger}\hat{H} - \hat{H}\hat{\rho}^{RED\dagger}$$ 1st TERM IN RHS

REPLACING

$$\frac{d}{dt}\hat{\rho}^{RED\dagger} = \frac{\overbrace{(-i)^*}^{(-i)}(-1)}{\hbar}[\hat{H}, \hat{\rho}^{RED\dagger}]$$

$$+ \sum_{i=1}^{N^2-1} \gamma_i^* \left(\hat{L}_i \hat{\rho}^{RED\dagger} \hat{L}_i^\dagger - \frac{1}{2}\left\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}^{RED\dagger}\right\} \right)$$

γ_i

$\hat{\rho}^{RED\dagger}$ SATISFIES SAME EQ.

b) $\frac{d}{dt}\left(\underbrace{\mathrm{Tr}\,\hat{\rho}^{\mathrm{RED}}}_{\text{CONST}}\right) \stackrel{?}{=} 0$

rhs: $\mathrm{Tr}\,[\hat{H}, \hat{\rho}^{\mathrm{RED}}] =$

$= \mathrm{Tr}\left(\hat{H}\hat{\rho}^{\mathrm{RED}} - \hat{\rho}^{\mathrm{RED}}\hat{H}\right) = 0$

$\mathrm{Tr}\left(\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i^\dagger \hat{L}_i \hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}_i^\dagger \hat{L}_i\right)$

$= \mathrm{Tr}\left(\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{\rho} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{\rho} \hat{L}_i^\dagger\right)$

$= 0$ ok.

c) POSITIVITY

LONG PROOF.

Ex.

$$|\psi_i t+dt\rangle = \begin{cases} e^{-i\hat{G}}\,|\psi_i t\rangle & p=\lambda dt \\ |\psi_i t\rangle & p=1-\lambda dt \end{cases}$$

$$\hat{G}=\hat{G}^{\dagger} \qquad \hat{U}=e^{-i\hat{G}} \qquad p=\lambda dt$$

$$\hat{\rho}(t+dt) = |\psi_i t+dt\rangle\langle\psi_i t+dt|$$

$$= (1-\lambda dt)\,\hat{\rho}(t) + \lambda dt\; e^{-i\hat{G}}\,\hat{\rho}(t)\,e^{i\hat{G}}$$

$$\frac{d\hat{\rho}(t)}{dt} = -\lambda\left[\hat{\rho}(t) - e^{-i\hat{G}}\,\hat{\rho}(t)\,e^{i\hat{G}}\right]$$

rhs is of LINDBLAD FORM:

$$\text{rhs} = \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}$$

$$= \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}(\hat{L}^\dagger\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L})$$

WITH $\hat{L} = \sqrt{\lambda}\, e^{-i\hat{G}}$ AND $\gamma = 1$

$$= \lambda\, e^{-i\hat{G}}\hat{\rho}\, e^{i\hat{G}} - \frac{1}{2}2\lambda\hat{\rho}$$

$$= -\lambda\hat{\rho} + \lambda\, e^{-i\hat{G}}\hat{\rho}\, e^{i\hat{G}}$$

OK

HOW DOES THE EQ LOOK LIKE IN MATRICIAL FORM ?

LAST TERM IN rhs:

$$\begin{pmatrix} e^{-ig_1} & 0 \\ 0 & e^{-ig_2} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} e^{ig_1} & 0 \\ 0 & e^{ig_2} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-ig_1} & 0 \\ 0 & e^{-ig_2} \end{pmatrix} \begin{pmatrix} \rho_{11} e^{ig_1} & \rho_{12} e^{ig_2} \\ \rho_{21} e^{ig_1} & \rho_{22} e^{ig_2} \end{pmatrix} =$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} e^{-i(g_1-g_2)} \\ \rho_{21} e^{i(g_1-g_2)} & \rho_{22} \end{pmatrix}$$

1st TERM in rhs $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

ADDING w/ sgn:

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & \rho_{12}(1-e^{-i(g_1-g_2)}) \\ \rho_{21}(1-e^{i(g_1-g_2)}) & 0 \end{pmatrix}$$

$$\dot{\rho}_{11} = \dot{\rho}_{22} = 0 \quad \Rightarrow \quad \rho_{11} = \text{cons} \qquad \rho_{22} = \text{const}$$

$$\dot{\rho}_{12} = -\lambda \rho_{12} \left(1 - \cos(g_1 - g_2) + i \sin(g_1 - g_2)\right)$$

$$\dot{\rho}_{21} = -\lambda \rho_{21} \left(1 - \cos(g_1 - g_2) - i \sin(g_1 - g_2)\right)$$

$$\frac{d \ln \rho_{12}}{dt} = -\lambda \left(1 - \cos(g_1 - g_2) + i \sin(g_1 - g_2)\right)$$

$$\rho_{12}(t) = \rho_{12}(0)\, e^{-\lambda \underbrace{\left(1 - \cos(g_1 - g_2)\right)}_{\geq 0} t}$$

$$\underbrace{e^{-i\lambda \sin(g_1 - g_2) t}}_{\text{etc. BOUNDED}}$$

EXP. RELAX. SAME FOR $\rho_{21}(t)$.

Ex 2

$$|\psi, t+dt\rangle = \begin{cases} |\psi, t\rangle & 1-\lambda dt \\ \hat{\sigma}^y |\psi, t\rangle & \lambda dt \end{cases}$$

$$\hat{\rho}(t+dt) = (1-\lambda dt)\,\hat{\rho}(t) + \lambda dt\,\hat{\sigma}^y\,\hat{\rho}(t)\,\hat{\sigma}^y$$

$$\frac{d\hat{\rho}(t)}{dt} = -\lambda\left[\hat{\rho}(t) - \hat{\sigma}^y\,\hat{\rho}(t)\,\hat{\sigma}^y\right]$$

$$= \gamma\left[\hat{L}\,\hat{\rho}\,\hat{L}^\dagger - \frac{1}{2}\left(\hat{L}^\dagger\hat{L}\,\hat{\rho} + \hat{\rho}\,\hat{L}^\dagger\hat{L}\right)\right]$$

$\gamma = 1 \qquad \hat{L} = \sqrt{\lambda}\,\hat{\sigma}^y$

$$= \lambda\left[\hat{\sigma}^y\,\hat{\rho}\,\hat{\sigma}^{y\dagger} - \frac{1}{2}\left(\hat{\sigma}^{y\dagger}\hat{\sigma}^y\,\hat{\rho} + \hat{\rho}\,\hat{\sigma}^{y\dagger}\hat{\sigma}^y\right)\right]$$

$$= \lambda \left[\hat{\sigma}^y \hat{\rho} \hat{\sigma}^y - \frac{1}{2} \cdot 2 \hat{\rho} \right]$$

$$= \lambda \left(\hat{\sigma}^y \hat{\rho} \hat{\sigma}^y - \hat{\rho} \right)$$

$$= -\lambda \left(\hat{\rho} - \hat{\sigma}^y \hat{\rho} \hat{\sigma}^y \right)$$

THE DENSITY MATRIX FOLLOW:

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$+ \lambda \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} +$$

$$+\lambda \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i\rho_{12} & -i\rho_{11} \\ i\rho_{22} & -i\rho_{21} \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} + \lambda \begin{pmatrix} \rho_{22} & -\rho_{21} \\ -\rho_{12} & \rho_{11} \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} \rho_{11}-\rho_{22} & \rho_{12}-\rho_{21} \\ \rho_{21}-\rho_{12} & \rho_{22}-\rho_{11} \end{pmatrix}$$

SO THESE IMPLY

$$\dot{\rho}_{11} = -\lambda(\rho_{11}-\rho_{22}) = -\dot{\rho}_{22}$$

$$\dot{\rho}_{12} = -\lambda(\rho_{12}-\rho_{21}) = -\dot{\rho}_{21}$$

$$\dot{\rho}_{21} = -\lambda(\rho_{21}-\rho_{12}) = -\dot{\rho}_{12}$$

$$\dot{\rho}_{22} = -\lambda(\rho_{22}-\rho_{11}) = -\dot{\rho}_{11}$$

$$\frac{d\rho_{11}}{dt} = -\lambda\,(\rho_{11} - \rho_{22})$$

WORKING WITH THE EQS. FOR ρ_{11}, ρ_{22}

$$\frac{d\rho_{22}}{dt} = -\lambda\,(\rho_{22} - \rho_{11})$$

ADD $\quad \frac{d}{dt}(\rho_{11} + \rho_{22}) = 0 \Rightarrow$

$$\rho_{11}(t) + \rho_{22}(t) = \text{CTE}$$

$$\rho_{22}(t) = \text{CTE} - \rho_{11}(t)$$

SUBTRACT $\quad \frac{d}{dt}(\rho_{11} - \rho_{22}) = -2\lambda\,(\rho_{11} - \rho_{22})$

$$\rho_{11}(t) - \rho_{22}(t) = \left(\rho_{11}(0) - \rho_{22}(0)\right) e^{-2\lambda t}$$

$$2\rho_{11}(t) - \text{CTE} = \left(2\rho_{11}(0) - \text{CTE}\right) e^{-2\lambda t}$$

$$\rho_{11}(t) = \frac{CTE}{2} + \left(\rho_{11}(0) - \frac{CTE}{2}\right) e^{-2\lambda t}$$

$$\xrightarrow[t \to \infty]{} \frac{CTE}{2}$$

$$\rho_{22}(t) = CTE - \rho_{11}(t) \longrightarrow \frac{CTE}{2}$$

WE WORK NOW WITH EQS FOR ρ_{12}, ρ_{21}

$$\dot{\rho}_{21} = -\lambda(\rho_{21} - \rho_{12}) = -\dot{\rho}_{12}$$

$$\dot{\rho}_{12} = -\lambda(\rho_{12} - \rho_{21})$$

ADD & SUBTRACT THEM

$$\frac{d}{dt}\left(\rho_{21}(t) + \rho_{12}(t)\right) = 0$$

$$\rho_{21}(t) + \rho_{12}(t) = CTE$$

$$\frac{d}{dt}\left(\rho_{21}(t)-\rho_{12}(t)\right)=-2\lambda\left(\rho_{21}(t)-\rho_{12}(t)\right)$$

$$\rho_{21}(t)-\rho_{12}(t)=\left(\rho_{21}(0)-\rho_{12}(0)\right)e^{-2\lambda t}$$

$$2\rho_{21}(t)-CTE'=\left(2\rho_{21}(0)-CTE'\right)e^{-2\lambda t}$$

$$\rho_{21}(t)=\frac{CTE'}{2}+\left(\rho_{21}(0)-\frac{CTE'}{2}\right)e^{-2\lambda t}$$

$$\longrightarrow \frac{CTE'}{2}$$

$$\rho_{12}(t)=CTE'-\rho_{21}(t)\longrightarrow\frac{CTE'}{2}$$

ALL ELEMENTS APPROACH $\frac{CTE'}{2}$

WHAT ARE THE CONST?

$$\mathrm{Tr}\,\rho = 1 = \rho_{11}(t) + \rho_{22}(t)$$

$$\lim_{t\to\infty}\left(\rho_{11}(t) + \rho_{22}(t)\right) = \text{CTE}$$

EX. OTDC

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}^z$$

$$\hat{L} = \sqrt{\gamma}\,\hat{\sigma}^z$$

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\,[\hat{H},\hat{\rho}]$$

$$+\gamma\left(\hat{\sigma}^z\hat{\rho}\hat{\sigma}^z - \hat{\rho}\right)$$

$$[\hat{H}, \hat{\rho}] = \frac{\hbar\omega}{2} [\hat{\sigma}^z, \hat{\rho}]$$

$$= \frac{\hbar\omega}{2} \left(\hat{\sigma}^z \hat{\rho} - \hat{\rho} \hat{\sigma}^z \right)$$

"DETERMINISTIC TERM"

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} \\ -\rho_{21} & -\rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & -\rho_{12} \\ \rho_{21} & -\rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

"JUMP OP."

$$\hat{\sigma}^z \hat{\rho} \hat{\sigma}^z - \hat{\rho}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & -\rho_{12} \\ \rho_{21} & -\rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

The eq is

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = \frac{-i}{\hbar} \hbar\omega \begin{pmatrix} 0 & 2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

$$+ \gamma \begin{pmatrix} 0 & -2\rho_{12} \\ -2\rho_{21} & 0 \end{pmatrix}$$

$$\dot{\rho}_{11} = 0 \quad \Longrightarrow \quad \rho_{11} = \text{CTE}$$

$$\dot{\rho}_{12} = -i\omega \, 2\rho_{12} - 2\gamma \rho_{12}$$

$$\dot{\rho}_{21} = +i\omega \, 2\rho_{21} - 2\gamma \rho_{21}$$

$$\dot{\rho}_{22} = 0 \quad \Longrightarrow \quad \rho_{22} = \text{CTE}$$

$$\rho_{12}(t) = \rho_{12}(0) \, e^{2(-\gamma - i\omega)t}$$

$$\rho_{21}(t) = \rho_{21}(0)\, e^{2(-\gamma + i\omega)t}$$

$$\rho_{12}(t) = \rho_{12}(0)\, e^{-2\gamma t}\,(\cos 2\omega t - i \sin 2\omega t)$$

$$\rho_{21}(t) = \rho_{21}(0)\, e^{-2\gamma t}\,(\cos 2\omega t + i \sin 2\omega t)$$

$$(\rho_{11} = \alpha E) + (\rho_{22} = \alpha E') = 1$$

$$\rho_{11} = 1 - \rho_{22}$$

OTOC

$$\langle \hat{\sigma}_x(t)\, \hat{\sigma}_z\, \hat{\sigma}_x(t)\, \hat{\sigma}_z \rangle$$

$$\propto e^{-2\gamma t}$$

DERIVATION OF LINDBLAD'S

$$\hat{H}_T = \hat{H}_S + \hat{H}_E + \hat{H}_{int}$$

$\hat{H}_S$ IN $\mathcal{H}_S$

$\hat{H}_E$ IN $\mathcal{H}_E$

$\hat{H}_{int}$ IN $\mathcal{H}_T$ $\quad \hat{H}_{int} = \alpha \sum_i \gamma_i \hat{S}_i \otimes \hat{E}_i$

$\alpha \in \mathbb{R}^+$ $\quad \gamma_i \in \mathbb{R}^+$ $\quad \hat{S}_i^\dagger = \hat{S}_i$ $\quad \hat{E}_i^\dagger = \hat{E}_i$

EVOLUTION OF $\hat{\rho}_T(t)$

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}_T, \hat{\rho}(t)]$$

WE WANT TO DERIVE AN EQ FOR $\hat{\rho}_S^{RED}(t)$

WE WILL USE SOME ASSUMPTIONS:

- WEAK COUPLING S-E LIMIT

- CORR S-E DEVELOP SLOWLY

- E is NOT MODIFIED BY S:
  EXC. OF E DECAY QUICKLY

- FAST OK. TERM WRT SYSTEM'S
  EVOL. TIME-SCALE ARE NEGLIGIBLE

⇒ MARKOV DYN.

IT'S AN APPROX !

WORK IN INTERACTION PICTURE

$$\hat{O}_I(t) = e^{\frac{i}{\hbar}(\hat{H}_S + \hat{H}_E)t}\, \hat{O}\, e^{-\frac{i}{\hbar}(\hat{H}_S + \hat{H}_E)t}$$

ONE PROVES THAT $\hat{\rho}_I(t)$ EVOLVES AS

IT WAS AN EX. LAST WEEK

$$\frac{d\hat{\rho}_T^I(t)}{dt} = -\frac{i}{\hbar}\alpha\left[\hat{H}_{int}^I(t), \hat{\rho}_T^I(t)\right] \qquad (*)$$

FORMAL INTEGRATION

$$\hat{\rho}_T^I(t) = \hat{\rho}_T^I(0) - \frac{i}{\hbar}\alpha\int_0^t ds\, \left[\hat{H}_{int}^I(s), \hat{\rho}_T^I(s)\right]$$

WE COULD TRY INT OPER DATA HERE

BUT ⟶ IMPOSSIBLE

& NON-MARKOVIAN

ITERATE THIS SOLUTION BY INSERTING IT IN EQ. (*)

$$\frac{d\hat{\rho}_T^I(t)}{dt} = \frac{-i}{\hbar}\alpha\left[\hat{H}_{int}^I(t),\ \hat{\rho}_T^I(0)\right]$$

$$\underbrace{-\frac{i}{\hbar}\alpha\left(\frac{-i\alpha}{\hbar}\right)}_{-(\alpha/\hbar)^2}\left[\hat{H}_{int}^I(t),\ \int_0^t ds\ \left[\hat{H}_{int}^I(s),\ \hat{\rho}_T^I(s)\right]\right]$$

$$-\int_0^t ds\ \left[\hat{H}_{int}^I(t),\ \left[\hat{H}_{int}^I(s),\ \hat{\rho}_T^I(s)\right]\right]$$

THIS ONE IS EXACT

ONCE MORE ⟶ $\hat{\rho}_T^I(0)$

$$+\ \mathcal{O}\left((\alpha/\hbar)^3\right) \longrightarrow 0$$

IF WE DROP IT :

$$\frac{d\hat{\rho}_T^I(t)}{dt} = \frac{-i}{\hbar}\alpha\left[\hat{H}_{int}^I(t), \hat{\rho}_T^I(0)\right]$$

$$-\left(\frac{\alpha}{\hbar}\right)^2\int_0^t ds\left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_T^I(0)\right]\right]$$

$$+\Theta\left((\alpha/\hbar)^3\right)$$

AN APPROX EQ.

BACK TO THE EXACT EQ ABOVE AND INT. OVER E:

$$\frac{d\hat{\rho}_{red,I}^{red,I}(t)}{dt} = \frac{-i}{\hbar}\alpha\,\mathrm{Tr}_E\left[\hat{H}_{int}^I(t), \hat{\rho}_T^I(0)\right]$$

$$-\left(\frac{\alpha}{\hbar}\right)^2\int_0^t ds\,\mathrm{Tr}_E\left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_T^I(s)\right]\right]$$

(**)

PROBLEM $\hat{\rho}_T^I(s)$ HERE

ASSUMPTIONS

$$t=0 \qquad \hat{\rho}_T(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$$

ALL PICTURES THE SAME

$$t=0 \qquad \hat{\rho}_S^{\text{RED}}(0) = \hat{\rho}_S(0)$$

WE CAN NOW TRACE OVER E THE 1st TERM IN rhs OF (**)
AND SHOW IT VANISHES IF

$$\mathrm{Tr}\left(\hat{E}_i(H)\,\hat{\rho}_E(0)\right) = 0$$

WHICH WE ASSUME – (TRANSL. OF $\hat{H}_T$)

THEN,

$$\frac{d\hat{\rho}_S^{red,I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_T^I(s) \right] \right]$$

Born approx

$\tau_{SE}, \tau_E \ll \tau_S$ LARGE BATH

$$\hat{\rho}_T(t) = \hat{\rho}_S^{RED}(t) \otimes \hat{\rho}_E(0)$$

SYSTEM'S STATE MODIFIED BY ENV.
BUT ENV IS NOT BY SYST

$$\frac{d\hat{\rho}_S^{red,I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[\hat{H}_{int}^I(t), \left[\hat{H}_{int}^I(s), \hat{\rho}_S^{RED,I}(s) \otimes \hat{\rho}_E(0) \right] \right]$$

STILL NON-MARKOV

MARKOV ASSUMPTION

$$\rho_S^{red.I}(s) \longrightarrow \rho_S^{red.I}(t)$$

IN INTEGRAL

$$\frac{d\rho_S^{red.I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds \; \mathrm{Tr}_E \left[H_{int}^I(t), \left[H_{int}^I(s), \hat{\rho}_S^{RED,I}(t) \otimes \tilde{\rho}_E(0) \right] \right]$$

$s \to t$

BORN-MARKOV EQ

ALREADY $\left(\frac{\alpha}{\hbar}\right)^2 \Longrightarrow \hat{\rho}_S^{RED,I}(s)$ DOESN'T CHANGE MUCH.

CHANGE INTEGRATION VARIABLES

$$s' = t - s \qquad s \in [0,t]$$

$$ds' = -ds \qquad s' \in [0,t]$$

$$\frac{d\rho_S^{red,I}(t)}{dt} =$$

$$-\left(\frac{\alpha}{\hbar}\right)^2 \int_0^t ds' \, \mathrm{Tr}_E \left[H_{int}^I(t), \left[\underbrace{H_{int}^I(t-s)}_{\text{MEMORY EFFECTS}}, \hat{\rho}_S^{red,I}(t) \otimes \hat{\rho}_E(0) \right] \right]$$

BUT DON'T DO IT HERE. GO ON WITH PREVIOUS EXPRESSION.

$$-\left(\frac{\hbar}{\alpha}\right)^2 \frac{d\rho_S^{red,I}(t)}{dt} =$$

$$\int_0^t ds \, \mathrm{Tr}_E \left[H_{int}^I(t), \left[H_{int}^I(s), \hat{\rho}_S^{red,I}(t) \otimes \hat{\rho}_E(0) \right] \right]$$

NOW, REPLACE $H_{int}^{I} = \sum_i \gamma_i \hat{S}_i \otimes \hat{E}_i$

OPEN THE COMMUTATORS, REARRANGE ALL AMONG TERMS

CORRELATIONS LIKE

$\mathrm{Tr}_E \; \hat{E}_i(s) \, \hat{E}_j^{\dagger}(t) \, \hat{\rho}_E(0)$ APPEAR

CALL THEM $\Gamma_{ij}(s,t) = \Gamma_{ij}(s-t)$

ASSUMING STATIONARITY OF BATH

IT GETS LONG AND NOT SO INTERESTING

FURTHER ASSUMPTION WHITE BATH

$\Gamma_{ij}(t,s) = h_{ij}\,\delta(t-s)$

VERY QUICK RELAX OF BATH

NO MEMORY

$$-\left(\frac{\hbar^2}{\alpha^2}\right) \frac{d\rho_S^{RED,I}(t)}{dt}$$

$$\sum_{ij} h_{ij} \left[-\hat{S}_j(t)\, \hat{\rho}_S^{RED,I}(t)\, \hat{S}_i^{\dagger}(t) + \frac{1}{2} \left\{ \hat{S}_i^{\dagger}(t)\, \hat{S}_j(t),\ \hat{\rho}_S^{RED,I}(t) \right\} \right]$$

GET RID OF $\sum_{ij}$ WITH A DIAGONALIZATION

$$\hat{L}_i(t) = \sum_k d_{ik}\, \hat{S}_k(t)$$

$$\Gamma_{ij} = \sum_{k\ell} d_{ik}\, h_{k\ell}\, d^{*}_{\ell j} = \Gamma_i\, \delta_{ij}$$

$\Rightarrow$ CUTS THE DOUBLE SUM INTO A SINGLE ONE OVER i

MOVING BACK TO SCHRÖDINGER'S PICT

$$\hat{\rho}_S^{\text{RED,S}}(t) = e^{\frac{i}{\hbar}\hat{H}_S t}\, \hat{\rho}_S^{\text{RED,I}}(t)\, e^{-\frac{i}{\hbar}\hat{H}_S t}$$

$\hat{H}_S$ SCHRÖD. PICT SYST'S HAMILT

$\Rightarrow$

$$\frac{d\hat{\rho}_S^{\text{RED}}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{H}_S, \hat{\rho}_S^{\text{RED}}(t)\right]$$

$$+ \frac{\alpha^2}{\hbar^2}\sum_i \Gamma_i \left(\hat{L}_i\, \hat{\rho}_S^{\text{RED}}(t)\, \hat{L}_i^\dagger - \frac{1}{2}\left\{ \hat{L}_i^\dagger \hat{L}_i, \hat{\rho}_S^{\text{RED}}(t) \right\} \right)$$

NO SHIFT OF $\hat{H}_S$

THIS IS A QUITE BRUTAL APPROX FOR BATH → WHITE

BETTER ONES:

ROTATING WAVE APPROX

$$S = \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} |\epsilon\rangle\langle\epsilon'|$$

$$\hat{H}_s |\epsilon\rangle = \epsilon |\epsilon\rangle$$

$$\hat{S}_I(t) = e^{\frac{i}{\hbar}\hat{H}_s t} \hat{S} e^{\frac{-i}{\hbar}\hat{H}_s t}$$

$$= e^{\frac{i}{\hbar}\hat{H}_s t} \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} |\epsilon\rangle\langle\epsilon'| e^{\frac{-i}{\hbar}\hat{H}_s t}$$

$$= \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} e^{\frac{i}{\hbar}\epsilon t} e^{\frac{-i}{\hbar}\epsilon' t} |\epsilon\rangle\langle\epsilon'|$$

$$= \sum_{\epsilon\epsilon'} S_{\epsilon\epsilon'} e^{\frac{i}{\hbar}(\epsilon-\epsilon')t} |\epsilon\rangle\langle\epsilon'|$$

$$= \sum_{\omega} \hat{S}(\omega) e^{-i\omega t}$$

$$\hat{S}(\omega) = S_{\epsilon\epsilon} |\epsilon\rangle\langle\epsilon'| \Big|_{\epsilon-\epsilon' = -\omega}$$

$$\hat{S}(\omega)^{\dagger} = \hat{S}(-\omega)$$

THE INT. HAM. IN INT PICT

$$\hat{H}^{I}_{int}(t) = \sum_i \gamma_i \sum_\omega e^{-i\omega t} \hat{S}_i(\omega) \otimes \hat{E}_i(t)$$

GOING BACK TO THE EQ. FOR $\hat{\rho}^{RED,I}_S(t)$

$$-\frac{\hbar^2}{\alpha^2} \frac{d\hat{\rho}^{RED,I}_S(t)}{dt} =$$

$$\sum_{ij} \sum_{\omega\omega'} \Gamma_{ij}(\omega)\, e^{i(\omega-\omega')t}$$

$$\Big(\hat{S}_i^{\dagger}(\omega)\, S_j(\omega')\, \hat{\rho}^{RED,I}_S(t) - \hat{S}_i(\omega)\, \hat{\rho}^{RED,I}_S(t)\, \hat{S}_j^{\dagger}(\omega')$$

$$+ \hat{\rho}^{RED,I}_S(t)\, \hat{S}_i^{\dagger}(\omega)\, \hat{S}_j(\omega') - \hat{S}_i(\omega')\, \hat{\rho}^{RED,I}_S(t)\, \hat{S}_j^{\dagger}(\omega) \Big)$$

ASSUMPTION $\longrightarrow$ $\omega = \omega'$ only.

WE HAVE TAKEN $t \to \infty$ IN LIMIT OF INT

RE-ARRANGING

$$\frac{d\rho_S^{RED.I}(t)}{dt} = \frac{-i}{\hbar} \sum_\omega \sum_{ij} \pi_{ij}(\omega) \left[\underbrace{\hat{S}_i^\dagger(\omega) \hat{S}_j(\omega)}_{\text{SHIFT}}, \hat{\rho}_S^{RED.I}(t) \right]$$

$$+ \sum_\omega \sum_{ij} \gamma_{ij}(\omega) \left(\hat{S}_j(\omega) \hat{\rho}_S^{RED.I}(t) \hat{S}_i^\dagger(\omega) \right.$$

$$\left. - \frac{1}{2} \left\{ \hat{S}_i^\dagger(\omega) \hat{S}_j(\omega), \hat{\rho}_S^{RED.I}(t) \right\} \right)$$

BACK TO SCHRÖD. PICTURE 1ST TERM

ADD

$\hat{H}_S$ TO COMM.