

Quantum Chaos and RMT

The BGS conjecture

Lorenzo Villassero

Universite Paris Saclay

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Definition of Classical Chaos

In Classical Mechanics (Hamiltonian) chaos is defined by:

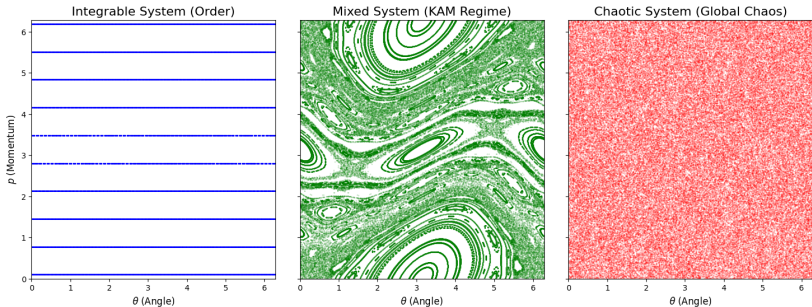
- ▶ Long term aperiodic behaviour
- ▶ Extreme sensitivity to initial conditions

Lyapunov exponent quantify the rate of separation of infinitesimally close trajectories in the system's phase space

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)} \quad (1)$$

In classical Mechanics we can have:

- ▶ Integrable system: N dof \Rightarrow N independent constant of motion in involution
- ▶ Mixed System: Chaotic or regular behaviour depending on the initial conditions
- ▶ Chaotic System: The only conserved quantity is the Energy (Hamiltonian), and trajectories fill the energy surface ergodically



Definition of Quantum Chaos

In Quantum Mechanics the concept of chaos is more subtle due to:

- ▶ The linearity of the Schrödinger equation: prevent the exponential divergence of nearby states.
- ▶ The uncertainty principle: no precise trajectories in phase space.

We can identify signatures of chaos in quantum systems by studying:

- ▶ Energy level statistics: chaotic quantum systems often exhibit level repulsion, where energy levels avoid crossing each other.
- ▶ Eigenstate properties: chaotic systems tend to have eigenstates that are delocalized.
- ▶ behaviour of quantum observables: the Loschmidt echo or the out-of-time-ordered correlators (OTOCs)

Loschmidt echo:

$$M(t) = |\langle \psi(0) | e^{iH_1 t/\hbar} e^{-iH_2 t/\hbar} | \psi(0) \rangle|^2 \quad (2)$$

OTOCs

$$C(t) = -\langle [W(t), V(0)]^2 \rangle \quad (3)$$

Conjectures

What is the relationship between classical and quantum chaos?

- ▶ **Berry-Tabor conjecture (77) [Berry and Tabor 1977]:** Quantum systems with classically integrable counterparts exhibit Poissonian energy level statistics.
- ▶ **Bohigas-Giannoni-Schmit conjecture (84) [Bohigas, Giannoni, and Schmit 1984]:** Quantum systems with classically chaotic counterparts exhibit energy level statistics that follow the predictions of random matrix theory (RMT).

Heavy nuclei and RMT

Derive the full spectrum for heavy nuclei is a complex problem



Wigner and Dyson (1950) , proposed to construct a statistical theory of energy levels.

The Hamiltonian describing the nuclei for an energy range significantly above the ground state should not differ significantly from an ensemble of random matrices.

Provided that:

- ▶ The quantum evolution should be unitary, implying that the corresponding Hamiltonian has to be hermitian
- ▶ All symmetries should be taken into account, and especially the symmetry under time reversal
- ▶ Beyond these symmetries, no information is contained in the ensemble

The details of the matrices distribution law would not matter much.

Examples of ensembles:

- ▶ Gaussian Orthogonal Ensemble (GOE): time-reversal invariant systems with integer spin
- ▶ Gaussian Unitary Ensemble (GUE): systems without time-reversal symmetry
- ▶ Gaussian Symplectic Ensemble (GSE): time-reversal invariant systems with half-integer spin

The joint eigenvalue probability density is given by:

$$P_{\beta}(\{\lambda_j\}) \propto \prod_{j < k} |\lambda_j - \lambda_k|^{\beta} e^{-\frac{\beta}{2} \sum \lambda_i^2}$$

GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$)

Spectral Statistics

Percival [Percival 1973] Spectral statistics serve as a fundamental criterion to distinguish dynamics.

Integrable Systems

- ▶ Constants of motion exist.
- ▶ Eigenstates associated with invariant tori (non-overlapping).
- ▶ **Berry & Tabor:** Energy levels are independent (uncorrelated).

Poisson Distribution

No repulsion.

$$P(s) = e^{-s}$$

Chaotic Systems

- ▶ Classical dynamics are ergodic and strongly mixing [Marklof 2001].
- ▶ Strong correlations between eigenstates.

Level Repulsion

Probability vanishes as $s \rightarrow 0$.

$$P(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow 0$$

The Breakthrough

Analyzed chaotic systems (e.g., *Sinai billiard*) and demonstrated that spectral fluctuations are fully described by **Random Matrix Theory (RMT)**.

This implies not only *level repulsion* but also long-range **spectral rigidity**.

$$P(s) = Cs^\beta e^{-as^2}$$

- ▶ $\beta = 1$: GOE (Time-reversal invariant)
- ▶ $\beta = 2$: GUE (Broken time-reversal)

Paradigm Shift: Universality of RMT arises strictly from underlying **classical chaos**, not merely system complexity. [Ullmo 2016]

Energy Spacings

Consider a sequence of ordered energy levels $E_1 \leq E_2 \leq \dots$ within an interval ΔE .

Definitions

- ▶ **Splittings:** The gap between consecutive levels:

$$S_n = E_{n+1} - E_n$$

- ▶ **Mean Spacing:** The average splitting over the interval:

$$D = \langle S_n \rangle$$

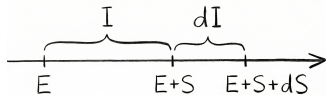
- ▶ **Normalized Variable:** We study the relative quantity:

$$s_n = \frac{S_n}{D}$$

Goal: Find the probability density function $p(s)$ for the normalized spacings. [Cugliandolo 2023]

Deriving the Probability Density I

Let $I = (E, E+S)$ and $p(S)dS$ be the probability of having the next level in $dI = (E + S, E + S + dS)$.



$$p(S)dS = \text{Prob}(\text{level in } dI \mid \text{no level in } I) \cdot \text{Prob}(\text{no level in } I)$$

One has:

$$\text{Prob}(\text{no level in } I) = \int_S^{S_{\max}} p(S')dS'$$

Defining the conditional probability $\mu(S)dS$ as:

$$\mu(S)dS \equiv \text{Prob}(\text{level in } dI \mid \text{no level in } I)$$

Solving the resulting differential equation yields the **general form**:

Deriving the Probability Density II

$$p(S) = C \cdot \mu(S) \cdot \exp\left(-\int_0^S \mu(S') dS'\right)$$

The distribution depends entirely on our assumption for $\mu(S)$.

Poisson vs. Wigner: The Choice of $\mu(S)$

1. Independent Levels If levels are uncorrelated, the probability of finding a level does not depend on the other interval.

$$\mu(S) = c$$

Substituting into the general form:

$$p(S) \propto c \cdot e^{-\int c dS'}$$

Result (Poisson):

$$p(s) = e^{-s}$$

No level repulsion.

2. Level Repulsion If levels repel, the probability is proportional to the gap size (linear repulsion).

$$\mu(S) \propto S$$

Substituting into the general form:

$$p(S) \propto S \cdot e^{-\int S dS'}$$

Result (Wigner-Dyson):

$$p(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}$$

Strong level repulsion.

Numerical Simulation

Our problems amount to solve the time independent Schrodinger equation on different 2-D geometries. [Cagliero and Rahmouni 2019]

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y) \quad (4)$$

where:

$$V(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \Omega \\ \infty & \text{if } (x,y) \in \partial\Omega \end{cases}$$

and

$$\psi(x,y) = 0 \quad \text{if } (x,y) \in \partial\Omega \quad (5)$$

Formulation

1. Background & Contrast Source

Assuming a constant background potential $V = V_0$, we define the wavenumber k_0 and the contrast source $J(\vec{r})$:

$$k_0^2 = \frac{2m(E - V_0)}{\hbar^2} \qquad J(\vec{r}) = \underbrace{\frac{2m}{\hbar^2}(V_0 - V(\vec{r}))}_{\chi(\vec{r})} \psi(\vec{r}) \qquad (6)$$

2. Integral Representation

We recast the original problem into:

$$\nabla^2 \psi + k_0^2 \psi = -J(\vec{r})$$

This is the Helmholtz equation, whose solution is:

$$\psi(\vec{r}) = \int_{\Omega} G_0(\vec{r} - \vec{r}') J(\vec{r}') dr' = S[J] \qquad (7)$$

3. Operator Forms

Defining the scattering operator $Sf(\vec{r}) = \int_{\Omega} G_0(\vec{r} - \vec{r}')f(\vec{r}')dr'$, we obtain:

State Equation:

$$(I - S\chi)\psi = 0$$

Source Equation:

$$\boxed{(I - \chi S)J = 0} \quad (8)$$

Now we have to discretize this integral equation over Ω and find the value of E such that:

$$\det(I - \chi S) = 0 \quad (9)$$

once this is done we can compute J and invert $J = \chi\psi$ to obtain the wavefunction over the computational domain.

Setup

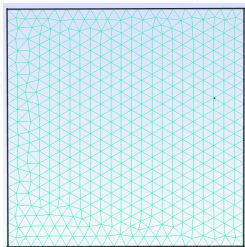


Figure 1: Mesh used for the square billiard

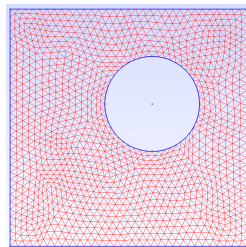


Figure 2: Mesh used for the Sinai billiard



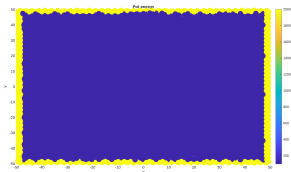


Figure 3: Potential used for the square billiard

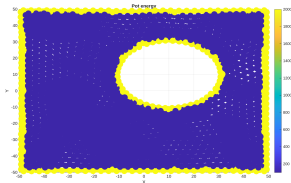


Figure 4: Potential used for the Sinai billiard

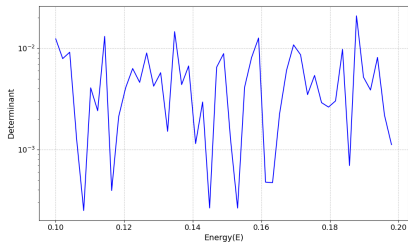


Figure 5: Determinant scan for the square billiard

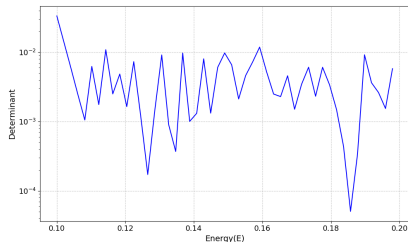


Figure 6: Determinant scan the Sinai billiard

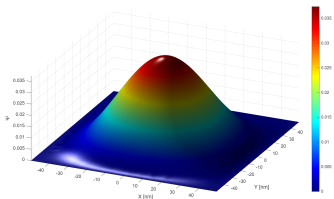


Figure 7: Gs for the square billiard

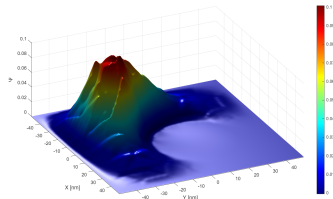


Figure 8: Gs the Sinai billiard

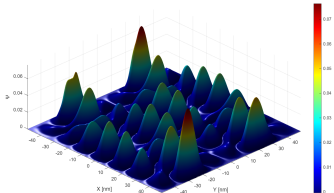


Figure 9: Exprencited level for the square billiard

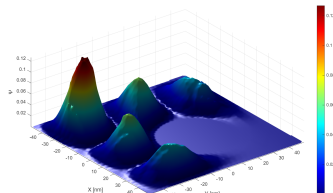


Figure 10: Exprencited level for the Sinai billiard

Spectrum Unfolding

- ▶ **Goal:** Study the *local properties* of the spectrum by separating them from global variations.
- ▶ Remove the spurious effects due to variations of the density


The Unfolding Procedure

Given a set of eigenvalues $\{x_1, x_2, \dots, x_n\}$, we map them to a new set $\{y_1, y_2, \dots, y_n\}$ such that the new sequence has:

- ▶ The same fluctuation properties.
- ▶ A mean density equal to **one**.¹

$$y_i = \langle N(x_i) \rangle \quad (10)$$

$\langle N(x_i) \rangle$: smoothed cumulative density

¹If the energy unit is the mean spacing $\bar{x} = \bar{s}$, the mean spacing between consecutive levels in the unfolded spectrum becomes **1**. 

Spacing Distribution

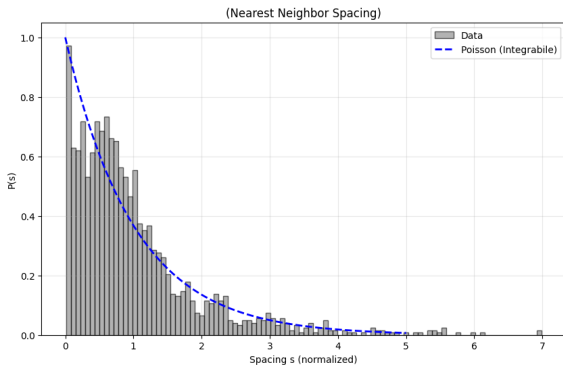


Figure 11: Spacing distribution for the unfolded spectrum of the square billiard

Correlation between spacings $C \approx 0$

Spacing Distribution

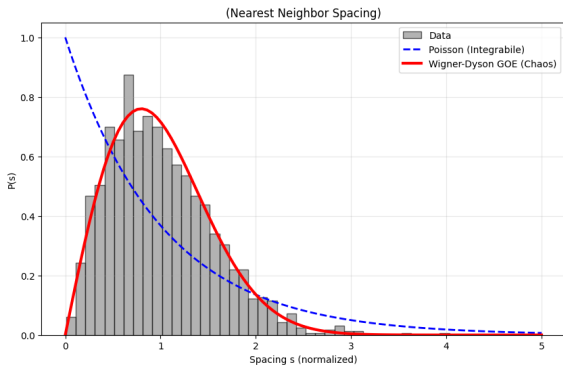


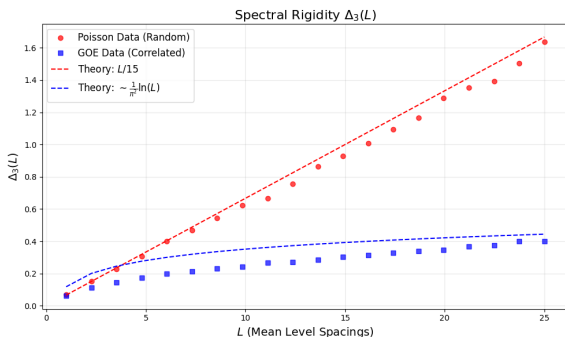
Figure 12: Spacing distribution for the unfolded spectrum of the Sinai billiard

Correlation between spacings $C \approx -0.2911$

Dyson-Metha statistics

A measure of spectrum rigidity is given by [Bohigas, Giannoni, and Schmit 1984]

$$\Delta_3(L) = \frac{1}{L} \min_{A,B} \int_{\alpha}^{\alpha+L} \left[N(E) - (AE + B) \right]^2 dE \quad (11)$$



Conclusion

We have discussed the success of the BGS conjecture; however, it provides only a partial description of Quantum Chaos.

- ▶ **A Binary Dichotomy:** The BGS conjecture suggests a sharp distinction between integrable (Poisson) and chaotic (RMT) systems.
- ▶ **Spectral Limitation:** While it successfully predicts energy levels statistics, it **fails to describe the eigenvectors** structure.

Eigenvectors I

Origin of the Structure

- ▶ The GOE Hamiltonian is invariant under orthogonal rotations ($H \rightarrow O^T H O$).
- ▶ This implies that there is **no preferred basis** in the Hilbert space.
- ▶ The eigenvectors must be uniformly distributed over the N -dimensional "hypersphere" defined by normalization ($\sum |c_i|^2 = 1$). They are "purely random" vectors.

Eigenvectors II

For a large system ($N \rightarrow \infty$), the component c_i of an eigenvector $\psi = \sum c_i |i\rangle$ in a generic basis follows a Gaussian distribution :

$$P(c) = \sqrt{\frac{N}{2\pi}} \exp\left(-\frac{N}{2}c^2\right)$$

(The distribution for the probabilities $y = |c|^2$ is often used, known as the **Porter-Thomas distribution**: $P(y) \sim y^{-1/2} e^{-Ny/2}$).





Summary: The Limits of the BGS Conjecture

Real physical eigenvectors possess **more structure** than pure random noise.




- ▶ **Global Structure Issue (Thermodynamics):** Pure RMT vectors are "too random" and statistically identical at all energies. They fail to encode the energy dependence ($T(E)$) necessary for thermalization (**Need for ETH**).
- ▶ **Local Structure Issue (Dynamics):** RMT vectors are uniformly distributed. They fail to capture the concentration of probability along classical periodic orbits, known as **Quantum Scars**.

Conclusion: Chaos makes the spectrum universal (BGS), but the eigenvectors retain specific physical information.

References I

-  Berry, M. V. and M. Tabor (1977). “Level clustering in the regular spectrum”. In: *Proc. R. Soc. Lond. A* 356, pp. 375–394.
-  Bohigas, O., M. J. Giannoni, and C. Schmit (1984). “Spectral properties of the Laplacian and random matrix theories”. In: *J. Phys. Lett.* 45, pp. 1015–1022.
-  Cagliero, Andrea and Lyes Rahmouni (2019). *Symmetric Galerkin boundary element method for computing the quantum states of the electron in a piecewise-uniform mesoscopic system*. arXiv: 1909.06596 [physics.comp-ph]. URL: <https://arxiv.org/abs/1909.06596>.
-  Cugliandolo, Leticia F. (2023). *Advanced Statistical Physics: Random matrices*. Lecture Notes, Sorbonne Université / LPTHE.

References II

-  Marklof, Jens (2001). “The Berry-Tabor Conjecture”. In: *European Congress of Mathematics*. Ed. by C. Casacuberta et al. Vol. 202. Progress in Mathematics. Birkhäuser, Basel. DOI: 10.1007/978-3-0348-8266-8_36.
-  Percival, I. C. (1973). “Regular and irregular spectra”. In: *J. Phys. B: Atom. Mol. Phys.* 6, p. L229. DOI: 10.1088/0022-3700/6/9/002.
-  Ullmo, Denis (2016). “Bohigas-Giannoni-Schmit conjecture”. In: *Scholarpedia* 11.9, p. 31721. URL: http://www.scholarpedia.org/article/Bohigas-Giannoni-Schmit_conjecture.