The Quantum TAP equations for the quantum p-spin glass

A visual introduction to the paper

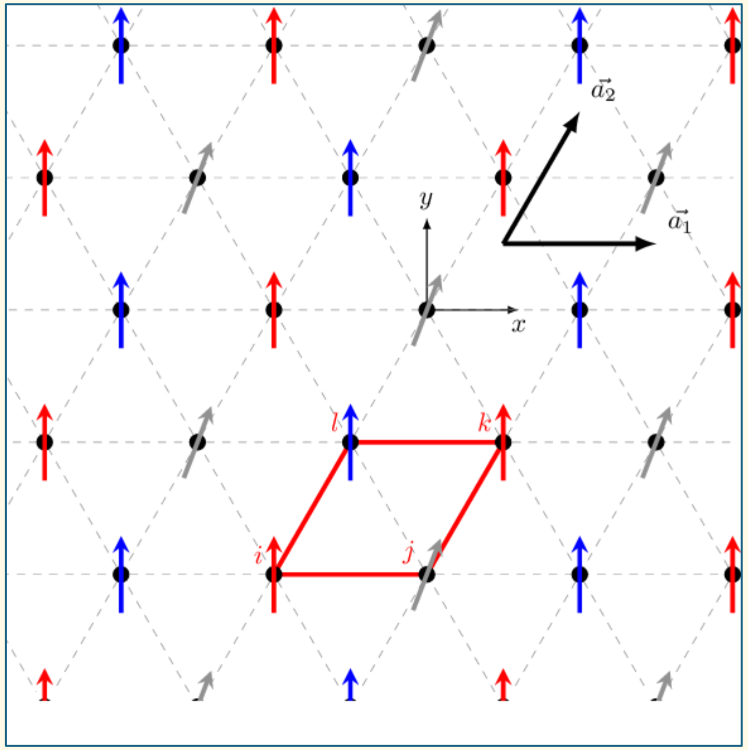
Quantum Thouless-Anderson-Palmer equations for glassy systems
G. Biroli and L. F. Cugliandolo, Phys. Rev. B 64, 014206 (2001)

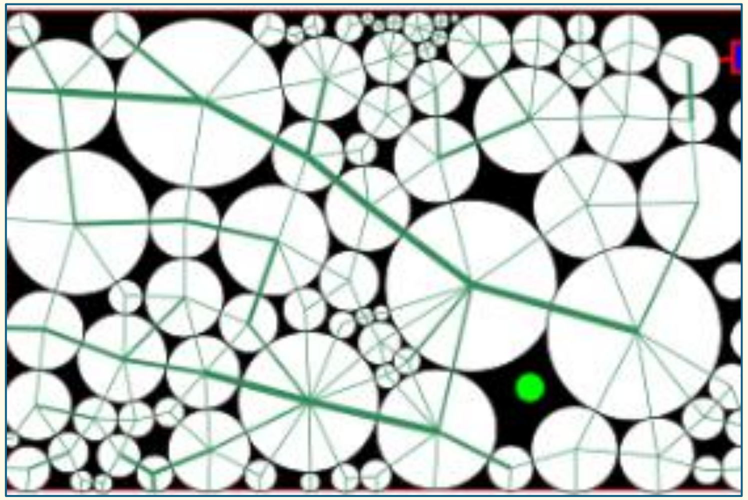
When is a system *glassy*?

- Disorder is not always glassy!
- Glassy is not always disordered!

- Experimentally: extremely slow, out-of-equilibrium dynamics below Tc («aging»)

- Theoretically: a non trivial complexity (exponential number of stationary points for the energy)





The classical TAP equations

- Usually, the free energy is written in terms of the fields. But local magnetizations are better!
- This is done via the Legendre transform

$$\mathrm{F}\left(\beta,\left\{J_{i_1\ldots i_p},m_i\right\}\right)=\max_{\{h_i\}}\left[\sum\nolimits_i h_i m_i-\mathrm{F}\left(\beta,\left\{J_{i_1\ldots i_p},h_i\right\}\right)\right]$$

- The TAP equations are the Legendre conditions of stationarity

$$\frac{-\partial\beta\mathrm{F}\left(\beta,\left\{J_{i_1\ldots i_p},m_i\right\}\right)}{\partial m_i}=h_i$$

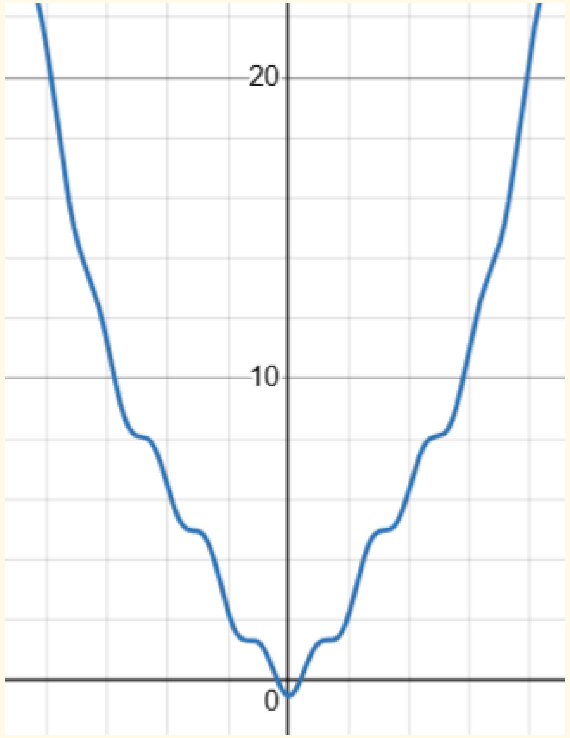
- The solutions $\{m_i^a\}$ of the TAP equations are stationary points of the TAP free energy for $h_i=0$!
- If $\{m_i^a\}$ are stable ⇒ “pure states” in which Z decomposes [1]:

$$Z=\sum\nolimits_a \exp -\beta\mathrm{F}(\beta,\{m_i^a\})$$

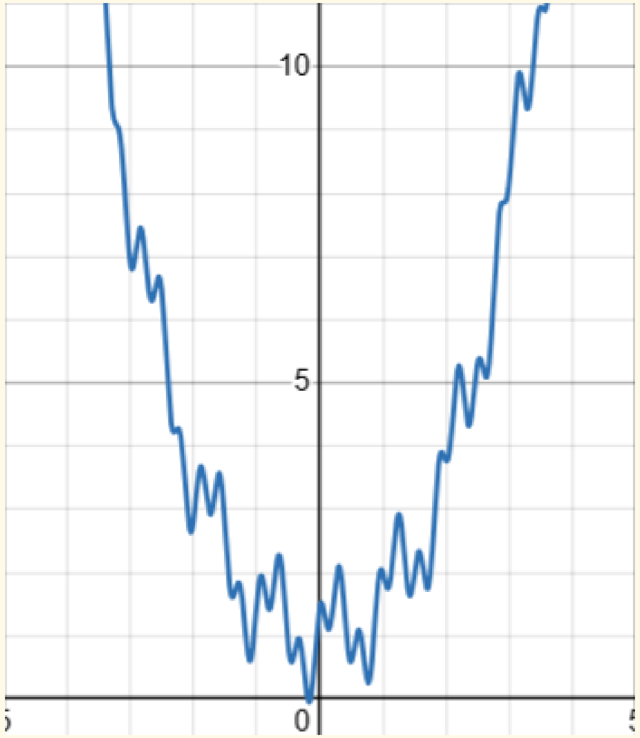
- By saddle point, the relevant (“typical”) configurations minimize βf −σ(β,f) with σ(β,f) an entropy of configurations with fixed TAP free energy.

[1] C. de Dominicis and A. P. Young, J. Phys. A16, 2063 (1983)

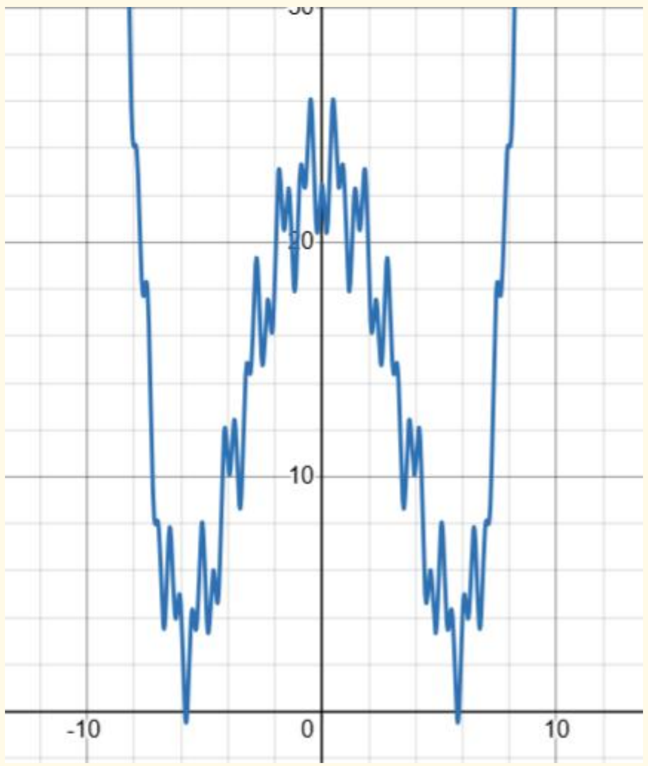
The TAP landscape



- $T_d < T$: a single stable minimum, the paramagnetic TAP solution $m_i = 0$



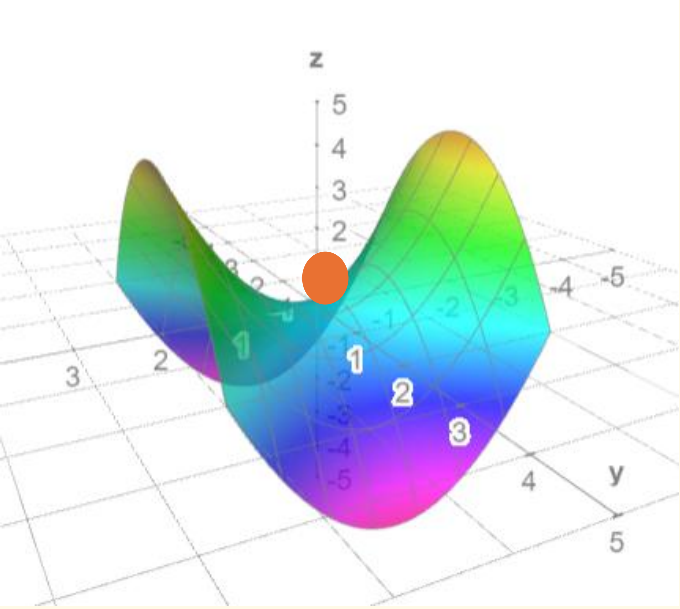
- $T_s < T < T_d$: exponentially many metastable states of comparable energy.

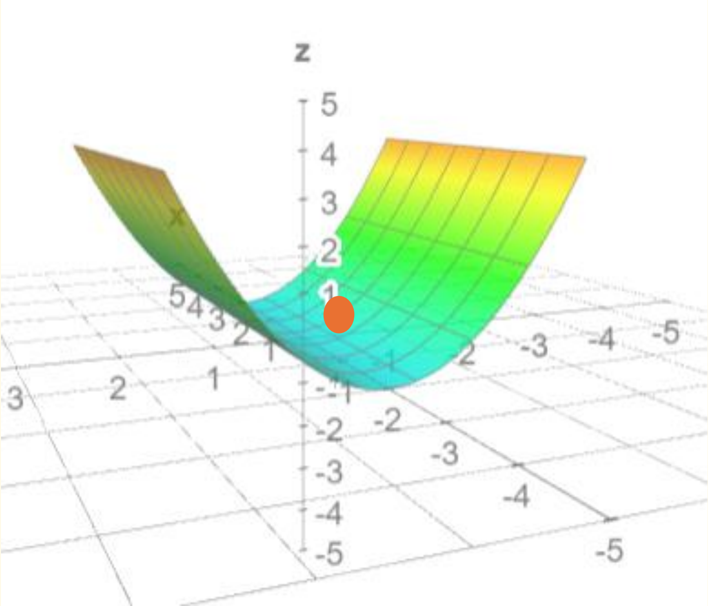


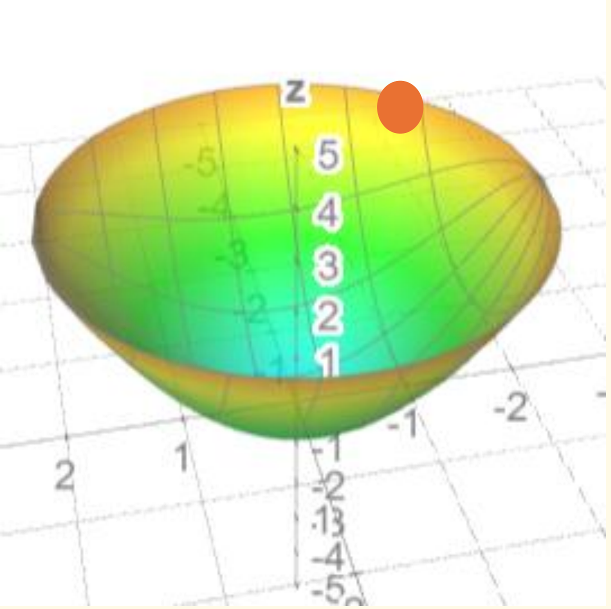
- $T < T_s$: the complexity of the TAP solutions is 0.

  Only few minima are relevant.

Dynamics in saddles, gutters, and valleys







- Below T_d, the out of equilibrium dynamics is dominated by the threshold states.
- Marginality is the cause of aging: gutters allow for diffusion in flat directions, with no relaxation or a characteristic time scale.

Why going quantum? From 2° to 1° order

- How do quantum fluctuations modify the *real time* dynamics?
- Experimental evidence: many quantum glassy systems (e.g $LiHO_xY_{1-x}F_4$) display a QPT from a paramagnetic to a glassy state near T=0. [1][2][3]
- This transition is first order!

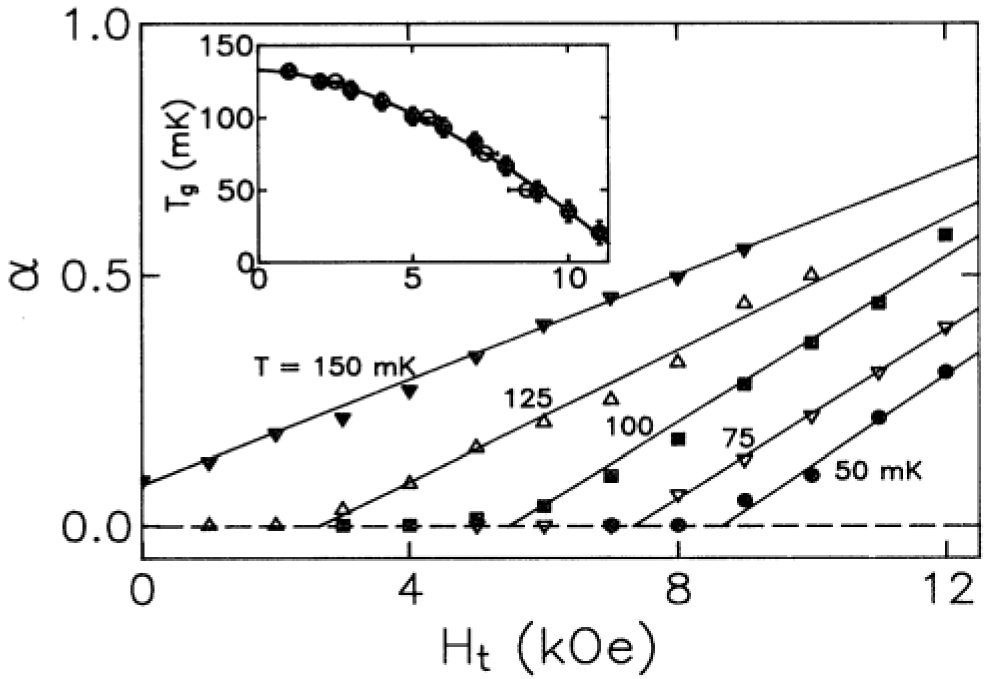


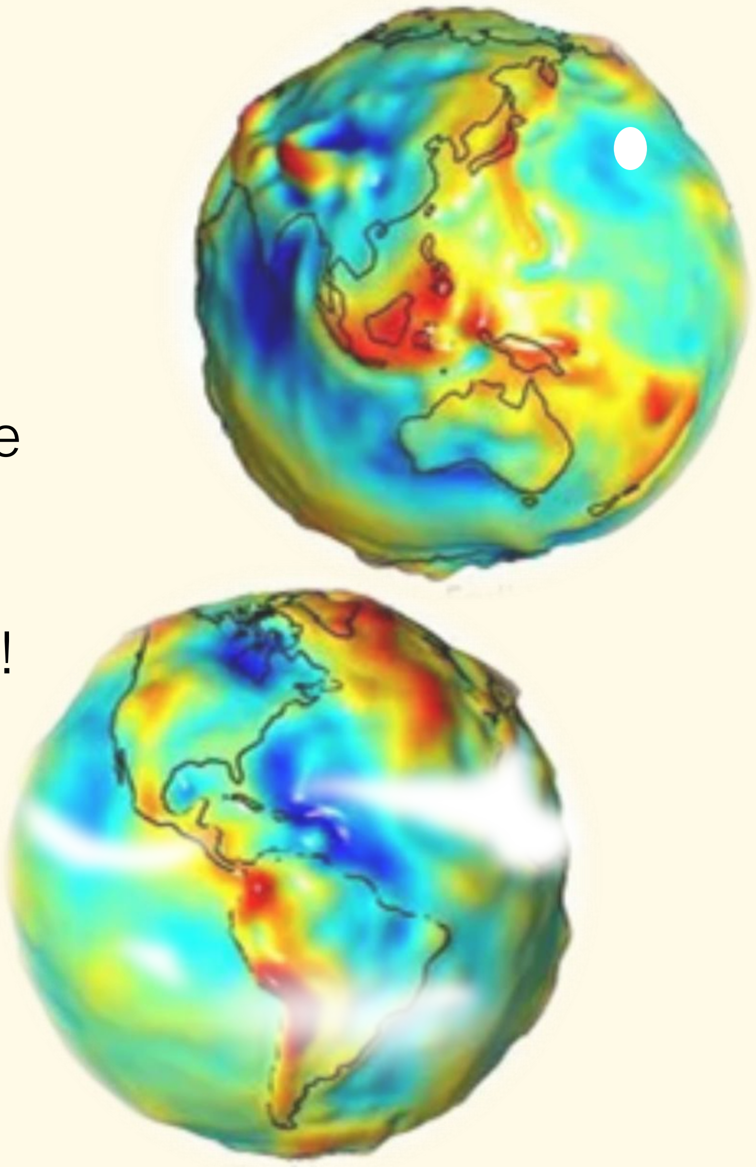
FIG. 5. The power in the form f^α fitted to the $f \to 0$ limit of $\chi''(f)$ vs transverse field at five different temperatures. We define spin-glass freezing when $\alpha \to 0$. Inset: Depression with H_t of the spin-glass transition temperature T_g so determined. Solid line is a least-squares fit with $T_g(0) - T_g(H_t) \sim H_t^{1.7 \pm 0.1}$ (see text).

[1] W. Wu, B. Ellmann, T. F. Rosenbaum, G. Aeppli and D. H. Reich, Phys. Rev. Lett. 67, 2076 (1991).
[2] W. Wu, D. Bitko, T. F. Rosenbaum and G. Aeppli, Phys. Rev. Lett. 71, 1919 (1993).
[3] T. F. Rosenbaum, J. Phys. C8, 9759 (1996). J. Brooke al, Science 284, 779 (1999).

The quantum p-spin model

$$\mathrm{H} = \sum_i \frac{p_i^2}{2m} + \sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p} s_{i_1} \ldots s_{i_p} \quad \text{with } \left[s_i, p_j\right] = -i\hbar\delta_{ij}$$

- The spherical p-spin model can be interpreted as a particle moving on a rugged hypersphere [1].
- In a classical system, the noise is external. To cross an energy barrier, you must be supplied energy from the bath!
- Instead, in quantum systems the Hamiltonian itself forces motion in the Hilbert space.
- $\Gamma = \frac{J\hbar^2}{2M}$ is the strength of quantum fluctuations.



[1] L. F. Cugliandolo, D. R. Grempel and C. A. da Silva Santos, Phys. Rev. Lett. (2000)

Imaginary time (and polymer rings)

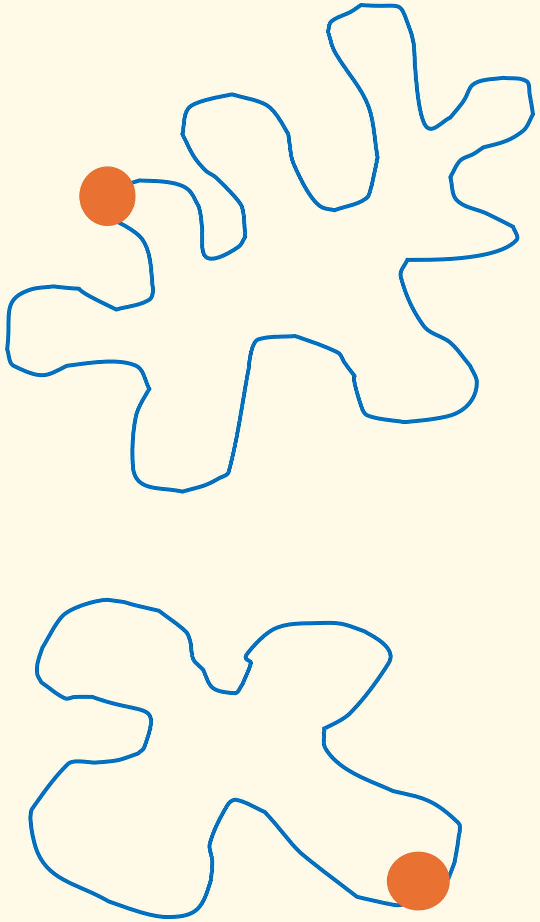
Recipe to build the TAP landscape:

- We map the quantum mechanics problem in D dimensions to a statistical mechanics problem in D + 1:

$$Z = Tr\, e^{-\beta H} = \int \mathcal{D}s\, e^{-\frac{i}{\hbar}S[s]} \qquad \text{with } s(0) = s(\beta\hbar)$$

- TAP states become time dependent objects $s(\tau)$.

- This can be mapped to a polymer in a disordered N-dimensional medium, with $N \to \infty$[1].



[1] Z. Ovadyahu, A. Vaknin and M. Pollack, Phys. Rev. Lett. 84, 3402 (2000)

Solving the Quantum TAP Equations

- Quantum TAP equations are self-consistency conditions for local trajectories

$$m_i(\tau) =< \sigma_i(\tau) >$$

- Quantum fluctuations appear as retarded self-interactions: each quantum state acts like its own bath!

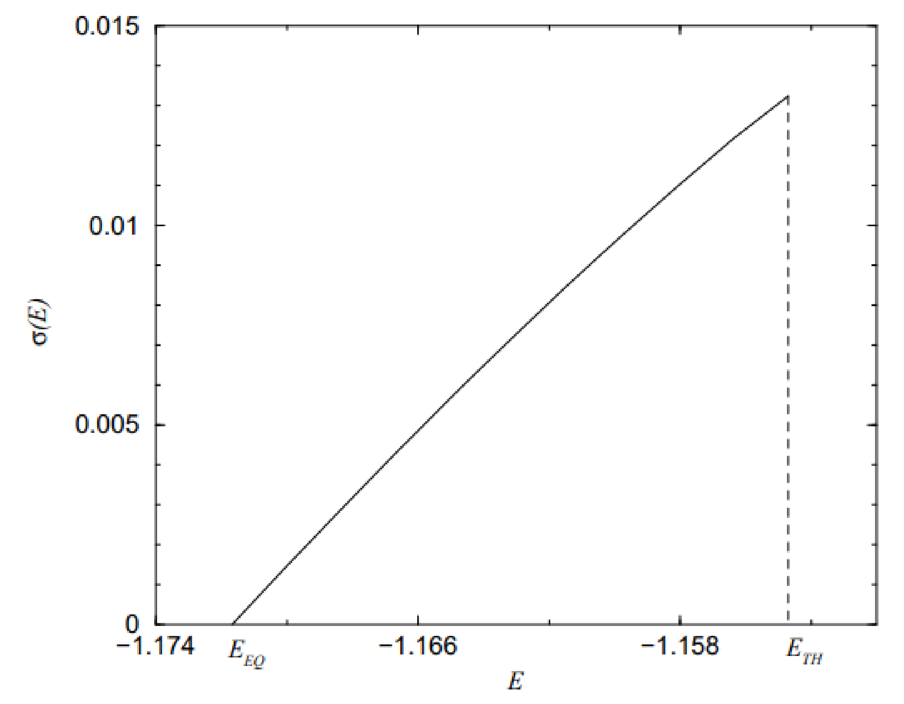
- Interactions with the rest of the system are encoded in an effective dynamical field, the Weiss function

$$G(\tau,\tau') \propto p\, C^{p-1}(\tau,\tau')$$

$$C(\tau,\tau') = \frac{1}{N}\sum_i < \sigma_i(\tau)\sigma_i(\tau') >$$

- Non-local in imaginary time: contains the memory

$$\mathcal{E}(\sigma) \equiv -\frac{1}{N}\sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p}\,\sigma_{i_1}\ldots\sigma_{i_p}$$

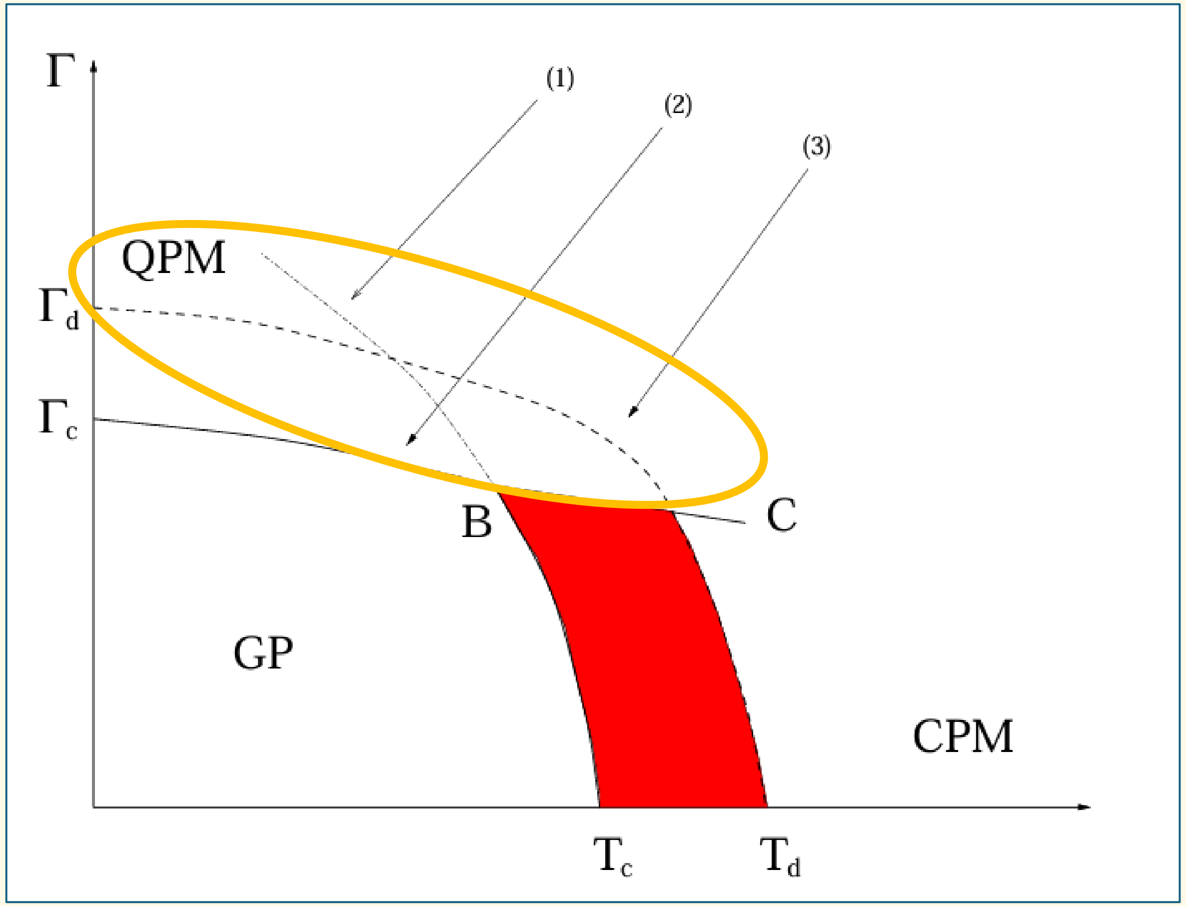


Complexity as a function of the potential energy.

The phase diagram of quantum p-spin

Solving the TAP equations, we find different complexities in the (Γ, T) plane.

- At zero temperature and low Γ the system is in the glassy phase (GP).
- At high Γ quantum fluctuations destroy the glassy phase and the system is a quantum paramagnet.
- At T=0, the jump in the EA parameter is not absorbed by the vanishing complexity, and the transition must be first order.



Conclusions

- The TAP analysis generalizes to quantum systems via the introduction of a time-dependent states.
- Quantum fluctuations depress the transition temperatures T_c, T_s until they reach 0 and a Quantum Phase Transition (QPT) occurs.
- In doing so, the order of the phase transition goes from 2nd to 1st.

References

1. 29 pag G. Biroli and L. F. Cugliandolo, *Quantum Thouless-Anderson-Palmer equations for glassy system*s, Phys. Rev. B 64, 014206 (2001) .
2. 49 pag L. F. Cugliandolo, D. R. Grempel, C. A. da Silva Santos*, Imaginary-time replica formalism study of a quantum spherical p-spin-glass mode*l, Phys. Rev. B 64, 014403 (2001).
3. 4 pag L. F. Cugliandolo, D. R. Grempel, C. A. da Silva Santos, *From second to first order transitions in a disordered quantum magnet*, Phys. Rev. Lett. 85, 2589 (2000)
4. 56 pag L. F. Cugliandolo and G. S. Lozano, *Real-time non-equilibrium dynamics of quantum glassy systems*, Phys. Rev. B 59, 915 (1998).
5. 4 pag L. F. Cugliandolo and G. S. Lozano*, Quantum aging in mean-field models*, Phys. Rev. Lett. 80, 4979 (1997)

The end. Thank you!

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