QUANTUM BROWNIAN MOTION

Impurity dynamics in 1D quantum liquids

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Outline

Introduction

Quantum Brownian Motion

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QBM is a standard framework in open quantum systems for modelling dissipation and fluctuations.

Multiple approaches to QBM present varying issues

Reduced density matrix calculation

- Functional integral approach $\rightarrow$ Generic initial conditions, but less than simple
- Master equation $\rightarrow$ Only exact equations for factorising and thermalised initial conditions
- Quantum Langevin equation $\rightarrow$ Only special initial conditions can be successfully treated

Generating Functionals Approach

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Non-equilibrium correlation function calculation

Generating functionals $\rightarrow$ Generic non-factorising Gaussian initial conditions

This approach will be applied to impurity dynamics in a 1D quantum fluid.

QBM Model

Hamiltonian

The system (particle + bath) is described by $\hat{\mathcal{H}} = \hat{\mathcal{H}}_S + \hat{\mathcal{H}}_B + \hat{\mathcal{H}}_{SB}$.

$$\hat{\mathcal{H}}_S[\hat{q}, \hat{p}] = \frac{\hat{p}^2}{2M} + V(\hat{q}; t) - H(t)\hat{q} \tag{1}$$

$$\hat{\mathcal{H}}_B[\{\hat{x}_n, \hat{p}_n\}] = \sum_{n=1}^{\infty} \frac{\hat{p}_n^2}{2m_n} + \frac{m_n\omega_n^2}{2}\hat{x}_n^2 \tag{2}$$

$$\hat{\mathcal{H}}_{SB}[\{\hat{x}_n\}, \hat{q}] = -\hat{q}\sum_{n=1}^{\infty} c_n\hat{x}_n + \hat{q}^2 \sum_{n=1}^{\infty} \frac{c_n^2}{2m_n\omega_n^2} \tag{3}$$

Notes

- Bilinear coupling of bath oscillators and particle positions
- Renormalisation term in (3) ensures that $V(\hat{q}; t)$ is the effective potential

Initial density matrix

Initial conditions are encoded in the initial density matrix $\hat{\rho}_0$.

- Equilibrium dynamics $\rightarrow \hat{\rho}_0 = \exp(-\beta\hat{\mathcal{H}})$
- Separate but isothermal equilibrium $\rightarrow \hat{\rho}_0 = e^{-\beta\hat{\mathcal{H}}_S} \otimes e^{-\beta\hat{\mathcal{H}}_B}$
- Non-equilibrium dynamics $\rightarrow \hat{\rho}_0 = \hat{\rho}_{S0} \otimes \hat{\rho}_{B0}$
  Unrealistic for most experimental setups.

We consider initial conditions of a pure state particle $\hat{\rho}_{S0} = |\psi\rangle\langle\phi|$, with $\langle q|\phi\rangle = \exp\left(-\frac{(q-q_m)^2}{4\Delta^2}\right)$ and $\langle q|\psi\rangle = \exp\left(-\frac{(q'-q'_m)^2}{4\Delta^2}\right)$

Spectral density

The spectral density contains information on the frequencies of the modes and their coupling to the particle:

$$S(\omega) = \pi \sum_{i=1}^{N} \frac{c_i^2}{2m_i\omega_i^2}\delta(\omega - \omega_i)$$

.

For $N \to \infty$ oscillators, $S(\omega)$ is a continuous function.

Ohmic vs Super(sub)-Ohmic dissipation

- $S(\omega) \sim \omega \to$ Ohmic dissipation
- $S(\omega) \sim \omega^{\alpha}\,, \alpha > 1 (\alpha < 1) \to$ Super(sub)-Ohmic dissipation

Generating functional

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The derivation of the generating functional will not be demonstrated.

A functional $\mathcal{J}[F, G]$ is constructed, allowing for easy calculations of non-equilibrium correlators:

$$\langle \hat{q}(t)\hat{q}(t')\rangle = \frac{\hbar}{i}\frac{\delta}{\delta G(t)}\frac{\hbar}{i}\left[\frac{\delta}{\delta G(t')} + \frac{\delta}{2\delta F(t')}\right] e^{\mathcal{J}[F,G]}\Big|_{F,G=0}$$

Impurity dynamics

Impurity dynamics

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Aim: Apply the generating functional toolkit to impurity dynamics in a Luttinger liquid.

Impurity-LL system experimentally realised by Catani et al. [2].

Experimental realisation

- Impurity is initially trapped at the centre of a 1D harmonic potential with the LL
- Impurity is released once the whole system equilibrates
- Position variance of impurity shows damped oscillations $\rightarrow$ effective quantum Brownian particle in a quantum heat bath

Luttinger liquid

The system (impurity + LL) is described by:

$$\hat{\mathcal{H}}_S = \frac{\hat{p}^2}{2M} + \frac{M\omega^2}{2}\hat{q}^2 \tag{4}$$

$$\hat{\mathcal{H}}_B = \frac{\hbar}{2\pi}\int \mathrm{d}x \left[\frac{uK_L}{\hbar^2}(\pi\hat{\Pi}(x))^2 + \frac{u}{K_L}(\nabla\hat{\phi}(x))^2\right] \tag{5}$$

$$\hat{\mathcal{H}}_{SB} = \int \mathrm{d}x\mathrm{d}y\ U(x-y)\hat{\rho}(y)\delta(x-\hat{q}) \tag{6}$$

In Fourier space:

$$\hat{\mathcal{H}}_{SB} = \frac{1}{\sqrt{L}}\sum_n \tilde{U}_{k_n}\ e^{ik_n\hat{q}}\left[-\frac{ik_n}{\pi}\tilde{\phi}(k_n)\right], \tag{7}$$

where L is the length of the confinement trap, and $k_n = n\pi/L$ are quantised wave numbers.

Luttinger liquid

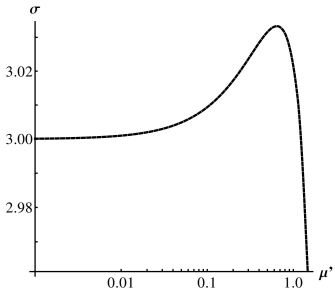
Many approximations and assumptions later, the generating functional approach of QBM may be applied in the impurity + LL system.

- Steep external potential → We can expand $e^{ik_n\hat{q}}$ up to second order in k (thus treatable as harmonic oscillators).
- Initial localisation of impurity is treated as an initial position measurement.
  Quenching potential viewpoint is also possible, but both are loosely equivalent in high temperature regime ($\hbar\beta\omega \ll 1$).

Signature of a Luttinger liquid

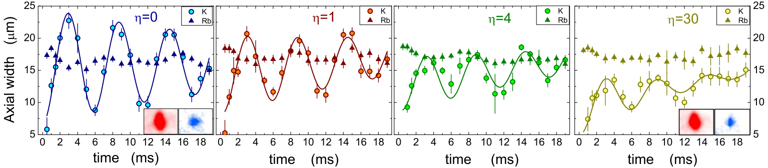
The Luttinger bath presents a spectral density of $S(\omega) \sim \omega^3$ for low $\omega \rightarrow$ Super-Ohmic dissipation.

- Frequency of position variance increases with increasing impurity-bath coupling strength
- Increase of frequency is followed by a steep decrease due to the bath
- Distinctly super-Ohmic behaviour

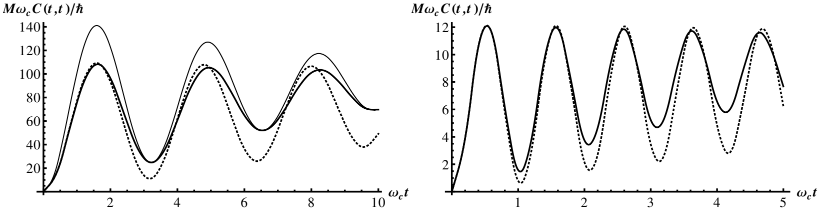


Frequency σ of equal-time correlation oscillations in function of the impurity-bath coupling μ'.[1]

Comparison to experimental results



Oscillations of the impurity's $\sqrt{C(t,t)}$ for different coupling strengths η.[2]



Theoretical results of the impurity position variance $C(t,t)$, for varying coupling strengths in the high-temperature regime.[1]

Conclusion

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- **Method:** generating functional $\rightarrow$ non-equilibrium correlations $C(t, t')$
- **Model:** QBM = impurity + bath
- **Mapping to QBM:** Luttinger liquid $\rightarrow$ effective harmonic bath
- **Result:** Luttinge bath is **super-Ohmic** ($S(\omega) \sim \omega^3$)
- **Observable:** impurity width $\langle x(t)^2 \rangle$
- **Limits:** works in low-energy regime; approximations matter

References

References I

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