Quantum Many-Body Scars:
Weak Ergodicity Breaking and Persistent Revivals

Understanding Non-Thermal Dynamics in Rydberg Atom Arrays

- ETH
- Exeptions to ETH
- What started it all?
- Why scars?
- How to make sense of scarred states
- Why should we care

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ETH

- The assumptions of **ergodicity** are well-motivated in **classical mechanics** as a result of dynamical chaos since a chaotic system will in general spend equal time in equal areas of its phase space. If we prepare an isolated, chaotic, classical system in some region of its phase space, then as the system is allowed to evolve in time, it will sample its entire phase space, subject only to a small number of conservation laws
- This argument cannot be straightforwardly extended to **quantum systems**, because time evolution of a quantum system does not uniformly sample all vectors in Hilbert space with a given energy.

$$\overline{A} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \langle \Psi(t) | \hat{A} | \Psi(t) \rangle \, dt$$

$$= \sum_{\alpha=1}^{D} |c_\alpha|^2 A_{\alpha\alpha} + i\hbar \lim_{\tau\to\infty} \left[\cancel{\sum_{\alpha\neq\beta}^{D} \frac{c_\alpha^* c_\beta A_{\alpha\beta}}{E_\beta - E_\alpha} \left(\frac{e^{-i(E_\beta - E_\alpha)\tau/\hbar} - 1}{\tau} \right)} \right]$$

ETH

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1. The diagonal matrix elements $A_{\alpha\alpha}$ vary smoothly as a function of energy, with the difference between neighboring values, $A_{\alpha+1,\alpha+1}$- $A_{\alpha\alpha}$, becoming exponentially small in the system size .
2. The off-diagonal matrix elements $A_{\alpha,\beta}$, with $\alpha \neq \beta$, are much smaller than the diagonal matrix elements, and in particular are themselves exponentially small in the system size.

$$\langle A\rangle_{\mathrm{mc}} = \frac{1}{\mathcal{N}} \sum_{\alpha'=1}^{\mathcal{N}} A_{\alpha'\alpha'} \approx \frac{1}{\mathcal{N}} \sum_{\alpha'=1}^{\mathcal{N}} A = A$$

$$\overline{A} = \sum_{\alpha=1}^{D} |c_\alpha|^2 A_{\alpha\alpha} \approx A \sum_{\alpha=1}^{D} |c_\alpha|^2 = A$$

ETH

The ETH is extended by demanding that if the global system is in an energy eigenstate, the entanglement entropy and the thermodynamic entropy for the subsystem A are equal

Notice that when the ETH holds, the **entanglement entropy** should scale as the **volume of the subsystem A** just like the microcanonical entropy .

ARE THERE EXCEPTIONS TO ETH?

Integrable Systems

dynamical system with sufficiently many conserved quantities, that its motion is confined to a submanifold of much smaller dimensionality than that of its phase space.

Many Body Localization

The system retains memory of the initial conditions

ARE THERE EXCEPTIONS TO ETH?

Integrable Systems

Quantum Scars

Many Body Localization

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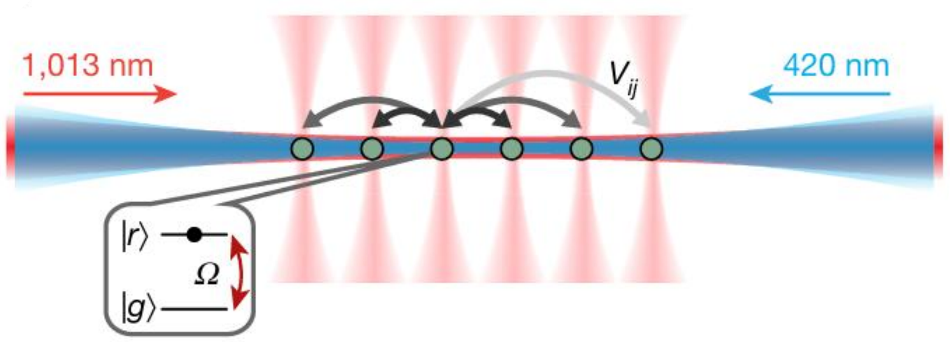
WHAT STARTED IT ALL

2018, an experiment on a new family of Rydberg-atom quantum simulators [1]



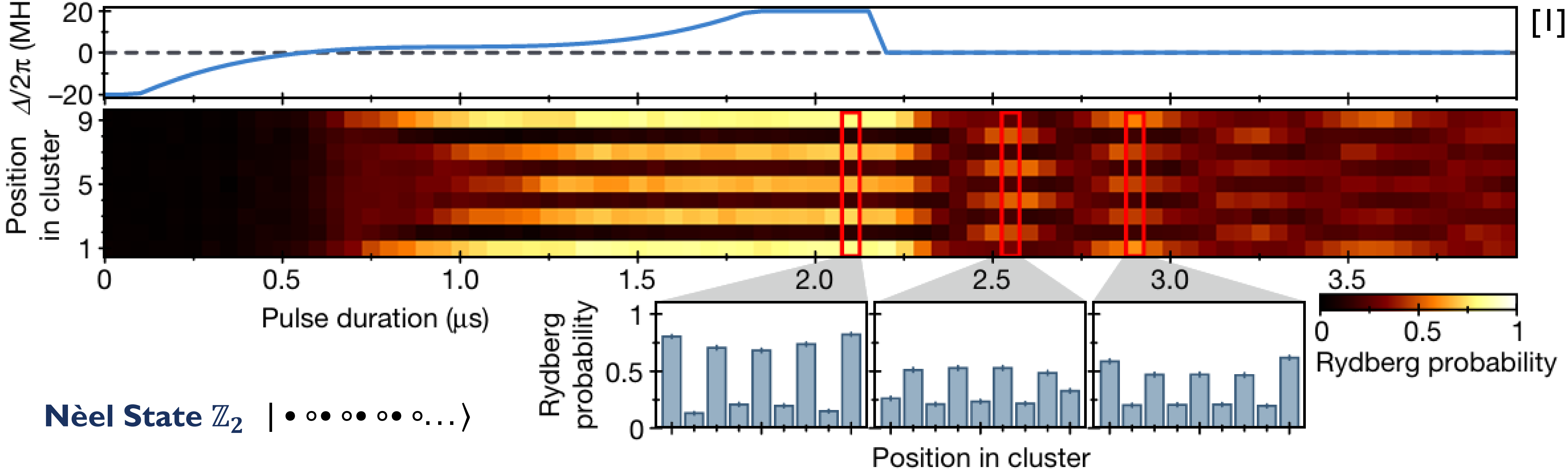
The strong, coherent interactions between Rydberg atoms provide an effective coherent constraint that prevents simultaneous excitation of nearby atoms into Rydberg states

Nèel State $\mathbb{Z}_2$ $|\bullet\circ\bullet\circ\bullet\circ\bullet\circ\ldots\rangle$

$$H_{prep} = \sum_i \left(\frac{\Omega}{2} \sigma_i^x - \Delta(t<0) n_i \right) + \sum_{i<j} V_{ij} n_i n_j \xrightarrow{\textbf{quench}} H_{post} = \sum_i \frac{\Omega}{2} \sigma_i^x + \sum_{i<j} V_{ij} n_i n_j$$

WHAT STARTED IT ALL

quench



[1]

Nèel State $\mathbb{Z}_2$ $|\bullet\circ\bullet\circ\bullet\circ\bullet\circ\ldots\rangle$

$$H_{prep} = \sum_i \left(\frac{\Omega}{2}\sigma_i^x - \Delta(t<0) n_i \right) + \sum_{i<j} V_{ij} n_i n_j \quad \Rightarrow \quad H_{post} = \sum_i \frac{\Omega}{2}\sigma_i^x + \sum_{i<j} V_{ij} n_i n_j$$

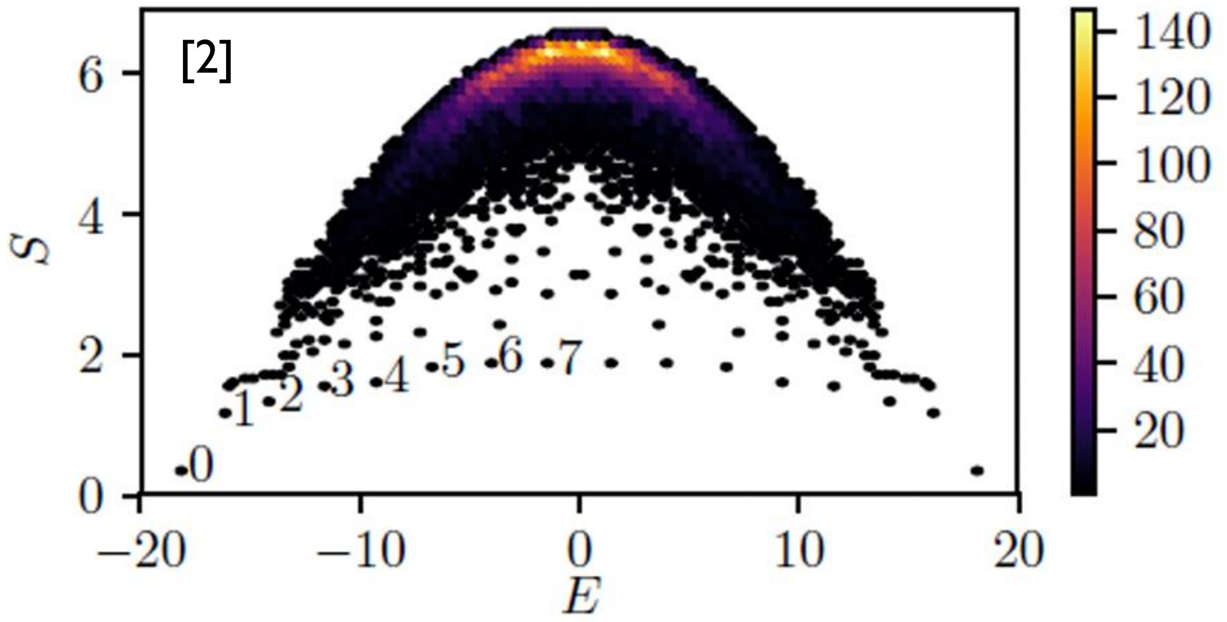
WHAT STARTED IT ALL

$$H_{post} = \sum_i \frac{\Omega}{2}\sigma_i^x + \sum_{i<j} V_{ij} n_i n_j \quad \approx \quad H_{PXP} = \sum_i P_{i-1}\sigma_i^x P_{i+1}$$

$\sigma_i^x = |\circ\rangle_i\langle\bullet|_i + |\bullet\rangle_i\langle\circ|_i$ Pauli x matrix responsible for Rabi oscillations of an individual atom on site i

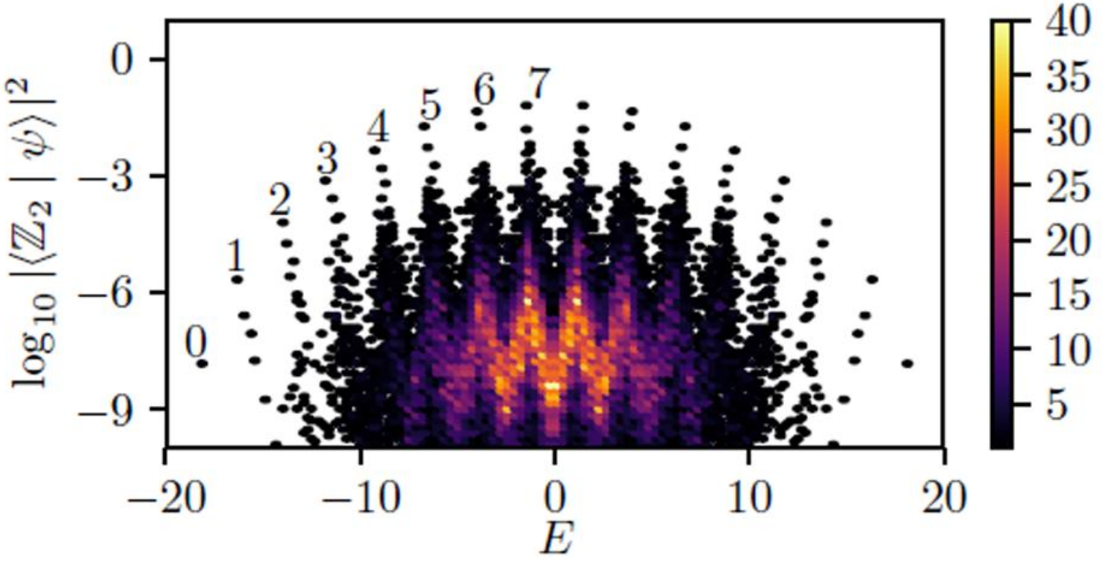
- our Hamiltonian, with or without long-range interactions, is far from any known integrable system, and features **neither strong disorder nor explicitly conserved quantities** [2]
- Neel state corresponds to an infinite-temperature ensemble for the atoms in the Rydberg blockade. Thus the **quench dynamics is not a priori limited to a small part of the energy spectrum** and cannot be attributed to weakly interacting low-energy excitations in the system

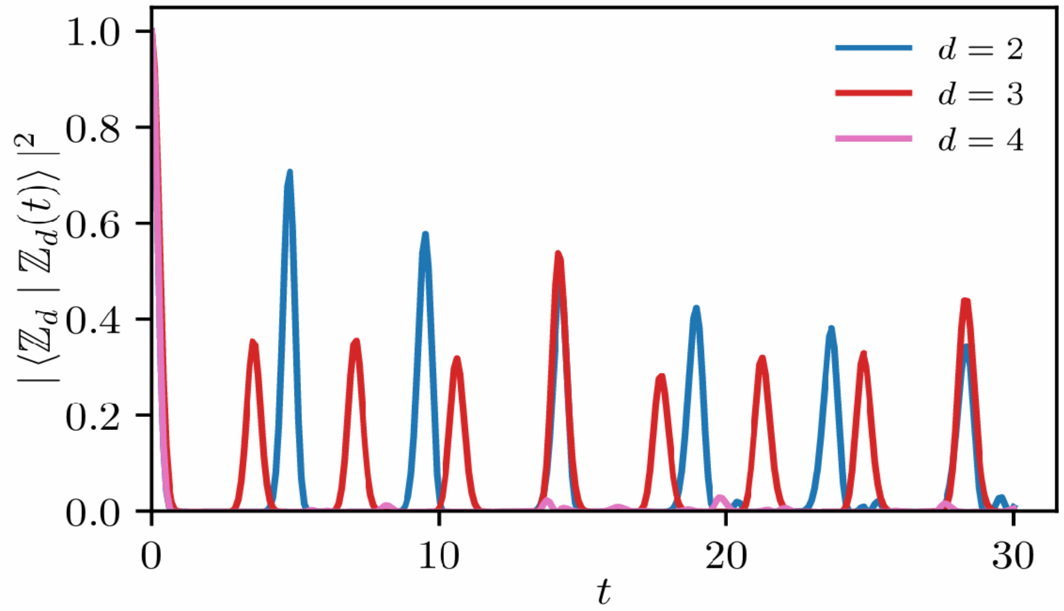
- **their frequency did not coincide with the bare Rabi frequency**, signalling the importance of many-body effects. [2]



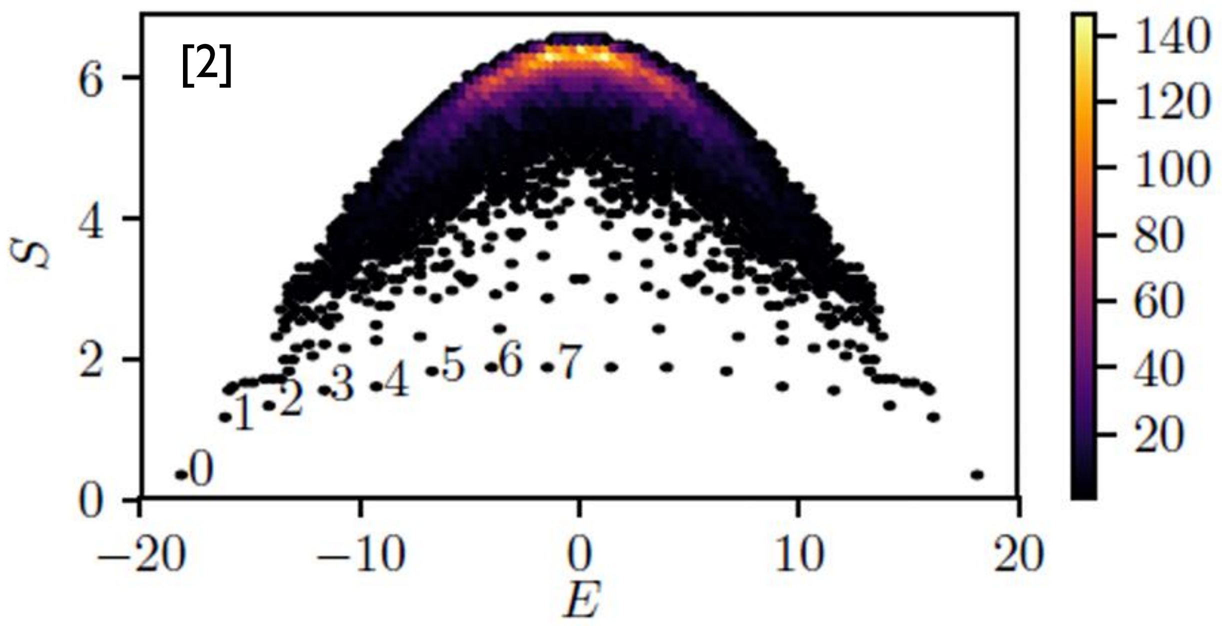
the special eigenstates are ‘embedded’ at roughly **equidistant energies** in an otherwise thermalizing spectrum of the PXP model, thus accounting for the dynamical revivals with a single dominant frequency

In **ETH** systems, the entanglement entropy of eigenstates scales **extensively** with the **size** of the subsystem. By contrast, **scarred eigenstates** have **entanglement** that increases only **logarithmically** with system size.





[4]



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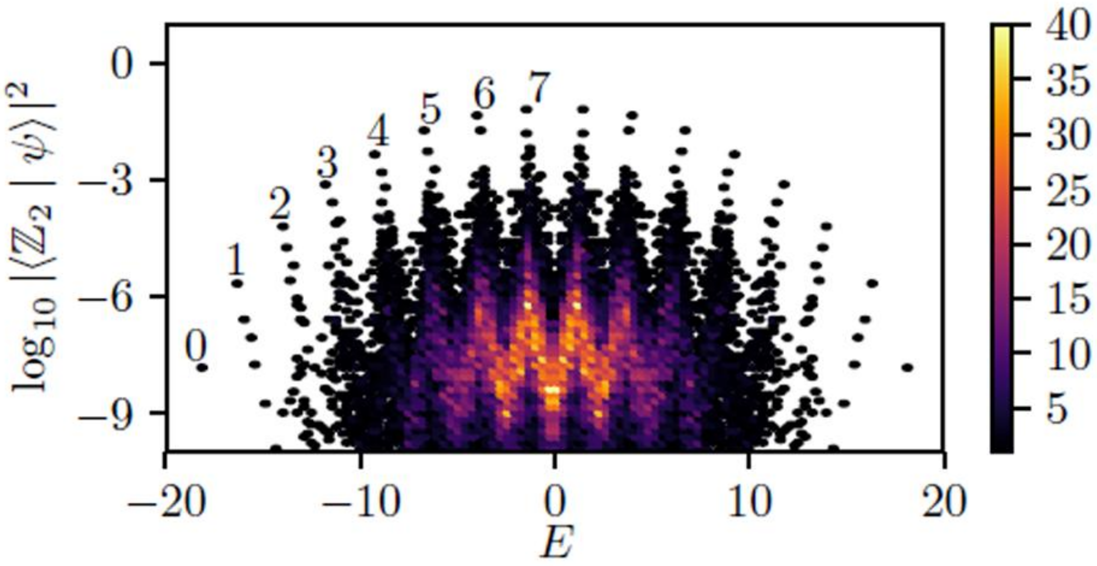
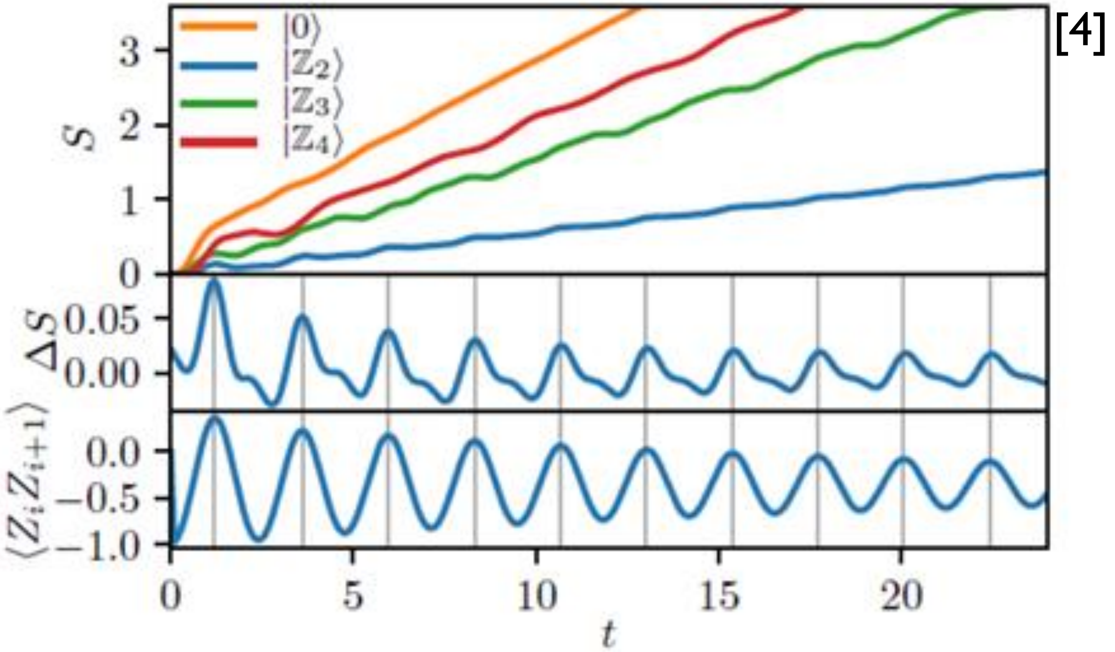
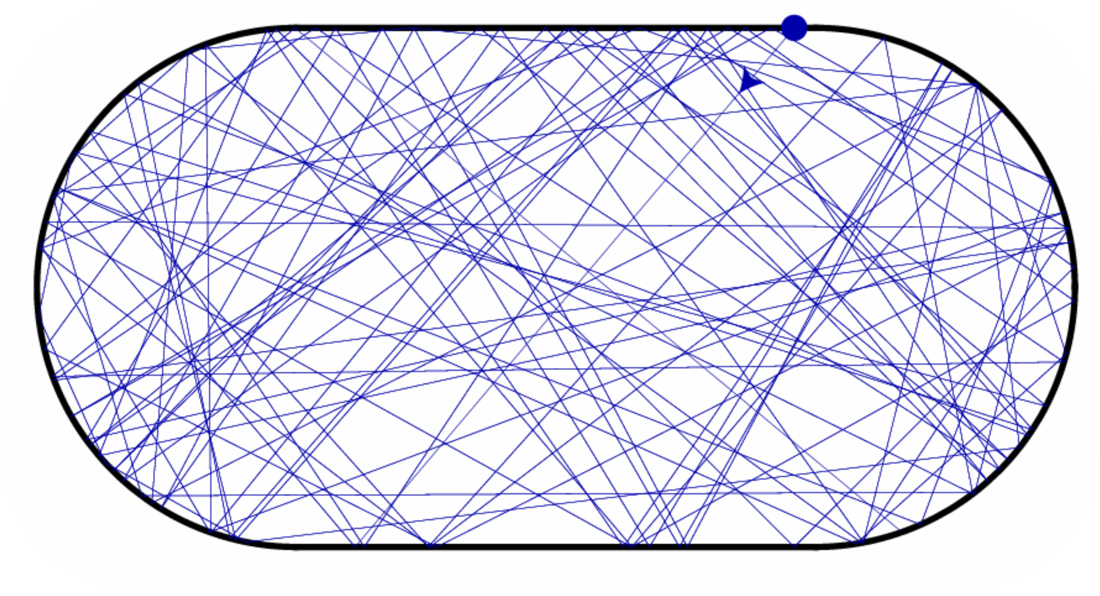


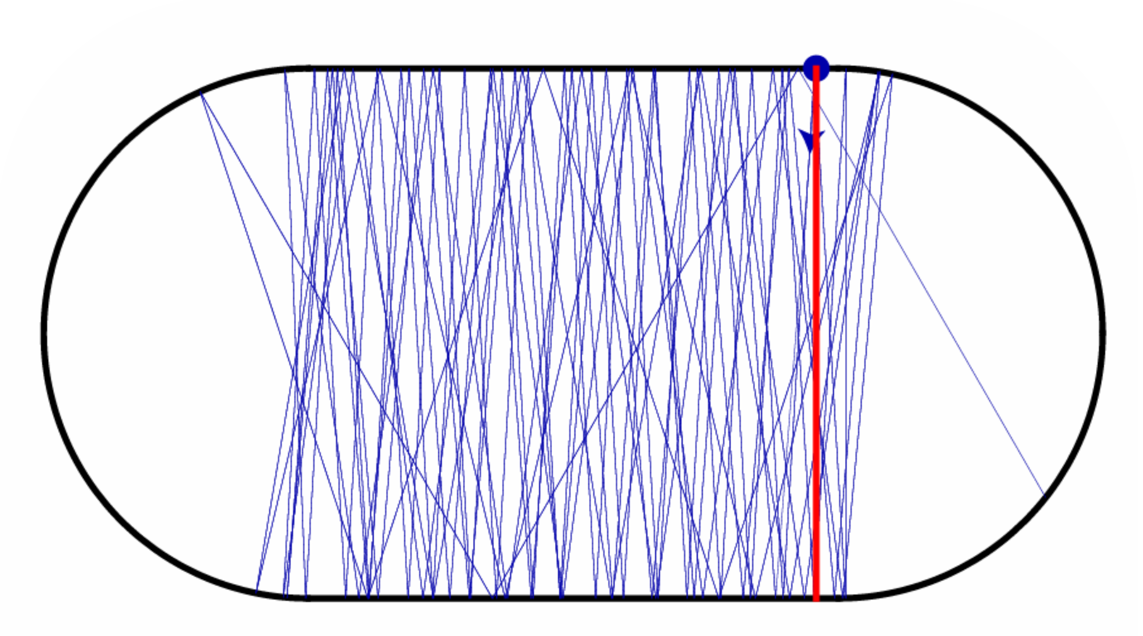


WHY SCARS?

BUNIMOVICH STADIUM



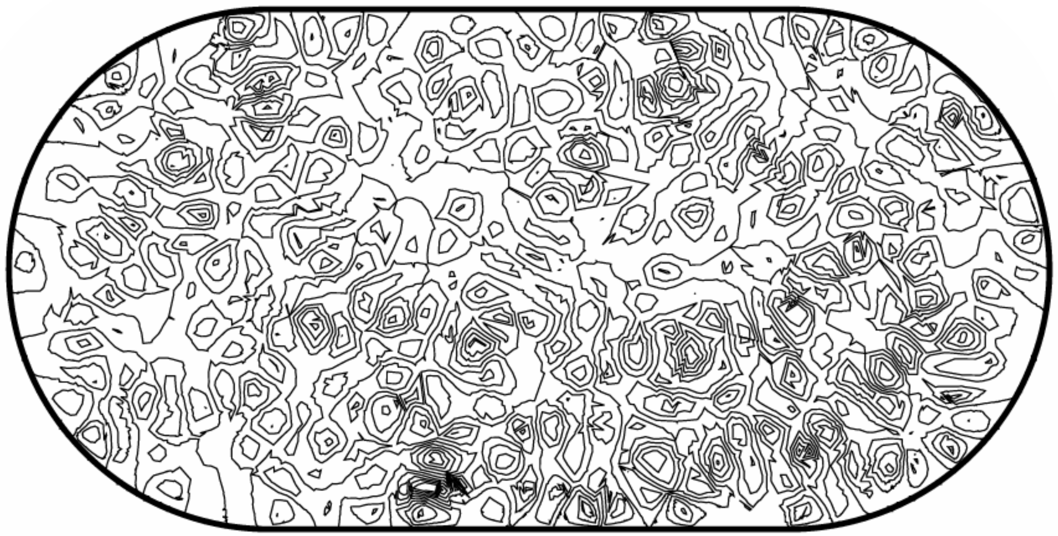
- A classical particle initialized away from a periodic trajectory displays **chaotic motion**

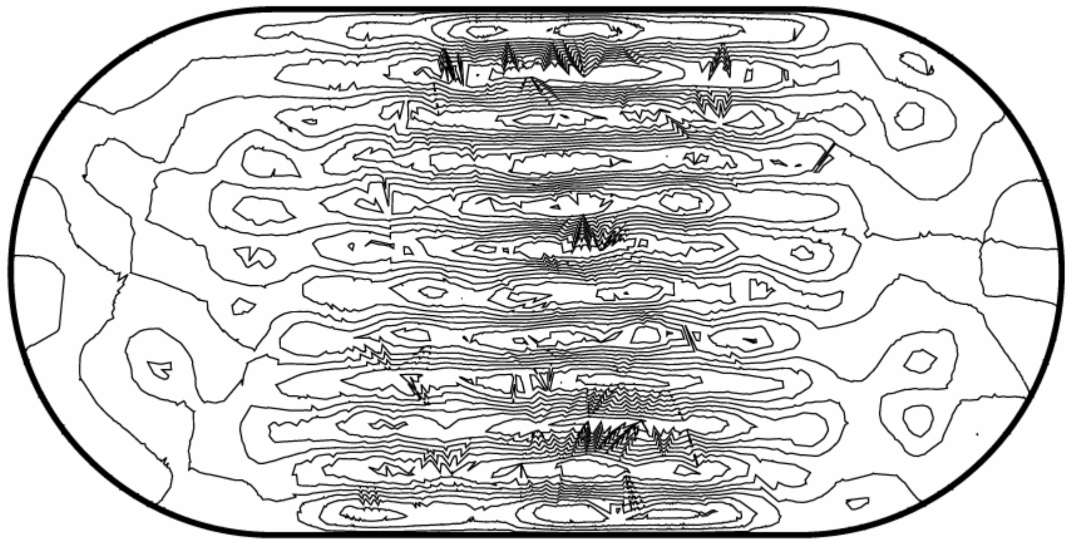


- In contrast, when launched **near** a ‘bouncing ball’ trajectory (shown in red, but there exist more), the particle spends a long time in its vicinity before escaping. Those are called **unstable periodic orbits.**

BUNIMOVICH STADIUM

The eigenstates of the billiard are obtained by solving the Schrödinger equation with the wavefunction vanishing at the boundary

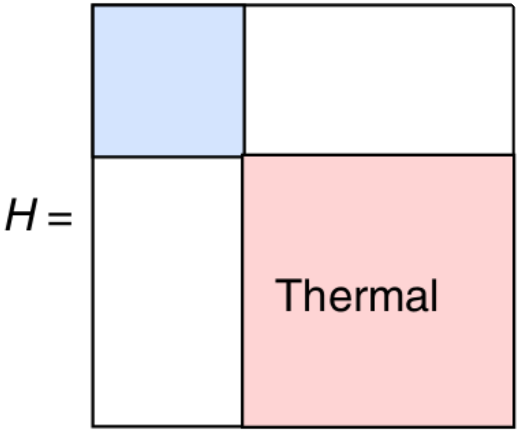




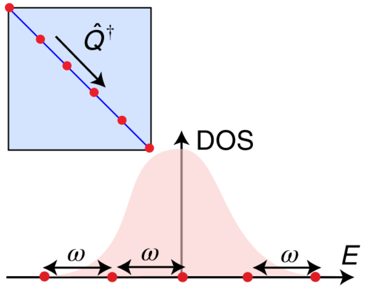
- The probability density of a typical highly excited eigenstate of the billiard resembles a collection of **random plane waves**.
- The probability density of a **quantum-scarred** eigenstate looks very different from a collection of plane waves, it looks like a standing wave. [2]

Recent theoretical studies have identified non-thermal eigenstates in other non-integrable quantum models. A common trait of all these models is **the emergence of a decoupled subspace** within the many-body Hilbert space [2]

$$H \approx H_{\text{scar}} \bigoplus H_{\text{thermal}}$$



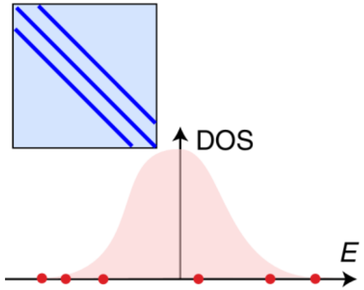
Spectrum-Generating Algebras (SGA):



Eg. extended Hubbard model, a spin-1 XY magnet and others.

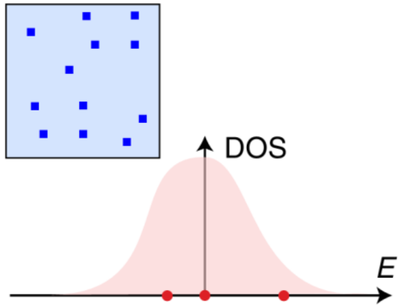
For the PXP it is a bit more complicated

exact Krylov subspace :



Eg. models of the fractional quantum Hall effect in a quasi-1D limit and in models of bosons with constrained hopping on optical lattices

Projector Embedding (Shiraishi-Mori)



HOW TO MAKE SENSE OF SCARRED STATES

The **Lanczos** procedure is an algorithm to put an Hermitian matrix H into a tridiagonal form. Following the article we start with the **free paramagnet** (no kinetic constraint) model for which the method is exact.

$$\boxed{H = \sum_{i=1}^{N} \sigma_i^x \xrightarrow[\sigma_\mathrm{i}^- = |\circ\rangle\langle\bullet|]{\sigma_\mathrm{i}^+ = |\bullet\rangle\langle\circ|} \begin{array}{l} H^+ = \sum_{i=1}^{L} \sigma_{2i-1}^- + \sigma_{2i}^+ \\ H^- = \sum_{i=1}^{L} \sigma_{2i-1}^+ + \sigma_{2i}^- \end{array}}$$

$$\beta_{i+1} |i+1\rangle = H^+ |i\rangle + \cancel{H^-} |i\rangle - \beta_i \cancel{|i} - 1\rangle$$

$$\Big\downarrow$$

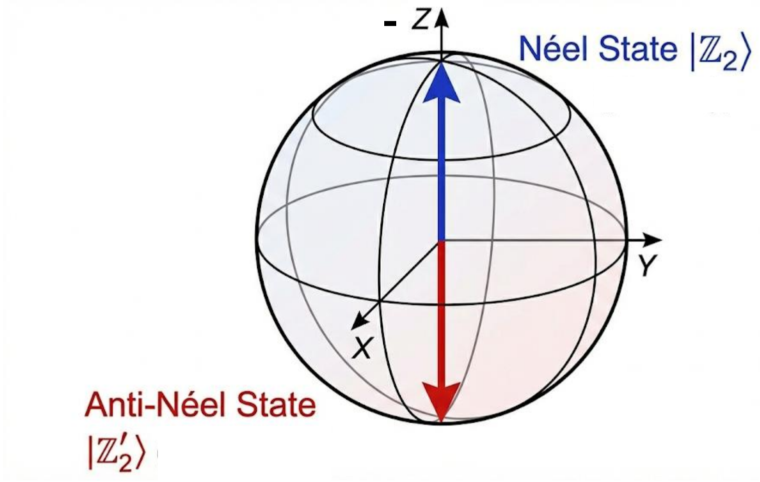
$$\beta_1 |1\rangle = H^+ |0\rangle = |\circ \circ \bullet \circ \bullet \cdots\rangle + |\bullet \bullet \bullet \circ \bullet \cdots\rangle + |\bullet \circ \circ \circ \bullet \cdots\rangle + \cdots$$

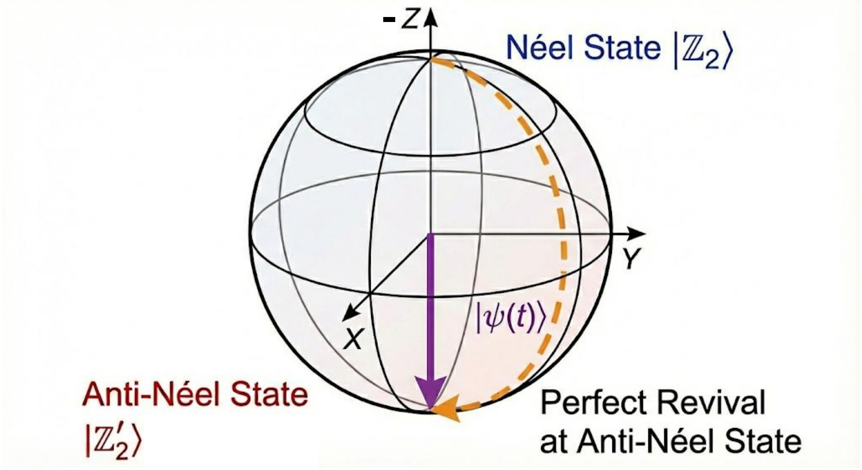
Neel state

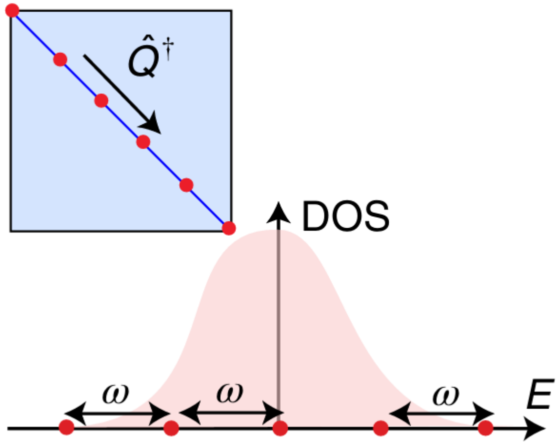
The algorithm then continues along the same pattern. After N steps the algorithm stops as we reach the anti Neel state ($\circ\bullet\circ\bullet\cdots$) which is annihilated by H+.

$$H = \sum_{n=0}^{N-1} \beta_{n+1} (|n\rangle \langle n+1| + |n+1\rangle \langle n|) \qquad \beta_j = \sqrt{j(N-j+1)}$$

Within this model we have a **perfect $SU(2)$ algebra** with the Hamiltonian representing the spin operator $2S^x$ for a spin of size L/2.
We can then interpret the many-body **oscillation** as the **precession** of a large spin initialized in an eigenstate of Sz (Neel state) and evolving under the operator S^x







Because of this perfect algebra and the subsequent equispacing between energies, the revivals would be periodic, but this is not the case

FSA

The FSA is a modification of the Lanczos procedure.

$$H^+ = \sum_{i=1}^{L} \tilde{\sigma}^-_{2i-1} + \tilde{\sigma}^+_{2i}$$

$$H^- = \sum_{i=1}^{L} \tilde{\sigma}^+_{2i-1} + \tilde{\sigma}^-_{2i}$$

$$\tilde{\sigma}^x_i = P_{i-1}\sigma^x_i P_{i+1}, \quad \tilde{\sigma}^z_i = P_{i-1}\sigma^z_i P_{i+1}, \quad \tilde{\sigma}^\pm_i = P_{i-1}\sigma^\pm_i P_{i+1}$$

$$\boxed{\beta_{i+1}\left|i+1\right\rangle = H^+\left|i\right\rangle}$$

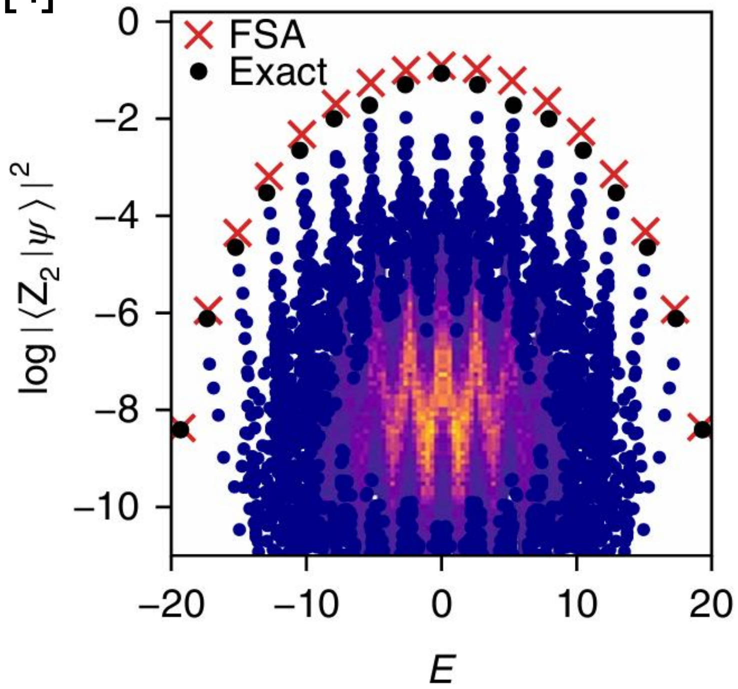
Justified by numerical evidence

$$H_{\mathrm{FSA}} = \sum_{n=0}^{N-1} \beta_{n+1}(\left|n\right\rangle\left\langle n+1\right| + \left|n+1\right\rangle\left\langle n\right|)$$

$$\boxed{\cancel{H^-\left|i\right\rangle = \beta_i\left|i-1\right\rangle}} \quad \Longrightarrow$$

Now the FSA hamiltonian is just an approximation to the PXP hamiltonian. By this approximation we now only have L+1 base vectors instead of the whole Hilbert space dimesion!

[4]



Let’s compare with the exact ones (diagonalization L=30)

We would expect that a basis that has only L + 1 states, each concentrated in small parts of the Hilbert space, would provide an extremely poor approximation for a generic highly excited eigenstate of a thermalizing system of size L.
However the **error is actually ≈2.6%** [4]

The link between the "inexactness" of an algebra and thermalization lies in the **breakdown of dynamical confinement**.

There are terms in the Hamiltonian that "leak" out of the algebra. These terms act as a **coupling** between the scar sector and the **"ergodic bath"** (the vast majority of other states in the spectrum that obey ETH).
The fact that the algebra is **"quasi-exact"** in the PXP model explains why we observe **revivals for a finite duration**.

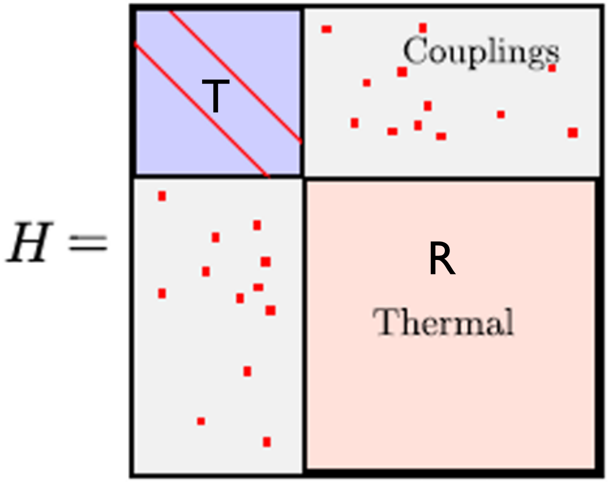
HOW TO MAKE SENSE OF THE DECAY

We can show that the PXP Hamiltonian can be written effectively as a sum of one operator acting only on T and which looks like a spin Sx operator, one coupling the subspaces T and R and one acting on R only. This decomposition is strongly reminiscent of the treatment of the **interaction of a small system with its environment**. We can thus reinterpret the dynamic of the PXP model after the quench as the **rotation of a spin coupled to an ergodic bath** and identify the form of the **coupling between the spin and the bath** [3]

$$[H^z, H^+] = H^+ + \delta^+, \quad [H^z, H^-] = -H^- + \delta^-$$

$$H = \Big[\beta_1\big(|1\rangle\langle 0| + |0\rangle\langle 1|\big) + \beta_2\big(|2\rangle\langle 1| + |1\rangle\langle 2|\big) + \cdots\Big] + \Big[\Big(-\frac{2}{\beta_3\beta_2}\delta^+|1\rangle\Big)\langle 3| + \mathrm{h.c} + \cdots\Big]$$
$$+ \Big[\frac{12}{(\gamma_4\beta_2\beta_3)^2}(H^+\delta^+|1\rangle)(\delta^+|1\rangle)^\dagger + \mathrm{h.c} + \cdots\Big]$$

$$H = H_S + H_{SB} + H_B$$



WHY SHOULD WE CARE?

- This special behaviour of the PXP model was found to be in general destroyed by perturbations, which can make this model integrable, frustration free or thermalizing, although **a certain degree of robustness was demonstrated with respect to disorder**. Moreover, it was argued that for weak perturbations, the dynamical signatures of scars survive for parametrically long times [2]

- there is strong experimental and practical interest in quantum many-body scars. Many-body scarred revivals provide a mechanism for **maintaining coherence**, despite the presence of interactions that normally scramble local quantum information. In particular, scars in Rydberg chains have already been used for the preparation of specific entangled states [2]

This suggests that scars may have a wider range of applications, for example, in protected state trans fer on quantum networks or in quantum sensing

QUANTUM MANY-BODY SCARS:
WEAK ERGODICITY BREAKING AND PERSISTENT REVIVALS

UNDERSTANDING NON-THERMAL DYNAMICS IN RYDBERG ATOM ARRAYS

[1]Bernien, H., Schwartz, S., Keesling, A., Levine, H., Omran, A., Pichler, H., Choi, S., Zibrov, A. S., Endres, M., Greiner, M., Vuletić, V., & Lukin, M. D. (2017). Probing many-body dynamics on a 51-atom quantum simulator. *Nature*, *551*(7682), 579–584.

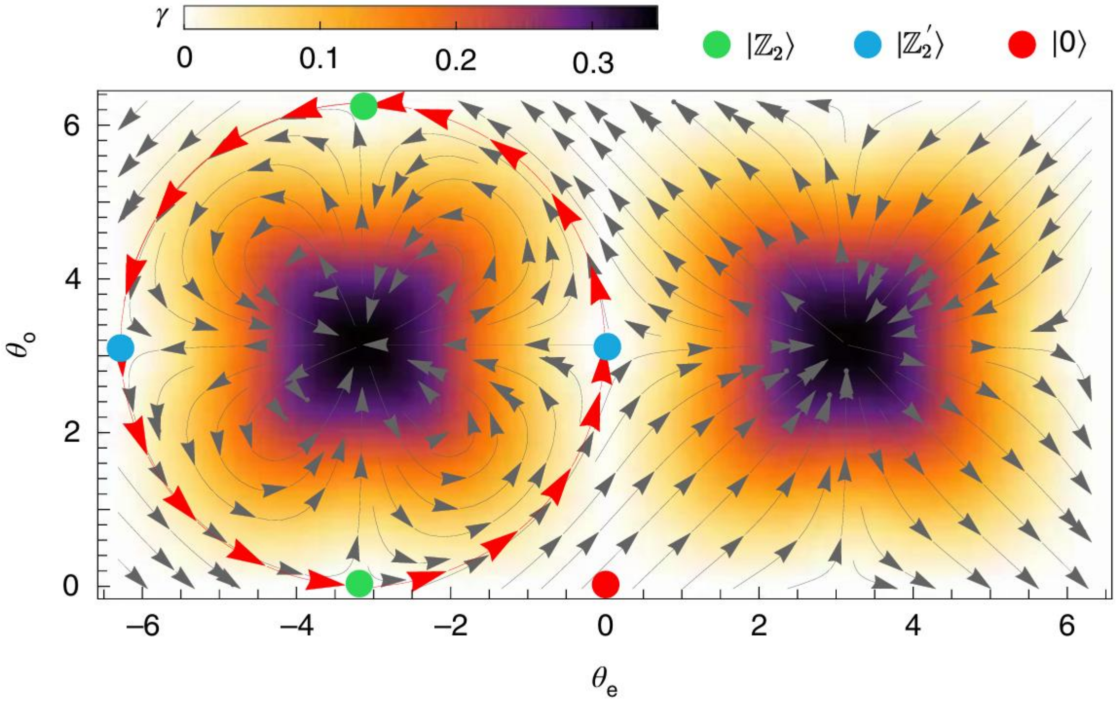
[2]Serbyn, M., Abanin, D.A., & Papić, Z. (2021) . Quantum many-body scars and weak breaking of ergodicity . *Nature Physics* , *17*(6), 675–685 .

[3]Larzul, A., & Schiró, M. (2020, Giugno) . *Spin-Bath Model for Quantum Many-Body Scars*. JEIP, USR 3573 CNRS, Collège de France, PSL Research University.

[4]Turner, C. J., Michailidis, A. A., Abanin, D. A., Serbyn, M., & Papić, Z. (2018). *Quantum scarred eigenstates in a Rydberg atom chain: entanglement, breakdown of thermalization, and stability to perturbations*. arXiv:1806.10933 [cond-mat.quant-gas].

BONUS on «why scars»

Initialization of a quantum wave packet on or near such an orbit leads to dynamical recurrences in which the particle tends to **cluster around the orbit**, a phenomenon called **quantum scarring**. Moreover, the periodic orbit leaves an imprint on the eigenfunctions of a particle, which exhibit anomalous centration in the vicinity of the periodic orbit, rather than being spread uniformly across the billiard.



[4]
Phase diagram obtained from the variational ansatz with two degrees of freedom reveals an unstable periodic trajectory, which is responsible for the revivals in the PXP.
(The variational principle determines the best direction of evolution within the manifold, such that the difference between exact and projected dynamics is minimal)