Advanced Statistical Physics: Disordered Systems

Exam

January, 2023

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English, French, Italian or Spanish, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but especially the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The answers must be written neatly within the boxes.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

Bareme: 19 pt (questions) + 1.5pt (clarity of exposition) = 20.5

The random field Ising model

Consider a single Ising spin s under a magnetic field h. The whole system is coupled to a thermal bath, which is in equilibrium at inverse temperature β.

1. The time evolution of the spin is ruled by stochastic dynamics and it flips with time-scale $\tau_s = 10^{-12}$ sec. The magnetic field dynamics are also stochastic with time-scale for changes τ_h. Give a condition on τ_h so that the field can be considered to be of quenched random nature.

1 – $\tau_h \gg \tau \geq \tau_s$ with τ the observation time scale. **1pt**

We now take the magnetic field h to be quenched random and distributed uniformly between $-h_m$ and h_m with $h_m > 0$.

2. Write the Hamiltonian of this system.
3. Calculate the annealed free-energy.
4. Calculate the quenched free-energy.
5. Are they equal?
6. Is there a limit in which they coincide? Comment.

2 – The Hamiltonian is **0.5pt**

$$H = -hs \tag{1}$$

3 – The annealed free-energy is **1pt**

$$\begin{aligned} -\beta f_{\text{annealed}} &= \ln\left[Z_h\right] = \ln \int_{-h_m}^{h_m} dh\, \frac{1}{2h_m} \sum_{s=\pm 1} e^{\beta h s} = \ln \int_{-h_m}^{h_m} dh\, \frac{1}{2h_m}\, 2\cosh(\beta h) \\ &= \ln\left(\frac{2}{\beta h_m} \sinh(\beta h_m)\right) = \ln 2 - \ln(\beta h_m) + \ln \sinh(\beta h_m) \end{aligned} \tag{2}$$

where we extracted the (adimensional) parameter βh_m dependence.

4 – The quenched free-energy is **1pt**

$$-\beta f_{\text{quenched}} = [\ln Z_h] = \int_{-h_m}^{h_m} dh\, \frac{1}{2h_m}\, \ln \sum_{s=\pm 1} e^{\beta h s} = \int_{-h_m}^{h_m} dh\, \frac{1}{2h_m}\, \ln\left(2\cosh(\beta h)\right)$$
$$= \ln 2 + \frac{1}{\beta h_m}\int_0^{\beta h_m} dy\, \ln\left(\cosh y\right) \qquad (3)$$

where we changed variables $y = \beta h$ and now the (adimensional) parameter dependence is clear.

5 – For generic adimensional parameterβh_m values they are not equal. From Jensen's inequality we know $F_{\text{quenched}} \geq F_{\text{annealed}}$ **1pt**

6 – If we take $\beta \to 0$ while keeping $h_m > 0$, or we take $h_m \to 0$ with β finite, so that in both cases $\beta h_m \to 0$, then $-\beta f_{\text{quenched}} = -\beta f_{\text{annealed}} = \ln 2$ (given just by $1/k_B$ times the entropy of the Ising spin). In the infinite temperature limit, or the vanishing width of the random field distribution that is the deterministic limit for the field, the kind of disorder average becomes irrelevant and the two coincide. **1.5pt**

Consider now an ensemble of N Ising spins placed on a regular cubic lattice with linear size L in d dimensions. The nearest-neighbours on the lattice are ferromagnetically coupled and each spin is also coupled to a local quenched random field taken from a probability distribution $p(h_i)$. The random fields on different sites are independent.

7. Write the Hamiltonian of this system.
8. Is the free-energy of this system expected to be self-averaging? Explain the reasoning that leads to your conclusion.
9. Can you envision a case in which the argument used in the previous question fails?

7 – $H = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i h_i s_i$ **0.5pt**

I take the convention so that the sum runs over each nearest neighbour couple only once.

8 – Yes. **2pt**

We cut the system in pieces of linear length $a \ll \ell \ll \xi$ with ξ the correlation length.

Since the interactions are on nearest-neighbours on the lattice, they are short ranged. Moreover, the local fields are $i.i.d$ with no long-range correlations either.

We neglect the surface contribution to the total energy, since it scales as L^{d-1} instead of L^d. The total energy becomes a sum over independent terms.

We separate the partition sum in products over these boxes.

The log in the free-energy becomes a sum over independent random terms.

Then we use the central limit theorem to claim that the variance of the free-energy over its average squared vanishes in the $N \gg 1$ limit.

9- For example, at the critical point, where the correlation length diverges and one cannot consider the boxes to be independent. Or when there are long-range interactions. **1pt**

In Fig. 1 we reproduce a plot of the ratio between the variance, σ_w^2, and the average squared, μ_w^2, of different sample-to-sample (different disorder realization) fluctuating quantities w:

$$R_w \equiv \frac{\sigma_w^2}{\mu_w^2} \; . \tag{4}$$

The plot is taken from N. G. Fytas and A. Malakis, `arXiv:1011.4823` The data are taken at the pseudo critical parameters (T_c, h_c) of the finite L random field Ising model in three dimensions.

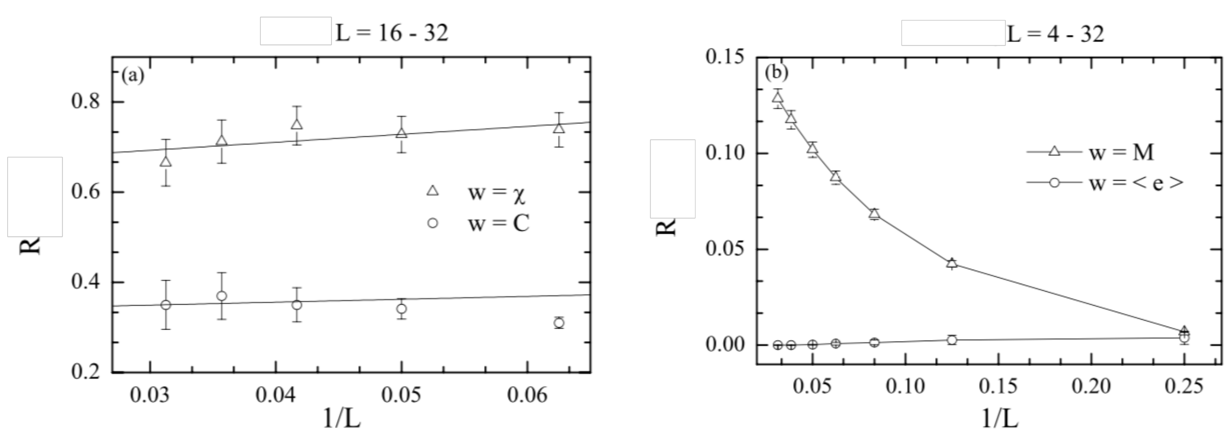

Figure 1: The ratio (4) for the magnetic susceptibility maximum χ and the specific heat C (a), the magnetization M and the mean energy per spin which is denoted $\langle e \rangle$ (b), as a function of inverse linear size $1/L$ of the system. In all cases the data are taken at the pseudo critical parameters (T_c, h_c) of the finite L system. The lines are fits.

10. What do you conclude about the self-averageness of these quantities?

11. Which could be the reason for lack of self-averageness in these measurements?

10 – Only the averaged energy per spin in (b) tends to zero for $1/L$. If this is a thermal average, then it does not correspond to R_e and we cannot conclude about its fluctuations. If by "mean" one refers to just energy density, and the points represent R_e then they tend to zero for $1/L$ and one concludes that it is a self-averaging quantity.

The R_M increases with $1/L \to 0$ which tells us that it does not self-average.

In (a) the straight lines may go to zero but if they do it's very very slowly, so conclusions are not clear. **1pt**

11 – The system is at the critical point, the correlation length diverges. **1pt**

12 - Show with a simple argument that ferromagnetic order is not expected to exist in $d \leq 2$ for the RFIM. Help yourself with a drawing and a plot to explain it.

1.5pt (argument) + 1.5pt (drawings)

It's the Imry Ma argument. Order the system in one ferromagnetic state, turn round a bubble with linear size ℓ and assume it is compact, as well as its interface. The free-energy change at very low T is

$$\Delta F(\ell) \sim \Delta E(\ell) = -h\ell^{d/2} + J\ell^{d-1} \tag{5}$$

for $d > 1$, where I ignored numerical factors of geometric origin and normalization too.

The form of this function depends on d and there is a change in $d = 2$.

If $d < 2$, ΔF vanishes at $\ell = 0$, takes negative values at $\ell \gtrsim 0$, attains a minimum at

$$\ell_{\min} \sim \left(\frac{h}{J}\right)^{2/(d-2)} = \left(\frac{J}{h}\right)^{2/(2-d)} \tag{6}$$

vanishes at an $\ell_0 > \ell_{\min}$ which scales with h/J in the same way, and then diverges towards infinity at $\ell \to \infty$.

Thus, for $d < 2$ it is favourable to make domains with linear size $\ell \stackrel{<}{\sim} (J/h)^{2/(2-d)}$ and disorder the sample.

For $d > 2$, the form of the curve is the opposite. It starts positive from zero, reaches a maximum and then turns round to minus infinity at $\ell \to \infty$.

One does not disorder the sample since the $\Delta F(\ell)$ is positive at small ℓ and one cannot create fluctuations of this sort. Extremely large sizes would be needed to disorder the sample at very low temperatures.

See the lectures for the plots

Take a simple harmonic oscillator with just potential energy

$$H = \frac{1}{2} m\omega^2 x^2 \ , \tag{7}$$

and consider that the frequency $\omega > 0$ is quenched and distributed according to

$$p(\omega) \quad = \quad \begin{cases} \frac{\omega}{\sigma^2} \, e^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}} & \omega > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

13. Calculate the partition function $\mathcal{Z}_\omega$ at fixed ω.
14. Express the free-energy F_ω at fixed ω.
15. Calculate the disorder averaged free-energy $[F_\omega]$.

13 - The Partition function is **0.5pt**

$Z_w = \int_{-\infty}^{\infty} dx \; e^{-\beta \frac{1}{2} m\omega^2 x^2} = \left(\dfrac{2\pi}{\beta m \omega^2} \right)^{1/2}$

14 - The free-energy **0.5pt**

$-\beta F_w = \ln \left(\dfrac{2\pi}{\beta m \omega^2} \right)^{1/2}$

15 - and the disorder average **1pt**

$-\beta [F_w] = \int_0^\infty \frac{d\omega}{\sigma} \frac{\omega}{\sigma} \, e^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}} \;\ln \left(\dfrac{2\pi}{\beta m \omega^2} \right)^{1/2} = \int_0^\infty dy \, y \, e^{-\frac{1}{2} y^2} \;\ln \left(\dfrac{2\pi}{\beta m \sigma^2 y^2} \right)^{1/2}$

that simplifies to $-\beta [F_w] = \ln \left(\dfrac{2\pi}{\beta m \sigma^2} \right)^{1/2} - \frac{1}{2} \int_0^\infty dy \; y \; e^{-\frac{1}{2} y^2} \;\ln y^2$

and changing variables $u = y^2$

$-\beta [F_w] = \ln \left(\dfrac{2\pi}{\beta m \sigma^2} \right)^{1/2} - \frac{1}{4} \int_0^\infty du \; e^{-\frac{1}{2} u} \;\ln u = \ln \left(\dfrac{2\pi}{\beta m \sigma^2} \right)^{1/2} - \frac{1}{4} \left(-2\gamma + \ln 4 \right)$

with the Euler constant $\gamma \equiv - \int_0^\infty dy \; e^{-y} \ln y$

Finally, $-\beta [F_w] = \ln \left(\dfrac{\pi}{\beta m \sigma^2} \right)^{1/2} + \frac{1}{2}\gamma$ Note that I wrote the arguments of the ln's always in dimensionless form.

Now, we study this problem with the replica trick.

16. Give the expression of the averaged free-energy as computed with the replica method.
17. Calculate the average over disorder of the n times replicated partition function, $[Z_w^n]$. Useful equations are

$$\prod_{a=1}^{n} \int dq_a \, f(\sum_a q_a^2) = S_{n-1} \int_0^\infty dr \, r^{n-1} \, f(r^2)$$

$$S_{n-1} = 2\pi^{n/2} / \, \Gamma(n/2)$$

$$\int_0^\infty dr \, \frac{r^{n-1}}{1+r^2} = \frac{1}{2} \Gamma(n/2)\Gamma(1-n/2) \qquad \text{and} \qquad \Gamma(1) = 1$$

18. Does $[F_w]$ obtained with the replica calculation coincide with expression found in point 15., following the direct calculation?

16 – $-\beta[F_w] = \lim_{n\to 0} \frac{[Z_w^n]-1}{n}$ **0.5pt**

17 – The disorder averaged replicated partition function is **1.5pt**

$$[Z_w^n] = \int_0^\infty \frac{d\omega}{\sigma} \frac{\omega}{\sigma} \, e^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}} \prod_{a=1}^{n} \int_{-\infty}^{\infty} dx_a \, e^{-\beta \frac{1}{2} m\omega^2 \sum_a x_a^2} = \prod_{a=1}^{n} \int_{-\infty}^{\infty} dx_a \, \int_0^\infty \frac{d\omega}{\sigma} \frac{\omega}{\sigma} \, e^{-\frac{1}{2}\frac{\omega^2}{\sigma^2}} \, e^{-\beta \frac{1}{2} m\omega^2 \sum_a x_a^2}$$

$$[Z_w^n] = \prod_{a=1}^{n} \int_{-\infty}^{\infty} dx_a \, \int_0^\infty dy \, y \, e^{-\frac{1}{2} y^2 (1+\beta m \sigma^2 \sum_a x_a^2)}$$

$$[Z_w^n] = \prod_{a=1}^{n} \int_{-\infty}^{\infty} dx_a \, \frac{1}{1+\beta m\sigma^2 \sum_a x_a^2} = \left(\beta m \sigma^2\right)^{-n/2} \prod_{a=1}^{n} \int_{-\infty}^{\infty} dq_a \, \frac{1}{1+\sum_a q_a^2}$$

Now, we change variables to spherical coordinates $r^2 = \sum_a q_a^2$,

$$[Z_w^n] = \left(\beta m \sigma^2\right)^{-n/2} \int d\Omega_{n-1} \int_0^\infty dr \, r^{n-1} \, \frac{1}{1+r^2} = \left(\beta m \sigma^2\right)^{-n/2} S_{n-1} \int_0^\infty dr \, \frac{r^{n-1}}{1+r^2}$$

that becomes

$$[Z_w^n] = \left(\beta m \sigma^2\right)^{-n/2} \frac{2\pi^{n/2}}{\Gamma(n/2)} \, \tfrac{1}{2} \Gamma(n/2)\Gamma(1-n/2)$$

that cancelling in numerator and denominator the common factor becomes

$$[Z_w^n] = \left(\frac{\pi}{\beta m \sigma^2}\right)^{n/2} \, \Gamma(1-n/2)$$

Taking now the $n \to 0$ limit of these two factors:

$$\left(\frac{\pi}{\beta m \sigma^2}\right)^{n/2} \simeq 1 + \frac{n}{2} \ln\left(\frac{\pi}{\beta m \sigma^2}\right) + \mathcal{O}(n^2)$$

and

$$\Gamma\left(1 - \frac{n}{2}\right) = \Gamma(1) - \frac{n}{2}\,\Gamma'(1) + \mathcal{O}(n^2) = 1 + \frac{n}{2}\,\gamma + \mathcal{O}(n^2)$$

with the same Euler constant as in the previous calculation.

Putting the two together

$$[Z_w^n] = \left[1 + n \ln\left(\frac{\pi}{\beta m \sigma^2}\right)^{1/2}\right]\left[1 + \frac{n}{2}\,\gamma\right] + \mathcal{O}(n^2)$$

$$= 1 + n \ln\left(\frac{\pi}{\beta m \sigma^2}\right)^{1/2} + \frac{n}{2}\,\gamma + \mathcal{O}(n^2)$$

and we finally get

$$\lim_{n \to 0} \frac{[Z_w^n] - 1}{n} \mapsto \ln\left(\frac{\pi}{\beta m \sigma^2}\right)^{1/2} + \frac{\gamma}{2}$$

(It was OK if you did not remember the relation between the γ constant and the $\Gamma'(1)$, and just focused on the parameter dependence of the result.)

18 – which is the same result as with the direct calculation **0.5pt**