Homework: Analytical and Numerical Study of Active Brownian Motion and the Active Ornstein-Uhlenbeck Process

February 3, 2026

The goal of this homework is to study the motion of an Active Brownian Particle (ABP) and an Active Ornstein-Uhlenbeck Particle (AOUP) both analytically and numerically. You will derive key statistical properties such as mean-squared displacement (MSD) and velocity autocorrelation, and implement simulations to verify your theoretical results.

1 Analytical Study

1.1 Active Brownian Particle (ABP)

The ABP model describes a self-propelled particle subject to stochastic noise. In two dimensions, the equations of motion are:

$$\frac{d\mathbf{x}}{dt} = v_0\mathbf{e}(t) + \sqrt{2D}\boldsymbol{\xi}(t), \tag{1}$$

$$\frac{d\theta}{dt} = \sqrt{2D_r}\eta(t), \tag{2}$$

where: $\mathbf{x} = (x, y)$ is the position of the particle on the plane, v_0 is the constant self-propulsion speed, $\mathbf{e}(t) = (\cos\theta, \sin\theta)$ is the propulsion direction, D is the translational noise strength, D_r is the rotational diffusion constant, $\boldsymbol{\xi}(t)$ and $\eta(t)$ are independent Gaussian white noise terms with zero mean and unit variance.

1. What is the average position of the particle $\langle\mathbf{x}(t)\rangle$?

2. Compute the mean squared displacement (MSD) $\langle|\mathbf{x}(t) - \mathbf{x}(0)|^2\rangle$. Identify four time regimes (ballistic, diffusive, ballistic and diffusive) and the parameter dependencies of the characteristic times which separate them. Do it numerically and analytically.

3. Determine the velocity autocorrelation function $C_v(t) = \langle\mathbf{v}(t)\cdot\mathbf{v}(0)\rangle$. Do you find different regimes as well?

1.2 Active Ornstein-Uhlenbeck Process (AOUP)

The AOUP model replaces the persistent velocity with an Ornstein-Uhlenbeck process:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t), \tag{3}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\tau}\mathbf{v} + \sqrt{2D_A}\boldsymbol{\xi}(t), \tag{4}$$

where τ is the persistence time and D_A is the effective noise strength.

1. What is the average position of the particle $\langle \mathbf{x}(t) \rangle$?
2. Derive the mean squared displacement $\langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle$ for different time regimes.
3. Compute the velocity autocorrelation function $C_v(t)$.

2 Numerical Simulation

1. Implement a numerical simulation for both ABP and AOUP using the Euler-Maruyama method.
2. Simulate and plot example trajectories for both models. How do they depend on the parameters v_0 and τ?
3. Compute and plot MSD and velocity autocorrelation from simulations.
4. Compare numerical results with analytical predictions.

3 Bonus Question

Investigate the behavior of ABP and AOUP in a confined environment (e.g., harmonic potential) and compare the steady-state distributions.

References

P. Romanczuk, M. Bär, W. Ebeling, B. Lindner, L. Schimansky-Geier, L. *Active Brownian particles*, The European Physical Journal Special Topics. **202**, 1-162 (2012).

D. Martin, J. O'Byrne, M. E. Cates, E. Fodor, C. Nardini, J. Tailleur, and F. van Wijland, *Statistical mechanics of active Ornstein-Uhlenbeck particles*, Phys. Rev. E **103**, 032607 (2021).