Homework
Simulation of a 1D Random Walk using the Langevin Equation with White Noise

February 3, 2026

Problem Statement

In this assignment, you will use the Langevin equation with white noise to simulate the motion of a 1D unbiased and biased random walk. You will analyze how the variance of the position evolves over time and compare your numerical results with theoretical expectations.

The **Langevin equation** describes the motion of a particle under stochastic forces. In one dimension, it is given by

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -\gamma v + F + \eta(t), \tag{1}$$

where $x(t)$ is the position of the particle, $v(t)$ is its velocity, γ is a friction coefficient, F is an external force (for biasing the motion), $\eta(t)$ is a Gaussian white noise term with mean zero $\langle \eta(t) \rangle = 0$ and correlation:

$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$$

where D is the noise strength. The last expression means that the noise is memoryless.

For a simple random walk, we assume the over-damped limit (no inertia), so the equation reduces to:

$$\frac{dx}{dt} = F + \eta(t).$$

For the unbiased case, set $F = 0$, while for the biased case, use a constant force $F \neq 0$.

The average observables are now calculated as averages over the distribution of the random noise η.

1 Numerical Simulation

Use the Euler-Maruyama method to discretize the Langevin equation:

$$x_{n+1} = x_n + (F\Delta t) + \sqrt{2D\,\Delta t}\,\eta_n,$$

where n is an integer denoting the time step and η_n is a standard normal random variable $\eta_n \sim \mathcal{N}(0, 1)$.

Implement a code to solve this equation numerically in Python or another programming language of your choice.

1. Simulate the 1D random walk for both the unbiased case ($F = 0$) and biased case ($F \neq 0$) for multiple realizations of the noise. Choose appropriate parameters: $\Delta t, D, F$, and total simulation time.

2. Plot example trajectories.

3. Compute the ensemble average particles position as function of time and trace it.

4. Compute the variance of the position as function of time and trace it.

5. Plot the probability distribution of the particle position at different times t.

6. Compare your results to what you found for the unbiased ($p = 1/2$) and biased ($p \neq 1/2$) random walks in homework 1.

2 Analytic results

Working with the stochastic Langevin equation (1), time is continuous.

1. Compute the mean displacement $\langle x(t) \rangle$ and compare with the numerical results.

2. Compute the variance $\text{Var}(x(t)) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$ and compare with the numerical results.

References

L. F. Cugliandolo, *Introduction to Langevin stochastic processes*, Advances in Physics 73, 3-80 (2024).