Homework Assignment 2:
Random Walk with Quenched Disorder

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Problem Statement

A particle performs a discrete-time random walk on a one-dimensional lattice with quenched disorder. The probability of moving from site i to site $i+1$ is denoted as p_i, which is randomly assigned for each site and remains fixed throughout the walk, that is, it is time independent. (This is the meaning of the adjective "quenched" in the title.) The probability of moving to site $i-1$ is therefore $q_i = 1 - p_i$. The probabilities to go right on each site i are independent and identically distributed (i.i.d.). This means that the joint probability distribution of all $\{p_i\}$ is

$$P(\{p_i\}) = \underbrace{\prod_i}_{\text{independence}} \underbrace{\pi(p_i)}_{\text{indentically distributed}} \tag{1}$$

The goal of this assignment is to:

1. Simulate the random walk numerically and analyze its behavior.
2. Compute or estimate key observables analytically.

1 Numerical Simulation

1.1 Implementation Details

Simulate a random walk on a 1D lattice with L sites, choosing quenched transition probabilities p_i randomly and independently for each site. Because of being probabilities, p_i are real valued and $0 \leq p_i \leq 1$. For convenience, we choose a uniform probability density, Uniform$(0, 1)$, also called "box distribution".

1. Consider a lattice of size L with (a) reflecting boundaries at $i = 0$ and $i = L$, (b) periodic boundary conditions, or (c) open boundary conditions. The last one is the simplest one, so let us choose open boundary conditions.

2. Assign each site i a probability $p_i \sim \text{Uniform}(0,1)$ and keep it fixed. We exclude the values 0 and 1 since they would constitute unsurmountable walls for motion to the right or to the left, respectively. What is the probability density $\pi(p_i)$ so that the distribution is normalized? Calculate the average and variance of p_i, which we will denote $[p_i]$ and $\sigma^2_{p_i} = [(p_i - [p_i])^2] = [p_i^2] - [p_i]^2$.

3. We will start the walker at $i = 0$ ($x_0 = 0$) and perform N steps for each trajectory. There are two ways of generating different trajectories:

   (a) Keep the disorder fixed (the p_is) and use different thermal noise realizations (the choices of going right or left).

   (b) Use just one realization of the thermal noise and change the disorder realization, that is the $\{p_i\}$s.

   Make two plots with some trajectories generated in these two ways.
   These two sources of randomness will give rise to two kinds of averages to consider. Over thermal noise or over the quenched randomness. We usually denote $\langle \ldots \rangle$ the former (as in homework 1 where there was no quenched disorder) and $[\ldots]$ the latter. The two averages can be taken, in which case one uses the notation $[\langle \ldots \rangle]$.

4. Plot the histograms of the particle's position at different times in the two cases.

5. Calculate analytically the average position $\langle x_t \rangle$ and the variance $\sigma^2_{x_t}$ assuming that the motion at each time step is independent. Do it at fixed realization of the disorder and also further averaging over disorder, $[\langle x_t \rangle]$ and $[\sigma^2_{x_t}]$.

6. Compute, numerically, the same quantities. Plot the variance $\sigma^2_{x_t}$ as a function of $(\ln N)^4$ (Sinai's diffusion) and check whether this guess describes the data better than the naive theoretical calculation of the previous item. Discuss.

2 Discussion and Conclusion

Summarize the key findings from simulations and analytical estimates. Discuss possible refinements such as different distributions for p_i, absorbing boundary conditions, or correlations between step probabilities.

Ya. G. Sinai, *The limiting behavior of a one-dimensional random walk in a random environment*, Theory of Probability & Its Applications **27**, 256 (1982).

F. Solomon, *Random walks in a random environment*, Annals of Probability **3**, 1 (1975).

B. D. Hughes, *Random Walks and Random Environments*, Volume 1: Random Walks (Oxford University Press, 1995).