Homework Assignment 1:
Random Walk and Diffusion

January 19, 2026

Problem Statement

A particle performs a discrete-time random walk on a one-dimensional lattice. The probability of moving from site i to site $i+1$ is denoted as p. The probability of moving to site $i-1$ is therefore $q = 1-p$. The goal of this assignment is to study this problem numerically and analytically.

For this and all other homeworks, write a Latex file with the results of your work, and extract from it a pdf including the plots required. Send the pdf with name homework1-2026.pdf to:
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1 Random Walk Simulation

Consider a one dimensional random walk on a periodic or open lattice (as you prefer). Place the particle at the position $x_0 \neq 0$ at time $t = 0$. Assume that at each time step, Δt, the particle moves either to the left, $x_n = x_{n-1} - \Delta x$, or to the right, $x_n = x_{n-1} + \Delta x$, with probability p and $1-p$, respectively, n labels here the time step, $n = 1, 2, \ldots$ Work with the cases $p = 1/2$ on the one hand, and $p \neq 1/2$ on the other hand.

Define the average of any observable $O(x_t)$, with x the particle's position at time t, as

$$\langle O(x_t) \rangle \equiv \frac{1}{\mathcal{N}} \sum_{a=1}^{\mathcal{N}} x_t^{(a)} \tag{1}$$

where $\mathcal{N}$ is the number of runs (repetitions of the numerical experiment with different choices of the random left-right moves). The discrete time t is $t = n\Delta t$, for some integer n and Δt is time step with dimension of time.

For both $p = 1/2$ and $p \neq 1/2$, perform the following tasks:

1. Simulate $\mathcal{N} = 1000$ trajectories, each consisting of $T = 100$ time steps.
2. Plot several trajectories (e.g., 10) on the same graph to visualize the random walk.
3. Fix a time $t = 10\Delta t$ and construct the histogram of the position of the particle at this time. Do it for $\mathcal{N} = 1,\ 10,\ 100,\ 1000,\ 10000,\ \ldots.$ What do you observe? Estimate the form that the probability density will have for $\mathcal{N} \to \infty$.
4. Repeat the analysis of the previous item for later times $t = 100\Delta t$, $1000\Delta t$, $\ldots.$ Analyze the results and conclude about the expected analytic form of $P(x, t)$.
5. Calculate, from the data points, the following observables:
   (a) Average position $\langle x_t \rangle$, defined as the average over many runs of the position, as a function of time. Do it for several $\mathcal{N}$ and conclude about the $\mathcal{N}$ dependence.
   (b) Variance of the position $\langle x_t^2 \rangle - \langle x_t \rangle^2$ as a function of time. Plot this quantity as a function of time and extract the asymptotic behavior. It is called *normal diffusion*. Do it for several $\mathcal{N}$ and conclude about the $\mathcal{N}$ dependence.
   (c) Extra: Calculate the averaged number of local minima of the trajectory x_n and estimate how scales with n. Construct the probability distribution. (See the 2024 manuscript by Kundu, Majumdar & Schehr) What do you expect for the local maxima?

2 Analytical Results

Derive the following results analytically for an unbiased 1D random walk:

1. The average position $\langle x_t \rangle$.

2. The variance of the position $\langle x_t^2 \rangle - \langle x_t \rangle^2$.

3. Repeat for a biased random walk with probabilities p to go right Δx and $q = 1 - p$ to go left Δx at each time step.

4. Reflect upon the role played by the sum of random variables and the central limit theorem in the derivations and conclusions.

3 Derivation of the Diffusion Equation

Focus on $p = 1/2$, and derive the diffusion equation for the probability $P(x,t)$ of finding the particle at position x at time t. Start from the master equation for the random walk and take the continuum limit.

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

with $D \equiv \langle (\Delta x)^2 \rangle / (2\Delta t)$, Δx the infinitesimal increase in position and Δt the infinitesimal increase in time.

Extend to the case $p \neq 1/2$.

A. Kundu, S. N. Majumdar, and G. Schehr, *Universal distribution of the number of minima for random walks and Lévy flights*, Phys. Rev. E **110**, 024137 (2024).