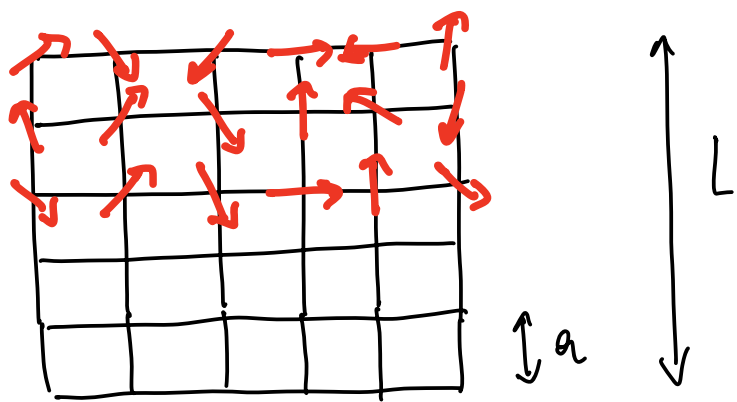


TD2: XY model and Spin waves



$$H = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'}$$

$$|S_{\vec{r}}| = 1$$

$$\vec{S}_{\vec{r}} \equiv (\cos \theta_{\vec{r}}, \sin \theta_{\vec{r}})$$

Phenomenological analysis

1. $J > 0$ ferromagnetic interaction $\rightarrow$ favours the alignment of the spins
2. $T_c \sim J$ (if a critical temperature exists for this model)
3. $T \gg J \rightarrow$ disordered configuration
Paramagnetic phase
Each spin points in a random direction

Can we perform a mean-field analysis?

Fully connected approximation $\rightarrow H = -\frac{J}{2N} \sum_{i,j} \vec{S}_i \cdot \vec{S}_j = -\frac{J}{2N} \left(\sum_i \vec{S}_i \right)^2$

$$\frac{1}{N} \sum_i \vec{S}_i = \vec{m} = m_x \vec{e}_x + m_y \vec{e}_y$$

$$\vec{m} \cdot \vec{m} = m_x^2 + m_y^2 = \left(\frac{1}{N} \sum_i \cos(\theta_i) \right)^2 + \left(\frac{1}{N} \sum_i \sin(\theta_i) \right)^2$$

$$Z = \int \prod_{i=1}^N \pi d\vec{S}_i e^{\frac{\beta J N}{2} \left(\frac{1}{N} \sum_i \cos \theta_i \right)^2 + \frac{\beta J N}{2} \left(\frac{1}{N} \sum_i \sin \theta_i \right)^2}$$

$$= \frac{\beta J N}{2\pi} \int_{-\infty}^{+\infty} dm_x \int_{-\infty}^{+\infty} dm_y e^{-\frac{\beta J N}{2} (m_x^2 + m_y^2)}$$

$$\times \int \prod_i d\theta_i e^{\beta J m_x \cos \theta_i + \beta J m_y \sin \theta_i}$$

$$Z_1 = \int dm_x dm_y e^{-\beta N f(m_x, m_y)}$$

$$f(m_x, m_y) = \frac{J}{2} (m_x^2 + m_y^2) - \frac{1}{\beta} \ln \int_0^{2\pi} d\theta_i e^{\beta J (m_x \cos \theta_i + m_y \sin \theta_i)}$$

$$I_0(x) \simeq 1 + \frac{x^2}{4} + \frac{1}{64} x^4$$

$$\ln I_0(x) \simeq \frac{x^2}{4} + \frac{x^4}{64} - \frac{1}{2} \frac{x^4}{16}$$

$$2\pi I_0(\beta J \sqrt{m_x^2 + m_y^2})$$

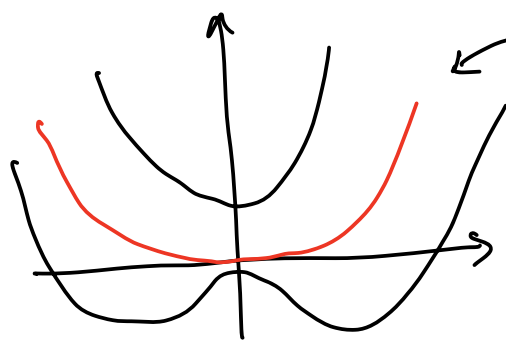
modified Bessel function of the first kind

$$f(|m|) = \frac{J}{2} |m|^2 - \frac{1}{\beta} \ln (2\pi I_0(\beta J |m|))$$

expand around $|m|=0$

$$f(|m|) \simeq \frac{J}{2} |m|^2 - \frac{1}{\beta} \left(\ln 2\pi + \frac{(\beta J)^2}{4} |m|^2 - \frac{(\beta J)^4}{64} |m|^4 + \dots \right)$$

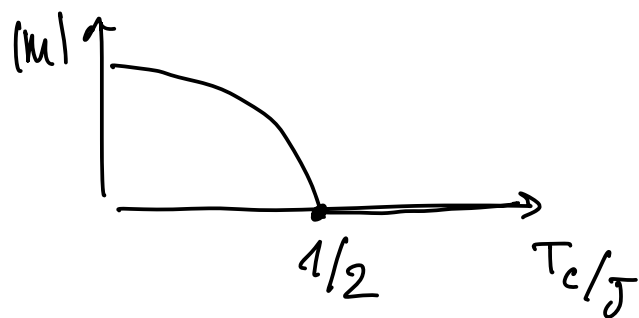
$$f(|m|) \simeq -\frac{1}{\beta} \ln 2\pi + \frac{J}{2} \left(1 - \frac{\beta J}{2} \right) |m|^2 + \frac{\beta^3 J^4}{64} |m|^4$$



$$1 - \frac{\beta_c J}{2} = 0$$

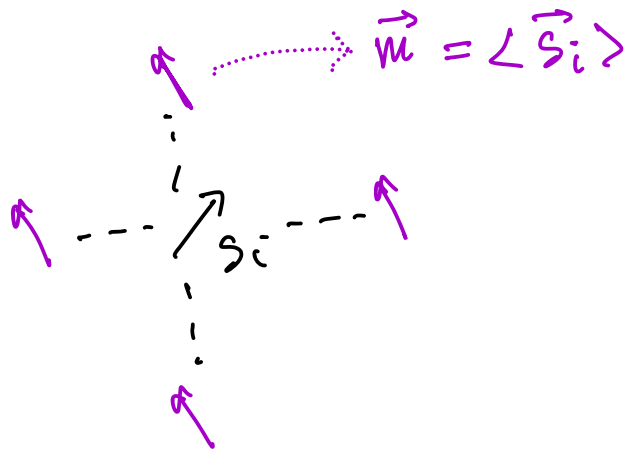
$$\frac{T_c}{J} = \frac{1}{2}$$

critical temperature



Curie-Weiss approximation

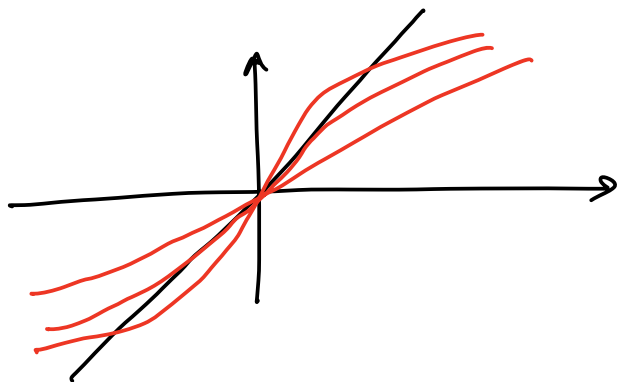
$$M_{cw} \approx -JZ\vec{m} \cdot \vec{S}_i$$



$$Z_{cw} = \int_0^{2\pi} d\theta e^{\beta J Z m \cos(\theta)} = 2\pi I_0(\beta J Z m)$$

$$\langle |\vec{S}_i| \rangle = \langle \cos \theta \rangle = \frac{\partial \ln Z_{cw}}{\partial (\beta J Z m)}$$

$$|m| = \frac{I_1(\beta J Z |m|)}{I_0(\beta J Z |m|)} \approx \frac{\beta J Z |m|}{2} - \frac{(\beta J Z |m|)^3}{16}$$



$$\frac{\beta_c J Z}{2} = 1 \quad T_c = \frac{Z}{2}$$

$$\vec{S}_i = \vec{m} + \delta \vec{S}_i$$

$$\begin{aligned} \vec{S}_i \cdot \vec{S}_j &= (\vec{m} + \delta \vec{S}_i) \cdot (\vec{m} + \delta \vec{S}_j) \\ &= m^2 + \vec{m} \cdot \delta \vec{S}_i + \vec{m} \cdot \delta \vec{S}_j + \delta \vec{S}_i \cdot \delta \vec{S}_j \\ &= m^2 + \vec{m} \cdot (\vec{S}_i - \vec{m}) + \vec{m} \cdot (\vec{S}_j - \vec{m}) + \delta \vec{S}_i \cdot \delta \vec{S}_j \\ &\approx -m^2 + \vec{m} \cdot (\vec{S}_i + \vec{S}_j) \end{aligned}$$

However the mean-field approximation is wrong for the XY model in 2d

- Many low-energy excitations that destroy the ferromagnetic order

Similar to the Ising model in 1d

$\uparrow \uparrow \uparrow \uparrow \dots \uparrow \uparrow \uparrow$

→ Ground state

$\uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \dots \downarrow \downarrow \uparrow \uparrow \uparrow$

→ excited state

$$\Delta E = 4J$$

$$S = k_B \ln N$$

→ the free energy at any finite T

For the xy model in 2d the low-energy excitations that destroy the ferromagnetic order are the "spin waves"



Low-T expansion

$$1. H = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos(\theta_{\vec{r}} - \theta_{\vec{r}'})$$

Ground state $\rightarrow \theta_{\vec{r}} = \theta_0 \quad \forall \vec{r} \quad E = -2JN$
(infinitely degenerate)

2. Low-T $\rightarrow$ "close" to the ground state

$$\theta_{\vec{r}} \simeq \theta_0 + \delta\theta_{\vec{r}}$$

$$\cos(\theta_{\vec{r}} - \theta_{\vec{r}'}) \simeq 1 - \frac{1}{2} (\theta_{\vec{r}} - \theta_{\vec{r}'})^2 + \dots$$

$$H_{sw} = -2JN + \frac{J}{2} \sum_{\langle \vec{r}, \vec{r}' \rangle} (\theta_{\vec{r}} - \theta_{\vec{r}'})^2$$

$$3. \frac{df}{dx} = \frac{f(x+a/2) - f(x-a/2)}{a}$$

$$\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial f}{\partial x}(x+a/2, y) - \frac{\partial f}{\partial x}(x-a/2, y) \right) / a$$

$$= \frac{1}{a} \left(\frac{f(x+a, y) - f(x, y)}{a} - \frac{f(x, y) - f(x-a, y)}{a} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f(x+a, y) + f(x-a, y) - 2f(x, y)}{a^2}$$

$$\nabla^2 f = \frac{f(x+a, y) + f(x-a, y) + f(x, y+a) + f(x, y-a) - 4f(x, y)}{a^2}$$

4. $-a \nabla^2 G_r = \delta_{\vec{r}, \vec{0}}$

Green's function of the Laplacian on the square lattice

$$\hat{G}_{\vec{q}} = \sum_{\vec{r}} e^{i\vec{q} \cdot \vec{r}} G_{\vec{r}}$$

$$\vec{q} = \frac{2\pi}{L} (n_x, n_y)$$

$$G_{\vec{r}} = \frac{1}{N} \sum_{\vec{q} \neq \vec{0}} e^{-i\vec{q} \cdot \vec{r}} G_{\vec{q}}$$

$$-\frac{L}{2a} \leq n_{x,y} \leq \frac{L}{2a}$$

$$-a^2 \nabla^2 G_{\vec{r}} = 4G(x, y) - G(x+a, y) - G(x-a, y) - G(x, y+a) - G(x, y-a)$$

$$G(x+a, y) = \frac{1}{N} \sum_{\vec{q} \neq \vec{0}} e^{-i\vec{q} \cdot \vec{r} - iq_x a} G_{\vec{q}}$$

$$G(x-a, y) = \frac{1}{N} \sum_{\vec{q} \neq \vec{0}} e^{-i\vec{q} \cdot \vec{r} + iq_x a} G_{\vec{q}}$$

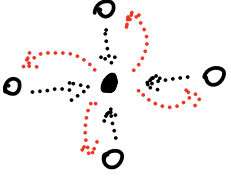
$$-a^2 \nabla^2 G_{\vec{r}} = \frac{1}{N} \sum_{\vec{q} \neq \vec{0}} e^{-i\vec{q} \cdot \vec{r}} G_{\vec{q}} (4 - 2\cos(q_x a) - 2\cos(q_y a)) = \delta_{\vec{r}, \vec{0}}$$

$$G_{\vec{q}} = \frac{1}{4 - 2\cos(q_x a) - 2\cos(q_y a)}$$

$$G_{\vec{0}} \approx \frac{1}{2\pi} \ln \frac{L}{a}$$

$$G_{|\vec{r}| \gg a} - G_0 \approx -\frac{1}{2\pi} \ln \frac{|\vec{r}|}{a} - C + o(1) \quad C = \frac{1}{2\pi} \left(8 + \frac{3}{2} \ln 2 \right)$$

$$Z_{SW} = \int \prod_{\vec{r}} d\theta_{\vec{r}} e^{\beta H_{SW}} = e^{2\beta J N} \int \prod_{\vec{r}} d\theta_{\vec{r}} e^{-\frac{K}{2} \sum_{\langle r, r' \rangle} (\theta_r - \theta_{r'})^2}$$

$$\sum_{\langle r, r' \rangle} (\theta_r - \theta_{r'})^2 = \frac{1}{2} \sum_r \left[(\theta_r - \theta_{r+ae_x})^2 + (\theta_r - \theta_{r-ae_x})^2 + (\theta_r - \theta_{r+ae_y})^2 + (\theta_r - \theta_{r-ae_y})^2 \right]$$


$$= \frac{1}{2} \sum_r \left[\theta_r^2 + \theta_{r+ae_x}^2 - 2\theta_r \theta_{r+ae_x} + \theta_r^2 + \theta_{r-ae_x}^2 - 2\theta_r \theta_{r-ae_x} + \theta_r^2 + \theta_{r+ae_y}^2 - 2\theta_r \theta_{r+ae_y} + \theta_r^2 + \theta_{r-ae_y}^2 - 2\theta_r \theta_{r-ae_y} \right]$$

$$= \sum_r \left[4\theta_r^2 - \theta_r \theta_{r+ae_x} - \theta_r \theta_{r-ae_x} - \theta_r \theta_{r+ae_y} - \theta_r \theta_{r-ae_y} \right]$$

$$\parallel \theta_r (-a \nabla^2 \theta_r)$$

$$Z_{SW} \propto \int \mathcal{D}\theta e^{-\frac{K}{2} \sum_r \theta_r (-a \nabla^2 \theta_r)}$$

Alternative route: (continuum limit)

$$\sum_{\langle r, r' \rangle} (\theta_r - \theta_{r'})^2 = \frac{1}{2} \sum_{\vec{r}} \sum_{\vec{r}' \in \partial \vec{r}} (\theta_r - \theta_{r'})^2$$

$$= \frac{1}{2} \sum_{\vec{r}} \left[(\theta_{\vec{r}} - \theta_{\vec{r}-a\vec{e}_x})^2 + (\theta_r - \theta_{r+a\vec{e}_x})^2 \right. \\ \left. + (\theta_r - \theta_{r-a\vec{e}_y})^2 + (\theta_r - \theta_{r+a\vec{e}_y})^2 \right]$$

$$\theta_r - \theta_{r-a\vec{e}_x} \simeq a \frac{\partial \theta_r}{\partial x}$$

$$= \frac{1}{2} \sum_r \left(2 \left(a \frac{\partial \theta_r}{\partial x} \right)^2 + 2 \left(a \frac{\partial \theta_r}{\partial y} \right)^2 \right)$$

$$= \frac{1}{2} \cdot 2 a^2 \sum_r (\nabla \theta_r)^2$$

$$Z_{sw} = e^{i\beta J N} \int D\theta e^{-\frac{k}{2} \sum_r a^2 (\nabla \theta_r)^2}$$

$$Z_{sw} \simeq e^{2\beta J N} \int D\theta_r e^{-\frac{k}{2} \int d^2 \vec{r} (\nabla \theta_r)^2}$$

$$\int d^2 \vec{r} (\vec{\nabla} \theta_r) (\vec{\nabla} \theta_r) \simeq - \int d^2 \vec{r} \theta_r \nabla^2 \theta_r = \int d^2 \vec{q} q^2 |\theta_q|^2$$

$$Z_{sw} \propto \int D\theta_q e^{-\frac{k}{2} \int d^2 \vec{q} q^2 |\theta_q|^2} \Rightarrow \langle \theta_q \theta_{q'} \rangle = \frac{1}{k q^2} \delta_{q+q', 0}$$

$$\langle \theta_r \theta_{r'} \rangle = \frac{1}{Z_{SW}} \int \mathcal{D}\theta \, \theta_r \theta_{r'} e^{-\frac{k}{2} \sum_{i,j} \theta_i (-\alpha \nabla^2) \theta_j}$$

Gaussian integral $\langle \theta_r \theta_{r'} \rangle = K (-a \nabla^2)^{-1}_{r,r'}$

[illegible]

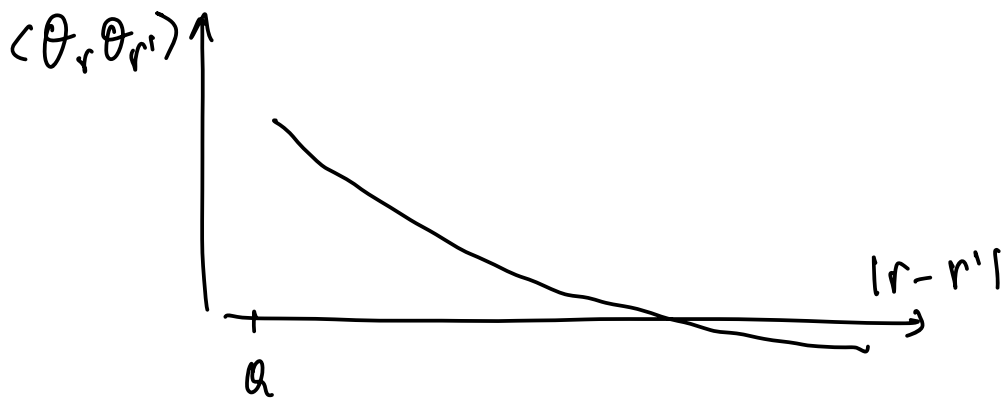
$$-a \nabla^2 \cdot G_{\vec{r}} = \delta_{\vec{r}, \vec{0}}$$

$$-a \nabla^2 G_{|r-r'|} = \delta \vec{r}, \vec{r}' = \underline{1}$$

$$(-a \nabla^2)^{-1}_{r,r'} = G_{r,r'} = G_{|r-r'|}$$

$$\langle \theta_r \theta_{r'} \rangle = K^{-1} G_{|r-r'|} \approx K^{-1} \left(G_0 - \frac{1}{2\pi} \ln \frac{|r-r'|}{a} - c + o(1) \right)$$

$$\langle \theta_r \theta_{r'} \rangle \approx \frac{1}{2\pi K} \ln \frac{L}{a} - \frac{1}{2\pi K} \ln \frac{|r-r'|}{a} - \frac{c}{K}$$

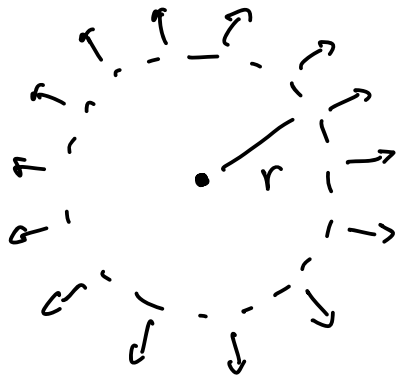


Poisson equation (electrostatic in 2d)

$$\nabla^2 V = -\rho / \epsilon_0$$

$$\rho(\vec{r}) = \delta(\vec{r})$$

Gauss theorem



$$2\pi r |E(r)| = \frac{1}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \vec{e}_r = -\vec{\nabla} V = -\frac{dV}{dr} \vec{e}_r$$

$$V = -\frac{1}{2\pi\epsilon_0} \ln r$$

5. $\langle \theta_{\vec{r}} \rangle = 0$

$$\langle \vec{S}_{\vec{r}} \rangle = \langle \cos \theta_{\vec{r}} \rangle \vec{e}_x + \langle \sin \theta_{\vec{r}} \rangle \vec{e}_y$$

$$\langle \cos \theta_{\vec{r}} \rangle = \langle \text{Re } e^{i\theta_{\vec{r}}} \rangle = \text{Re} \langle e^{i\theta_{\vec{r}}} \rangle$$

$$\vec{J} = (0, 0, \dots, 0, i, 0, 0, \dots, 0)$$

$\uparrow$
 r-th position

$$J_{\vec{r}, \vec{r}} = i \delta_{\vec{r}, \vec{r}}$$

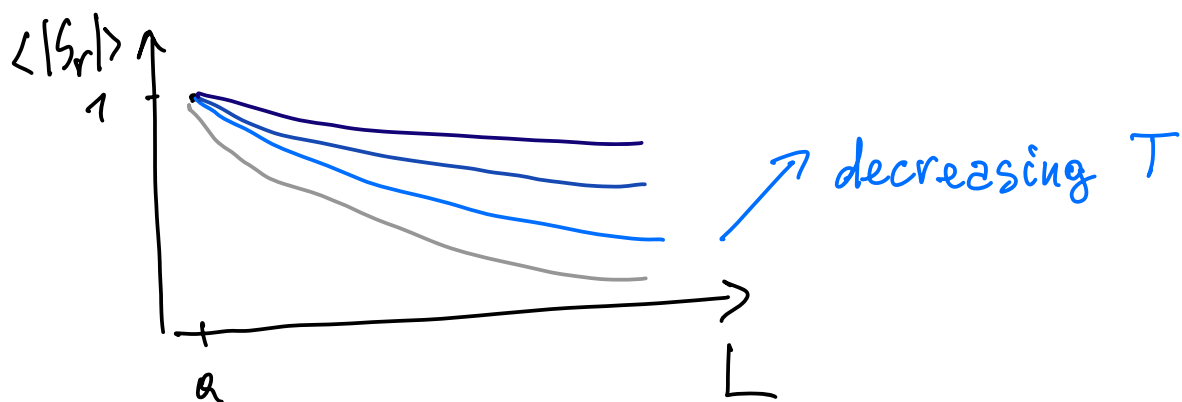
$$e^{i\theta_{\vec{r}}} = e^{\sum_{\vec{r}'} J_{\vec{r}, \vec{r}'} \cdot \theta_{\vec{r}'}} \quad \langle \cos \theta_r \rangle = \text{Re} \langle e^{J \cdot \theta} \rangle$$

$$\langle e^{J \cdot \theta} \rangle = \frac{1}{Z_{SW}} \int D\theta \, e^{-\frac{\kappa}{2} \theta \cdot (-a \nabla^2) \cdot \theta + J \cdot \theta}$$

$$= e^{\frac{1}{2\kappa} J \cdot (-a \nabla^2)^{-1} \cdot J} = e^{\frac{1}{2\kappa} J \cdot G \cdot J}$$

$$\langle \cos \theta_r \rangle = \text{Re } e^{\frac{1}{2k} \sum_{r,r'} J_r G_{r,r'} J_{r'}} = \text{Re } e^{-\frac{1}{2k} G_0} \simeq e^{-\frac{1}{2k} \cdot \frac{1}{2\pi} \ln \frac{L}{a}}$$

$$\langle \cos \theta_r \rangle \simeq e^{-\frac{1}{4\pi k} \ln(\frac{L}{a})} = e^{\frac{1}{4\pi k} \ln(\frac{a}{L})} = \left(\frac{a}{L}\right)^{\frac{k_B T}{4\pi J}}$$



No finite magnetization
in the thermodynamic limit but there is a "residual"
magnetization for L finite

$$6. \langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \langle \cos(\theta_r - \theta_{r'}) \rangle = \text{Re} \langle e^{i(\theta_r - \theta_{r'})} \rangle$$

$$\vec{J} = (000 \dots 0 \overset{\uparrow}{1} 00000 \overset{\uparrow}{-i} 00 \dots 00)$$

$\uparrow$ r $\uparrow$ r'

$$\vec{J}(w) = i(\delta_{\vec{r}, \vec{w}} - \delta_{\vec{r}', \vec{w}})$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \text{Re} \langle e^{\vec{J} \cdot \vec{\theta}} \rangle = \frac{1}{Z_{SW}} \int D\theta e^{-\frac{k}{2} \theta \cdot (-a \nabla^2) \theta + J\theta}$$

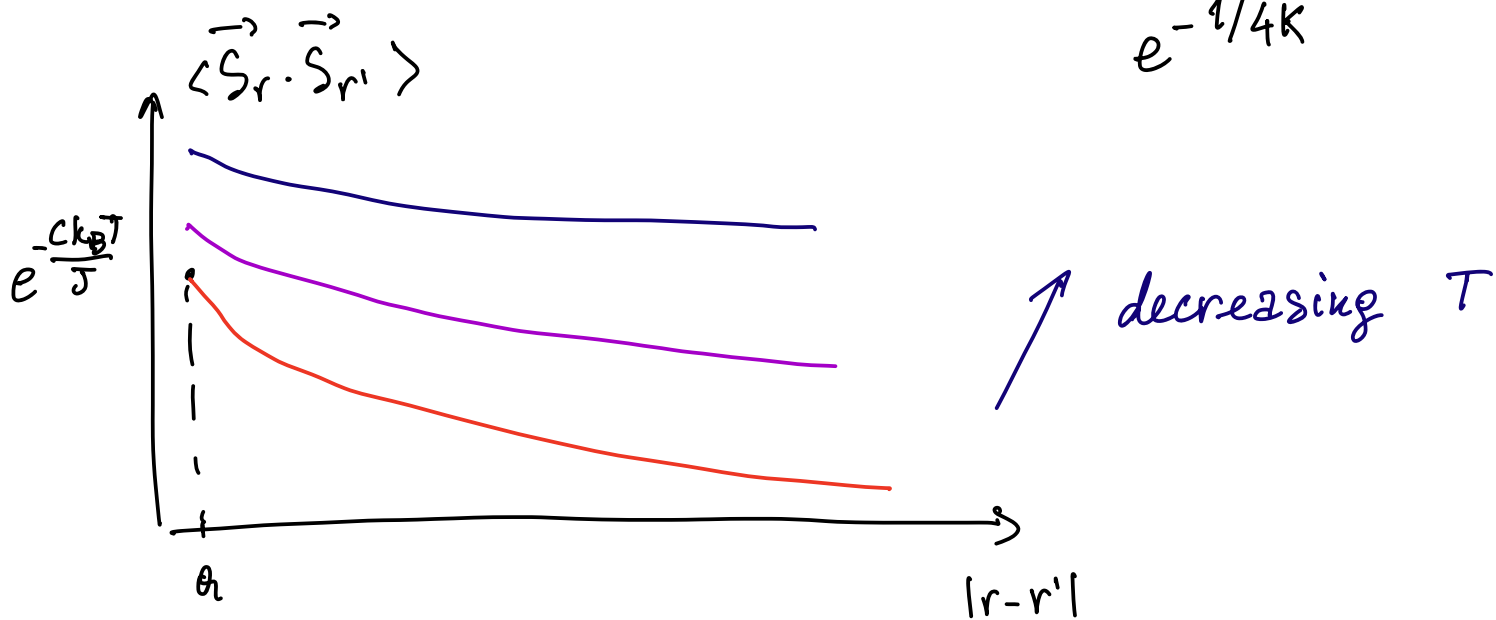
$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \text{Re } e^{\frac{1}{2k} J \cdot G \cdot J} = e^{\frac{1}{2k} \sum_{a,b} J_a G_{a,b} J_b}$$

$$\begin{aligned}
\sum_{a,b} J_a G_{a,b} J_b &= - \sum_{a,b} (S_{ar} - S_{ar'}) G_{a,b} (S_{br} - S_{br'}) \\
&= - \sum_{a,b} \left(S_{ar} S_{br} + S_{ar'} S_{br'} - S_{ar} S_{br'} - S_{ar'} S_{br} \right) G_{ab} \\
&= - (G_{rr} + G_{r'r'} - G_{rr'} - G_{r'r}) \\
&= -2 (G_0 - G_{|r-r'|}) = 2 (G_{|r-r'|} - G_0)
\end{aligned}$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = e^{\frac{1}{k} (G_{|r-r'|} - G_0)} \simeq e^{-\frac{1}{2\pi k} \ln \frac{|r-r'|}{a} - \frac{c}{k}}$$

$$= e^{-\frac{c}{k}} e^{\frac{1}{2\pi k} \ln \frac{a}{|r-r'|}} = \underbrace{e^{-c/k}}_{\simeq e^{-1/4k}} \left(\frac{a}{|r-r'|} \right)^{\frac{k_B T}{2\pi J}}$$

power-law correlations



7. $\xi = \infty$

8. $\chi = \left| \frac{d \vec{m}}{d h} \right|$

$$H = -J \sum_{\langle r r' \rangle} S_r \cdot S_{r'} - \vec{h} \cdot \sum_{\vec{r}} \vec{S}_r \quad \vec{h} = h \vec{e}_x$$

$$H_{sw} \approx -2JN + \frac{J}{2} \sum_{\langle r r' \rangle} (\theta_r - \theta_{r'})^2 - h \sum_{\vec{r}} \cos(\theta_r)$$

$$\frac{\partial \ln Z}{\partial (\beta h)} = \left\langle \sum_{\vec{r}} \vec{S}_r \right\rangle = \vec{M}$$

$$\frac{\partial^2 \ln Z}{\partial (\beta h)^2} = \begin{cases} \frac{\partial}{\partial (\beta h)} \vec{M} = k_B T \chi \\ \left\langle \left(\sum_{\vec{r}} S_r \right)^2 \right\rangle - \left(\left\langle \sum_{\vec{r}} S_r \right\rangle \right)^2 \end{cases}$$

$$k_B T \chi = \sum_{r, r'} \langle S_r S_{r'} \rangle - N^2 \langle S_r \rangle^2$$

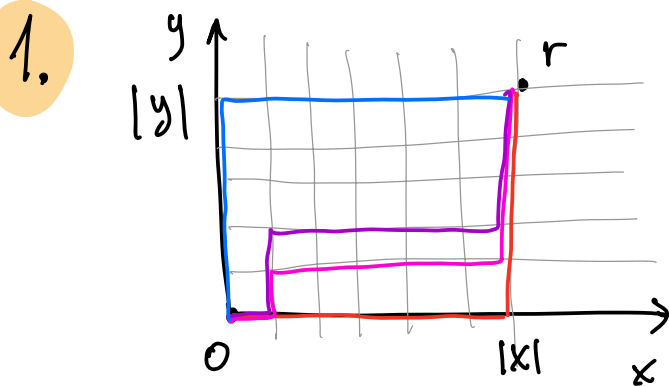
$$k_B T \chi = N \sum_r \langle S_0 S_r \rangle - N^2 \langle S_r \rangle^2$$

$$\begin{aligned} \left(\frac{L}{a} \right)^2 & \int \frac{d^2 r}{a^2} e^{-c/k} \left(\frac{a}{r} \right)^{k_B T / 2\pi J} \approx 2\pi e^{-c/k} \int_a^L \left(\frac{a}{r} \right)^{\frac{k_B T}{2\pi J}} \left(\frac{r}{a} \right) \frac{dr}{a} \\ & = 2\pi e^{-c/k} \int_a^{L/a} x^{1 - k_B T / 2\pi J} dx \\ & = 2\pi e^{-c/k} \frac{x^{2 - k_B T / 2\pi J}}{2 - \frac{k_B T}{2\pi J}} \Big|_a^{L/a} \\ & \approx \frac{2\pi e^{-c/k}}{2 - \frac{k_B T}{2\pi J}} \left(\frac{L}{a} \right)^{2 - \frac{k_B T}{2\pi J}} \end{aligned}$$

$$\frac{k_B T \chi}{N} \simeq \frac{2\pi e^{-c/k}}{2 - \frac{k_B T}{2\pi J}} \left(\frac{L}{a} \right)^{2 - \frac{k_B T}{2\pi J}} - \left(\frac{L}{a} \right)^{2 - \frac{k_B T}{2\pi J}} \propto \left(\frac{L}{a} \right)^{2 - \frac{k_B T}{2\pi J}}$$

High-T expansion

Diverges at low enough T in the thermodynamic limit
($T < 4\pi J$)



|x| steps right
|y| steps up
with any order

$$W(\vec{r}) = \binom{|x|+|y|}{|x|} = \binom{|x|+|y|}{|y|} = \frac{(|x|+|y|)!}{|x|! |y|!}$$

|y|=0

$W(r) = 1$

$W(r)$ is maximal when $|x|=|y|$

$$W(r) = \frac{(2|x|)!}{(|x|!)^2} = \frac{(2x)^{2x} \cancel{e^{-2x}} \sqrt{4\pi x}}{(x^x \cancel{e^{-x}} \sqrt{2\pi x})^2} = \left(\frac{2x}{x} \right)^{2x} \cancel{\frac{1}{\sqrt{\pi x}}}$$

$$W(r) \simeq 2^{2|x|} = 2^{\|r\|_1}$$

In general $W(r) \leq 2^{\|r\|_1}$

2.

$$\int_0^{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) d\theta_2 = \pi \cos(\theta_1 - \theta_3)$$

$$\left(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \right) \left(\cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \right)$$

$$\int_0^{2\pi} \cos^2 \theta_2 d\theta_2 = \int_0^{2\pi} \sin^2 \theta_2 d\theta_2 = \pi$$

$$\int_0^{2\pi} \cos \theta_2 \sin \theta_2 d\theta_2 = \int_0^{2\pi} \frac{\sin(2\theta_2)}{2} d\theta_2 = -\frac{\cos(2\theta_2)}{4} \Big|_0^{2\pi} = 0$$

$$\begin{aligned} \int_0^{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) d\theta_2 &= \pi \cos \theta_1 \cos \theta_3 + \pi \sin \theta_1 \sin \theta_3 \\ &= \pi \cos(\theta_1 - \theta_3) \end{aligned}$$

Alternative method

$$\theta = \theta_1 - \theta_3 \Rightarrow \theta_3 = \theta_1 - \theta$$

$$\int_0^{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_1 + \theta) d\theta_2 = \pi \cos(\theta)$$

$$\theta' = \theta_2 - \theta_1$$

$$\int_0^{2\pi} \cos \theta' \cos(\theta' + \theta) d\theta' = \pi \cos(\theta)$$

$$\int_0^{2\pi} \cos(\theta') \cos(\theta' + \theta) d\theta' \equiv f(\theta)$$

$$f(0) = \int_0^{2\pi} \cos^2(\theta') d\theta' = \pi$$

$$f'(\theta) = -\int_0^{2\pi} \cos(\theta') \sin(\theta' + \theta) d\theta' \quad f'(0) = 0$$

$$f''(\theta) = -\int_0^{2\pi} \cos(\theta') \cos(\theta' + \theta) d\theta' = -f(\theta)$$

$$f''(\theta) + f(\theta) = 0 \Rightarrow f(\theta) = A \cos \theta + B \sin \theta$$

$$f'(0) = 0 \Rightarrow B = 0$$

$$f(0) = \pi \Rightarrow A = \pi$$

3. $T \gg J$

$\beta J \ll 1$

$$Z = \int D\theta_r e^{\beta J \sum_{\langle rr' \rangle} \cos(\theta_r - \theta_{r'})} = \int D\theta_r \prod_{\langle rr' \rangle} e^{\beta J \cos(\theta_r - \theta_{r'})}$$

$$Z \simeq \int D\theta_r \prod_{\langle rr' \rangle} (1 + \beta J \cos(\theta_r - \theta_{r'}) + \dots)$$

$$\simeq \int D\theta_r \left(1 + \sum_{\langle rr' \rangle} \beta J \cos(\theta_r - \theta_{r'}) + \dots \right)$$

$$= (2\pi)^N + \beta J \sum_{\langle rr' \rangle} \int d\theta_r d\theta_{r'} \cos(\theta_r - \theta_{r'})$$

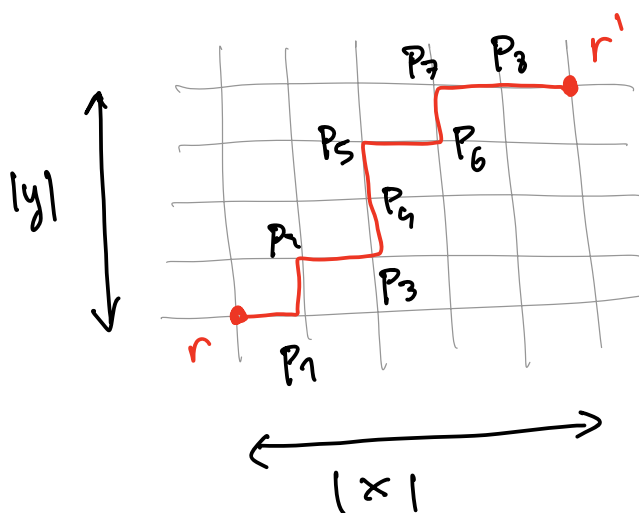
$$+ (\beta J)^2 \sum_{\substack{\langle rr' \rangle \\ \langle pp' \rangle}} \int d\theta_r d\theta_{r'} d\theta_p d\theta_{p'} \cos(\theta_r - \theta_{r'}) \cos(\theta_p - \theta_{p'})$$

+ ...

↑ only the term with $\langle rr' \rangle = \langle pp' \rangle$ survives

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle \equiv C(|\vec{r} - \vec{r}'|) = \frac{1}{Z} \int D\theta_r \cos(\theta_r - \theta_{r'}) \prod_{\langle pp' \rangle} e^{K \cos(\theta_p - \theta_{p'})}$$

$$C(|r-r'|) = \frac{1}{Z} \int D\theta_r \cos(\theta_r - \theta_{r'}) \prod_{\langle pp' \rangle} (1 + K \cos(\theta_p - \theta_{p'}))$$



the only terms that survive (at the lowest order in K) are of the form

$$K^{|r-r'|} \int D\theta_r \cos(\theta_r - \theta_{r'}) \cos(\theta_r - \theta_{P_1}) \cos(\theta_{P_1} - \theta_{P_2}) \times \dots \times \cos(\theta_{P_7} - \theta_{P_8}) \cos(\theta_{P_8} - \theta_{r'})$$

$$\int \cos(\theta_r - \theta_{p_1}) \cos(\theta_{p_1} - \theta_{p_2}) d\theta_{p_1} = \pi \cos(\theta_r - \theta_{p_2})$$

$$\int \cos(\theta_r - \theta_{p_2}) \cos(\theta_{p_2} - \theta_{p_3}) d\theta_{p_2} = \pi \cos(\theta_r - \theta_{p_3})$$

⋮

$$\int \cos(\theta_r - \theta_{p_g}) \cos(\theta_{p_g} - \theta_{r'}) d\theta_{p_g} = \pi \cos(\theta_r - \theta_{r'})$$

$$\int d\theta_r d\theta_{r'} \cos^2(\theta_r - \theta_{r'}) = 2\pi^2$$

$$\langle \vec{S}_r \cdot \vec{S}_{r'} \rangle = \frac{1}{(2\pi)^N} (2\pi)^{N - \|r - r'\|_1} \pi^{\|r - r'\|_1} k^{\|r - r'\|_1} \mathcal{N}(\|r - r'\|)$$

$$\mathcal{C}(\|r - r'\|) \simeq \left(\frac{k}{2}\right)^{\|r - r'\|_1} \mathcal{N}(\|r - r'\|)$$

$$\mathcal{C}(\|r - r'\|) \leq k^{\|r - r'\|_1} = k^{|x| + |y|}$$

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y| \geq |x|^2 + |y|^2 = r^2$$

$$k < 1 \quad k^{|x| + |y|} \leq |k|^r$$

$$\mathcal{C}(\|r - r'\|) \leq k^r = e^{r \ln k} = e^{-r/\xi}$$

$$\xi \simeq -\frac{1}{\ln k} = -\frac{1}{\ln J/T}$$



Low-T

power law
correlations

$$\xi = \infty \quad \chi = \infty$$

critical phase

$$\langle m \rangle \propto \left(\frac{1}{L} \right)^{\frac{K_B T}{4\pi J}} \xrightarrow{L \rightarrow \infty} 0$$

High-T

exponential correlations
"standard" paramagnetic
phase