2 - CLASSICAL PHASE TRANSITIONS

2.1 PHENOMENON

- USUALLY, SYSTEM IN CONTACT W/ ENVIRONMENT
  (BUT ALSO CLOSED POSSIBLE)

- GLOBAL OBSERVABLES
  eg. $\frac{\text{ENERGY}}{N}$, $\frac{\text{MAGN}}{N}$, ETC
  DENSITIES

- CONTROL PARAMETERS
  eg. TEMP., MAGN FIELDS, ETC

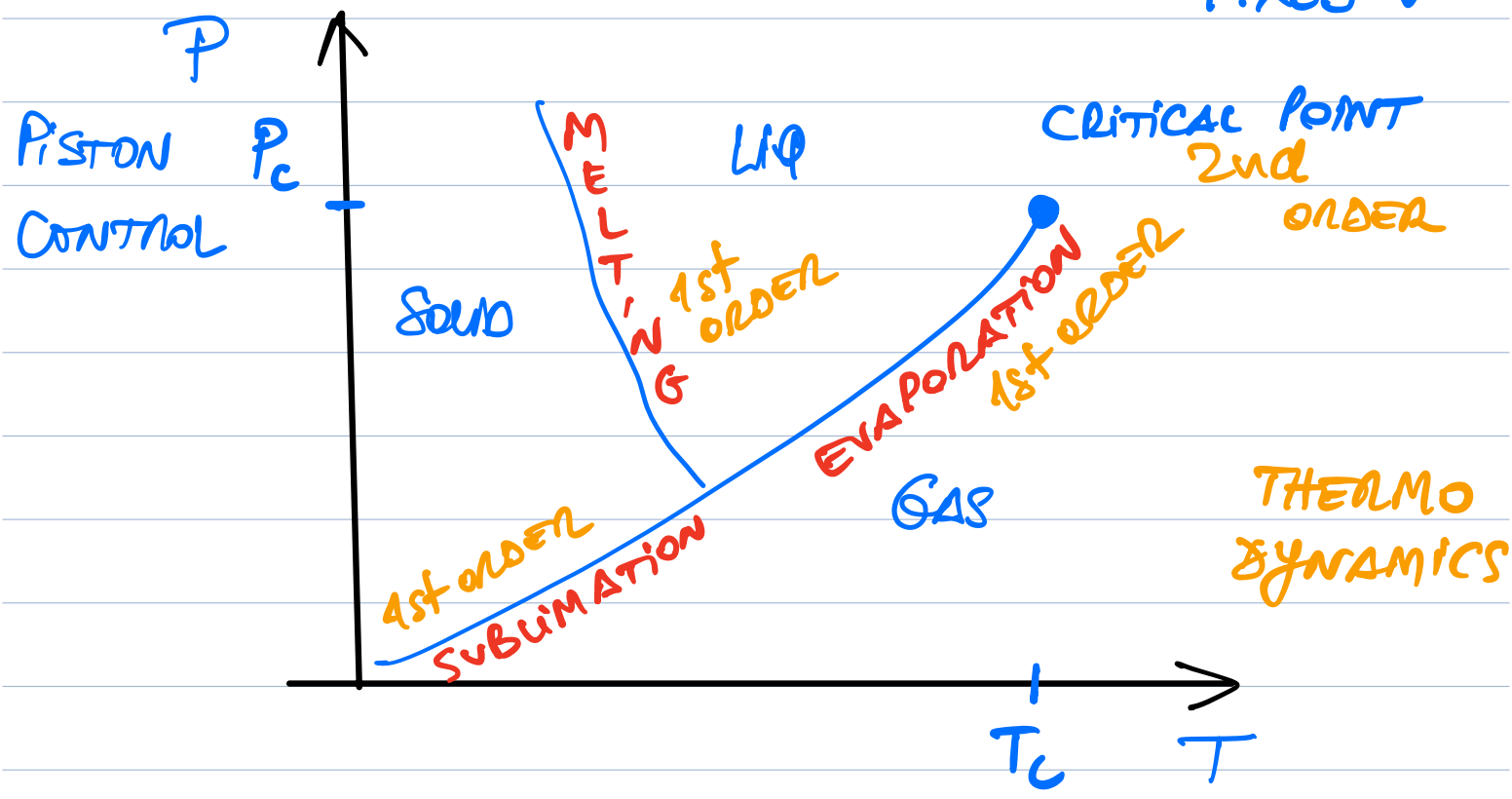
TUNE CONTROL PARAM $\longrightarrow$ MEASURE OBS

PHASE TRANS $\equiv$ SHARP CHANGES IN MACROSC BEHAV.

COLLECT PHENOMENA IN $N \to \infty$

PHASE DIAGRAMS (3d syst)

FIXED V



IN ISOLATION, E FIXED (REPLACES T)

VOLUME ALSO FIXED

CHANGES IN PARAM IN HAMILTONIAN

PHASES IDENTIFIED BY PARAM DEP. OF ORDER PARAMETERS

↓

GLOBAL OBS. JUDICIOUSLY CHOSEN

PHASE TRANSITIONS

SHARP CHANGES (INTUITIVE)

NON-ANALYTICITIES of RELEVANT THERMODYN POTENTIALS WITH RESPECT TO CONTROL PARAM. (FORMAL)

e.g. FREE-ENERGY $[\beta J] = 1$

$$-\beta f(\beta J) = \frac{1}{N} \ln Z(\beta J)$$

$$Z(\beta J) = \sum_{C} e^{-\beta H_J(C)}$$

LOCATE W/ THEM THE CRITICAL CURVES

GOALS

- ORDER PARAM AS fcts of CONTROL PARAM
- CRITICAL CURVES IN PH. DIAG.
- CRITICAL BEHAVIOURS
- THERMODYN PROPS $PV = nRT$

METHODS

- MEAN FIELD
  - QUALITATIVE UNDERSTANDING ✓
  - QUANTITATIVE APPROX ONLY ~
  - CRIT BEH TYP. WRONG ✗
- SCALING TH
  - PHENOM of CRIT PHEN ✓
- RENORM. GROUP
  - CRIT BEH ✓

BEYOND STANDARD PH. TRANS

TOPOLOGICAL ONES

QUANTUM PHASE TRANS (LATER)

DYN PHASE TRANS. (NOT DISCUSSED)

CONTEXT

PART: IN INTERACT

SPINS IN INTERACT

EASIER TO TREAT

2.2 ISING MODELS & EXTENSIONS

SIMPLIFIED REPRESENTATION OF A MAGN. SYST

MAGN MOM.  $\vec{S_i}$ LOCALIZED ARROW W/ n-COMP.

$$H[\{\vec{S_i}\}] = -\frac{J}{2}\sum_{ij} \vec{S_i}\cdot\vec{S_j} - \sum_i \vec{h}\cdot\vec{S_i}$$

$J>0$ FM // INTERACTIONS

$J<0$ AF ANTI- //

$\sum_{ij}$ DECIDE WHETHER SPINS SIT ON A LATTICE / A GRAPH

$\vec{h}$ UNIFORM MAGN FIELD

d DIM. OF SPACE n DIM OF SPIN

FACTS — SIMPLEST MODEL ISING $J>0$

- SOLVED IN $d=1,\ 2\ (h=0),\ \infty$
  UNSOLVED ANALYTICALLY IN $d=3$
- $d \geq d_l = 2$ PH. TRANS AT $0 < \beta_c J < \infty$ $(h=0)$
  BTW FM & PM
- d_u UPPER CRIT DIM : MF OK $d > d_u = 4$
  FAILS BELOW.
- SCALING & RG AT WORK ✓

OTHER CASES, SAY STILL $J>0$ $\forall ij$

- $n=1$ ISING
  $n=2$ XY
  $n=3$ HEISENBERG

$n \to \infty$ $O(n)$

$$\sum_{a=1}^{n} \left(S_i^a\right)^2 = n$$

MORE DIFFICULT / INTERESTING CASES

- FRUSTRATION AF ON $\triangle$ LATT.
- QUENCHED DISORDER $J \to J_{ij}$ pdfs
  $h \to h_i$

eg. RANDOM BOND FM $J_{ij} > 0$ SPIN-GLASS $J_{ij} \gtrless 0$

- STOCHASTIC DYNAMICS / KINETIC ISING OR
  N COMP MODELS / DOMAIN GROWTH

DYNAMICS — WE'LL NOT TREAT THEM

2.3 CONCEPTS

THERMODYN LIMIT

$N \to \infty$ NECESSARY TO GET NON-ANAL $f(\beta J, \beta h)$

OTHERWISE JUST $\sum_{\text{config.}}$ POSIT. TERM $= \mathcal{Z}$

AND THIS IS CONTINUOUS IN THE PARAM.
NO NON-ANALYTICITIES

STRESS

ROLE OF ONSAGER'S SOL 2d IM
A PROOF IN A SOLVABLE CASE

ORDER PARAMETERS

A MEASURE OF THE DEGREE OF ORDER

STATISTICAL AVERAGE OF AN OBSERVABLE

$\langle \theta \rangle$

WHICH TAKES Diff VALUES IN $\neq$ PHASES

eg. $\tilde{\rho}(\vec{k}) = \int d^d r \; \rho(\vec{r}) \, e^{i\vec{k}\vec{r}}$

LOCAL PART DENSITY

FOR SOLID LIQUID

$\vec{m} \equiv \left\langle \frac{1}{N} \sum_i \vec{s}_i \right\rangle$ FOR FM - PM

SUBTLETY ABOUT SYMM TO BE DISCUSSED LATER

TRANSITION CONTINUOUS VS. DISCONT.

↓ CONT DEPARTS FROM 0

↘ JUMP IN $\langle \theta \rangle$

SYMMETRIES

CONTINUOUS $O(n)$ SYMMETRY

$\vec{h} = 0$ UNIFORM ROTATION OF ALL SPINS

$$H[\{\vec{S}_i'\}] = -\frac{J}{2}\sum_{ij}\vec{S}_i'\cdot\vec{S}_j'$$

$$= -\frac{J}{2}\sum_{ij}^{N}\sum_{abc}^{n} R^{ab} S_i^b R^{ac} S_j^c$$

$$= -\frac{J}{2}\sum_{ij}^{N}\sum_{bc}^{n}\underbrace{\sum_{a}(R^t)^{ba} R^{ac}}_{\delta^{bc}} S_i^b S_j^c$$

$$R R^t = R^t R = \mathbb{1} \implies \delta^{bc}$$

$$= -\frac{J}{2}\sum_{ij}\sum_{b} S_i^b S_j^b = H[\{\vec{S}_i\}]$$

DISCRETE $\mathbb{Z}_2$ UP DOWN SYMM

FOR $n=1$, THE ISING CASE EVEN SIMPLER

$$H[\{s_i\}] = -\frac{J}{2}\sum_{ij} s_i s_j$$

$$= -\frac{J}{2}\sum_{ij} (-s_i)(-s_j)$$

$$= H[\{s_i\}]$$

- NOTE THAT THESE SYMM HOLD FOR ALL LATTICES / GRAPHS
- BROKEN BY $\vec{h} \neq 0$ OR $h \neq 0$

FOR PARTICLE SYST AND SOLID - LIQUID INVARIANCE IS SPACE TRANSLATIONS.

$$\vec{r}' = \vec{r} + \vec{\Delta}$$

SINCE, TYPICALLY, $V(\vec{r}_i, \vec{r}_j) = V(|\vec{r}_i - \vec{r}_j|)$

PINNING FIELD

CHECK CANDIDATE ORDER PARAM

$m = \frac{1}{N} \sum_i \langle s_i \rangle$

$$\langle s_i \rangle = \sum_{\{s_j = \pm 1\}} \frac{s_i \, e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

CHANGE of NAME $\{s_i \to s_i'\}$

$$= \sum_{\{s_j' = \pm 1\}} \frac{s_i' \, e^{-\beta H(\{s_j'\})}}{Z(\beta)}$$

CHANGE VAR. $s_i' = -s_i$

NOTE THAT $H(\{s_i\}) = H(\{-s_i\})$

$$= \sum_{\{s_j = \pm 1\}} \frac{(-s_i) \, e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

$$= - \langle s_i \rangle$$

$$\Rightarrow \quad \langle s_i \rangle = 0 \qquad \forall i$$

So, HOW CAN m BE $\neq 0$?

PINNING FIELD, WHEN GOING ACROSS CRITICALITY

RESIDUAL OR INTENTIONAL

THE ACTUAL PROCEDURE IS

$$\langle s_i \rangle = \lim_{h \to 0^+} \lim_{N \to \infty} \langle s_i \rangle_h$$

(or $h \to 0^-$) AND THIS FORM YIELDS

$\langle s_i \rangle \gtrless 0$ AT $(T/J) <$ CRIT $h \to 0^{\pm}$

$\langle s_i \rangle = 0$ AT $(T/J) >$ CRIT

BROKEN ERGODICITY

WE DEFINED ERGOD FROM

$$\overline{A} = \langle A \rangle$$

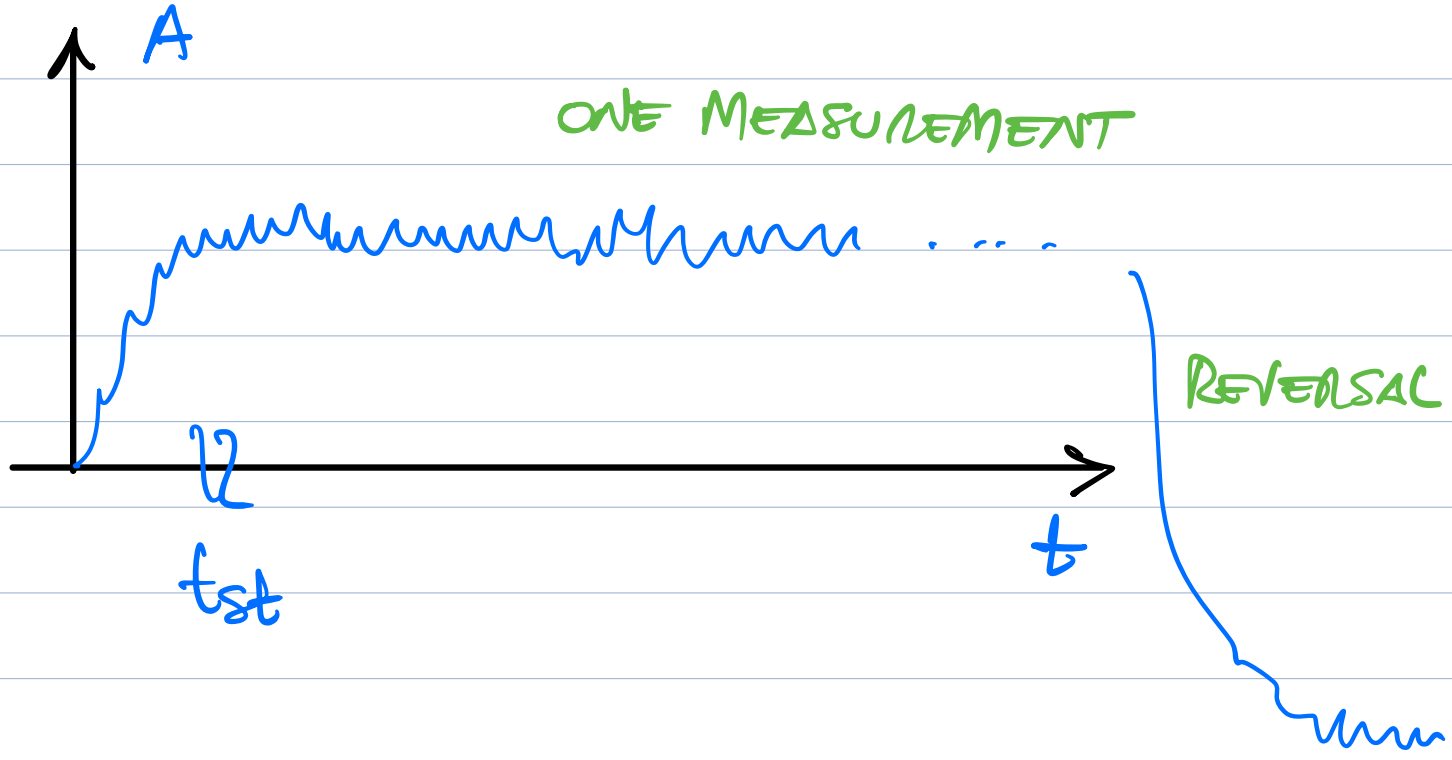
FOR GLOBAL OBSERV.

NOW, WHAT HAPPENS W/ THIS EQUAL. IS A QUESTION OF TIME SCALES AND DEFINITION OF $\langle \cdot \rangle$

EXPERIMENT

$$\bar{A} = \lim_{\tau \gg t_0} \lim_{t > t_{st}} \frac{1}{\tau} \int_{t_{st}}^{t_{st}+\tau} dt' \, A(t')$$

IN THE MAGNETIC (NICKEL) EXAMPLE A = GLOBAL MAGN.



IF WE COMPARE !

- $\overline{A}$ WITH $\tau < t_{rev}$ TO $\langle A \rangle$ WITH NO PINNING FIELD

$$\overline{A} \neq \langle A \rangle = 0$$

- $\overline{A}$ WITH $\tau \gg t_{rev}$ TO $\langle A \rangle$ WITH NO PINNING FIELD

$$\overline{A} = \langle A \rangle = 0$$

- $\overline{A}$ WITH $\tau < t_{rev}$ TO $\langle A \rangle$ W/ h

$$\overline{A} = \langle A \rangle \neq 0$$

- $\overline{A}$ WITH $\tau \gg t_{rev}$ TO $\langle A \rangle$ W/ h

$$0 = \overline{A} \neq \langle A \rangle \neq 0$$

ERGODICITY DEPENDS ON TIME SCALES AND HOW THE STAT AVERAGES ARE COMPUTED

ORDER OF LIMITS $h \to 0^{\pm}$ AND $N \to \infty$ MATTERS

SPONTANEOUS SYMM BREAKING

BY CHOOSING ONE AMONG TWO (OR MORE) EQUIL. PHASES WHILE GOING ACROSS THE PH. TRANS

$\Longrightarrow$ SPONTANEOUS SYMM BROKEN

$$\lim_{h\to 0^+} \lim_{N\to\infty} \langle s_i \rangle = - \lim_{h\to 0^-} \lim_{N\to\infty} \langle s_i \rangle$$

$$\neq 0$$

ENTROPY VS. ENERGY — PEIERLS

FROM THERMODYN WE KNOW THAT IN EQUIL

- ENERGY TENDS TO BE MINIMIZED
- ENTROPY MAXIMIZED.

IN OPEN CANONICAL CONDITIONS THE RELEVANT THERMODYN. POTENTIAL TO BE MINIMIZED IS THE FREE-ENERGY

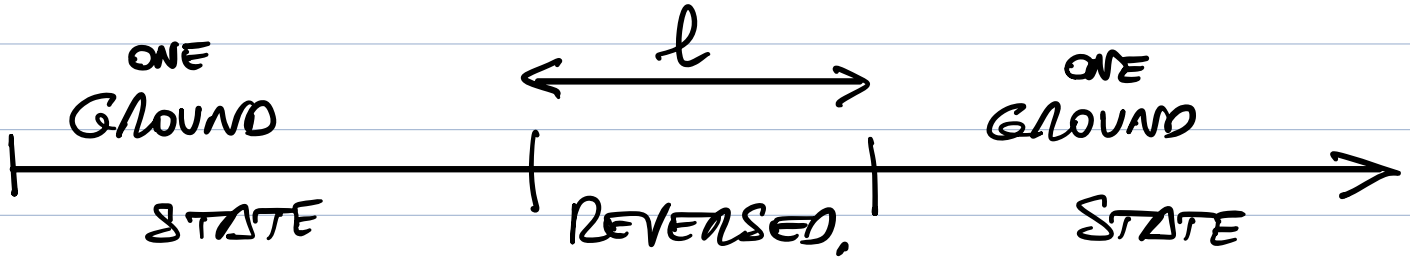
$$F = U - TS$$

$U = \langle H \rangle$ INTERNAL ENERGY

S ENTROPY

$$H = -J \sum_i S_i S_{i+1}$$ EACH BOND ONCE

EX. 1d ISING CHAIN W/ PBC



IC THE OTHER GROUND STATE

WE STUDY WHETHER $T=0$ GROUND STATE $\{S_i = +1\}$ IS STABLE AGAINST TEMP FLUCT

LOWEST ENERGY EXCITATION : ONE DOMAIN, TWO DOMAIN WALLS

TO SIMPLIFY DISCUSSION TAKE ONE BACKGROUND - SAY ALL UP - AND REVERSE A DOMAIN TO DOWN WITHIN

$F_0 = U_0 - TS_0$ GROUND STATE

$F_2 = U_2 - TS_2$ LOWEST EXC.

(THE SUBSCRIPT 2 $\equiv$ 2 DWS)

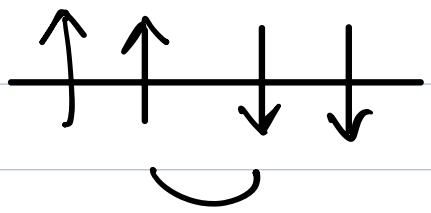
$\Delta f = F_2 - F_0 \gtrless 0$?

IF $< 0 \Rightarrow$ ORDERED WILL BE DESTROYED BY FLUCT

$\Delta f = \Delta U - T\Delta S$

$$\Delta U = U_2 - U_0 = 2 \cdot 2J = 4J$$

↖ THE TWO WALLS OF INTERVAL ℓ



DW $\Rightarrow$ $+J$

NO DW $\Rightarrow$ $-J$

$\Big\}$ DIFF: $2J$

ON ALL OTHER LINKS $U_2 - U_0 = 0$ LOCALLY

ΔS ONE CONF WITHOUT DW $S_0 = k_B \ln 1$

N CONFS W/ ONE DOMAIN OF LENGTH ℓ

$$S_2 = k_B \ln N$$

$$\Delta S = k_B \ln N - k_B \ln 1$$

$$\longrightarrow k_B \ln N$$

$T \neq 0$

$$\Delta f = 4J - k_B T \ln N \longrightarrow -k_B T \ln N$$

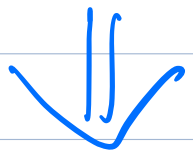
$N \gg 1$

J FINITE

if $T \neq 0$ Δf IS NEGATIVE AND LOGARITHMICALLY LARGE IN N

$\Longrightarrow$ IT'S CONVENIENT TO HAVE THE REVERSED DOMAIN

NEXT REPEAT THE ARGUMENT W/ ANOTHER REVERSED DOMAIN



1d IM DISORDERS AS SOON AS $T > 0$

BUT THERE CAN BE ORDER AT FINITE T:

1d IM W/ POWER LAW DECAYING INTERACTIONS & PBC

RECALL $\alpha > d = 1$ ARGUED IN CH. 1 WITH ENERGY INTEGRAL TO HAVE SHORT-RANGE INTER.

SEE CALC. IN THE NOTES

FOR SUFF. LONG-RANGE INT $\alpha < 1$

LONG-RANGE ORDER IS POSSIBLE

MANY MATH PAPERS W/ COND FOR ABSENCE OF ORDER

2.4 OTHER METHODS

- LOW TEMP EXPANSIONS
  - IDENTIFY GROUND STATES (MINIMA OF THE ENERGY) AND 1st EXCITED STATES & SUM OVER THESE CONFS ONLY
  - MORE SYSTEMATIC

$$e^{-\beta H} \qquad \beta \to \infty \quad (T \to 0)$$

  - SADDLE POINT APPROX
  - EXPAND AROUND THE MINIMUM & TREAT NON-QUAD. TERMS AS PERTURB W/ WICK.

- HIGH TEMP EXPANSION

  EXPLOIT SOME RE-WRITING

$$e^{\beta J S_i S_j} = a(1 + b\, S_i S_j)$$

AND ORGANIZE THE PARTITION SUM

$\Rightarrow$ CLUSTER EXP

- DUALITY

  MAPPING $g \to \frac{1}{g}$

  STRONG $\leftrightarrow$ WEAK COUPLING

- TRANSFER MATRIX

  SPECIFIC TO 2d -

WE WILL SEE IT WHEN DISCUSSING QUANTUM CHAINS AND THEIR MAPPING TO CLASSICAL 2d MODELS.

WE WILL NOT TREAT LOW / HIGH T EXP. NOR DUALITIES.

TRANSFER MATRIX METHODS $\to$ QUANTUM

2.5 MEAN-FIELD

WE USE SPIN MODELS SINCE THEY ARE SIMPLER TO TREAT THAN PARTICLE ONES

YOU NEED TO LEARN CONCEPTS AND METHODS, AND HOW TO USE THEM

BACK TO SPIN MODELS, THE ONES WITH FM INTERACTIONS ARE EASIER.

ONE CAN GUESS THEIR HIGH / LOW T PHASES

BUT SOLVING THEM, e.g. FINDING ANALYT. EXPRESSIONS FOR T_c, ORDER PARAM. CRITICAL BEHAVIOUR : DIFFICULT

DONE IN $d = 1, 2, \infty$

MF METHODS GET APPROX RESULTS FOR T_c, ORDER PARAMETERS, CRITICAL BEHAVIOUR

THESE RESULTS ARE GENERICALLY CORRECT

QUALITATIVELY

AND TO A CERTAIN EXTENT ALSO

QUANTITATIVELY

BUT, e.g. CRITICAL BEHAVIOUR WRONG BELOW d_u

SOMETIMES PHASE TRANS. PREDICTED WHICH $\nexists$, etc.

GENERIC MODEL

p-SPIN WITH FM INTER.

$$H(\{s_i\}) = -J \sum_{i_1 \cdots i_p} s_{i_1} \cdots s_{i_p} - h \sum_i s_i$$

FOCUS ON $p=2$, BUT $p \geq 2$ ALSO INTERESTING

POPULAR, EXTENSIONS $p>2$
GLASSY PHYSICS
OPTIM. PBM
EVEN BLACK HOLE PHYSICS
IF QUANTUM

$p=2$ TWO BODY INTERACTION

AND KEEP J_{ij} & h_i LOCAL
IN DERIVATION BELOW

$$H = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j$$

KEEP IT GENERAL

DOUBLE SUM NO SELF-COUPLING

$$\frac{1}{2} \sum_{\langle ij \rangle} \bullet = \frac{1}{2} \sum_i \sum_{\partial i} \bullet$$

EQUIVALENT NOTATIONS

$1/2$ ENSURES THAT EACH PAIR $\Rightarrow$ ONCE

FACTORS AND NOTATION

$$\sum_{ij} \longmapsto \sum_{i \neq j}$$

SINCE THERE'S NO SELF-INTERACTION

$i \neq j$ only ONCE

$$\boxed{\frac{1}{2} \sum_{i \neq j} \ldots = \frac{1}{2} \sum_{i} \sum_{\partial i}}$$

TO AVOID SUMMING TWICE EACH PAIR

eg $d = 1$

$$\sum_{i \neq j_{nn}} s_i s_j = \sum_i s_i s_{i+1} = \ldots s_1 s_2 + s_2 s_3 + \ldots$$

$$\frac{1}{2} \sum_i \sum_{j=\partial i} s_i s_j = \frac{1}{2} \Big[\underline{s_1} (s_0 + \underset{nn}{s_2}) + \underset{nn}{s_2} (\underline{s_1} + s_3) + s_3 (\underset{nn}{s_2} + s_4) \Big] +$$

TERMS WITH s_2

$s_1 s_2$ APPEARS TWICE IN THE SUM

THAT'S WHY WE NEED THE $1/2$

STANDARD INTRODUCTION OF MF

$$S_i = m_i + \delta S_i \qquad \delta S_i \ll m_i$$

NON FLUCT → will ask $m_i = \langle S_i \rangle$

FLUCT

CANDIDATE ORDER PARAMETER

eg. TWO BODY INTERACTION

$$S_i S_j = (m_i + \delta S_i)(m_j + \delta S_j)$$

$$\simeq m_i m_j + m_i \delta S_j + m_j \delta S_i$$

2nd ORDER IN δS DROPPED

USING AGAIN $\delta S_i = S_i - m_i$

$$S_i S_j \simeq m_i m_j + [m_i (S_j - m_j) + i \leftrightarrow j]$$

$$= m_i S_j + m_j S_i - m_i m_j$$

USE GENERIC TWO BODY COUP. J_{ij} & FIELDS h_i

APPROXIMATION

$$-\frac{1}{2}\sum_{\langle ij\rangle} J_{ij}\, s_i s_j \simeq$$

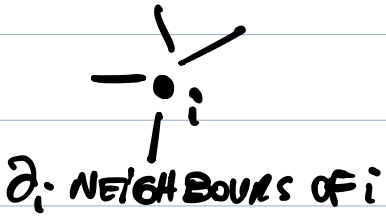
$$-\frac{1}{2}\sum_{\langle ij\rangle} J_{ij}\,(\underbrace{m_i s_j + m_j s_i}_{\text{SPINS IN A FIELD}} - \underbrace{m_i m_j}_{\text{CONST WRT } \{s_i\}})$$

IN BOLTZMANN FACTOR $\exp\left\{-\beta\left[(-\frac{1}{2})\sum_{i\neq j} J_{ij} s_i s_j - \sum_i h_i s_i\right]\right\}$

$$Z(\beta J_{ij}, \beta h_i) = e^{-\frac{\beta}{2}\sum_{\langle ij\rangle} J_{ij} m_i m_j} \underbrace{\sum_{\{s_i=\pm 1\}} e^{+\beta\sum_{\langle ij\rangle} J_{ij} m_i s_j} e^{+\beta\sum_i h_i s_i}}$$

JUST INDEP SPINS UNDER LOCAL FIELDS

$$h_i^{loc} = \underbrace{\sum_{j=\partial i} J_{ij} m_j}_{\text{GENERATED BY THE OTHER ONES}} + \underbrace{h_i}_{\text{THE EXT. FIELD}}$$



PINNING SOURCE

IT IS AS IF THE HAMILTONIAN BECAME AN EFFECTIVE ONE

$\mathcal{H}_{eff} = -\sum_i h_i^{loc} s_i$ AND THE $\{s_i\}$-DEP BOLTZMANN WEIGHT

$$e^{\beta \sum_i h_i^{loc} s_i}$$

USING NOW $\sum_{s_i = \pm 1} e^{\beta h_i^{loc} s_i} = 2 \operatorname{ch}(\beta h_i^{loc})$ FOR EACH i

$$\mathcal{Z}(\{\beta J_{ij}\}, \{\beta h_i\}) = e^{-\frac{\beta}{2} \sum_i \sum_{j \in \partial i} J_{ij} m_i m_j} \prod_i 2 \operatorname{ch} \beta \left(\sum_{j \in \partial i} J_{ij} m_j + h_i \right)$$

RECALL NOTATION

$\sum_{\partial i}$: SUM OVER NEIGHBS OF SITE i

THE FREE-ENERGY DENSITY IS

$$-\beta f = \frac{1}{N} \ln \mathcal{Z}$$

SO TAKING THE $\ln$:

$$-\beta f\left(\{\beta J_{ij}\},\{\beta h_i\}\right) = -\frac{1}{N}\,\frac{\beta}{2}\sum_i \sum_{j=\partial i} J_{ij}\, m_i m_j + \frac{1}{N}\sum_i \ln\left[2\cosh\beta\left(\sum_{j=\partial i} J_{ij} m_j + h_i\right)\right]$$

THE SELF-CONSISTENCY IS

$$\langle s_i \rangle = m_i$$

WITHIN THIS APPROX

QUANTITY $\mathcal{O}(1)$, SINGLE SITE ONE

$$\langle s_i \rangle = \frac{\sum_{\{s_j=\pm 1\}} s_i\, e^{\beta \sum_j h_j^{loc} s_j}}{\sum_{\{s_j=\pm 1\}} e^{\beta \sum_j h_j^{loc} s_j}}$$

IN h_j^{loc} THERE'S THE EXT. h_j AS ONE TERM

A SOURCE

IN GENERAL $\forall$ $H - \sum_i h_i s_i$

$$\mathcal{Z} = \sum_{\{s_i\}} e^{-\beta H + \beta \sum_i h_i s_i}$$

$$\frac{\partial \ln \mathcal{Z}}{\partial h_i} = \frac{1}{\mathcal{Z}} \sum_{\{s_i\}} e^{-\beta H + \beta \sum_i h_i s_i} \beta s_i$$

$$= \beta \langle s_i \rangle$$

$$-\beta f = \frac{1}{N} \ln \mathcal{Z}$$

$$\frac{\partial (-\beta f N)}{\partial h_i} = \beta \langle s_i \rangle$$

$$-\frac{\partial (fN)}{\partial h_i} = \langle s_i \rangle$$

ONLY ASSUMPTION h_i IN 2nd TERM BUT NOT IN FIRST H TERM.

METHOD OF SOURCES

ADD TO $H \to H \quad J \cdot \mathcal{O}$

SOURCE J

OBSERVABLE $\mathcal{O}$

$$\mathcal{Z} = \sum_{conf} e^{-\beta H} \to$$

$$\mathcal{Z}[J] = \sum_{conf} e^{-\beta H + \beta J \mathcal{O}}$$

$$\frac{1}{\beta} \frac{\partial \ln \mathcal{Z}[J]}{\partial J} = \frac{1}{\mathcal{Z}} \sum_{conf} e^{-\beta H + \beta J \mathcal{O}} \mathcal{O}$$

$$= \langle \mathcal{O} \rangle$$ ← CALCULATED WITH J

$$-\beta f = \ln \mathcal{Z} \Rightarrow -\frac{\partial f}{\partial J}\Big|_{J=0} = \langle \mathcal{O} \rangle_{NO\ J}$$

$$\frac{1}{\beta}\frac{\partial \ln Z(\{\beta J_{ij}, \beta h_i\})}{\partial h_i} = \langle S_i \rangle$$

THE ln IS HERE TO GET THE NORM. & $1/\beta$ TO GET RID OF βS_i

$\leftarrow$ ACTING AS A SOURCE

USING NOW

$$-\beta f = \frac{1}{N} \ln Z$$

$$\Rightarrow \boxed{-\frac{\partial (Nf)}{\partial h_i} = \langle S_i \rangle}$$

$\Theta(1)$

AVERAGE UNDER h_i

IF WANTED AT $h_i = 0$

$\Rightarrow$ EVALUATE $h_i = 0$ AFTER TAKING $\partial/\partial h_i$

WE THEREFORE HAVE

$$m_i = \langle S_i \rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial h_i} = -\frac{\partial (Nf)}{\partial h_i}$$

NOW, APPLY THIS TO THE MEAN-FIELD $(-\beta f)$

BACK TO THE EXPRESSION ABOVE FOR $(-\beta f)$ (1)

THE 1st TERM IS INDEP. OF h_i

THE 2nd TERM DOES DEP. ON h_i

WE FIND*

$$m_i = \frac{1}{\beta}\frac{\partial}{\partial h_i}\sum_k \ln\left[2\cosh\beta\left(\sum_{j=\partial k} J_{kj} m_j + h_k\right)\right]$$

$$= \frac{1}{\beta}\sum_k \frac{2\sinh\beta\left(\sum_{j=\partial k} J_{kj} m_j + h_k\right)}{2\cosh\beta\left(\sum_{j=\partial k} J_{kj} m_j + h_k\right)}\,\delta_{ki}\,\beta$$

$$\boxed{m_i = \tanh\beta\left(\sum_{j=\partial i} J_{ij} m_j + h_i\right)}$$

IN THE CONTEXT OF DISORDERED SYSTEMS, WHERE J_{ij} DEPEND ON ij AND TAKEN FROM A PDF :

NAIVE MF EQS (TAP)

A GENERALIZATION OF THE USUAL FORM, RECOVERED FOR $J_{ij} = J$ FOR $m_i = m$ $h_i = h$ WHICH

$$\boxed{m = \tanh\left(\beta J \sum_{\partial i} m + \beta h\right)} \qquad (2)$$

$\sum_{\partial i} \bullet = z$ # NEIGHBOURS OF SITE i DEP ON LATTICE, eg. $z=4$ 2d □.

* COMMENT / CLARIFICATION

WE GET AN EQ FOR m_i THE SOL. OF WHICH DEPENDS ON h_i.

WHY HAVEN'T WE CONSIDERED $\frac{\partial m_j}{\partial h_i}$ AS WELL WHEN TAKING $\frac{\partial f}{\partial m_i}$?

THIS IS THE EXP. FOR f:

$$-\beta f\left(\{\beta J_{ij}, \beta h_i\}\right) = \frac{-1}{N}\,\frac{\beta}{2}\sum_i\sum_j J_{ij} m_i m_j + \frac{1}{N}\sum_i \ln\left[2\cosh\beta\left(\sum_j J_{ij} m_j + h_i\right)\right]$$

$$\frac{\partial f}{\partial h_i} = \underbrace{\frac{\partial f}{\partial h_i}^{\text{EXPLICIT}}}_{\text{WHAT WE CALCULATED}} + \sum_k \underbrace{\frac{\partial f}{\partial m_k}}_{\hookrightarrow\ \text{BUT } = 0 \text{ ON THE EQ. ITSELF.}} \frac{\partial m_k}{\partial h_i}$$

Proof:

$$N\frac{\partial f}{\partial m_k} = \beta \sum_{i=\partial k} J_{ki} m_i - \beta \sum_i \frac{\cancel{2}\sinh\beta\left(\sum_j J_{ij} m_j + h_i\right)}{\cancel{2}\cosh\beta\left(\sum_j J_{ij} m_j + h_i\right)} \frac{\partial}{\partial m_k}\left(\sum_j J_{ij} m_j + h_i\right)$$

$$\sum_{j=\partial i} J_{ij}\,\delta_{jk}$$

CUTS THE SUM OVER $j=\partial i$ TO BE ONLY k

$\Rightarrow$ CUTS SUM OVER i TO BE CONNECTED TO k

$$N\frac{\partial f}{\partial m_k} = \beta \sum_{i=\partial_k} J_{ki}\, m_i$$

$$-\beta \sum_{i=\partial_k} J_{ik} + h\,\beta\left(\sum_{\partial i} J_{ij} m_j + h_i\right)$$

$= 0$ ON THE EQ. ITSELF.

NB. IT'S $\beta \sum_{i=\partial_k}$ APPLIED TO WHAT WE GET FROM

$\frac{\partial f}{\partial h_i}$ EXPLICIT DERIVATIVE

SO, ONE CAN TAKE

$$\boxed{-N\frac{\partial f}{\partial h_i}^{\text{EXPLICIT}} = m_i}$$

TO YIELD THE EQ. FIXING m_i. $\exists$ OTHER WAYS TO JUSTIFY IT TOO.

SUMMARY I AND II

- PHENOMENOLOGY
- ERGODICITY $\overline{\Theta(t)} = \langle \Theta \rangle$
- CONCEPTS

  $N \to \infty$
  ORDER PARAMETERS
  PHASE DIAGRAMS
  PHASE TRANSITIONS
  CRITICAL LINES & POINTS
  SYMMETRY
  SPONT SYMM BREAKING
  EXPLICIT SYMM BREAKING

MEAN-FIELD

$$s_i = m_i + \delta s_i \qquad m_i = \langle s_i \rangle$$

DROP $\delta s_i \, \delta s_j$ $\Rightarrow$ SPIN DECOUPLING

CALCULATE

$$Z(\beta J_{ij}, \beta h_i ; m_i)$$

$$Z(\beta J_{ij}, \beta h_i) = e^{-\frac{\beta}{2} \sum_i \sum_{j=\partial_i} J_{ij} \, m_i \, m_j} \prod_i 2 \cosh \beta \left(\sum_{j=\partial_i} J_{ij} \, m_j + h_i \right)$$

IMPOSE SELF-CONSISTENCY

$$m_i = \langle s_i \rangle$$

"NAIVE MEAN-FIELD EQS"

$$m_i = \tanh\left(\beta \sum_{j \in \partial i} J_{ij} \cdot m_j + \beta h_i\right)$$

IF $J_{ij} = J$ $\quad h_i = h \quad \Longrightarrow$

$$m = \tanh\left(\beta z J m + \beta h\right)$$

WE HAVE $\quad Z\left(\beta J, \beta h; m\right)$

$\Longrightarrow$

$$-\beta f = \ln Z$$

$$f = \frac{J z m^2}{2} - k_B T \ln\left(2 \cosh\left(\beta J z m + \beta h\right)\right)$$

FREE-ENERGY LANDSCAPE

ONE CAN DEF. AN ORDER PARAM. AND CONTROL PARAM. DEP. FUNCTION

$$f(m;\beta J,\beta h)$$

FROM WHICH THE EQ. (2) CORRESP TO

$$\frac{\partial f(m;\beta J,\beta h)}{\partial m}=0$$

$$f(m)=\frac{Jzm^2}{2}-k_BT\ln 2\cosh(\beta Jzm+\beta h)$$

I WRITE ONLY THE ORDER PARAM. DEP. IN f

IN THE FOLLOW. $J_i=J$ & $h_i=h$ TO SIMPLIFY DISCUSSION.

LET'S CHECK IT:

$$\frac{\partial f(m)}{\partial m} = 0 =$$

$$\frac{2JzM}{2} - \cancel{hBJ} \frac{2\cancel{sh}(\beta JMz + \beta h)}{\cancel{2}ch(\beta JzM + \beta h)} \cancel{\beta Jz}$$

$$\Rightarrow \quad m = th(\beta JMz + \beta h)$$

AND IT IS OK

IT CAN BE GENERALIZED TO

$$f(\{m_i\}; \{\beta J_{ij}\}, \{\beta h_i\})$$

IMPORTANT MF

WE REPLACED THE TWO-BODY INTER BY A ONE SITE PROBLEM DECOUPLED FROM OTHERS, SEEN "ON AVERAGE"

MEAN-FIELD EQ

ANALYSIS

THE NUMERICAL FACTORS. — A CHECK

Always CHECK LIMITS !

- LET'S CHECK WHETHER WE GET $m = \pm 1$ FOR $T \to 0$ AND $h = 0$ FROM THE EQ.

$$m = \text{th}\left(\beta J z m + \cancel{\beta h}\right) \to 0$$

— FOR $\beta J \to \infty$, $m > 0$ $\quad \text{th}\, y \to 1$

$\Longrightarrow \quad m = 1$ ok

— FOR $\beta J \to \infty$, $m < 0$ $\quad \text{th}\, y \to -1$

$\Longrightarrow \quad m = -1$ ok

- DO WE GET $m \to 0$ FOR $\beta \to 0$ AND ANY h?

$$m = \tanh(\beta J z m + \beta h)$$

$$= \tanh 0 = 0 \Rightarrow m = 0 \quad \underline{\text{ok}}.$$

- m IS IN $[-1, 1]$ SINCE rhs IN $[-1, 1]$

- NOTE THAT THE "TRICKY" FACTOR

$$\beta J z$$

IS EASY TO CHECK. ONE MNEMOTECHNIQUE RULE

$$z \to N-1 \qquad J \to J/N$$

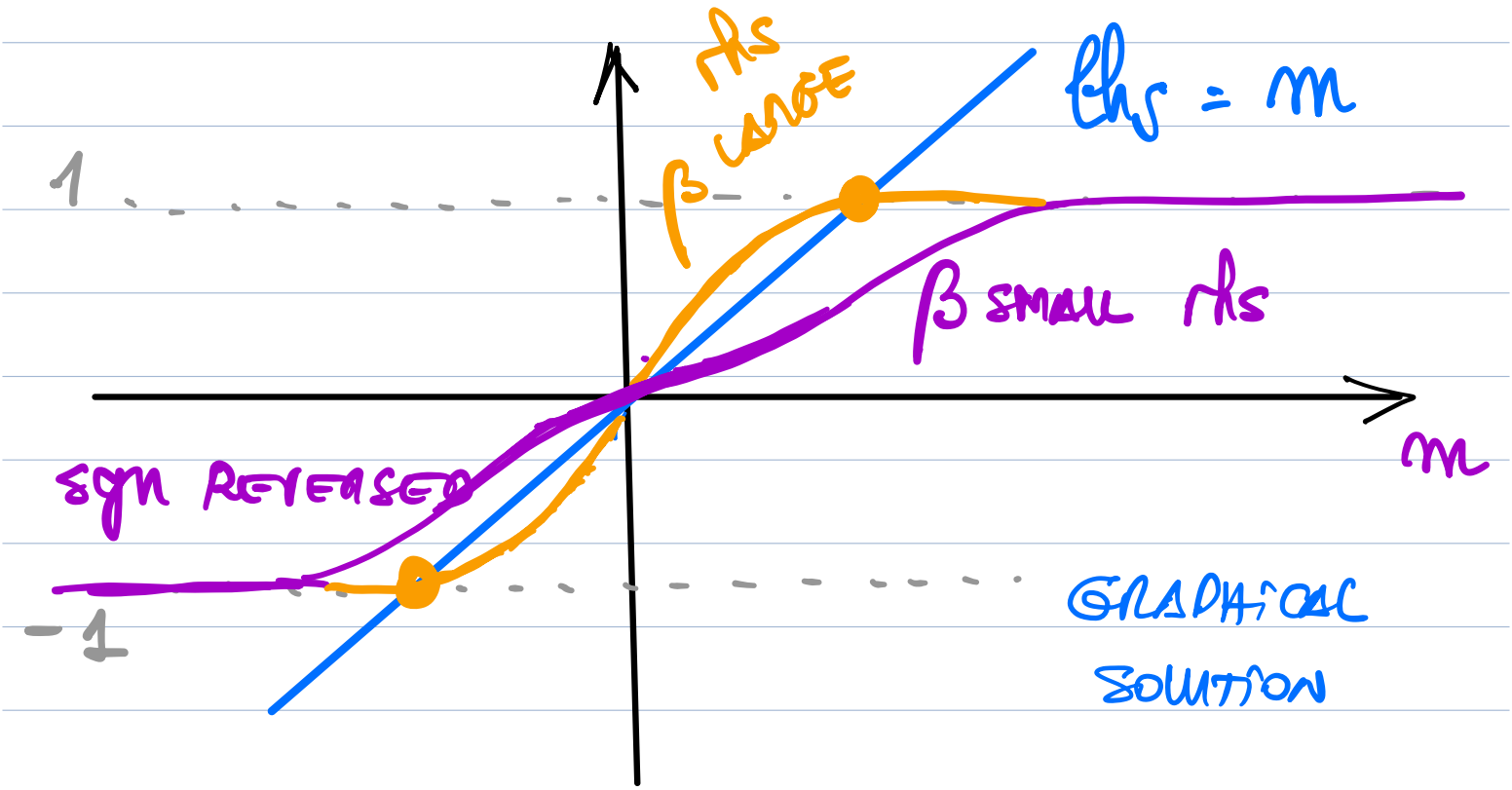
$$\beta J z \to \beta J \Rightarrow T_c = J$$

FULLY CONNECTED MODEL WITH EACH PAIR COUNTED ONCE.

MEAN FIELD EQUATION ANALYSIS

$$m = \tanh(\beta z J m + \beta h)$$

$h = 0$



CHANGE IN # SOLS WHEN SLOPE AT $m \simeq 0$ OF lhs = SLOPE rhs

- ONE SOLUTION $m = 0$ if $zJ\beta < 1$
- THREE SOLUTIONS $m = 0$, $m_+ = -m_-$ OTHERWISE

$$2J\beta_c = 1$$

NB NUM. FACTORS DEP ON LATTICE (z) AND CONNECTION

PROPERTIES

- HIGH T only $m=0$ NO ORDER
- LOW T symm $m \leftrightarrow -m$

WHICH ONE WINS? $|m| \neq 0$ OR $m=0$

WHO IS STABLE?

ITERATIVE SOLUTION & CONVERGENCE

$$m = \text{th}(\beta J z m) \qquad h=0$$

ITERATIVE SOLUTION

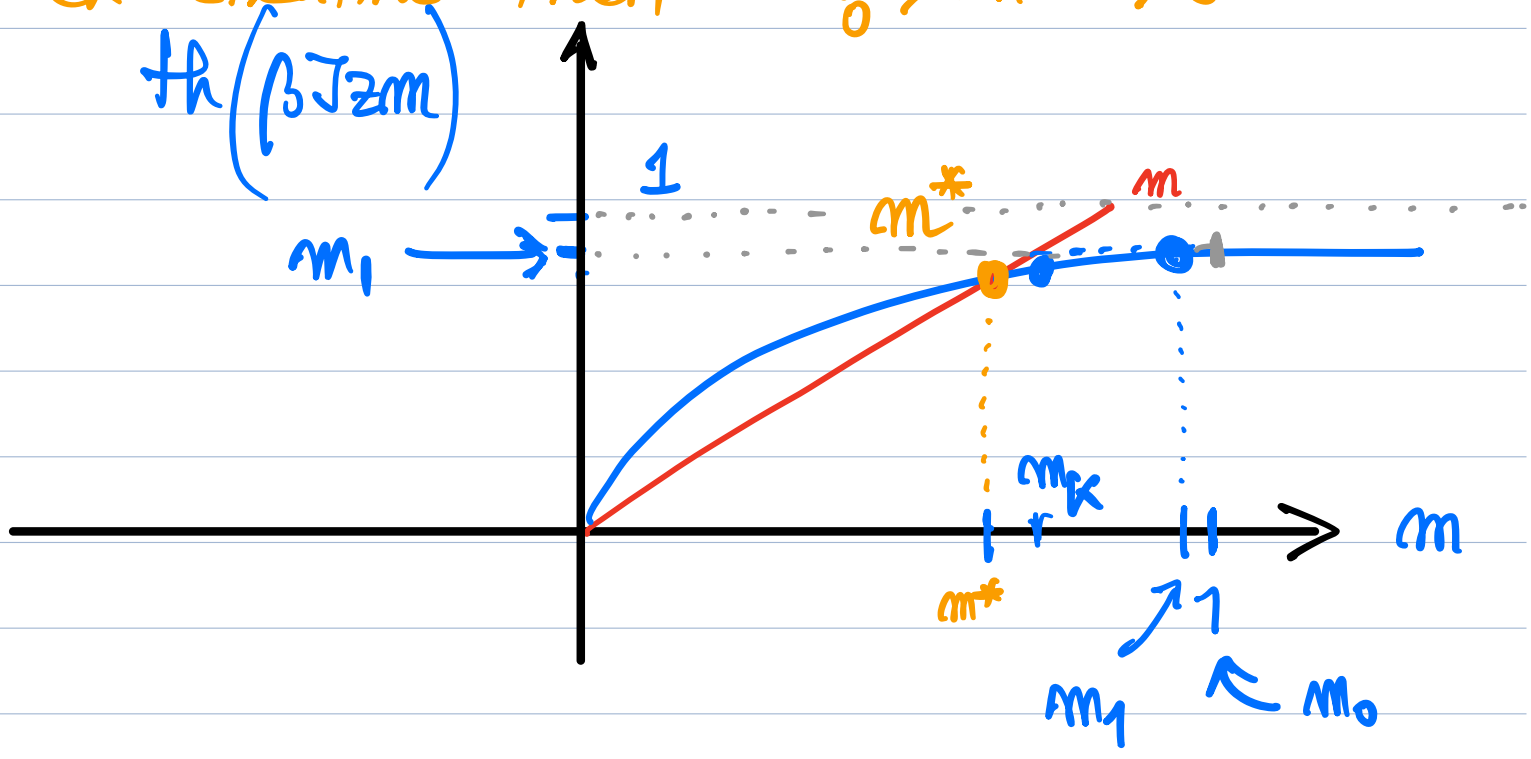
SAY YOU START FROM $m_0 = 1$

$$m_1 = \text{th}(\beta J z m_0) < 1 \qquad \text{if } \beta J z > 1$$

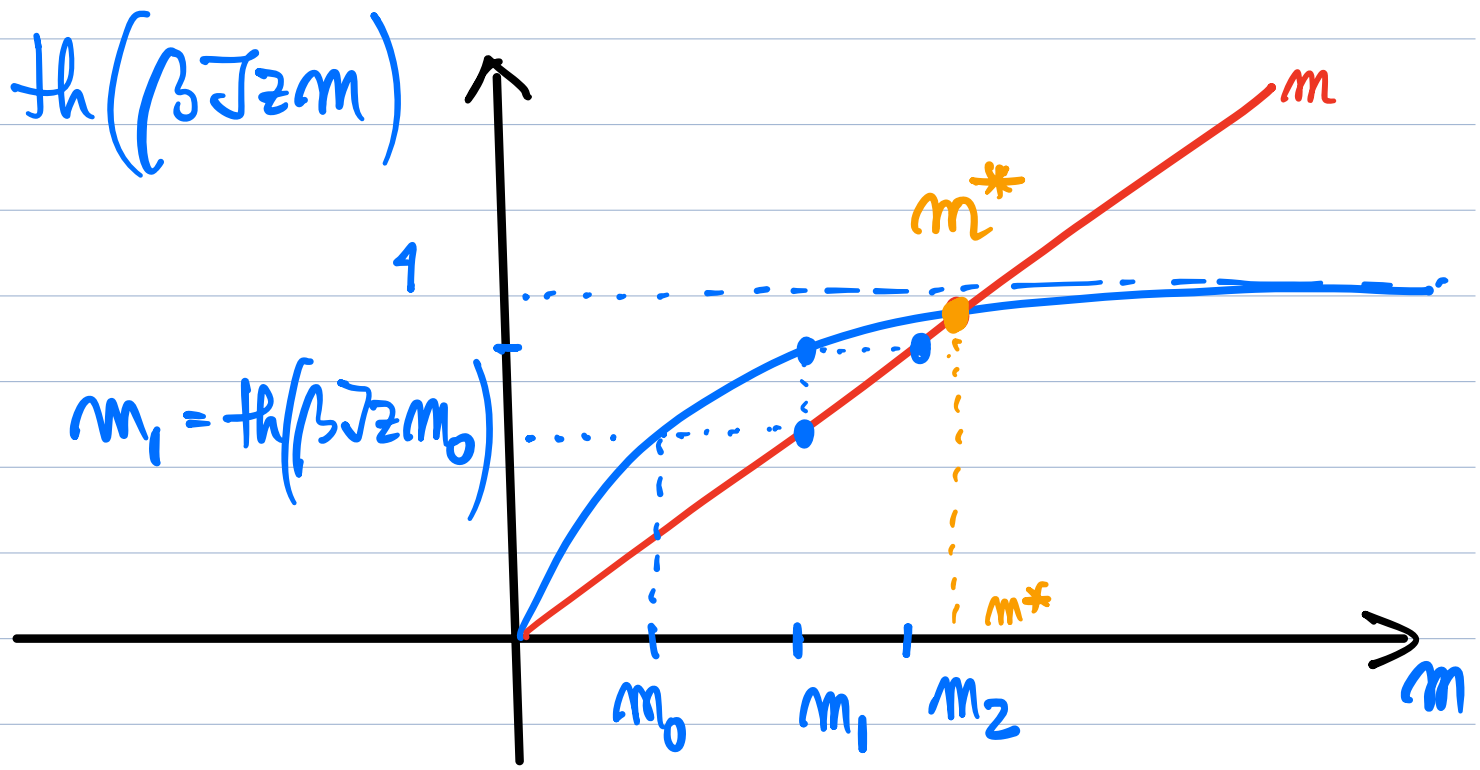
$m_2 = th(\beta J z m_1) < m_1$ ETC UNTIL

FIXED POINT REACHED

ex STARTING FROM $m_0 > m^* > 0$

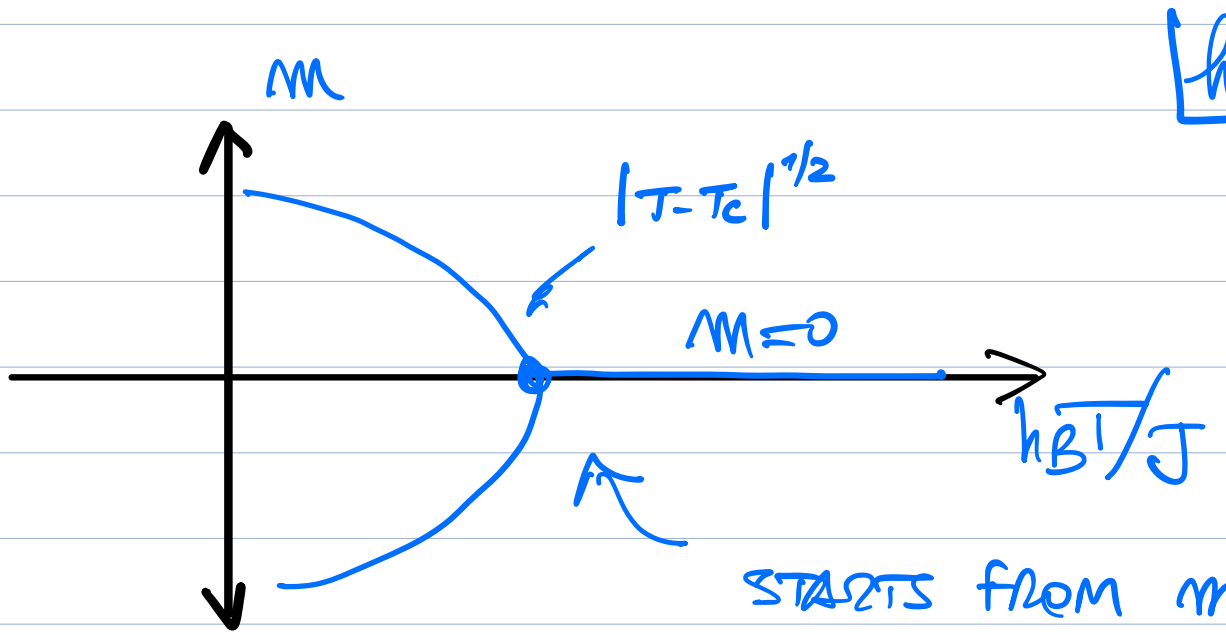


NOW, FROM $0 < m_0 < m^*$



STARTING FROM A NEGATIVE VALUE $m_0 < 0 \Rightarrow m^* < 0$

THE CONTROL PARAM DEP. OF THE ORDER PARAM. m



STARTS FROM $m=0$

CONTINUOUS TRANSITION

IF WE INTERPRET CHANGE IN # SOL AS THE PHASE TRANSITION

BUT IF WE SUM OVER TWO BRANCHES NEEDED AT LOW T $\Rightarrow$ 0 AGAIN!

SEE PINNING FIELD BELOW

EXPERIMENTS WERE SHOWING MEASUREM WITH SMALLER VALUES OF β IN $d=3$ SAMPLES

CASTING DOUBTS ABOUT MF THEORY CORRECTNESS CLOSE TO T_c

THE EXTERNAL PINNING FIELD

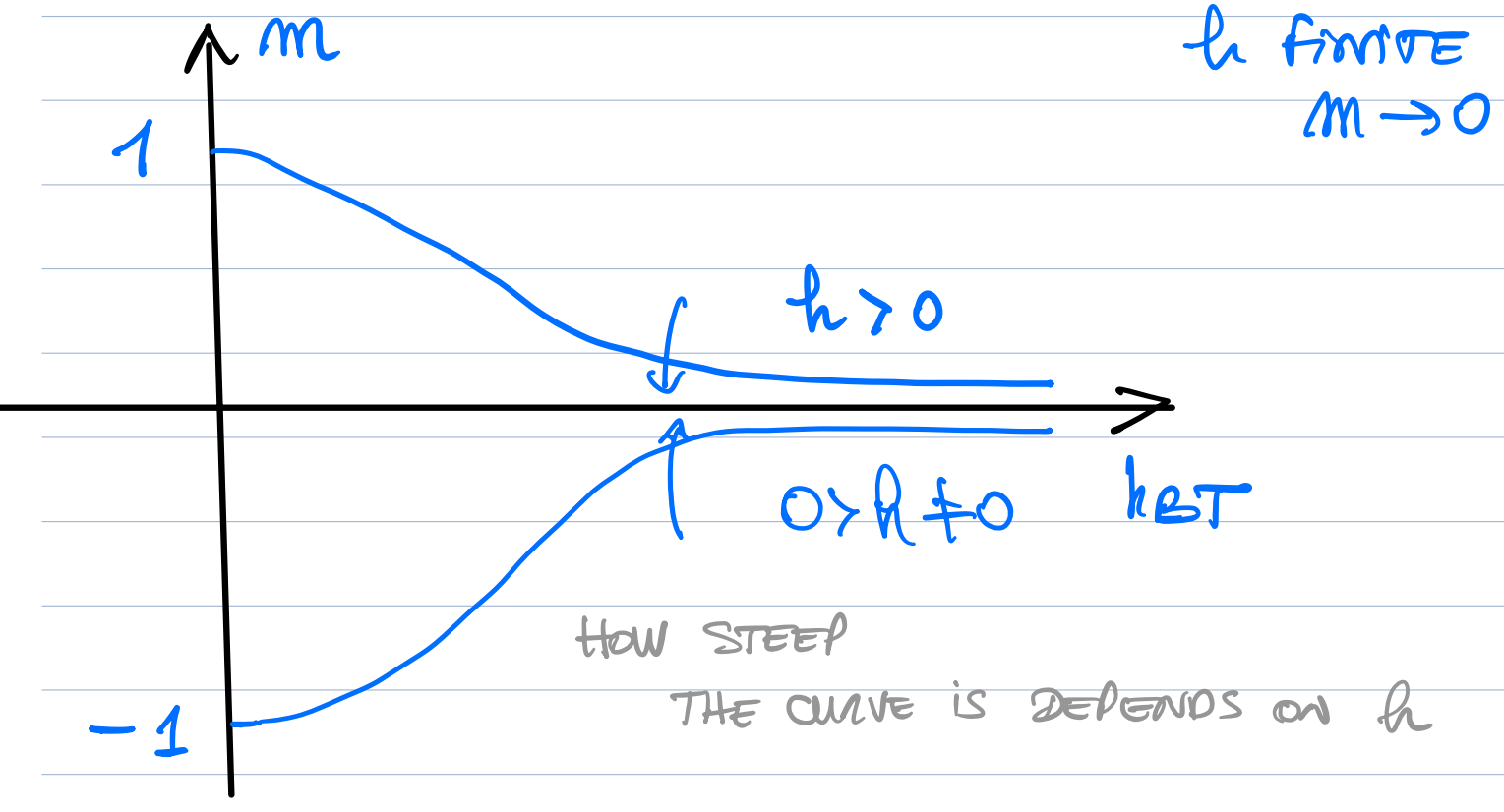
$$m = \tanh \beta (Jzm + h)$$

IS $m=0$ A SOLUTION IF $h \neq 0$? NO!

$$0 = \tanh \beta h \neq 0$$

$$\Rightarrow \boxed{m \neq 0} \quad \forall\ k_B T < \infty,\ h \neq 0$$

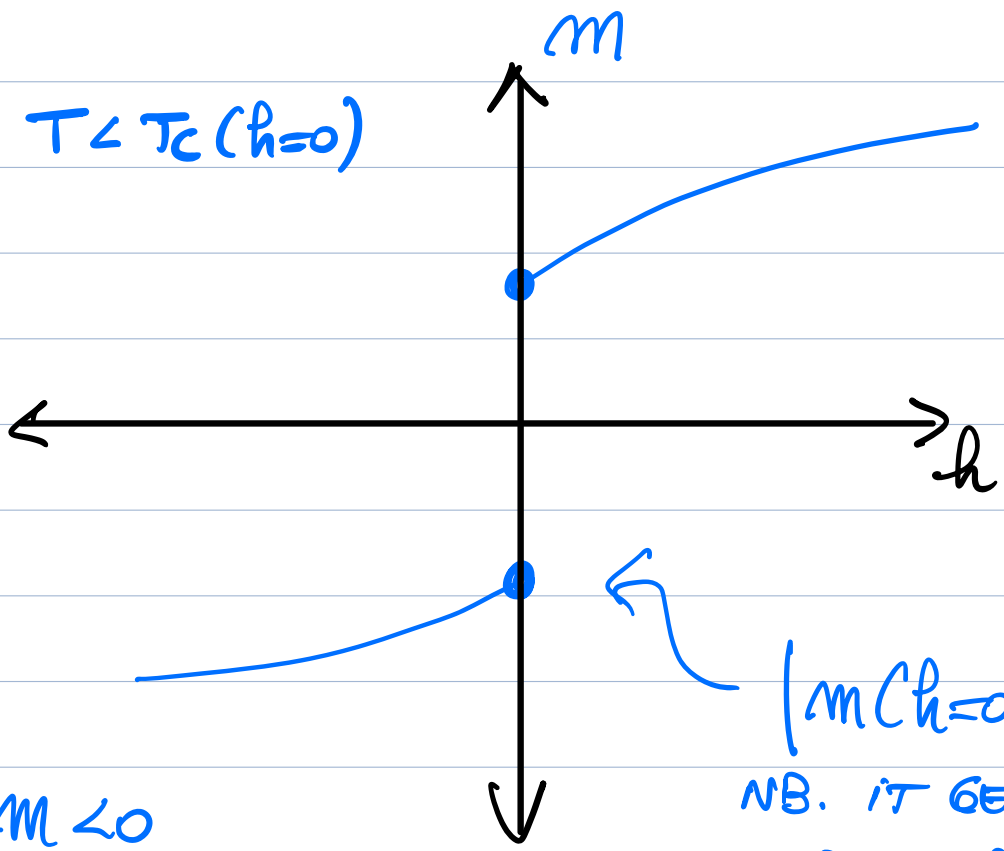
only for
$k_B T \to \infty$
h finite
$m \to 0$



HOW STEEP
THE CURVE IS DEPENDS ON h

$$\lim_{h \to 0^+} m(h) = m > 0 \qquad T < T_c(h=0)$$

SIMIL. FOR $h \to 0^-$ $\quad m < 0$



$T < T_c(h=0)$

$m > 0$ FOR $h > 0$

$|m(h=0, T)|$

NB. IT GETS CLOSER TO ZERO FOR $T \to T_c$.

$m < 0$ FOR $h < 0$

NOTE DISCONTINUITY OF m AT $h=0$

HOW DOES $m \longrightarrow 0$ WITH $h \to 0$ AT PRECISELY T_c ? LATER

MEANING OF THE EQUILIBRIUM STATE

WITH A POSITIVE PINNING FIELD

- AT $T=0$ THERE IS A SINGLE CONF W/ $s_i = +1$ AND $m = 1$ WHICH IS THE GROUND STATE

- AT $T = \varepsilon > 0$ FLUCTUATIONS ARE SWITCHED ON & $m \lesssim 1$

ALREADY N CONFS CONTRIBUTE TO $m = 1 - 2/N$

THE NON-VANISHING T EQUIL. STATES ARE LARGE ENSEMBLES OF MICRO CONFS.

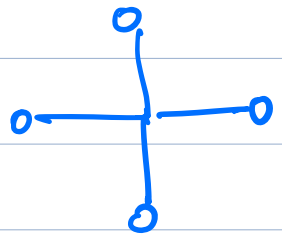
MACROSTATES VS. MICROSTATES

HOW DOES T_c DEPEND ON LATTICE?

$$k_B T_c = zJ \qquad \nearrow \text{ if } J \nearrow \text{ ok}$$

IF MODEL ON A SQUARE LATTICE

$z = 4$



$T_c \nearrow$ IF $z \nearrow$ WHICH IS EXPECTED $\Rightarrow$ ORDERED FAVOURED BY MORE INTER.

BUT $z = 2$ IN $d = 1$ $\Rightarrow$ $k_B T_c = 2J$

$\Rightarrow$ PROBLEM ! $T_c = 0$

MF FAILS AT LOW d !

THE ORDER PARAM ($h=0$) — β EXPONENT

$$m = \tanh\beta(Jzm + h)$$

$h \to 0$

$$m \simeq \beta Jzm - \frac{1}{3}(\beta Jzm)^3$$

$(\beta Jzm) \to m^2$

$$m^2 \simeq \frac{3}{(\beta Jz)^3}\left[-1 + \beta Jz\right]$$

$\beta Jz \to k_B T_c$; $\beta \to \frac{1}{k_B T}$

$\beta = \beta_c \to$ $(\beta Jz)^3 \to 1$

$$\frac{1}{k_B T}\left(k_B T_c - k_B T\right) \geq 0 \ \checkmark$$

$$\boxed{m \sim \sqrt{3}\,\beta_c \left(k_B T_c - k_B T\right)^{1/2}}$$

THE NUMERICAL PREFACTOR $\sqrt{3}$ IS NOT OF INTEREST

$[m] = 1 \Rightarrow$ ok

CRITICAL EXPONENT CONTROLS HOW THE ORDER PARAM VANISHES

$$\boxed{\beta = 1/2}$$

$$\underbrace{\ln m}_{y} \sim \text{ct.} + \frac{1}{2}\ln\left(k_B T_c - k_B T\right) = \text{ct}' + \frac{1}{2}\ln\left(1 - T/T_c\right)$$

THE MAGN. SUSCEPT - γ EXPONENT

VARIATION OF THE ORDER PARAM WRT CONJUGATE FIELD

$$m = \tanh \beta \left(J z m + h \right)$$

THE SOLUTION IS $m(h)$

COMPUTE $\dfrac{\partial m(h)}{\partial h}$ AND TAKE $h \to 0$

CALLING $m(h) = m_h$ & TAKING $\dfrac{\partial m_h}{\partial h}$:

$$\frac{\partial m_h}{\partial h} = \frac{\beta \left(J z \dfrac{\partial m_h}{\partial h} + 1 \right)}{\cosh^2 \beta \left(J z m_h + h \right)} \Longrightarrow$$

$$\frac{\partial m_h}{\partial h} = \frac{\beta}{\left[\cosh^2 \beta \left(J z m_h + h \right) - \beta J z \right]}$$

DIMENSIONS $[m] = 1 \quad [h] = [k_B T] \Rightarrow [\chi] = [\beta]$

RECALL $k_B T_c = J z = 1/\beta_c$ AT $h=0$

AND WE FOCUS ON $T \to T_c (h=0)$ ±

THAT IS SLIGHTLY ABOVE OR BELOW.

- $\underline{T > T_c}$ χ_+ $m_{h=0}$, $h=0$

DEN. $d^2 0 - \beta J z = 1 - \beta J z$

$$= 1 - \frac{J z}{k_B T} = \frac{k_B T - k_B T_c}{k_B T_c}$$

NUM $\beta = \frac{1}{k_B T} \to \frac{1}{k_B T_c}$

DIMENSION OF FACTOR IS 1

$$\chi_+ = \left.\frac{\partial m_h}{\partial h}\right|_{h=0} = \frac{1}{k_B T - k_B T_c}$$

$\boxed{\gamma_+ = 1}$ PREFACTOR $a_+ = 1$

CAREFUL THIS IS THE EXPONENT →

- $T < T_c$ $\quad \chi_- \quad m_h \propto (T_c - T)^{\beta}$

$$\chi_- = \left.\frac{\partial m_h}{\partial h}\right|_{h=0} \simeq \quad \text{EXPAND } m_h \simeq 0$$

$$\cosh x \sim \frac{e^x + e^{-x}}{2} =$$

$$= \frac{1}{2}\left[\left(1 + x + x^2/2\right) + \left(1 - x + x^2/2\right)\right] + \dots$$

$$= 1 + \frac{x^2}{2} + \dots$$

$$\cosh^2 x \simeq \left(1 + \frac{x^2}{2} + \dots\right)^2 \simeq 1 + x^2 + \dots$$

$$\frac{\partial m_h}{\partial h} = \frac{\beta}{1 + \beta^2 \left(Jz\, m_h + h\right)^2 - \beta J z}$$

$m_h = 0$, $h = 0$

LET'S PROPOSE $T = T_c - \varepsilon = \frac{2}{3} - \varepsilon$ AND

$$m \sim \sqrt{3\beta_c}\,(T_c - T)^{1/2} =$$

$$= \sqrt{\frac{3}{k_B T_c}}\,\varepsilon^{1/2} \qquad h = 0$$

REPLACE

$$\frac{\partial m_h}{\partial h} = \frac{\beta_c}{1 + \beta_c^2 \left(T_c \sqrt{3\beta_c}\,\varepsilon^{1/2}\right)^2 - \frac{T_c}{T_c - \varepsilon}}$$

$$= \frac{\beta_c}{1 + 3\varepsilon - \left(1 + \frac{\varepsilon}{T_c}\right)}$$

$$\chi_- = \frac{1}{2\varepsilon} = \frac{1}{2(T_c - T)}$$

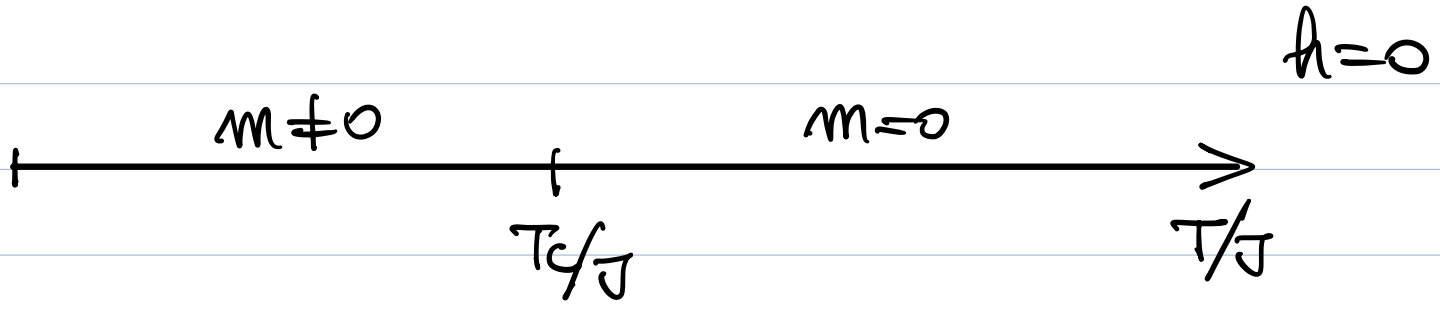
$$\boxed{\gamma_- = 1}$$

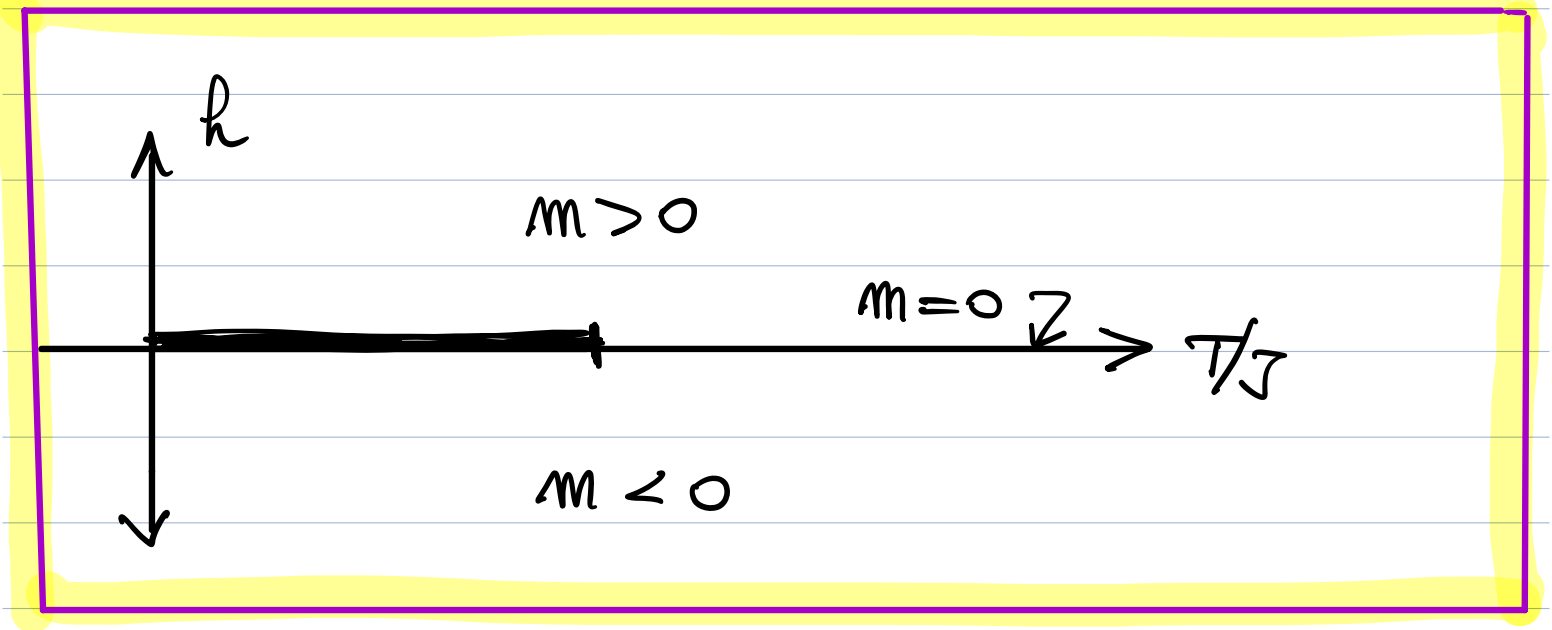
PREFACTOR $a_- = 1/2$

$$\boxed{\chi_\pm = \frac{a_\pm}{|T - T_c|}}$$

WITH $\neq$ PREFACTORS ABOVE AND BELOW T_c

DEPENDENCE ON FIELD - δ EXPONENT





WHAT HAPPENS GOING ACROSS $h=0$ AT FIXED $T < T_c(h=0)$?

A JUMP IN $m>0$ TO $m<0$

OR $m<0$ TO $m>0$

↓ DISCONTINUOUS TRANSITION

JUST AT $T_c(h=0)$

$m = \text{th}\,\beta(Jzm+h)$ EXPAND m, h SMALL

$$m \sim \beta(Jzm+h) - \frac{\beta^3}{3}(Jzm+h)^3 \qquad \beta=\beta_c$$

$$0 \sim h - \frac{\beta_c^2}{3}(Jzm+h)^3$$

DROP h WRT m :

if $m \sim h^{1/\delta}$ AND $1/\delta < 1$ $\Rightarrow$

$$h \sim \frac{\beta_c^2}{3}(Jz)^3 m^3 \Rightarrow \boxed{m \sim h^{1/3}} \qquad \boxed{\delta = 3}$$

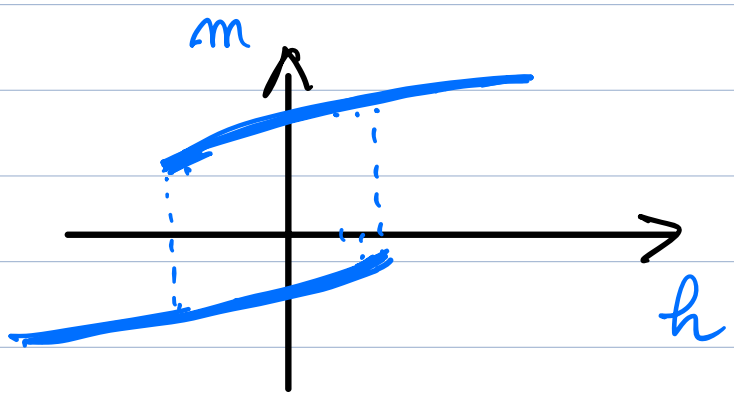
CONSISTENTCY $1/\delta < 1$

PREFACTOR $(3/\beta_c)^{1/3}$

$[m] = [\beta_c h]^{1/3} = 1$ ok.

HYSTERESIS

IN AN ACTUAL EXP., SINCE THE "INITIAL" STATE IS STILL STABLE* AFTER GOING ACROSS THE TRANS. $\Rightarrow$ ABRUPT CHANGE CAN OCCUR LATER



* LATER f

SUMMARY OF MEAN-FIELD (SO FAR)

- MEAN-FIELD FOR UNIFORM m

(IN NOTES, DONE FOR POSSIBLY LOCAL m_i IF J_{ij} NOT ALL THE SAME, IE HETEROGENEOUS)

$m_i + \delta s_i = s_i \qquad \langle \delta s_i \rangle = 0$

DROP $\delta s_i \delta s_j$ TERM IN BOLTZMANN WEIGHT

- $f(m) = \frac{Jzm^2}{2} - hm - k_B T \ln\left[2\cosh\beta(Jzm+h)\right]$

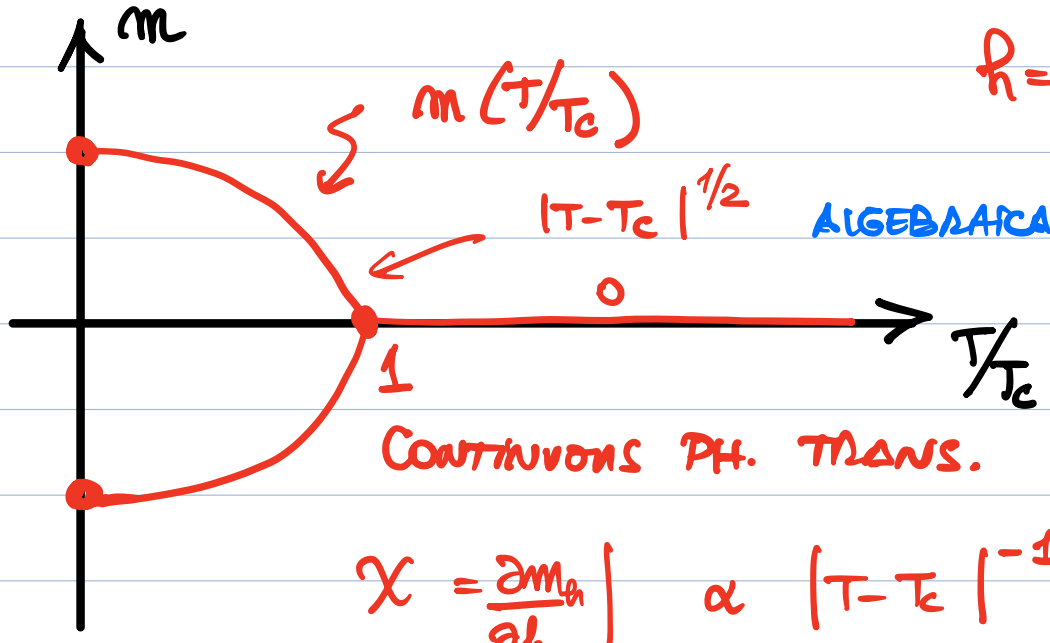
- MF EQ FROM $\frac{\partial f(m)}{\partial m} = 0$

SO FAR WE SAW $f(m)$ AS DETERM BY CONDITION

- MF EQ $m = \text{th}\left(\beta J z m + \beta h\right)$

2.5.3 ORDER PARAMETER

$k_B T_c(h=0) = Jz$



$h = 0$

ALGEBRAICALLY $\to 0$

$\beta = 1/2$

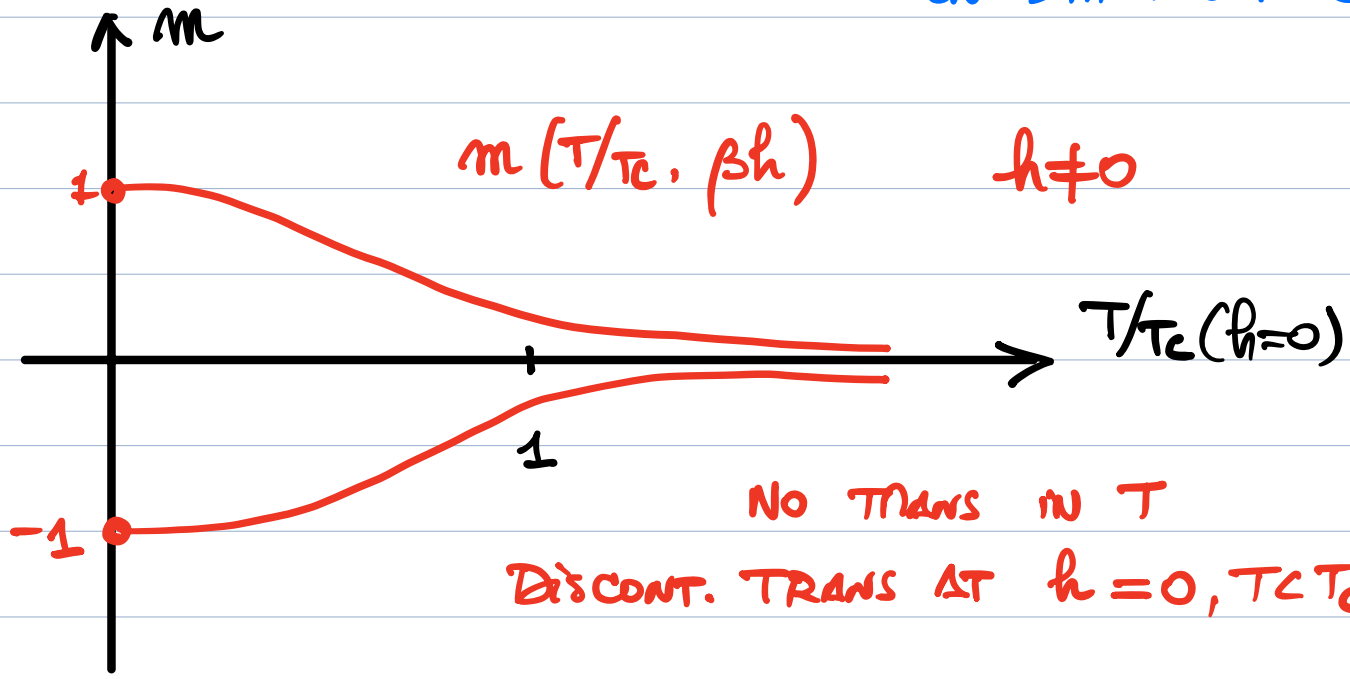
CONTINUOUS PH. TRANS.

$$\chi = \left.\frac{\partial m_h}{\partial h}\right|_{h\to 0} \propto |T - T_c|^{-1}$$

ALGEBRAIC DIVERGENCE

$\gamma = 1$

ON BOTH SIDES OF T_c



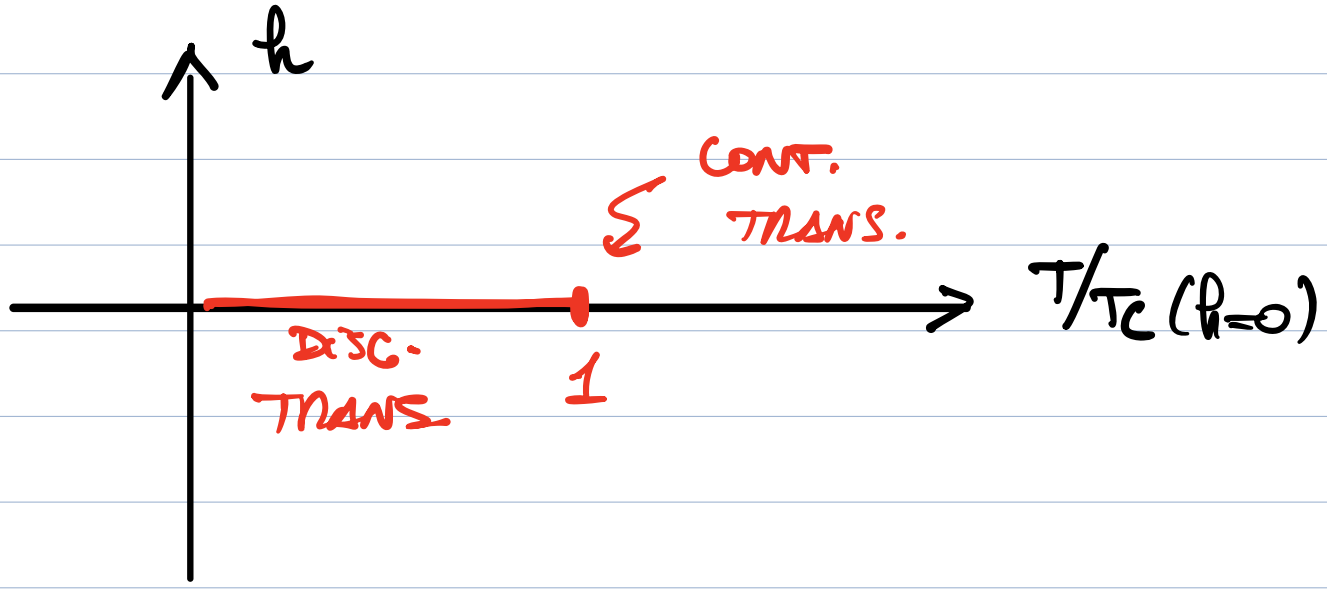
$m(T/T_c, \beta h)$ $\quad h \neq 0$

NO TRANS IN T

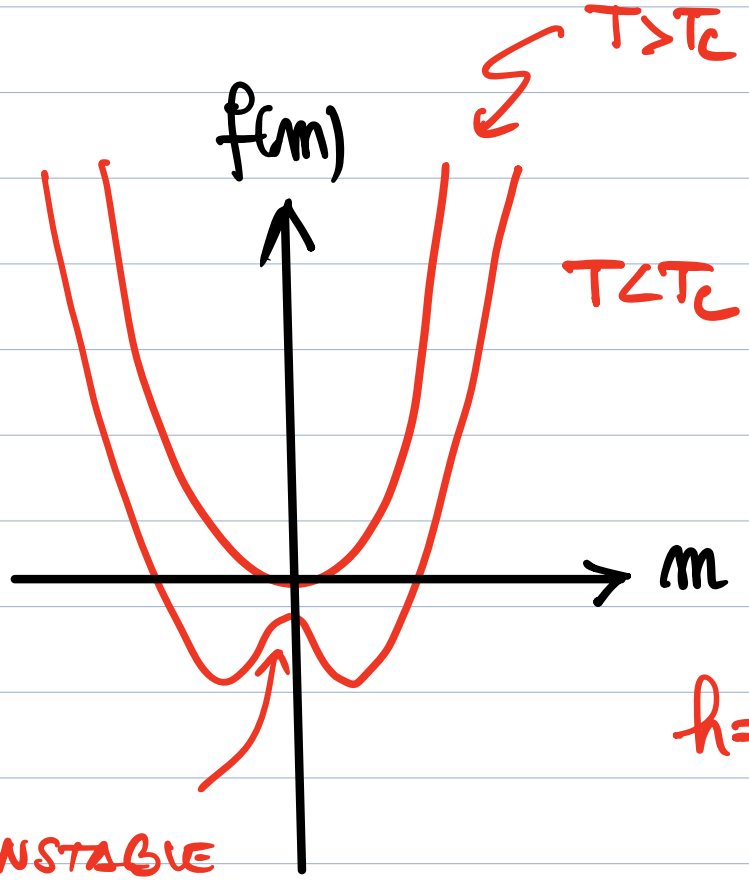
DISCONT. TRANS AT $h = 0, T < T_c$

$$m \underset{h\to 0}{\sim} h^{1/3} \qquad T = T_c \qquad \delta = 3$$

PHASE DIAGRAM



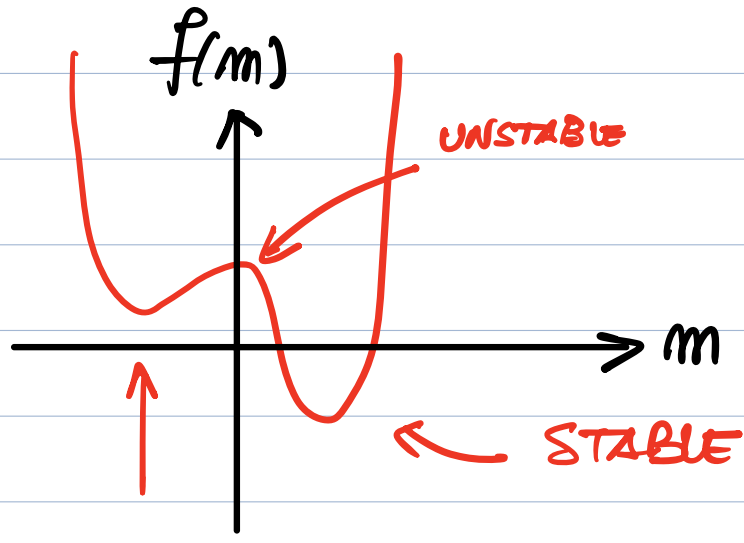
FREE-ENERGY DENSITY AS fct ORDER PARAM



AND GETS TILTED BY h

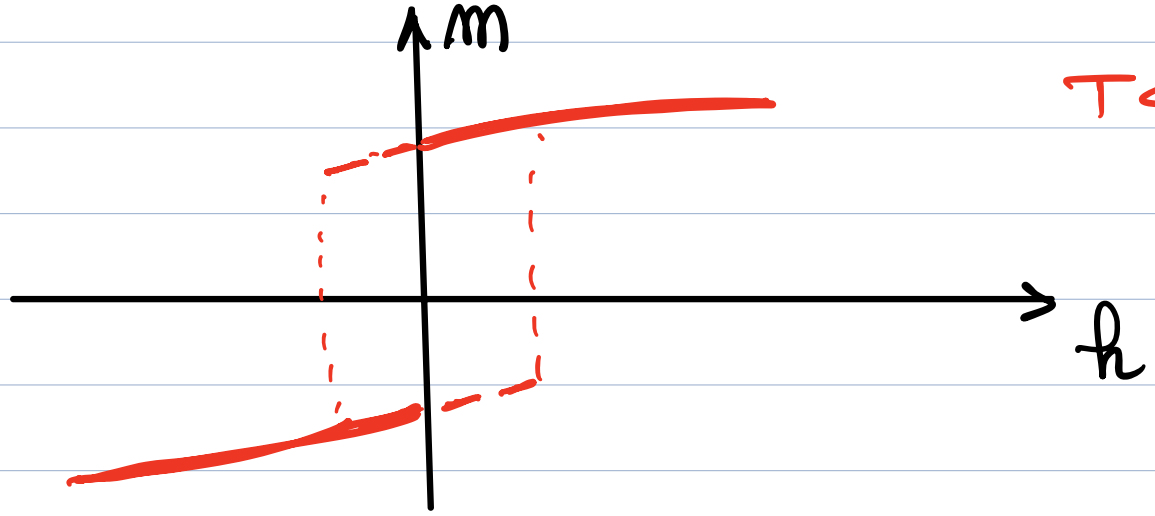
EFFECT OF A FIELD

- DISCONTINUOUS TRANS
- HYSTERESIS



$h > 0$

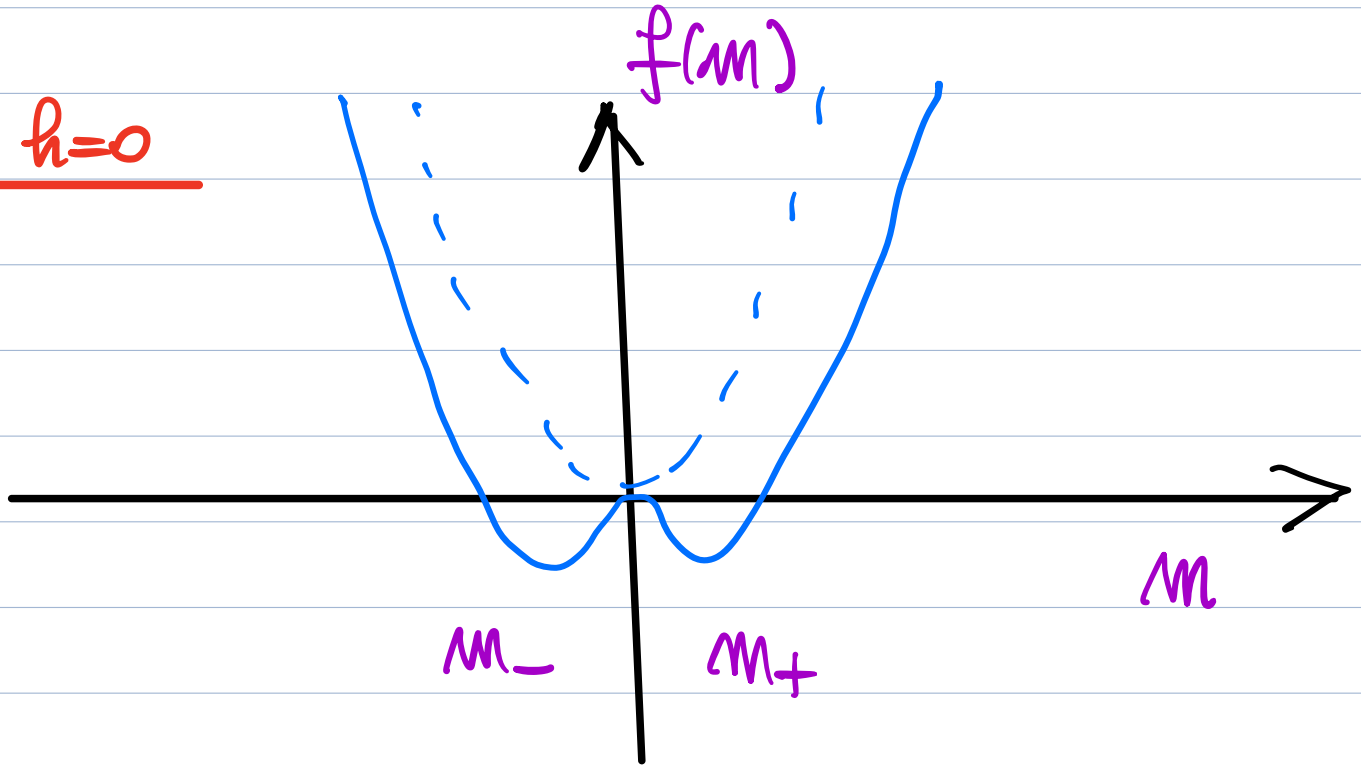
THE SYST CAN BE NEGATIV. MAGN. UNDER $h > 0$ IF PREPARED AT $m < 0$



$T < T_c(h=0)$

THE FULL FREE-ENERGY LANDSCAPE

$h=0$



HERE $f(m)$ IS A FUNCT OF ALL POSSIB m VALUES, IE $m \in [-1,1]$

BELOW T_c $|m| \neq 0$ ARE BELOW $m=0$
$\Rightarrow$ NON TRIVIAL EQ. SOLUTIONS

$$f(m) = \frac{J}{2} z m^2 - k_B T \ln 2 \cosh \beta (J z m + h)$$

THE PINNING FIELD ROLE

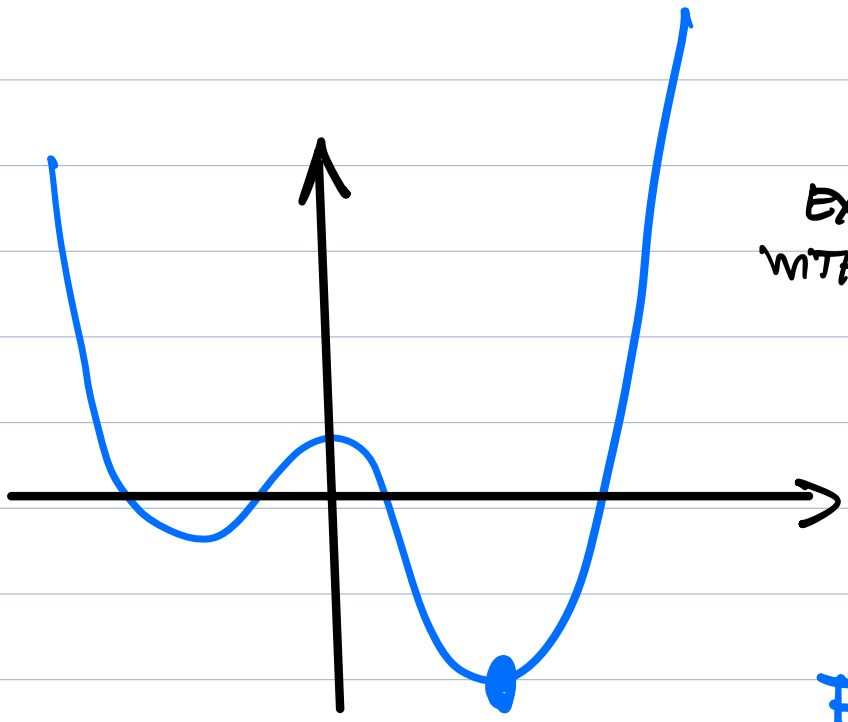
EFFECT of FIELD $-hm$ $\Rightarrow$ TILT $f(m)$ TO MAKE $m>0$ LOWER if $h>0$

$m<0$ LOWER if $h<0$

$h \to 0$ SPONTANEOUS SYMM BREAKING

IT TILTS THE FREE-ENERGY

$f(m)$



EX. PLOT IT WITH SOME GRAPHICAL TOOL.

FAVOURED

THE $\lambda\phi^4$ POTENTIAL

$$f(m) = \frac{zJ}{2} m^2 - k_B T \ln 2 \cosh \beta (Jzm + h)$$

IF WE ASSUME $m \gtrsim 0$ CLOSE TO T_c (ie $T \lesssim T_c$, $h = 0$)

TAYLOR EXPAND 2nd TERM

$$\ln \cosh y = \ln \left(1 + \frac{y^2}{2} + \frac{y^4}{24}\right) \simeq y^2/2 - y^4/12$$

$$f(m) \simeq \frac{zJ}{2} m^2 - \frac{k_B T}{2} (\beta J z m)^2 + \frac{k_B T}{12} (\beta J z m)^4 - k_B T \ln 2$$

$$= \frac{zJ}{2} m^2 \left[1 - \beta J z\right] + \frac{\beta^3 J^4 z^4}{12} m^4 - k_B T \ln 2$$

1 AT T_c

$$\boxed{f(m) \simeq \frac{z\beta J}{2} (k_B T - k_B T_c) m^2 + \frac{\lambda m^4}{4!}}$$

$h = 0$

THE CURVATURE CLOSE TO $m = 0$ CHANGES AT T_c

+ ADDITIVE CONST $- k_B T \ln 2$

WHAT HAPPENS IF WE KEEP h ?
ALSO CLOSE TO T_c OF COURSE

$\Rightarrow$ QUADRATIC TERM

$$-\frac{k_B T}{2}\left(\beta(Jzm+h)\right)^2$$

$$\sim -\frac{k_B T}{2}\underbrace{\beta^2 (Jz)^2}_{1} m^2$$

$$-\frac{k_B T}{2}\overbrace{2\beta^2 Jz}^{1} mh \qquad + \Theta(h^2)$$

DROP

$$= -\frac{k_B T}{2} m^2 - mh$$

MAGN. FIELD CONTRIB.

Do it at home

THE SPECIFIC HEAT

DISCONTINUOUS AT T_c

$$C_V = \frac{\partial E}{\partial T} = -\frac{1}{k_B T^2}\frac{\partial^2(\beta F)}{\partial \beta^2} = -2k_B\frac{\partial \beta F}{\partial T} - k_B T^2\frac{\partial^2(\beta F)}{\partial T^2}$$

$$\ln Z = -\beta F = -\beta N f \qquad E = \langle H \rangle$$

$$f(m) \sim -k_B T \ln 2 + \frac{1}{2}\left(k_B T - k_B T_c\right) m^2$$

DROPPED $O(m^4)$ TERMS

WHAT IS f AS A FUNCT. OF T?

- $T > T_c \quad m = 0$

$$\beta f(m(T)) = -\ln 2$$

$$\Rightarrow \frac{\partial^2 \beta f(m(T))}{\partial T^2} = 0$$

- $T \lesssim T_c \quad m \sim (T_c - T)^{\beta}$

$$\beta f(m(T)) \sim -\ln 2 - \frac{1}{2}\beta\,(T_c - T)^{1+2\beta}$$

$1+2\beta = 2 \Leftarrow$ EXP. $\beta = 1/2$

$$\frac{\partial(\beta f)}{\partial T} = +\frac{1}{2}\frac{(T_c-T)^2}{(k_B T)^2} - \frac{1}{2}\beta\, 2\,(T_c - T) = 0 \qquad T = T_c$$

$$\frac{\partial^2(\beta f)}{\partial T^2} = \frac{1}{2}(-2)\frac{(T_c-T)^2}{(k_B T)^3} + \frac{1}{2}\frac{2\,(T_c-T)(-1)}{(k_B T)^2}$$

$$+ \frac{1}{(k_B T)^2}(T_c - T) + \frac{1}{k_B T}(-1)$$

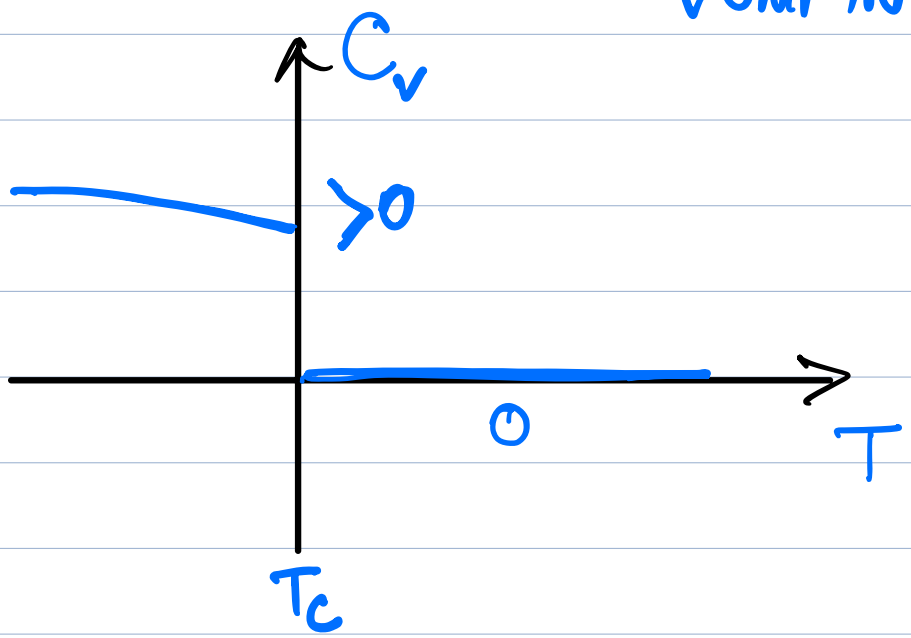
AT $T \lesssim T_c$: $\quad = -\beta_c$

$$C_V = \beta_c^2 \frac{\partial^2 \ln Z}{\partial T^2} = \beta_c^2 \frac{\partial^2(-\beta N f)}{\partial T^2}$$

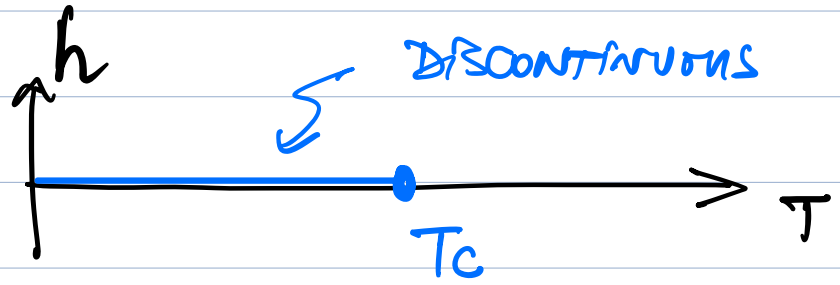
$$\downarrow$$

$$\text{AT } T_c \quad = \begin{cases} 0 & T > T_c \\ \text{CONST} & T \lesssim T_c \end{cases}$$

JUMP IN C_v AT T_c



2.5.7 HINTS ON UNIVERSALITY





VERY SIMILAR

MOREOVER SPIN MODEL LATTICE GAS

S_i^0 $\quad$ $n_i^0 = \frac{S_i^0 + 1}{2}$

± 1 $\quad$ $1, 0$

VERY ≠ PROBLEMS W/ SIMILAR PHYSICS

$\Rightarrow$ UNIVERSALITY? SAME EXPONENTS $\alpha, \beta, \gamma, \delta$ MORE LATER

WHAT DO WE MISS WITH THE MF APPROX?

2.5.1 ABSENCE OF COOP

CONNECTED CORRELATIONS: $C_{ij}^{CONN} \equiv$

$$\langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle = \langle \delta s_i \, \delta s_j \rangle$$

UNDER THE ASSUMPTIONS MADE

RECALL WE DROPPED THIS QUAD. TERM

DO WE GET ZERO FOR THIS CORR ?

SEVERAL WAYS TO DO IT. BY DEF: $(h_i = 0)$:

$$C_{kl}^{CONN} = \frac{\sum_{\{s_i\}} s_k \, s_l \, e^{-\beta H(m_i, s_i)}}{\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}} - \frac{\sum_{\{s_i\}} s_k \, e^{-\beta H(m_i, s_i)}}{\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}} \cdot \frac{\sum_{s_j} s_l \, e^{-\beta H(m_j, s_j)}}{\sum_{s_j} e^{-\beta H(m_j, s_j)}}$$

WITH $H(m_i, s_i) = -\frac{J}{2} \sum_{i \neq j} (m_i m_j - 2 s_i m_j)$

FIRST, NOTICE THAT ONLY THE TERMS IN THE SUMS WHERE S_k AND S_ℓ APPEAR. REMAIN.
ALL OTHERS CANCEL BTW NUM & DEN.
FOR EX.

$$\frac{\sum_{\{S_i\}} S_k e^{-\beta H(\{S_i, m_i\})}}{\sum_{\{S_i\}} e^{-\beta H(\{S_i, m_i\})}} = \frac{\sum_{S_k} S_k e^{-\beta h_k^{loc} S_k}}{\sum_{S_k} e^{-\beta h_k^{loc} S_k}}$$

IN 2nd TERM AND ONLY

$$\sum_{S_k, S_\ell} \frac{S_k S_\ell \; e^{-\beta h_k^{loc} S_k - \beta h_\ell^{loc} S_\ell}}{\sum_{S_k S_\ell} e^{-\beta h_k^{loc} S_k - \beta h_k^{loc} S_\ell}}$$

REMAINS IN 1st TERM

RECALL $h_i^{loc} = J_{ij} m_j + h_i$

MOREOVER, IF k AND l ARE $\neq$

$\Rightarrow$ FACTORIZE 1st SUM

$\Rightarrow$ TWO TERMS IN COSH ARE THE SAME AND RESULT VANISHES IDENTICALLY

$$C^{conn}_{kl} = 0$$

NO CORRELATION BEYOND LATTICE DISTANCE

EVEN MORE CLEARLY SEEN IN THE FOLLOWING WAY OF OBTAINING THE SAME DERIVATION:

2.5.2

ANOTHER WAY TO OBTAIN THE SAME EQS.

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

$$p_i(s_i) = \frac{(1+m_i)}{2}\delta_{s_i,1} + \frac{(1-m_i)}{2}\delta_{s_i,-1}$$

NOTATION: WE KEEP A POSSIBLE i-DEP. IN m_i IN CASE WE WANT TO CONSIDER J_{ij} HETEROG.

FOR USUAL ISING $J_{ij} = J$ & $m_i = m$

CHECK $\sum_{\{s_i=\pm 1\}} p_i(s_i) = 1$ NORM.

$\sum_{\{s_i=\pm 1\}} s_i\, p_i(s_i) = m_i$ DEF of $\{m_i\}$

$F = U - TS$ FREE-ENERGY

WHAT IS U ? $\langle H \rangle_P$

WHAT IS S ? $-k_B \langle \ln P \rangle_P$

FIND $F(\{m_i\})$ sgn $P<1$, $\ln P<0$, $S>0$

ASK IT TO BE A MINIMUM WRT $\{m_i\}$

$\Rightarrow$ FIND EQS ABOVE FOR $\{m_i\}$

$$\langle H \rangle_p = -\frac{1}{2}\sum_{\langle ij \rangle} J_{ij} \langle s_i s_j \rangle_p = -\frac{1}{2}\sum_{\langle ij \rangle} J_{ij} \langle s_i \rangle_p \langle s_j \rangle_p$$

$$= -\frac{1}{2}\sum_{\langle ij \rangle} J_{ij}\, m_i\, m_j$$

IF ALSO LOCAL FIELDS $\left\langle \sum_i h_i s_i \right\rangle_p = \sum_i h_i \langle s_i \rangle_p = \sum_i h_i m_i$

$$\langle \ln P \rangle_p = \left\langle \sum_i \ln\left(\frac{1+m_i}{2}\,\delta_{s_i,1} + \frac{1-m_i}{2}\,\delta_{s_i,-1} \right) \right\rangle_p =$$

$$= \frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2} \quad (<0)$$

$$\beta f(\{m_i\}) = -\frac{\beta}{2N}\sum_{\langle ij \rangle} J_{ij}\, m_i m_j - \frac{\beta}{N}\sum_i h_i m_i + \frac{1}{N}\sum_i \left(\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2} \right)$$

LOOKS A BIT DIFF. BUT IT'S EQUIVALENT

SHOW

$$p_i(s_i) = \frac{e^{\beta h_i^{loc}(m_i)\, s_i}}{2\cosh \beta h_i^{loc}(m_i)}$$

FOR WE HAD ABOVE

EXERCISE

$$h_i^{loc} = h_i^{eff} + h$$

PROOF

IN THE PROPOSAL $P(\{s_i\}) = \prod_i p_i(s_i)$:

$$p_i(s_i = 1) = \frac{1 + m_i}{2}$$

WITH $m_i = \text{th}(\beta h_i^{loc}) \Rightarrow$

$$p_i(s_i = 1) = \frac{1}{2}\left(1 + \text{th}\,\beta h_i^{loc}\right)$$

$$= \frac{1}{2}\,\frac{\text{ch}\,\beta h_i^{loc} + \text{sh}\,\beta h_i^{loc}}{\text{ch}\,\beta h_i^{loc}}$$

$$= \frac{1}{4}\left[\frac{e^{\beta h_i^{loc}} + e^{-\beta h_i^{loc}} + e^{\beta h_i^{loc}} - e^{-\beta h_i^{loc}}}{\text{ch}\,\beta h_i^{loc}}\right]$$

$$= \frac{1}{2}\,\frac{e^{\beta h_i^{loc}}}{\text{ch}\,\beta h_i^{loc}}$$ ✓ similarly for $s_i = -1$

ARE THE TWO EXPRESSIONS OF $F(m)$ THE SAME?

DO IT FOR $J_{ij} = J$ $h_i = h$ FOR SIMPLICITY

$$-\frac{1}{2} J z m^2 - hm + k_B T \left[\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

$$\stackrel{?}{=}$$

$$\frac{Jz}{2} m^2 - hm - k_B T \ln 2 \cosh(\beta J z m + \beta h)$$

- CHECK $\frac{\partial f(m)}{\partial m} = 0$ FROM 1st EXPRESSION

  SET $h = 0$ TO MAKE EXPRESSIONS SHORTER

$$0 = -Jzm + k_B T \left[\frac{1}{2} \ln \frac{1+m}{2} - \frac{1}{2} \ln \frac{1-m}{2} \right]$$

$$+ k_B T \left[\frac{1+m}{2} \frac{1/2}{\frac{1+m}{2}} + \frac{1-m}{2} \frac{(-1/2)}{\frac{1-m}{2}} \right]$$

$$= 0$$

$$0 = -Jzm + \frac{k_B T}{z} \ln \frac{\frac{1+m}{z}}{\frac{1-m}{z}}$$

$$\frac{1+m}{1-m} = \exp\left(\frac{-2Jzm}{k_B T}\right)$$

$$1+m = (1-m)\exp\left(-2\beta Jzm\right)$$

$$1 - e^{-2\beta Jzm} = m\left(e^{-2\beta Jzm} + 1\right)$$

$$m = \frac{1 - e^{-2\beta Jzm}}{1 + e^{-2\beta Jzm}}$$

$$m = \tanh\left(\beta Jzm\right) \quad \checkmark$$

- Check now $\frac{\partial f(m)}{\partial m} = 0$ from 2nd exp.

$$\frac{J_z}{2} m^2 - k_B T \ln 2 \operatorname{ch}(\beta J z m)$$

$$\Rightarrow \quad J z m - k_B T \frac{2 \operatorname{sh}(\beta J z m)}{2 \operatorname{ch}(\beta J z m)} \cdot \beta J z = 0$$

$$m = \operatorname{th}(\beta J z m)$$

SAME EQ. ✓

OK

CAN ALSO TRANSFORM ONE $f(m)$ INTO ANOTHER. IT CAN BE DONE USING MF EQ.

SIMPLER

CHECK THEY HAVE THE SAME $m \sim 0$ EXPANSION

SUMMARY OF 1st THREE LECTURES

- PHENOMENOLOGY
- CONCEPTS
- SPIN MODELS
- MEAN-FIELD IN TWO WAYS

(1) $s_i = m + \delta s_i$

(2) $P(\{s\}) = \prod_i p_i(s_i)$

$$= \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1}$$

$m_i = \langle s_i \rangle$ IN BOTH

EQUIV. BTW THE TWO $\dfrac{e^{\beta h_i^{loc}(m_i)}}{2\cosh \beta h_i^{loc}(m_i)}$

IN THE TD

$$H_J[\{s\}] = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - h\sum_i s_i$$

$$\mathcal{Z} = \sum_{\{s_i\}} \cdots =$$

$$= \sum_x e^{-N\beta\left[f\phi(x) - k_B T s(x)\right]}$$

$$\to \int dx\; e^{-N\beta f(x)}$$

AND SADDLE-POINT $N \to \infty$ CLEARLY SEEN – DETAILS BELOW.

GO BACK TO CALC. of CRIT EXPONENTS NOW. p. 53 EXPLAIN β, γ, δ

2.5.5 THIRD WAY OF GETTING THE MEAN-FIELD EQS

- IN NOTES GENERALIZED TO p-SPIN MODELS
- IN TD IN DETAIL.

IN ISING SPIN MODEL

$$Z = \sum_{\{s_i = \pm 1\}} e^{\frac{\beta}{2} \sum_{ij} J_{ij} s_i s_j + \beta \sum_i h_i s_i}$$

$$J_{ij} = J \; \Delta \sum_{i \neq j}$$

FULLY CONNECTED GRAPH

SUM OVER ALL PAIRS ON COMPLETE GRAPH

$J \to J/N$ OTHERWISE $H = \Theta(N^2)$

$$\sum_{i \neq j} s_i s_j = \sum_{ij} s_i s_j - \sum_i s_i^2$$

$$= \left(\sum_i s_i\right)\left(\sum_j s_j\right) - \sum_i 1$$

$$= N \left(\frac{1}{N}\sum_i s_i\right)\left(\frac{1}{N}\sum_j s_j\right) - N$$

TRANSFORM SUM OVER $\{s_i = \pm 1\}$ IN SUM OVER

$$x = \frac{1}{N}\sum_i s_i \in [-1, 1], \quad \delta x = \frac{2}{N} \ll 1$$

$$x = -1, -1 + \frac{2}{N}, -1 + \frac{4}{N}, \ldots, 1$$

$$\mathcal{Z} = \sum_x \mathcal{N}\left(\{s_i\} / \frac{1}{N}\sum_i s_i = x\right) e^{\frac{\beta N}{2} x^2}$$

WHAT IS THE DEGENERATION $\mathcal{N}$?

$$\mathcal{N} = \binom{N}{N_+} \qquad N_+ : \#\,+\,\text{SPINS}$$

$$= \frac{N!}{N_+!\, N_-!}$$

STIRLING $\Longrightarrow$

$$\mathcal{N} = e^{N J(x)}$$

WITH

$$-J(x) = \frac{1+x}{2} \ln \frac{1+x}{2} + \frac{1-x}{2} \ln \frac{1-x}{2}$$

ENTROPY OF ISING SPINS

$$\mathcal{Z} \longrightarrow \int dx\, e^{-\beta N f(x)}$$

$$\boxed{f(x) = -\frac{x^2}{2} - hx - h\beta T J(x)}$$

THE SADDLE-POINT VALUES EQUAL THE AVERAGE

$$\mathcal{Z} \sim \sum_{\alpha} e^{-\beta N f(x_{sp}^{\alpha})}$$

α LABELING THE SADDLE POINTS (IF SEVERAL)

$$x_{sp} = -\frac{\partial f(x)}{\partial h}\bigg|_{x_{sp}} = \langle x \rangle$$

JUST FROM $f(x)$ & THE LINEAR COUPLING BTW h & x

USING h AS A SOURCE WITHIN PARTITION SUM

PROOF BELOW

$\Rightarrow$ $\boxed{x_{sp} = \langle x \rangle}$

IF MORE THAN ONE SADDLE POINT $\Rightarrow$ AVERAGE OVER THE CORRESP "ERGODIC COMPONENT" OR "STATE"

CAN SELECT THEM W/ PINNING FIELD

HOW DOES THE "SOURCE" METHOD WORK ?

$$\mathcal{Z} = \sum_\alpha e^{-\beta N f(x_{sp}^\alpha)} \Rightarrow e^{-\beta N f(x_{sp})}$$

JUST ONE SP TO SIMPLIFY

$$\frac{\partial \mathcal{Z}}{\partial h} = -\beta N \frac{\partial f(x_{sp})}{\partial h} e^{-\beta N f(x_{sp})}$$

if $f(x_{sp}) = \underbrace{f^{(0)}(x_{sp})}_{\text{NOT DEP ON } h} - \underbrace{h x_{sp}}_{\text{DEP. TERM}}$ THE FIELD

$$\frac{\partial f(x_{sp})}{\partial h} = -x_{sp}$$

NB NO "CHAIN RULE" CONTRIB SINCE

$$\underbrace{\frac{\partial f(x_{sp})}{\partial x_{sp}}}_{\text{"0" by def of } x_{sp}.} \frac{\partial x_{sp}}{\partial h} = 0$$

THEN $\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial h} = \frac{\beta N x_{sp} e^{-\beta N f(x_{sp})}}{e^{-\beta N f(x_{sp})}} = \beta N x_{sp}$

THE LEFT-HAND-SIDE IS $\dfrac{\partial \ln Z}{\partial h}$

BUT $Z = \int dx \; e^{-\beta N f^{(0)}(x) + \beta N h x}$

BEFORE SADDLE-POINT

$$\Rightarrow \frac{\partial \ln Z}{\partial h} = \frac{1}{Z} \int dx \; e^{-\beta N f(x)} \beta N x$$

$$= \beta N \langle x \rangle$$

FINALLY, COMPARING

$$\frac{\partial \ln Z}{\partial h} = \beta N \, x_{sp} = \beta N \langle x \rangle$$

$$\Rightarrow \quad x_{sp} = \langle x \rangle$$

IN THE TD YOU'LL SEE YET ANOTHER WAY OF GETTING THE MF $f(m)$ & EQ. ON m.

PROBLEM THE EXPONENTS MEASURED ARE NOT THE MEAN-FIELD ONES

SHOW $m \sim (T_c - T)^{\beta}$ $\beta \simeq 0.3$

IN $d=3$

SAMPLES $C \sim (T_c - T)^{\alpha}$ $\alpha \simeq 0.11$

ONSAGER VALUES IN $d=2$

$\beta = 1/8$ $\alpha = 0$

VS $\beta_{MF} = 1/2$ $\alpha_{MF} = 0$ $\forall d$

SO?

↳ SOLUTION LATER

NOT COMPLETELY WRONG

ORDER OK

PHASE DIAGRAM OK } $d=2,3$

$d \geq 4 = d_u$ EVERYTHING IS OK

WHAT HAVE WE DONE ?

- MEAN-FIELD WITH NO SPATIAL STRUCT
  FOUND PHASE TRANS

  CONT. (NT) → CRITICAL EXP INDEP of d

  DISC (NH)

SIMPLE FRACTIONS

SUMMARY 3rd LECTURE

- FINISHED ANALYSIS OF MF EQS & OBTAINED ALL EXPONENTS

$\alpha, \beta, \gamma, \delta$

- COMPARED TO EXP. DATA AND SAW THAT THE INTEGER VALUES - INDEP. of d - WERE NOT MEASURED

- SHOWED $\langle \delta S_i \, \delta S_j \rangle = 0$ $i \neq j$

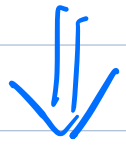
PLAN FOR 4th LECTURE

- LANDAU'S PRESCRIPTIONS
- HOMOGENEOUS S.P.
  DOMAIN WALL / KINKS
- CORR. OF FUCT $\langle \delta\phi(\vec{x})\, \delta\phi(\vec{y}) \rangle$

  CORR. LENGTH $\xi \sim |T - T_c|^{-\nu}$
- GINZBURG CRITERIUM

  $\langle \delta\phi^2_{coh} \rangle$ vs. $\langle \phi \rangle^2$
- 1st ORDER

- WE NEED SOME SPACE DEPENDENCE



2.6 GINZBURG - LANDAU THEORY

1 - LANDAU TH. $F(\phi)$

2 - CORR FCT. $G_{conn}(r)$

3 - GINZBURG CRIT

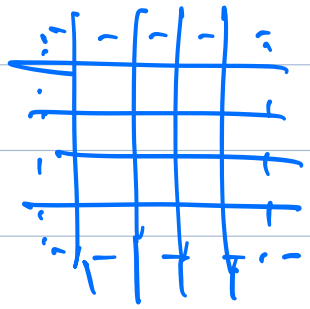
ESTIMATE "STRENGTH" OF FLUCTUATIONS

$d_u \leq d$ MF ok

2.6 LANDAU THEORY

- WE HAD NO SPATIAL INFO SO FAR
- LANDAU → GO A BIT BEYOND, SIMILAR APPROACH TO WHAT DONE FOR FULLY-CONNECTED

- COARSE-GRAIN



$$\phi(\vec{x}) = \frac{1}{n_{\vec{x}}} \sum_{i \in v_{\vec{x}}} s_i$$

SUM OVER SPINS IN BOX

ℓ

$$a \ll \ell \ll L$$

$n_{\vec{x}}$ = # SPINS IN BOX
$= (\ell/a)^d$

$\phi(\vec{x})$ TAKES DISCRETE VALUES BUT IN THIS LIMIT ⇒ CONTINUOUS FCT OF SPACE

OVERLAPPING BOXES $\delta x = a$

NON-OVERL. BOXES $\delta x = \ell$

$$\mathcal{Z} = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$$

SAME AS WHAT WE DID FOR FC MODEL, GROUP SPINS ACCORD. TO THE VALUES OF SOME AUX. VAR. & SUM OVER THEM. GENERIC α INDEX

$$= \sum_{\alpha} \sum_{\{s_i\}/\alpha} e^{-\beta H_\alpha(\{s_i\})}$$

APPLY TO α BEING ϕ

$$= \int \mathcal{D}\phi \sum_{\{s_i\}/\phi} e^{-\beta H_\phi(\{s_i\})}$$

{ ALL CONF OF SPINS YIELDING THE SAME ϕ

$$\boxed{\mathcal{Z} = \int \mathcal{D}\phi \; e^{-\beta F(\phi)}}$$

↓ ENTROPIC CONTRIB.

WRITTEN AS A FIELD DEP. (β FREE ENERGY)

GINZBURG-LANDAU FREE ENERGY (NB IT'S MACROSCOPIC)

$\int \mathcal{D}\phi$?

MEANS THAT ONE HAS TO SUM OVER ALL POSSIBLE VALUES OF ϕ AT EACH POINT IN SPACE $\vec{x}$ LABELS THE BOXES

$$\int \mathcal{D}\phi = \prod_{\vec{x}} \int_{-\infty}^{\infty} d\phi(\vec{x})$$

WE LIFT THE CONSTRAINT $-1 \leq \phi(\vec{x}) \leq 1$

THE BEST WAY TO UNDERSTAND A FIELD IS TO THINK ABOUT $\vec{x}$ AS A LABEL

$F(\phi)$ IS A MACRO. FREE ENERGY

FUNCTIONAL

(A FUNCTION of A FUNCTION)

F ϕ

THE INTEGRAL IS A FUNCTIONAL INTEGRAL

RECALL $x = \frac{1}{N}\sum_i s_i$ CURIE-WEISS

HERE $\phi(\vec{x}) = \frac{1}{n_{\vec{x}}} \sum_{i \in V_{\vec{x}}} s_i$ SPACE-DEP.

GUESSES FOR $F(\phi)$

1 – LOCAL & MACROSCOPIC $\quad F(\phi) = \int d^d x \, f(\phi(\vec{x}))$

2 – SPATIAL SYMMETRIES PRESERVED
TRANSL & ROTAT OF LATTICE
$\Rightarrow$ CONT TH. TOO

3 – SPIN REVERSAL $\Rightarrow$ $\phi \to -\phi$ SYMM.

$F(\phi) = F(-\phi)$

4 – ANALYTIC FUNCTIONAL
$F(\phi)$ TAYLOR EXPANDABLE IN POWERS OF $(\phi, \vec{\nabla}\phi)$

5 – SMOOTH VARIATIONS IN SPACE
KEEP ONLY $(\vec{\nabla}\phi)^2$

$$F(\phi) = \int d^d x \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + \frac{a}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

a CONTROLS SMALL ϕ BEHAVIOUR

$\bar{a} > 0 \quad T > T_c$

$\bar{a} < 0 \quad T < T_c$

$$\bar{a} = a \cdot \frac{T - T_c}{T_c}$$

MEAN-FIELD

cfr FULLY CONNECTED ISING MODEL

FOR SMALL ORDER PARAMETER (x)

THE SAME + $(\vec{\nabla}\phi)^2$ TERM

DIMENSIONS

$$[c]\left[\frac{\phi^2}{L^2}\right] = [\bar{a}][\phi^2] = [\lambda][\phi^4]$$

$$\Downarrow$$

$$[c] = [a] L^2 \qquad L = [x], \quad [\bar{a}] = [a]$$

$$[c] = [a]\, L^2$$

$$[\phi^2][\lambda] = [a]$$

$$\left[\frac{c}{a}\right]^{1/2} = L$$

NB THE RÔLE PLAYED BY N IS NOW PLAYED BY THE VOLUME OF SPACE $\int d^d x \rightarrow$ Volume

SADDLE POINT

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = 0$$

HOW DO YOU CALCULATE FUNCTIONAL DERIV. ?
HOW MANY OF YOU KNOW HOW TO DO IT?

MATH SUPPORT

$$F(\phi) = \int d^d x \, f(\phi(\vec{x}))$$

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = \int d^d x \, \frac{\delta f(\phi(\vec{x}))}{\delta \phi(\vec{y})}$$

if $f(\phi(\vec{x}))$ JUST A fct OF $\phi(\vec{x})$

$$= \int d^d x \; f'(\phi(\vec{x})) \underbrace{\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})}}_{\delta^d(\vec{x}-\vec{y})}$$

$$= f'(\phi(\vec{y}))$$

NB: IT SETS $\vec{x}$ TO $\vec{y}$ AND TAKES A PARTIAL (NORMAL DERIV.)

WHAT HAPPENS WITH TERMS WITH $\vec{\nabla}\phi(\vec{x})$?

eg. $F_{grad}(\phi) = \int d^d x \; \frac{1}{2}\left(\vec{\nabla}\phi(\vec{x})\right)^2$

$$= \int d^d x \; \frac{1}{2} \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}\phi(\vec{x})$$

$$\frac{\delta F_{grad}(\phi)}{\delta\phi(\vec{y})} = \int d^d x \left[\frac{1}{2} \vec{\nabla} \frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})} \cdot \vec{\nabla}\phi(\vec{x}) + \frac{1}{2} \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla} \frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})} \right]$$

TWICE THE SAME TERM

$$= \int d^d x \; \vec{\nabla} \delta^d(\vec{x}-\vec{y}) \cdot \vec{\nabla}\phi(\vec{x})$$

$$= - \int d^d x \; \delta^d(\vec{x}-\vec{y}) \; \nabla^2 \phi(\vec{x}) + \text{BORDER TERMS}$$

$$\propto \delta(\infty - \vec{y}) \; \text{"} \; 0$$

$$= - \nabla^2 \phi(\vec{y})$$

ALL TOGETHER

$$0 = \frac{\delta F(\phi)}{\delta \phi(\vec{y})} = -c\nabla^2\phi(\vec{y}) + \frac{\lambda}{3!}\phi^3(\vec{y}) + a\left(\frac{T-T_c}{T_c}\right)\phi(\vec{y})$$

$$\frac{\lambda}{3!}\phi_{sp}^3 + a\frac{T-T_c}{T_c}\phi_{sp} = 0$$

$\phi(\vec{x}) = \phi_{sp}$ CONST

ONE KIND OF SOLUTION

$$\phi_{sp} = 0 \quad \text{or}$$

$$\phi_{sp} = \pm \sqrt{\frac{3!}{\lambda} a \frac{(T_c - T)}{T_c}}$$

$\rightarrow T < T_c$

UNITS OK

UNTIL HERE $\rightarrow$ REALLY SIMILAR! SAME EXPONENTS

4th LECTURE PLAN

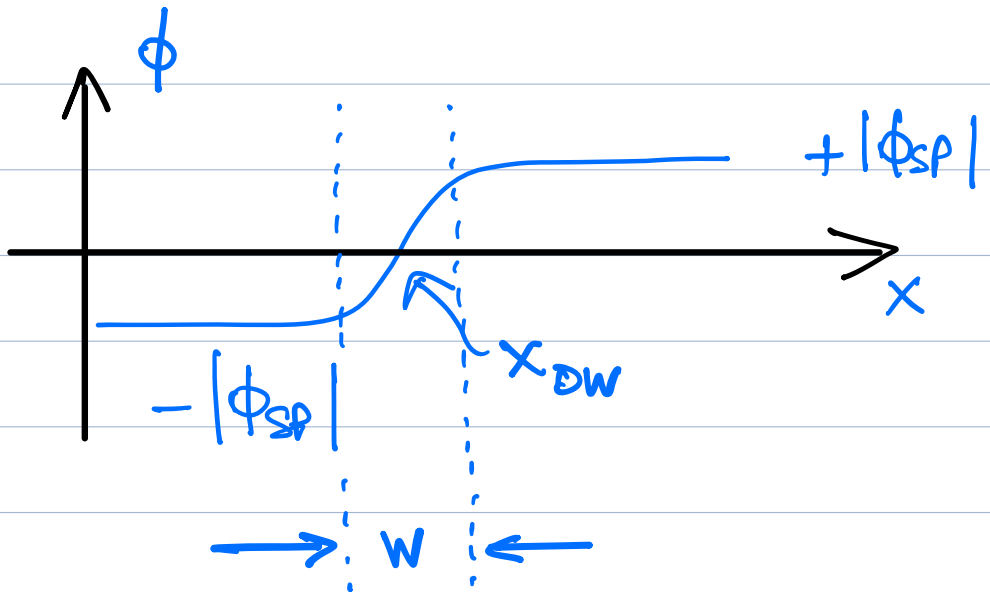
- DOMAIN WALLS 10'
- CORR of FUNCT CONN → ξ 20'
- GINZBURG CRIT $d_u = 4$ 10'
- DIVERGENCE OF χ 10'
- 1st ORDER PHASE TRANS. 5'
  OTHER MODELS WITH MORE FIELDS
- SHOW SLIDES ON EXP. & CONFS. 10'

SCALING

DOMAIN WALLS

SPATIAL VARIATION / INFO

EX. $d = 1$



ADDING $\nabla^2\phi$ TO THE SP. EQ

WITH $\lambda\phi^4$ "POTENTIAL"

$\Rightarrow$ SOLUTION VARIATION ALONG DIRECT. x

$$\phi_{DW}(x) = |\phi_{SP}| \tanh\left(\frac{x - x_{DW}}{w}\right)$$

$$c \frac{d^2\phi}{dx^2} = a \frac{(T-T_c)}{T_c} \phi + \frac{\lambda}{3!} \phi^3$$

CHECK

$$\text{lhs: } c\, |\phi_{sp}| \frac{d}{dx} \frac{1}{ch^2\left(\frac{x-x_{DW}}{w}\right)} \frac{1}{w} =$$

$$= c \frac{|\phi_{sp}|}{w} \frac{(-2)\, sh\left(\frac{x-x_{DW}}{w}\right)}{ch^3\left(\frac{x-x_{DW}}{w}\right)} \frac{1}{w}$$

$$\text{rhs: } a \frac{T-T_c}{T_c} |\phi_{sp}| th\left(\frac{x-x_{DW}}{w}\right) +$$

$$\frac{\lambda}{3!} |\phi_{sp}|^3 th^3\left(\frac{x-x_{DW}}{w}\right)$$

REMAINS

$$\frac{-2c}{w^2} \frac{1}{ch^2\left(\frac{x-x_{DW}}{w}\right)} =$$

$$a\frac{T-T_c}{T_c} + \frac{\lambda}{3!}|\phi_{sp}|^2 = a\frac{T_c-T}{T_c}$$

$$|\phi_{sp}|^2 = \frac{3!}{\lambda}\, a\frac{T_c-T}{T_c}$$

$\Rightarrow$

$$-\frac{2c}{w^2}\frac{1}{ch^2\left(\frac{x-x_{DW}}{w}\right)} =$$

$$= a\frac{T-T_c}{T_c}\,\frac{\left[ch^2\left(\frac{x-x_{DW}}{w}\right) - sh^2\left(\frac{x-x_{DW}}{w}\right)\right]}{ch^2\left(\frac{x-x_{DW}}{w}\right)}$$

$$= a\frac{T-T_c}{T_c}\,\frac{1}{ch^2\left(\frac{x-x_{DW}}{w}\right)}$$

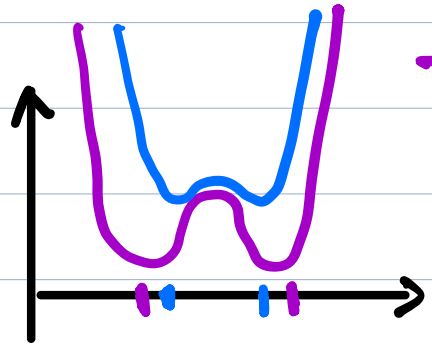
THUS $$\frac{2c}{w^2} = a\frac{T_c - T}{T_c}$$

$$w^2 = \frac{2c}{a}\frac{T_c}{T_c - T} \qquad > 0 \ \checkmark$$

INTERPRETATION

$$|\phi_{sp}|^2 = \frac{3!}{\lambda} a \frac{(T_c - T)}{T_c}$$

DECREASES FOR $T \to T_c^-$

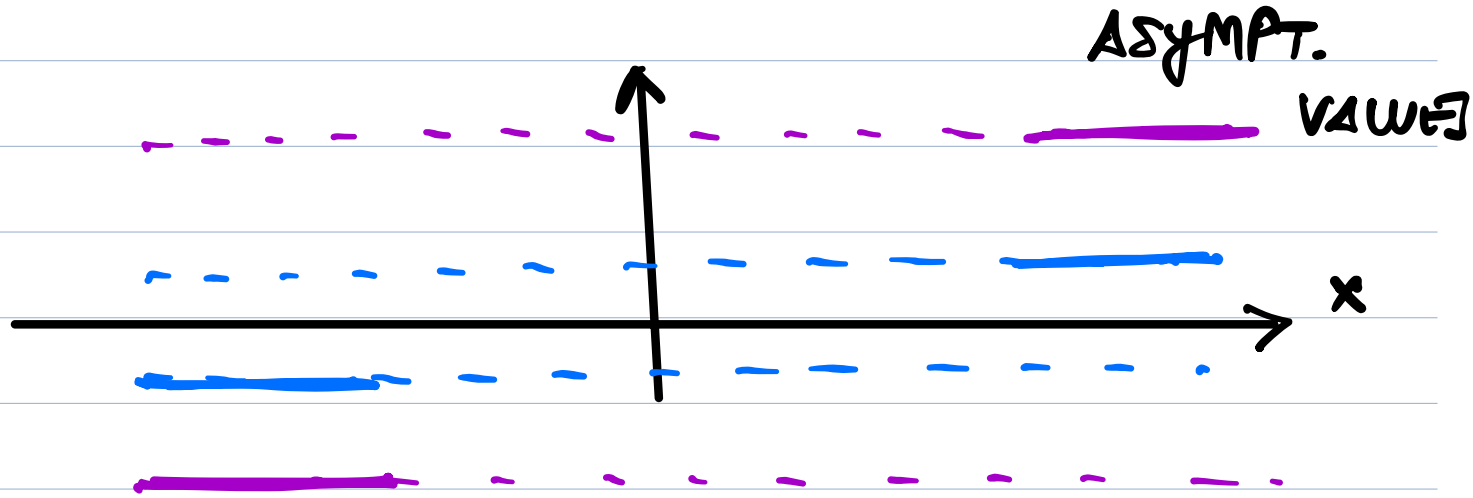


$T \ll T_c$

$T \lesssim T_c$

INDEP. OF c, JUST "POTENTIAL" FORM GIVES $|\phi_{sp}|^2$

$$|\phi_{sp}| \xrightarrow[T \to T_c^-]{} 0$$



MEANING OF WIDTH

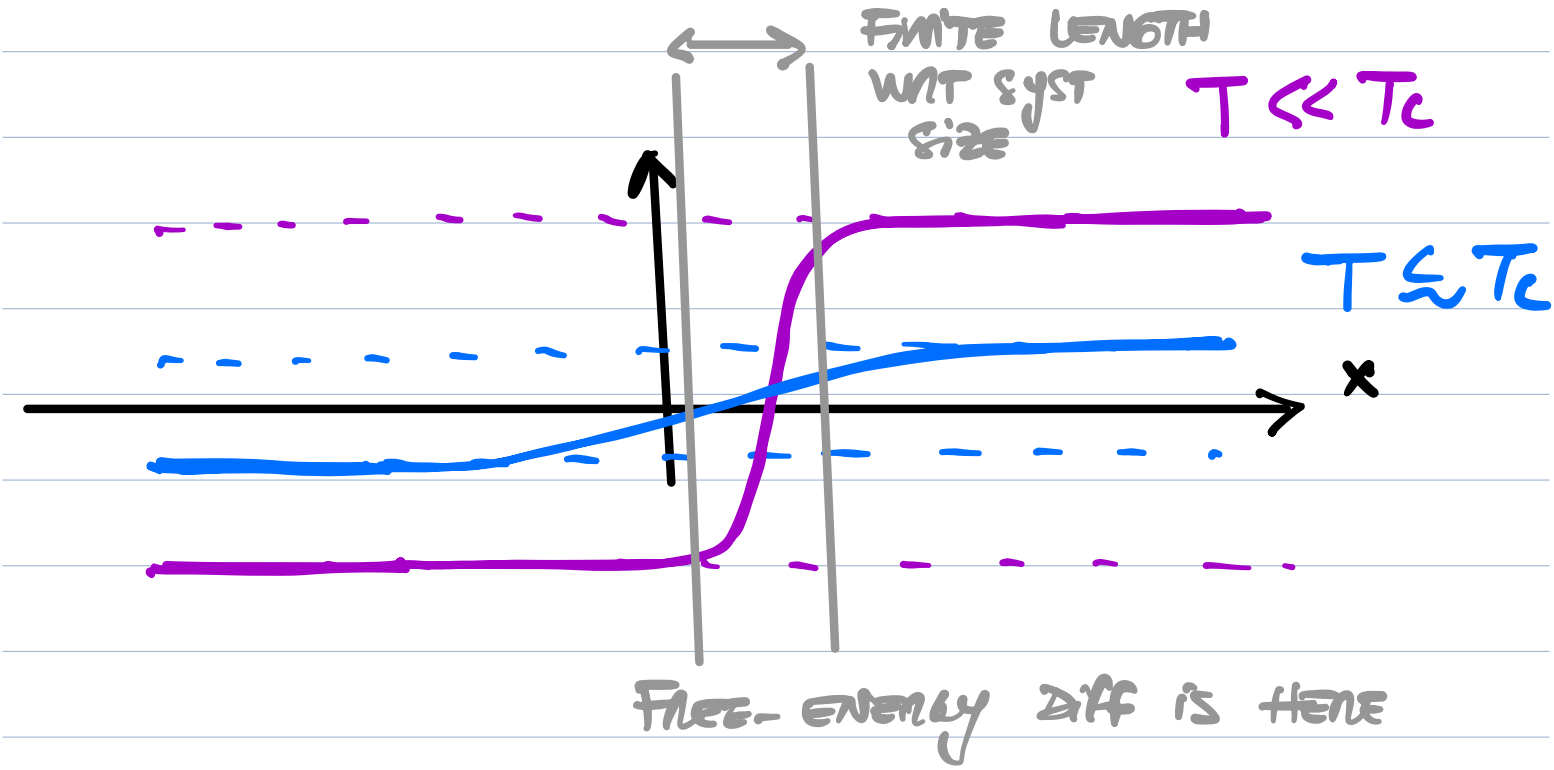
W^2 INCREASES WITH $\frac{\varepsilon T_c}{a(T_c - T)}$

WHEN ELASTICITY ↑ IT'S BETTER TO SPREAD THE FIELD VARIATION OVER LONG DISTANCES. NO SHARP WALLS.

APPROACHING T_c, $|\phi_{sp}| \to 0$

SO, EASY TO GO FROM POSIT TO NEG.

AND $W \to \infty$.



X_{DW} FREE , ϕ_{SP} THE SOL WITHOUT $\nabla^2\phi$

$$W = \left(\frac{c\,T_c}{a(T_c - T)}\right)^{1/2}$$ WIDTH

$\exists$ IF $T < T_c$

UNITS $\left(\frac{[c]}{[a]}\right)^{1/2} = \left(\frac{[a][L^2]}{[a]}\right)^{1/2} = [L^2]^{1/2} = [L]$

OH, IT'S A LENGTH.

W DIVERGES AT $T \to T_c^-$

DEPENDS ON C ELAST CONST.

DIVERGENCE OF W $\Rightarrow$ $\phi_{DW} \to 0$

FLATTER DW CONF $\Rightarrow$ $\phi_{DW} \sim$ ZERO EVERYWHERE

$$W \sim (T_c - T)^{-1/2}$$

$$\boxed{W \propto \xi}$$

ξ CORRELATION LENGTH

WE'LL SEE IT LATER FROM ITS DEF. FROM THE CORR. FUNCTION.

$$F[\phi_{sp}] = -\frac{3a^2}{2\lambda}\left(\frac{T_c - T}{T_c}\right)^2 L^d$$

$$\Delta F = F[\phi_{DW}] - F[\phi_{sp}]$$

$$\sim L^{d-1}\left[\frac{c}{a\lambda^4}\left(\frac{T_c - T}{T_c}\right)^3\right]^{1/2}$$

WHICH SOLUTION IS BETTER ?

ONE SHOULD COMPARE

$$F[\phi_{SP}] \gtrless F[\phi_{DW}]$$

THE EVALUATION YIELDS

$$F[\phi_{SP}=\text{CONST}] < F[\phi_{DW}]$$

CLEAR FROM THE FACT THAT

$$(\vec{\nabla}\phi_{SP})^2 = 0 \quad \text{WHILE} \quad (\vec{\nabla}\phi_{DW})^2 > 0$$

BUT BOTH ARE STABLE, IN THE SENSE THAT

$$\frac{\delta^2 F[\phi]}{\delta\phi(\vec{x})\,\delta\phi(\vec{y})} \quad \text{POSITIVE DEF AT} \quad \begin{cases} \phi_{SP} \\ \phi_{DW} \end{cases}$$

RECALL THAT WHEN DISCUSSING THE PEIERLS ARG. WE ALSO MENTION DOMAIN WALLS ↑↑↑↓↓↓↑↑↑

DW GET THICKER $\Rightarrow$ MORE FLUCT GETTING CLOSE TO T_c^-

LESS ORDER $\Rightarrow$ APPROACH TO CRITICALITY

$$F_{DW} - F_{SP} = \int_{-\infty}^{\infty} dx \left\{ \frac{\epsilon}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi) - V(\phi_{sp}) \right\}$$

$$V(\phi) - V(\phi_{sp}) = \frac{\lambda}{24}\left(\phi^2 - \phi_{sp}^2\right)^2$$

FROM EQ. MOTION $\phi'' = -V'(\phi)$ MULT. BY ϕ AND INT. OVER x FROM $-\infty$ TO x

$$= \frac{\epsilon}{2}\left(\frac{d\phi}{dx}\right)^2$$

$$F_{DW} - F_{SP} = c \int_{-\infty}^{\infty} dx \left(\frac{d\phi}{dx}\right)^2$$

$$= \frac{c\phi_{sp}^2}{w^2} \underbrace{\int_{-\infty}^{\infty} dx \ \operatorname{sech}^4\left(\frac{x}{w}\right)}_{4/3}$$

$$> 0$$

2.6.2 CORRELATION FUNCTIONS

THE CONNECTED CORRELATION

$$C_{CONN}(\vec{x},\vec{y}) = \langle \delta\phi(\vec{x})\, \delta\phi(\vec{y}) \rangle$$

$$\delta\phi(\vec{x}) = \phi(\vec{x}) - \langle \phi(\vec{x}) \rangle$$

OF THE FLUCTUATIONS CAN BE CALCULATED ADDING A SPACE DEP. FIELD AS A SOURCE

$$Z_h = \int \mathcal{D}\phi \; e^{-\beta F(\phi) + \beta \int d^d x\, h(\vec{x})\, \phi(\vec{x})}$$

RECALL PROPS OF CUMULANTS (MATH SUPPORT)

$$\left. \frac{\delta^2 \ln Z_h}{\delta h(\vec{x})\, \delta h(\vec{y})} \right|_{h=0} = \beta^2 \langle \delta\phi(\vec{x})\, \delta\phi(\vec{y}) \rangle$$

HOW DO WE ISOLATE THE h- DEP IN Z_h?

- WE DROP THE $\phi^4(\vec{x})$ TERM IN F_h ASSUMING WE WORK CLOSE TO T_c

- WE FOURIER TRANSFORM THE FIELD TO "DECOUPLE" THE GRADIENT TERM

$$\phi(\vec{k}) = \int d^d x \; e^{i \vec{k}\cdot\vec{x}} \; \phi(\vec{x})$$

$\phi(\vec{x}) \in \mathbb{R} \Longrightarrow \phi(\vec{k}) \in \mathbb{C}$ BUT

$$\phi^*(\vec{k}) = \phi(-\vec{k})$$

NO DOUBLING of d.o.f.

IN TERMS OF $\phi(\vec{k})$ THE FREE-ENERGY IS ($\phi(k) \simeq 0$)

$$F_h(\phi) = \int \frac{d^d k}{(2\pi)^d} \left[\frac{ck^2}{2} |\phi(\vec{k})|^2 + \frac{a(T-T_c)}{2\,T_c} |\phi(\vec{k})|^2 \right]$$

$$- \int \frac{d^d k}{(2\pi)^d} \; h(\vec{k}) \, \phi^*(\vec{k})$$



FOR THE MOMENT SHORT-HAND NOTATION

$$\xi^{-2} \equiv a \cdot \frac{T - T_c}{T_c}$$

$\nu = 1/2$

- WE IDENTIFY ξ

AS A FACTOR OF THE QUADRATIC TERM

WE'LL SEE LATER THAT IT IS THE

CORRELATION LENGTH

$T > T_c$ WAS ASSUMED.

EXPANSION AROUND $\phi = 0$

FOR $T < T_c$ SHOULD EXPAND AROUND

$|\phi_{sp}| \neq 0$

$$F_h(\phi) = \int \frac{d^d k}{(2\pi)^d} \; \frac{1}{2}\left(ck^2 + \xi^{-2}\right) |\phi(k)|^2$$

$$- \int \frac{d^d k}{(2\pi)^d} \; h(k)\, \phi^*(k)$$

NB EACH k ON ITS OWN $\Rightarrow$ DECOUPLED

- NOW, WE NEED TO EXTRACT THE $h(k)$ DEPENDENCE FROM THE FUNCT. INT. OVER $\phi(k)$

WE USE A TRICK:

TRANSLATE THE FIELD

$$\varphi(k) = \phi(k) - \frac{h(k)}{ck^2 + \xi^{-2}} \; A$$

A: NUM IR CONST TO FIX CONVENIENTLY

AND REWRITE F IN TERMS OF $\varphi(k)$:

$$F_h[\phi] = F[\varphi] + \text{TERM IN } h^2$$

$F[\varphi]$: NO h

THE SQUARE TERM

$$|\phi(k)|^2 = \left| \varphi(k) + \frac{h(k)\,A}{ck^2 + \xi^{-2}} \right|^2$$

$$= |\varphi(k)|^2 + \left(\frac{\varphi(k)\, h^*(k)}{ck^2 + \xi^{-2}} + \text{c.c.} \right) A$$

$$+ \frac{|h(k)|^2 A^2}{(ck^2 + \xi^{-2})^2}$$

THE FIELD-DEP. TERM

$$h(k)\,\phi^*(k) = h(k) \left[\varphi(k) + \frac{h(k)\,A}{ck^2 + \xi^{-2}} \right]^*$$

$$= h(k)\,\varphi^*(k) + \frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}$$

- PUTTING ALL TOGETHER: IN THE EXPONENTIAL

$$= (ck^2 + \xi^{-2})\,|\phi(k)|^2 - h(k)\,\phi^*(k) =$$

$$= (ck^2 + \xi^{-2}) \left[|\varphi(k)|^2 + \left(\frac{\varphi(k) h^*(k) + \text{cc}}{ck^2 + \xi^{-2}} \right) A \right.$$

$$\left. + \frac{|h(k)|^2 A^2}{(ck^2 + \xi^{-2})^2} \right]$$

$$- h(k)\,\varphi^*(k) - \frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}$$

$$= \underbrace{(ck^2 + \xi^{-2})\,|\varphi(k)|^2 - h(k)\,\varphi^*(k)}$$

SAME EXP AS ORIGINAL ONE BUT in $\varphi(k)$

$$+ \left(\varphi(k)\,h^*(k) + \varphi^*(k)\,h(k)\right) A$$

$\varphi^*(-k)\,h(k)$

$\Rightarrow \varphi(k)\,h(-k)$ → UNDER INT. OVER $\int d^d k$

$= \varphi(k)\,h^*(k)$

TWICE THE SAME

$$- \underbrace{\frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}}$$

IT DOESN'T DEP. ON $\varphi(k)$

$\Rightarrow$ GOES OUTSIDE INTEGRAL

WE CAN CANCEL THE TERMS IN THE BUBBLES TAKING $A = 1/2$

THOSE ARE THE ONES LINEAR IN $h(k)$ COUPLED TO $\varphi(k)$

THEN, WE HAVE

$$F_h(\varphi) = \underbrace{\frac{1}{2}\int d^dk \left[ck^2 |\varphi(k)|^2 + \tilde{\xi}^{-2} |\varphi(k)|^2 \right]}_{F[\varphi]} - \frac{1}{2}\int d^dk \, \frac{|h(k)|^2}{ck^2 + \tilde{\xi}^{-2}}$$

AND THE VARIATIONS WRT $h(\vec{x})$ ONLY TOUCH THE LAST TERM IN $F_h(\phi)$ WHICH IS INDEP. OF φ

$$Z_h = \underbrace{\int \mathcal{D}\varphi \, e^{-\beta F(\varphi)}}_{B} \underbrace{e^{\frac{\beta}{2}\int \frac{d^dk \, |h(k)|^2}{ck^2 + \tilde{\xi}^{-2}}}}_{\text{WHAT WE NEEDED}}$$

- NOW, FOURIER TRANSFORM BACK TO $\vec{x}$ SPACE

$$Z_h = B \, e^{\frac{\beta}{2}\int d^dx \int d^dy \, h(\vec{x}) \, G(\vec{x}-\vec{y}) \, h(\vec{y})}$$

$$\boxed{G(\vec{x}-\vec{y}) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-ik(\vec{x}-\vec{y})}}{ck^2 + \tilde{\xi}^{-2}}}$$

- Finally,

$$\ln Z_h = \ln B + \frac{\beta}{2} \int d^d x \, d^d y \; h(\vec{x}) \, G(\vec{x}-\vec{y}) \, h(\vec{y})$$

$$\frac{\delta^2 \ln Z_h}{\delta h(\vec{x}) \, \delta h(\vec{y})} = \beta \, G(\vec{x}-\vec{y}) = \beta^2 \langle \delta\phi(\vec{x}) \, \delta\phi(\vec{y}) \rangle$$

WE ALREADY KNEW IT

AND THEN

$$\langle \delta\phi(\vec{x}) \, \delta\phi(\vec{y}) \rangle = k_B T \, G(\vec{x}-\vec{y})$$

NOW WE HAVE TO GO BACK TO THE ~~DEF~~ OF

$$G(\vec{x}-\vec{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i k (\vec{x}-\vec{y})}}{c k^2 + \tilde{\xi}^{-2}}$$

AND TRY TO CALCULATE THE INTEGRAL OVER $\vec{k}$ EXPLICITLY

* HERE, FOLLOW THE CALC. IN THE LECTURE NOTES, VERY DETAILED ! ALSO IN TD.

$$G(\vec{x}-\vec{y}) \Rightarrow G(r) \qquad r = \|\vec{x}-\vec{y}\|$$

WITH

- $r \gg \xi$ $\quad G(r) \sim \exp(-r/\xi)$

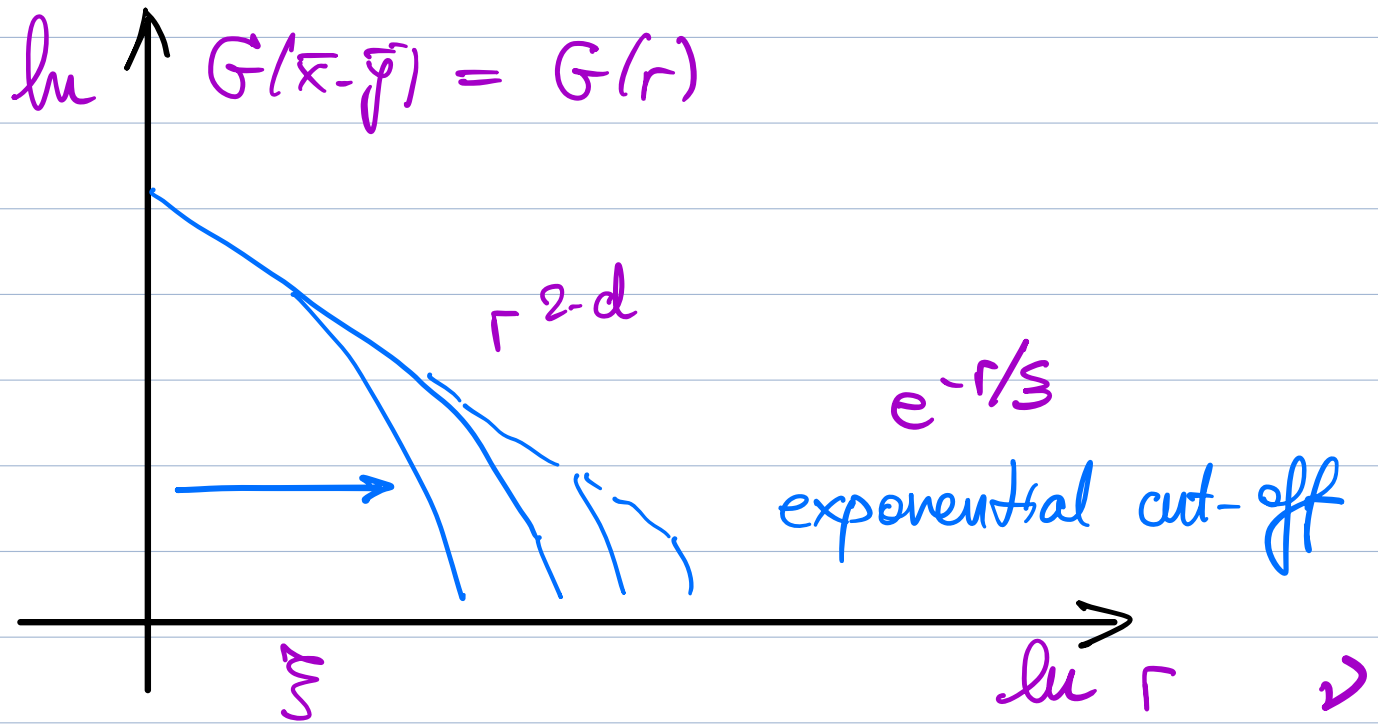
- $r \ll \xi$ $\quad G(r) \propto r^{2-d}$

AND NOW FOR BOTH $r \ll \xi$ AND $r \gg \xi$

PUTTING TOGETHER THE TWO LIMITS

$$G(r) \sim r^{2-d} e^{-r/\xi}$$

DISCUSS THE FORM OF $G(\vec{x}-\vec{y})$



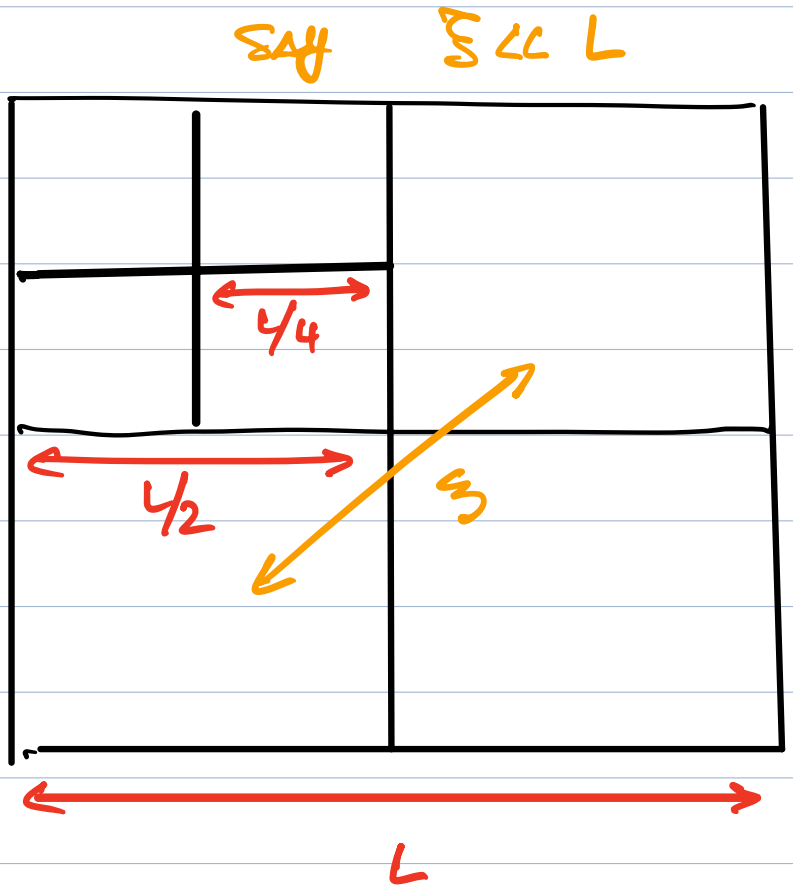
ξ DEPENDS ON $T-T_c$

$$\xi \sim (T-T_c)^{-\nu}$$

- FROM LEFT TO RIGHT T GETS CLOSER TO T_c
- FOR $T \to T_c^+$ JUST THE POWER LAW r^{2-d}
- WE SEE THAT $d=2$ PLAYS A SPECIAL ROLE IN THIS THEORY.
  THE POWER LAW IS A CONST.

THE CORRELATION LENGTH - MEANING

DISTANCE OVER WHICH THE FLUCT. OF THE ORDER PARAMETER $\delta\phi$ ARE QUASI LONG. RANGE CORRELATED



MEASURE OBSERV IN L^d, $(L/2)^d$, ...
SAME RESULT UNTIL

$$L \sim \xi$$

THEN, RESULTS SHOULD CHANGE.

FROM N INDEP BOXES, TO CORRELATED THINGS

WE SEE THERMAL FLUCTUATIONS IN SAN CONF - SNAPSHOTS

SEE SLIDES & LECTURE NOTES

THE AVERAGED SIZE OF THESE REVERSED DROPLETS GIVE AN IDEA OF

ξ

ALTHOUGH IT'S NOT EXACTLY EQUAL TO IT

CONSEQUENTLY

SYST WITH $\xi \to \infty$ HAVE DROPLETS OF ALL SIZES.

NO SCALE

$$\xi \to \begin{cases} \xi_0^+ (T - T_c)/T_c & \xi_0^+ = \sqrt{\frac{c}{a}} \\ \xi_0^- (T_c - T)/T_c & \xi_0^- = \sqrt{\frac{c}{2a}} \end{cases}$$

TRANSLATIONAL INVARIANCE $\longrightarrow$

SCALE INVARIANCE

SCALE INVARIANCE

$$P(\ell) \sim \ell^{-\tau}$$ NO CUT-OFF

PROB. OF A "DOMAIN" OF LINEAR SIZE ℓ

$$\ell \to b\ell$$ SCALE TRANSF.

$$P(b\ell) = (b\ell)^{-\tau} = b^{-\tau}\ \ell^{-\tau}$$

INVARIANT UP TO A PREFACTOR

$$\boxed{b^{\tau} P(b\ell) = P(\ell)}$$

LOOK AT SNAPSHOTS AT T_c.

MORE LATER

2.6.3.

GINZBURG CRITERIUM

THE RESULTS ABOVE ARE OK AS LONG AS lhs ≪ rhs

GAUSSIAN FLUCT AROUND THE HOMOG. S.P.

$$\langle \delta\phi^2_{coh} \rangle \sim \langle \phi \rangle^2 = \phi^2_{sp}$$

PROOF OF $\langle\phi\rangle = \phi_{sp}$ SEE NOTES, EXC.

$$\delta\phi_{coh} \equiv \frac{1}{V_{coh}} \int d^d x \, (\phi(\vec{x}) - \phi_{sp})$$

$$V_{coh} = \xi^d$$

WITH ξ CORREL. LENGTH

$$\langle \delta\phi^2_{coh} \rangle = \frac{1}{V^2_{coh}} \int d^d x \int d^d y \underbrace{\langle (\phi(\vec{x}) - \phi_{sp})(\phi(\vec{y}) - \phi_{sp}) \rangle}$$

DEFINITION OF $C^{CONN}(\vec{x}, \vec{y})$

FOR A HOMOGENEOUS PROBLEM

TRANSL. INV.

ISOTROPY

EXPECTED $\Rightarrow$

$$C^{\text{CONN}}(\vec{x},\vec{y}) = C^{\text{CONN}}(\|\vec{x}-\vec{y}\|)$$

CALL $r = \|\vec{x}-\vec{y}\|$ IT'S WHAT WE FOUND

$$\langle \delta\phi^2_{\text{coh}} \rangle = \frac{1}{V^2_{\text{coh}}} V_{\text{coh}} \int d^d r \; C^{\text{CONN}}(r)$$

$$= \frac{1}{V_{\text{coh}}} \Omega_d \int dr \, r^{d-1} \, C^{\text{CONN}}(r)$$

WE SHOWED IT ABOVE.

IF $\boxed{C^{\text{CONN}}(r) \sim r^{2-d-\eta} \, e^{-r/\xi}}$ CLOSE TO T_C WITH $\eta = 0$ WITHIN LANDAU

$$\langle \delta\phi^2_{\text{coh}} \rangle = \frac{\Omega_d}{\xi^d} \int dr \, r^{d-1} \, r^{2-d} \, e^{-r/\xi}$$

$$= \frac{\Omega_d}{\xi^d} \int \frac{dr}{\xi} \, \frac{r}{\xi} \, e^{-r/\xi} \, \xi^2$$

$$= \Omega_d \, \xi^{2-d} \underbrace{\int_{l/\xi}^{L/\xi} dy \; y \, e^{-y}}_{A'}$$

A NUMBER

$$\boxed{\langle \delta\phi^2_{coh} \rangle = \xi^{2-d} \; A}$$

CLOSE TO THE CRITICAL POINT

$$\langle \phi \rangle = \phi_{sp} \simeq |T_c - T|^{\beta} \qquad \text{(LANDAU)}$$

IF $\xi \sim |T - T_c|^{-\nu} \Longrightarrow |T - T_c| \sim \xi^{-1/\nu}$

$$\Longrightarrow \quad \phi_{sp} \sim \xi^{-\beta/\nu}$$

WITHIN THE LANDAU APPROX

$\beta = 1/2 \qquad \nu = 1/2$

$$\boxed{\langle \phi \rangle = \phi_{sp} \sim \xi^{-1}}$$

BACK TO THE GINZBURG CRITERIUM

$$\langle \delta\phi^2_{coh} \rangle \lesssim \langle \phi \rangle^2 \Rightarrow$$

$$\xi^{2-d} \lesssim \xi^{-2} \Rightarrow \xi^{4-d} \lesssim 1$$

$$\boxed{d \geq 4 \quad \& \quad d_u = 4}$$

ABOVE $d=4$ $\Rightarrow$ ok

- if $d \geq 4$ THEN $\langle \delta\phi^2_{coh} \rangle \lesssim \langle \phi \rangle^2$ Σ

MEAN FIELD OK

- if $d < 4$ $\Rightarrow$ NO! FOR $T \to T_c$

$d_u \equiv$ UPPER CRITICAL DIMENSION

STILL AT $d<4$ ONE CAN APPROACH T_c BUT NOT GET TOO CLOSE TO IT SO THAT $\xi^{4-d} < 1$ IF ξ REMAINS FINITE. $\Rightarrow$ AWAY FROM CRITICAL REGION

$\xi \sim |T-T_c|^{-\nu}$ $\Rightarrow (T-T_c)^{-\nu(4-d)} < 1$

2.6.4 FIRST ORDER PHASE TRANS. WITHIN LANDAU'S FIELD THEORY

- FIELD DRIVEN

$$-\int d^d x \; h(\vec{x})\,\phi(\vec{x})$$

- TEMP DRIVEN

$$\int d^d x \; \phi^3(\vec{x})$$

LIKE IN $p=3$ FM MODEL

THIS TERMS CHANGE THE FORM of $F[\phi]$ AND ALLOW FOR METASTABLE STATES, AND JUMPS IN ORDER PARAM. AT THE 1st ORDER TRANSITION

APPLICATIONS

- COND-MATTER
- HIGH ENERGY ϕ HIGGS FIELD
  A_μ FIELD
- etc.

EXTENSIONS TO VECTOR VARIABLES

- VECTOR FIELDS

$$\vec{\phi}(\vec{x})$$

LATER : 2dXY MODEL $\Longleftrightarrow$
$O(2)$ FIELD TH. IN
$d = 2$

27.2 CORR FUNCT & GLOBAL SUSCEPTIBILITY

BACK TO GENERICS. $\theta \Rightarrow$ ORDER PAR. $\langle\theta\rangle$

$$C(\vec{x},\vec{y}) = \langle \delta\theta(\vec{x})\,\delta\theta(\vec{y})\rangle$$ CORR fct

(I WILL NOT WRITE "CONN" ANYMORE)

$$\sim r^{2-d-\eta}\, f(r/\xi)$$

η ANOMALOUS EXP. — BEYOND WHAT WE'VE DONE SO FAR

f CUT-OFF FUNCT — TRANSL INV. / ISOTROPY

$$\xi \sim |T-T_c|^{-\nu}$$ CORR LENGTH

$$\chi(\vec{x},\vec{y}) \equiv \left.\frac{\delta\langle\theta(\vec{x})\rangle_h}{\delta h(\vec{y})}\right|_{h\to 0}$$ LINEAR SUSCEP.

$$H \to H - \int d^dx\, h(\vec{x})\,\theta(\vec{x})$$

RELATION BETWEEN χ AND C FDT

$$\chi(\vec{x},\vec{y}) \equiv$$

$$\frac{\delta}{\delta h(\vec{y})} \frac{\int \mathcal{D}\phi \; e^{-\beta H + \beta \int d^d x' \, h(\vec{x}') \phi(\vec{x}')} \phi(\vec{x})}{\int \mathcal{D}\phi \; e^{-\beta H + \beta \int d^d x' \, h(\vec{x}') \phi(\vec{x}')}}$$

HAVE TO TAKE VARIATION IN NUM & DEN AND THEN SET $h=0$

$$\text{NUM} \rightarrow \frac{\int \mathcal{D}\phi \; e^{-\beta H + \beta \int d^d x \, h(\vec{x}') \phi(\vec{x}')} \phi(\vec{x}) \phi(\vec{y})}{\text{DEN}}$$

$$\text{DEN} \rightarrow \frac{\int \mathcal{D}\phi e^{-\beta H_h} \phi(\vec{x}) \int \mathcal{D}\phi \, e^{-\beta H_h} \phi(\vec{y})}{\left(\int \mathcal{D}\phi \; e^{-\beta H_h} \right)^2}$$

$$\Rightarrow \quad \langle \phi(\vec{x}) \phi(\vec{y}) \rangle - \langle \phi(\vec{x}) \rangle \langle \phi(\vec{y}) \rangle$$

WITH AVERAGES AT $h=0$

$$\chi(\vec{x},\vec{y}) = \beta\, C(\vec{x},\vec{y})$$

LOCAL

FDT

RECALL, IT'S CONN. CORR. FUNCT.

DIVERGENCE of ξ $\Rightarrow$ DIVERGENCE of χ INTEGRATED OVER FULL VOLUME

Proof

GLOBAL OR TOTAL SUSC

$$\underbrace{\int d^d x \int d^d y}_{\text{INTEGRATING OVER VOLUME TWICE}} \underbrace{\chi(\vec{x},\vec{y})}_{\chi(r)} = \chi$$

HOM / ISOT

$$= V\Omega_d \int dr\, r^{d-1}\, \chi(r)$$

FDT

$$= V\Omega_d \beta \int dr\, r^{d-1}\, C(r)$$

ONE VOLUME FACTOR

ONE VOLUME INT

if $C(r) \sim r^{2-d-\eta} f(r/\xi) \Rightarrow$

$$\chi = V\Omega_d \beta \int dr\, r^{d-1}\, r^{2-d-\eta}\, f(r/\xi)$$

$$= V\Omega_d \beta \int dr\, r^{1-\eta}\, f(r/\xi)$$

$$= V\Omega d\, \beta\, \xi^{2-\eta} \underbrace{\int \frac{dr}{\xi} \left(\frac{r}{\xi}\right)^{1-\eta} f(r/\xi)}_{\text{CONST } A}$$

THIS ONE IS INTEGRATED TWICE OVER VOLUME

$$\boxed{\chi = V\Omega d A\, \beta\, \xi^{2-\eta}}$$

IF $\eta < 2 \Rightarrow$ ξ DIVERG. $\Rightarrow$ χ DIVERG.

POWER LAW DECAY OF CORR FOR $r < \xi \Rightarrow$ DIVERGENCE OF χ (TWICE INTEGRATED) WITH $\xi^{2-\eta}$

CHECK $\chi \sim V \xi^{2-\eta}$

$\downarrow$ ONE VOLUME L^d $\int \frac{\chi}{L^d} \sim \xi^{2-\eta}$

$\sim |T - T_c|^{-\nu(2-\eta)}$

DIVERGENCE CHANGED WRT $\chi \sim |T - T_c|^{-\gamma}$

RELATION BETWEEN EXPONENTS $\longrightarrow$

MORE TO COME

SHOW SLIDES HERE

EXPONENTS => UNIVERSALITY

$\alpha = \alpha' = 0$ $\beta = 1/2$ $\gamma = \gamma' = 1$ $\delta = 3$

$\nu = 1/2$ $\eta = 0$ MF

$\alpha = \alpha' = 0$ $\beta = 1/8$ $\gamma = \gamma' = 7/4$ $\delta = 15$

$\nu = 1$ $\eta = 1/4$ d=2 ONSAGER

$\alpha = \alpha' = 0.11$ $\beta = 0.325$ $\gamma = 1.24$ $\delta = 4.82$

$\nu = 0.63$ $\eta = 0.032$ d=3 APROX

THE SAME VALUES IN MANY MANY ≠ SYST

MICROSC DOESN'T MATTER

APART FROM FEATURES WE WILL SEE DO MATTER (DIM, DIM ORDER PARAM, SYMM.)

d n SYMM

SUMMARY

TO BE SHOWN WITH SLIDES

LANDAU $\phi(\vec{x})$

$F(\phi)$ BASED ON ASSUMPTIONS

↙ 2nd ↘ 1st

GINZBURG

LIMITS OF VALIDITY $d > d_u$ EXACT

$d \leq d_u$ FAR FROM T_c ok.

$$\xi \sim |T - T_c|^{-\nu}$$

$$C_{CONN}(\vec{r}) \sim r^{2-d-\eta} f(r/\xi)$$

$f(r/\xi)$ CUT-OFF $r \gg \xi$

$$\chi = \int d^d x \int d^d y \; \chi(\vec{x}, \vec{y}), \quad FDT,$$

$$\propto \beta \xi^{2-\eta}$$ DIVERGES WITH ξ

SET OF EXPONENTS

2.8 SCALING AND UNIVERSALITY

SHOW THAT THE SINGULAR PART OF f DEPENDS ON THE CONTROL PARAMETERS AS.

$$f_{SING} = |t|^{2-\alpha} \, g_f\left(\frac{h}{|t|^{\Delta}}\right)$$

$$t = \frac{T - T_c}{T_c}$$

THIS IS A HOMOGENEOUS FUNCTION OF THE PARAMETERS (t, h)

THIS ALLOWS US TO :

$\Rightarrow$ FIND RELATIONS BETWEEN EXPONENTS

$\alpha, \beta, \gamma, \delta, \nu, \eta$ NOT ALL INDEP.

IN CURIE-WEISS MODEL

$$f(t,h) = \min_m \left(\frac{t}{2} m^2 + u m^4 - h m \right)$$

ALREADY EXPANDED FOR $m \sim 0$

I MAY BE CHANGING NUMERICAL FACTORS:

IT IS NOT IMPORTANT

$$t \equiv \frac{T - T_c}{T_c} \qquad u = \frac{\lambda}{4!}$$

(IN PREVIOUS NOTATION)

USE THE EQ. FOR THE ORDER PARAMETER $\frac{\partial f}{\partial m} = 0$

$$t m + 4 u m^3 - h = 0 \quad \Rightarrow \quad m_{sol}(u, t, h)$$

NOT IMPORTANT

IMPORTANT

AND WRITE $f(m; t, h) = f(m_{sol}; t, h)$

$$= f(t, h)$$

FOR SING. PART

2) FOR $\boxed{t<0 \quad h=0}$ $\quad 4u m^3 + t m = 0 \Rightarrow$

$$m^2 = \frac{-t}{4u} \Rightarrow$$

$$f(t,h) = \frac{t}{2}\frac{(-t)}{4u} + u\frac{t^2}{16u^2}$$

$$\boxed{f(t,h) = \frac{-t^2}{16u}}$$

PARAMETER DEPENDENCE OF FREE-ENERGY DENSITY

3) FOR $\boxed{h \neq 0 \quad t=0}$ $\quad 4u m^3 - h = 0 \Rightarrow$

$$m = \left(\frac{h}{4u}\right)^{1/3}$$

$$f(t,h) = u\left(\frac{h}{4u}\right)^{4/3} - h\left(\frac{h}{4u}\right)^{1/3}$$

$$\boxed{f(t,h) = \frac{3}{4^{4/3}}\frac{h^{4/3}}{u^{1/3}}}$$

PUTTING THESE TOGETHER, IN CURIE WEISS:

$$f_{SING}(t,h) = |t|^2 \, g_f\left(\frac{h}{|t|^\Delta}\right)$$

WITH $g_f(y=0) = -\text{CONST}$ ✓ $(h=0)$

$$g_f(y \to \infty) = y^{4/3} \qquad |t| \to 0$$

$\Rightarrow$

FOR $y \to \infty$ THEN f_{SING} GOES AS

$$|t|^2 \frac{h^{4/3}}{|t|^{\frac{\Delta}{3}}} \sim h^{4/3} \Rightarrow 2 - \frac{4\Delta}{3} = 0 \Rightarrow \boxed{\Delta = \frac{3}{2}}$$

IMPOSE IT
SINCE WE SAW IT ABOVE
FOR $t=0$

f_{SING} IS A GENERALIZED HOMOG.
FUNCT OF ITS ARGUM.
THE CONTROL PARAM.

RECALL HOMOGENEOUS FUNCTION

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^k f(x_1, \dots, x_n)$$

GENERALIZED HOMOG. FCT

$$f(\lambda^{y_1} x_1, \lambda^{y_2} x_2, \dots, \lambda^{y_n} x_n) =$$

$$= \lambda^k f(x_1, x_2, \dots, x_n)$$

$$f_{SING}(t,h) = |t|^2 g_f\left(\frac{h}{|t|^\Delta}\right)$$

CURIE WEISS $\Delta = 3/2$

CHECK

$$f_{SING}(\lambda^{y_t} t, \lambda^{y_h} h) =$$

$$= |t|^2 \lambda^{2y_t} g_f\left(\frac{\lambda^{y_h} h}{\lambda^{y_t \Delta} |t|^\Delta}\right)$$

NEED $y_h - y_t \Delta = 0$ $\Delta = 3/2 \Rightarrow$

$$y_h - 3/2\, y_t = 0$$

THEREFORE

$$f_{sing}\left(\lambda^{y_t} t, \lambda^{\frac{3}{2}y_t} h\right) =$$

$$= \lambda^{2y_t} |t|^2 g_f\left(\frac{h}{|t|^\Delta}\right)$$

$$= \underbrace{\lambda^{2y_t}} \; f_{sing}(t, h)$$

↳ FACTOR.

SUGGEST TO WRITE THE

SINGULAR PART OF THE FREE-ENERGY DENSITY

AS A FUNCTION OF THE CONTROL PARAMETERS

IN CASES WITH $\alpha \neq 0$ AND $\alpha = \alpha'$ ON BOTH SIDES OF CRIT. POINT AS

$$f_{SING}(t,h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^{\Delta}}\right)$$

BEYOND CURIE-WEISS

JUSTIFICATION OF POWER $2-\alpha$ IN 1ST FACTOR $C_V \sim |t|^{-\alpha}$

RECALL $\langle H \rangle = -\dfrac{\partial \ln Z}{\partial \beta}$

$$C_V = \frac{\partial \langle H \rangle}{\partial T} = \frac{\partial}{\partial T}\left(-\frac{\partial \ln Z}{\partial \beta}\right) = \frac{\partial}{\partial T}\frac{\partial}{\partial \beta}(\beta F)$$

$$= \frac{\partial}{\partial T}\left[F + \beta \frac{\partial F}{\partial \beta}\right]$$

$$= \frac{\partial F}{\partial T} + \frac{\partial \beta}{\partial T}\frac{\partial F}{\partial \beta} + \underbrace{\beta \frac{\partial}{\partial T}\frac{\partial F}{\partial \beta}}_{\text{MOST SING. PART}}$$

$C_V \sim \beta \dfrac{\partial}{\partial T}\dfrac{\partial F}{\partial \beta} \Rightarrow$ THE MOST SING PART COMES FROM $\dfrac{\partial^2}{\partial T^2}$

$$C_V \sim \frac{\partial^2 F}{\partial T^2} \sim N \frac{\partial^2 f_{SING}}{\partial T^2}$$

$$C_V \sim |t|^{-\alpha} \Leftrightarrow f_{SING} \sim |t|^{2-\alpha} \underbrace{g\left(\frac{h}{|t|^{\Delta}}\right)}_{\text{THIS PART DOESN'T CONT}}$$

$h \to 0$: CONST

$\dfrac{\partial^2 f_{SING}}{\partial T^2} \sim |t|^{-\alpha}$ OK.

WE JUSTIFIED f_{SING} BEYOND CURIE-WEISS ($\alpha \neq 0$)

CONSEQUENCES: THE MAGN DENSITY

$$m = \frac{\partial f}{\partial h} = |t|^{2-\alpha} \, g_f'\left(\frac{h}{|t|^{\Delta}}\right) \frac{1}{|t|^{\Delta}}$$

$$= |t|^{2-\alpha-\Delta} \, g_m\left(\frac{h}{|t|^{\Delta}}\right)$$

WE KNOW $m \sim |t|^{\beta}$ FOR $\frac{h}{|t|^{\Delta}} \to 0$

$$\Rightarrow \boxed{2-\alpha-\Delta = \beta} \quad \& \quad g_m(y) \sim 1 \qquad y \to 0$$

WE ALSO KNOW $m \sim h^{1/\delta}$ FOR $\frac{h}{|t|^{\Delta}} \to \infty$

$$\Rightarrow g_m(y) \sim y^{1/\delta} \qquad y \to \infty$$

$$\boxed{2-\alpha-\Delta-\Delta/\delta = 0}$$

FROM NO DEP. ON $|t|$ LEFT

PUTTING THE TWO TOGETHER

$$\boxed{\Delta = \beta\delta} \quad (1)$$

$$2 - \alpha - \beta\delta = \beta \Rightarrow \boxed{\beta = \frac{2-\alpha}{1+\delta}} \quad (2)$$

IN THE MF CASE

$$\beta = 1/2 \quad \alpha = 0 \quad \delta = 3 : \quad \frac{1}{2} = \frac{2}{4} \quad \checkmark$$

SUSCEPTIBILITY

$$\chi = \left.\frac{\partial^2 f}{\partial h^2}\right|_{h\to 0} = \left.\frac{\partial m}{\partial h}\right|_{h\to 0}$$

BOTH m AND h ARE "INTENSIVE" NOT SCALING WITH L^d

$$= |t|^{2-\alpha}\left[g_m''\left(\frac{h}{|t|^{\Delta}}\right)\frac{1}{|t|^{\Delta}}\frac{1}{|t|^{\Delta}}\right]$$

$$= |t|^{2-\alpha-2\Delta}\, g_m''(0) \qquad h\to 0$$

SHOULD BE EQUAL TO $|t|^{-\gamma}$

$$\gamma = \alpha + 2\Delta - 2$$

$$\boxed{\gamma = \alpha + 2\beta\delta - 2} \quad \Rightarrow (3)$$

Proof of Widom's

FROM (2) $\beta = \dfrac{2-\alpha}{1+\delta} \Rightarrow \alpha = 2 - \beta(1+\delta)$

IN (3) $\gamma = 2 - \beta - \beta\delta + 2\beta\delta - 2 = \beta(\delta - 1)$

Widom's identity

$$\boxed{\delta - 1 = \gamma/\beta} \qquad (4)$$

IN CURIE-WEISS $\delta = 3,\ \gamma = 1,\ \beta = 1/2 \Rightarrow 3 - 1 = 1/1/2$ ✓

FINALLY, ANOTHER ONE WITH A NAME:

REPLACE WIDOM'S δ IN (3)

$$\gamma = \alpha + 2\beta\left(1 + \gamma/\beta\right) - 2$$

$$\gamma = \alpha + 2\beta + \gamma - 2$$

$$2 = \alpha + 2\beta + \gamma$$

RUSHBROOKE'S IDENTITY

$$\boxed{\alpha + 2\beta + \gamma = 2}$$

IN MF CURIE-WEISS $\alpha = 0$ $\beta = 1/2$ $\gamma = 1$

$0 + 1 + 1 = 2$ ok. ✓

RUSHBROOKE'S

WIDOM'S

$$\alpha + 2\beta + \gamma = 2$$

$$\delta - 1 = \gamma/\beta$$

THESE ARE, FOR THE MOMENT, INDEP OF d

FROM 6 EXPONENTS NOW WE ARE LEFT WITH 4 INDEP. ONES ONLY. BECAUSE WE HAVE TWO RELATIONS

SPATIAL INFO IN C AND ξ START USING ξ

ASSUME SCALING OF CORRELATION LENGTH

$$\xi(t,h) \sim |t|^{-\nu} g_\xi\left(\frac{h}{|t|^\Delta}\right)$$

$$g_S(y \to 0) = \text{ct.} \qquad g_S(y \gg 1) \sim y^{-\nu/\Delta}$$

$$\hookrightarrow \xi \sim h^{-\nu/\Delta}$$

$$-\beta f_{\text{sing}}(t,h) \sim L^{-d} \underbrace{\ln \mathcal{Z}}_{\text{ADIM}}$$

$$\sim L^{-d} \left(\frac{L}{\xi}\right)^d \quad \begin{matrix}\text{UNION OF} \\ (L/\xi)^d \text{ INDEP} \\ \text{PIECES}\end{matrix}$$

$$\sim \xi^{-d}$$

$$\sim |t|^{+d\nu} \left[g_\xi\left(\frac{h}{|t|^\Delta}\right)\right]^{-d}$$

REPLACING ξ BEHAVIOUR

THIS HAS TO BE EQUAL TO THE PREVIOUS FORM WE USED

$$f_{\text{sing}}(t,h) \sim |t|^{2-\alpha} g_f\left(\frac{h}{|t|^\Delta}\right)$$

$\Rightarrow$

$$2 - \alpha = d\nu$$

JOSEPHSON'S IDENTITY

A RELATION WITH d IN IT. CALLED

HYPERSCALING

STILL HAVE TO RELATE η TO THE OTHER EXPONENTS WE KNOW FROM FDT AND DIVERG OF GLOBAL χ :

$$C(r) \sim r^{2-d-\eta} \Rightarrow \frac{\chi}{L^d} \sim \xi^{2-\eta}$$

$$\frac{\chi}{L^d} \sim |t|^{-\gamma} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} \Rightarrow$$

$$\gamma = \nu(2-\eta)$$

WE SAW IT ALREADY WHEN WE CALCULATED χ

WE PROVED FOUR RELATIONS BETWEEN EXPONENTS

IN CONCLUSION, THERE ARE ONLY TWO INDEPENDENT EXPONENTS.

ALL OTHERS ONES CAN BE WRITTEN IN TERMS OF THEM.

eg. ν AND η

IT WAS BELIEVED THAT $\alpha = \alpha'$ $\gamma = \gamma'$ ON TWO SIDES OF TRANS. BUT THIS IS NOT NECESSARILY TRUE $\Rightarrow$ APPLY ABOVE ON EACH SIDE.

HOMOGENEOUS FUNCTIONS

ONE VARIABLE

TAKE A FUNCTION OF ONE VARIABLE $f(x)$

IT IS HOMOGENEOUS OF DEGREE p IF UNDER SCALING OF x BY A FACTOR b

$$x \to xb$$

THE FUNCTION GETS MULT. BY b^p:

$$f(bx) = b^p f(x)$$

b IS THE SCALING FACTOR

p IS THE DEGREE OF HOMOGENEITY

NB I COULD CALL $b \to b^{p_1}$ AND DO $x \to b^{p_1} x$, $f(b^{p_1} x) = b^{p} f(x)$ EQUIV.

MANY VARIABLES x_i

THEY ARE THE CONTROL PARAM OF THE TRANSITION e.g H) & h IN OUR EXAMPLE

$$b^{p_f}\ g(x_1, x_2, \dots, x_n) =$$

$$= g(b^{p_1} x_1,\ b^{p_2} x_2, \dots,\ b^{p_n} x_n)$$

$$\forall\ b \in \mathbb{R}$$

TAKE $x_n = b^{-p_n}$ $\Rightarrow$ nth ARGUMENT $= 1$

$$b = x_n^{-1/p_n}$$

THIS IS A CHOICE OF b

THE x's ARE THE "INDEP. VARIABLES."

$$g(x_1, x_2, \dots, x_n) =$$

$$x_n^{p_f/p_n}\ g\left(x_n^{-p_1/p_n} x_1,\ x_n^{-p_2/p_n} x_2, \dots, 1\right)$$

ONE LESS INDEP VARIABLE

& IS ALREADY AS

$$\frac{x_i}{x_n^{p_i/p_n}}$$

CALL $\dfrac{p_i}{p_n} = \Delta_i$ $\Rightarrow$

$$\frac{x_i}{x_n^{\Delta_i}}$$

→ THE ARGUMENTS OF g $n-1$ SCALING VARIABLES

g SCALING FUNCTION.

EXAMPLE CURIE WEISS g is $f_{SING}(h, |t|)$

$x_1 = h \qquad x_2 = |t|$

$$\frac{x_1}{x_n^{\Delta}} = \frac{h}{|t|^{\Delta}} \qquad \frac{p_1}{p_2} = \Delta$$

THERE IS ONLY ONE Δ SINCE ONLY ONE RATIO p_1/p_2

$$f_{SING} = |t|^2 \, g\left(\frac{h}{|t|^{\Delta}}\right) \qquad \Delta = \frac{3}{2} \; ; \; \frac{2}{p_1} = 2$$

WE WILL PROVE HOMOGENEITY OF f WITH RG BELOW.

WONDERS OF SCALING : { SAME EXPONENTS & SAME SCALING FUNCTIONS

WE SAW ABOVE THAT THE FREE-ENERGY DENSITY, SUSC., MAGN., ETC ARE HOMOGENEOUS FUNCT OF THE CONTROL PARAM.

PARTICULAR CASE:

IF THE RESULT REMAINS UNCHANGED UNDER SCALING

$$x \to bx$$

$$f(bx) = f(x)$$

$p=0$ IN THE ONE-VARIABLE CASE ABOVE

THE FREE-ENERGY AND SO ON DON'T HAVE $p=0$ BECAUSE THERE'S A FACTOR IN FRONT OF SCALING FCT.

IN SOME PLACES SCALE INV IS ASSOC. TO NO PRE-FACTOR. NOT IN CONT. PHASE TRANSITIONS

WHAT HAPPENS WITH LENGTH DEPEND. OBSERV. AT T_c ?

$$\xi(T) \sim |T - T_c|^{-\nu} \qquad h=0$$

$$\Rightarrow \quad \xi(bT) \sim |bT - bT_c|^{-\nu} = b^{-\nu} |T - T_c|^{-\nu}$$

$$\boxed{\xi(bT) \sim b^{-\nu} \xi(T)}$$

HOMOGENEITY OF SPATIAL CORR NEAR T_c

SCALE INVARIANCE USUAL WAY OF REFERRING TO.

$L \to \infty$, JUST ξ AS ONLY SCALING LENGTH.

BUT, AT CRIT POINT $\xi \to \infty$ $\Rightarrow$ NONE LEFT !

WHAT HAPPENS WITH $C(\vec{x},\vec{y})$?

$$C(\vec{x},\vec{y}) = C(|\vec{x}-\vec{y}|) = C(r)$$

$$C(r) \sim r^{2-d-\eta}$$

$$C(br) \sim b^{2-d-\eta}\, C(r)$$

BECAUSE IT'S A POWER LAW

WE CAN ALSO WRITE IT AS:

$$\boxed{b^p\, C(\vec{x},\vec{y}) = C(b\vec{x}, b\vec{y})}$$

$$p = 2-d-\eta$$

WE ZOOM IN & OUT & STATISTICALLY WE SEE THE SAME PICTURES

NOT ONLY AT CRIT POINT OF 2nd ORDER PHASE TRANS. BUT ALSO FRACTALS, ETC.

FINITE SIZE SCALING

$a \ll \xi \ll L$ IN REAL LIFE

SCALING $\frac{\xi}{a}, \frac{L}{\xi}$ ONLY RELEVANT

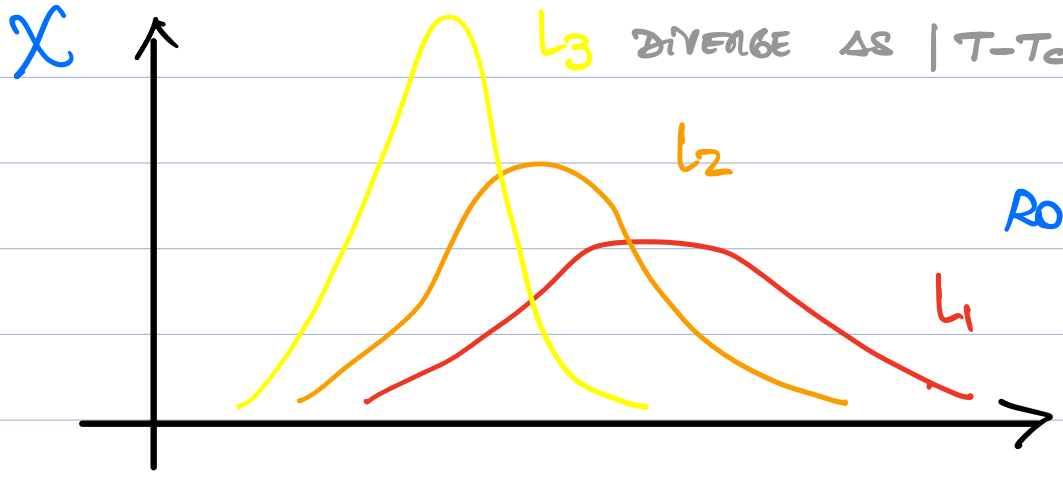
TYPICAL SITUATION IN EXPERIMENTS

$\frac{\xi}{a} \longrightarrow \infty$ $\frac{L}{\xi}$ FINITE

- QUESTION : WHAT HAPPENS WITH eq $\chi(t)$, A QUANTITY THAT WOULD DIVERGE AT T_c IF $L \longrightarrow \infty$?

OBSERVATIONS FROM EXPS & NUM SIMUL.

THIS IS THE LOCAL ONE, OR AN INTENSIVE ONE THAT SHOULD



DIVERGE AS $|T-T_c|^{-\gamma}$ FOR $L \to \infty$

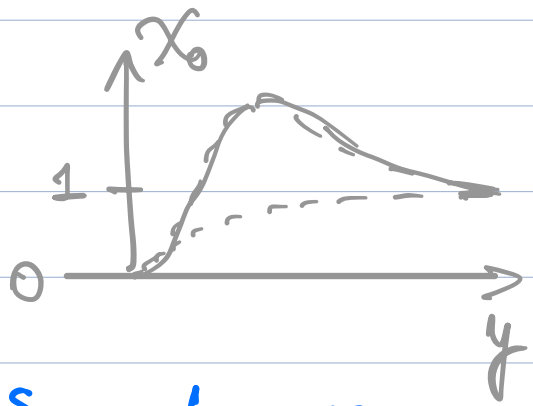
- MAXIMUM MOVES TO THE "LEFT" FOR INCREASING L
- WIDTH OF CURVE DIMINISHED WITH L ↑
- HEIGTH OF MAX ↑ WITH L ↑

ALL ~ POWER LAWS

HYPOTHESIS

$$\chi \sim \xi^{\gamma/\nu}\, \chi_0\left(\frac{L}{\xi}\right)$$

$$\chi_0(y) \sim \begin{cases} \text{cst} & y \gg 1 \\ y^{\gamma/\nu} & y \to 0 \end{cases}$$



ENSURING $\begin{cases} \xi^{\gamma/\nu} & \text{FOR } L \gg \xi\,; \quad L \to \infty \\ L^{\gamma/\nu} & \text{FOR } \xi \gg L \end{cases}$

IN TERMS OF $|t|$ AGAIN, USE $\xi \sim |t|^{-\nu}$

TO REWRITE χ AS A FCT OF $|t|$ AND L

ONE CAN GUESS IT

$$\chi(|t|, L) \to L^{\gamma/\nu}\, \tilde{\chi}\,(L\, |t|^{\nu})$$

$$\boxed{L^{\gamma/\nu}\, \tilde{\chi}\,(L^{1/\nu}\, |t|)}$$

WHERE WAS ξ NOW THERE'S L (IN FRONT) AND ALGIMENT CHANGE IS QUITE OBVIOUS

HOW TO MAKE A CHECK OF THIS ?

PLOT $L^{-\gamma/\nu}\, \chi_L$ VS. $L^{1/\nu}\, |t|$ FOR DIFF L.

DO THE DATA POINTS FALL ON TOP OF EACH OTHER ?

CAREFUL WITH NOTATION $\chi = \dfrac{\partial M}{\partial h}$ CAN BE A LOCAL QUANTITY OR A GLOBAL ONE : IF INTEGRATED OVER SPACE THERE'S AN L^d DIFF BTW THE TWO

HOMEWORK: ON FINITE SIZE SCALING

THE RENORMALIZATION GROUP (REAL SPACE)

- LARGE CORRELATED SCALES (ξ)
- MICROSC. DETAILS SHOULD NOT MATTER

$\Rightarrow$ COLLECTIVE PHENOM.
SCALE INV.
UNIVERSALITY

VIA BLOCK SPINS
FLOW OF COUPLING CSTS.

THE 1d ISING CHAIN

$K = \beta J$, $\beta h = 0$

$$\mathcal{Z} = \sum_{\{S_i\}} e^{-H_K(\{S_i\})} = \sum_{\{S'_I\}} e^{\underbrace{-H_{K'}(\{S_I\})}}$$

SAME FCT FORM BUT $\neq$ COUPLING STRENGTH

BOTH SUMS ON ISING VARIABLES

$i = 1, \ldots, N$
$I = 1, \ldots, N'$

$K = \beta J$ ADIMENSIONAL PARAMETER

H_K & $H_{K'}$ SAME FUNCT FORM BUT Diff COUPLING CONST.

N ODD eg $N = 7$

N-1 EVEN eg $N-1 = 6 \Rightarrow \frac{N-1}{2} = 3$

$$Z = \sum_{\{S_I\}} e^{K[S_1 S_2 + S_2 S_3] + K[S_3 S_4 + S_4 S_5] + \cdots}$$

THE EVEN SPINS APPEAR WITHIN THE [...] AND ONLY ONCE IN EACH

$\Rightarrow$ SUM OVER THEIR CONFS.

$$Z = \sum_{\{S_I\}} 2\cosh K(S_1 + S_3)\; 2\cosh K(S_3 + S_4) \cdots$$

REWRITE EACH FACTOR IN THE FORM OF THE 1DIM CANONICAL WEIGHT, e.g.

$$2\cosh K(S_1 + S_3) = f(K)\, e^{K' S_1 S_3}$$

$s_1 s_3 = 1$ TWO POSSIBILITIES OUT OF

$s_1 s_3 = -1$ $s_1 = \pm 1$ $s_3 = \pm 1$ $\Rightarrow$

$$2 \operatorname{ch} 2k = f(k) \exp k'$$

$$2 = f(k) \exp(-k')$$

TAKE THE RATIO BTW THE TWO EXPRESSIONS

$$\operatorname{ch} 2k = \exp(2k') \quad \Rightarrow$$

$$\boxed{k' = \frac{1}{2} \ln \operatorname{ch} 2k}$$

$$f(k) = 2 \exp k' = 2 \exp \tfrac{1}{2} \ln \operatorname{ch} 2k$$

$$\boxed{f(k) = 2 (\operatorname{ch} 2k)^{1/2}}$$

SO WE HAVE FOUND A REWRITING SUCH THAT

$$\mathcal{Z}(K,N) = \left(f(k)\right)^{\frac{N-1}{2}} \mathcal{Z}(k', \tfrac{N+1}{2})$$

$\mathcal{Z}(K,N)$: ORIGINAL COUPLING STRENGTH, # SPINS

$\left(f(k)\right)^{\frac{N-1}{2}}$: # DECIM. SPINS

$\mathcal{Z}(k', \frac{N+1}{2})$: NEW COUPLING, # REMAINING SPINS

$$\times \sum_{\{S_I = \pm 1\}} e^{k' \sum_I S_I S_{I+1}}$$

SUM OVER LESS SPINS

RELATION BETWEEN PARTITION FUNCTIONS

AFTER ONE STEP OF DECIMATION

YOU CAN CHECK IT WORKS FOR N EVEN & ODD

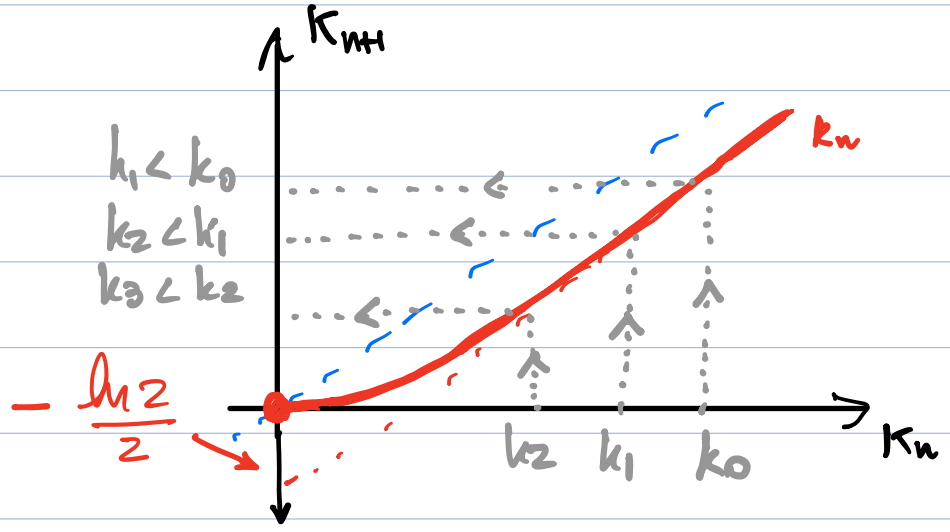
HOW IS THE EVOLUTION OF THE COUPLING CONSTANTS GOING ?

$$k' = \frac{1}{2} \ln \cosh 2k$$

IT CAN BE THOUGHT OF AS A RECURRENCE

$$k_{n+1} = \frac{1}{2} \ln \cosh 2k_n$$

WITH n LABELING THE STEP OF THE DECIMATION.



FUNCTIONAL FORM OF ITERATION

$k_n = 0 \Rightarrow k_{n+1} = 0$

$k_n \to \infty \Rightarrow$

$$k_{n+1} \to \frac{1}{2} \ln\left(\frac{e^{2k_n}}{2}\right) = \frac{1}{2}\left(2k_n - \ln 2\right)$$

$$= k_n - \frac{\ln 2}{2}$$

$$\frac{d}{dk_n}\left(\frac{1}{2}\ln\cosh 2k_n\right) = \frac{1}{2}\tanh(2k_n)\cdot 2$$

$$\xrightarrow[k_n \to 0]{} 0 \quad ; \quad \xrightarrow[k_n \to \infty]{} 1$$

TWO FIXED POINTS $k^* = 0$ ATTRACTIVE

$k^* \to \infty$ REPULSIVE

BACK TO THE PARTITION FUNCTION, TAKE $\ln$

$\ln Z(k, N) = \zeta N$ SHOULD BE $\propto N$

THAT IS $\zeta = \Theta(1)$

(IT'S PROP. OF FREE-ENERGY)

$$\ln Z(N,k) = \frac{N-1}{2}\ln f(k) + \ln Z\left(k', \frac{N+1}{2}\right)$$

↑ # DECIMATED SPINS

REMAINING SPINS

$\Rightarrow$ $N \gg 1$

$$\boxed{S(k) = \frac{1}{2}\ln f(k) + \frac{1}{2}S(k')}$$

WITH $f(k)$ WRITTEN ABOVE. FROM HERE, $S(k')$ OBTAINED FROM $S(k)$ AND k.

$$S(k') = 2S(k) - \ln f(k)$$

ITERATE IN OPPOSITE DIRECTION. START FROM k' VERY SMALL $\Rightarrow$ GET $k > k'$

(HIGHT) (LOWT)

APPROACH $T \to 0$ LIKE THIS & CONSTRUCT $S(k)$. EXCELLENT AGREEMENT W/ EXACT VALUES. SEE TABLE.

NOTE: WE HAVEN'T SEEN THE EFFECT OF $a \to 2a$ IN THIS CALC. BUT, IN $C(\vec{x}, \vec{y})$ WE'LL DO.

RECAP

WE SAW SCALING AND A VERY QUICK ANALYSIS OF THE 1D IM $\Rightarrow$ REPEAT IT

SCALING RELATIONS

$$f_{SING}(t,h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^{\Delta}}\right)$$

$$\xi(t,h) \sim |t|^{-\nu} g_\xi\left(\frac{h}{|t|^{\Delta}}\right)$$

FROM HERE WE DERIVED:

$\Rightarrow$

- RELATIONS BETWEEN EXPONENTS
- ONLY TWO INDEPENDENT EXPONENTS

FIRST ENCOUNTER WITH RG IDEAS

1d IM - $h=0$

DECIMATION $\Rightarrow$ INTEGRATE AWAY ONE OUT OF TWO SPINS

$$K_{n+1} = \frac{1}{2} \ln \cosh 2K_n$$

FLOW OF COUPLING CONSTANT W/ TWO FIXED POINTS

$K^* = 0$	$K^* \to \infty$
$T \to 0$	$T = 0$
STABLE	UNSTABLE
	$\Downarrow$
	CRITICAL POINT

$$\Im(K) = \frac{1}{2} \ln f(K) + \Im(K')$$

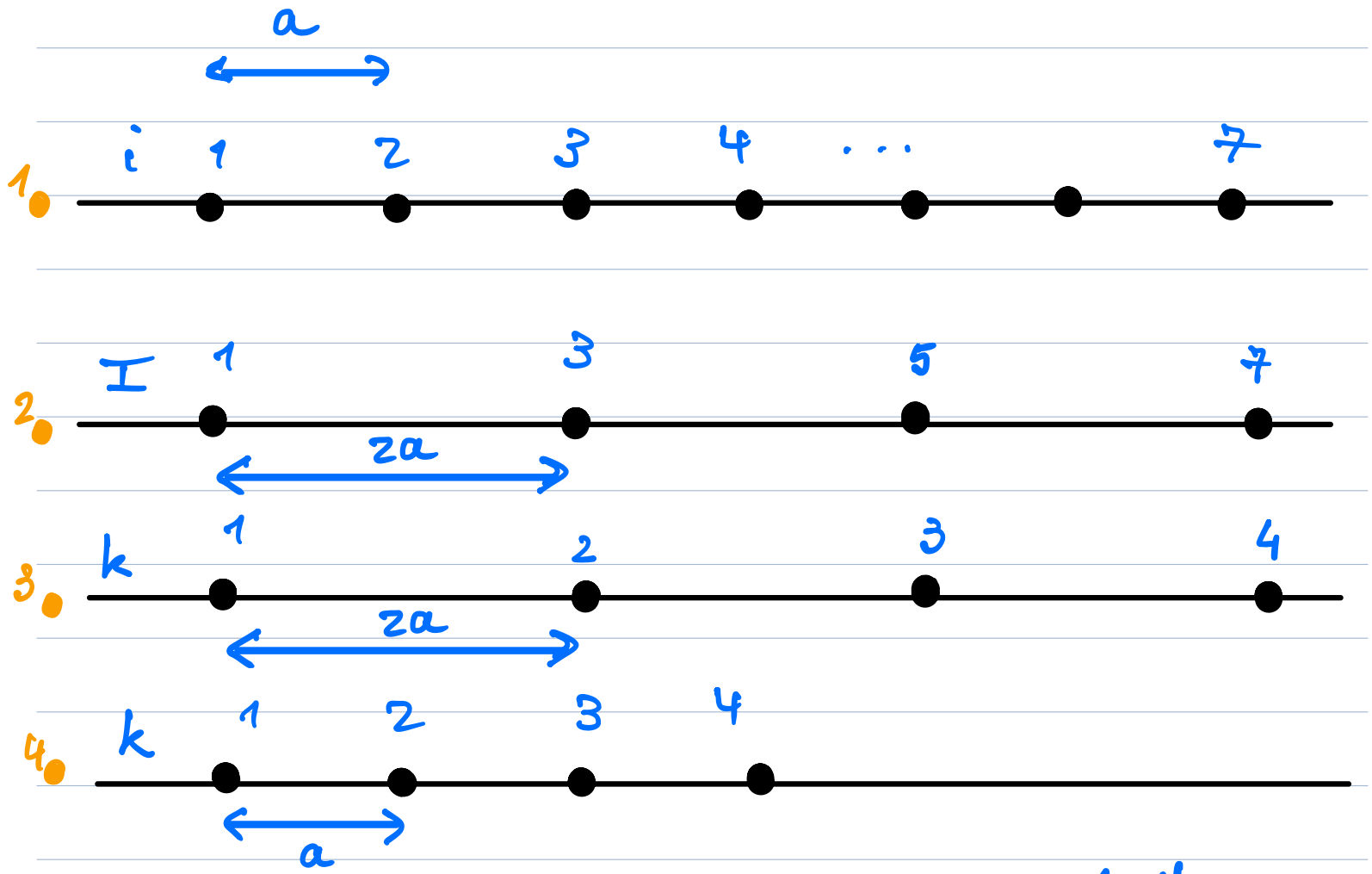
WHERE

$$\Im(K) = N \ln Z(K)$$

$$f(K) = 2[\cosh(2K)]^{1/2}$$

(NOTE $\Im(K) = -k_B T F(K)$)

THE SKETCH OF TRANSFORMATIONS



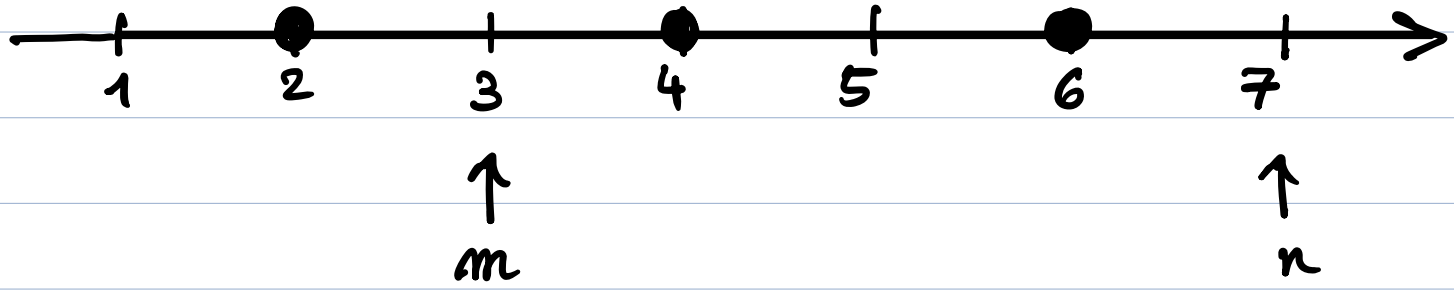
1. $i = 1, \dots, N$ SPINS — DECIMATE $\frac{1-N}{2}$ SPINS:

2. $I = 1, \dots, \frac{N+1}{2}$ SPINS — RELABEL REMAINING SPINS:

3. $k = \frac{I+1}{2} = 1, \dots, \frac{N+1}{2}$ SPINS — SPACE RESCALING:

4. $2a \longrightarrow a$

NOTE THAT IN THE LAST TRANSFORMATION THE DISTANCES HAVE BEEN DIVIDED BY 2



- DECIMATED

m, n LABELS OF SPINS TO BE CORRELATED

$$C_{mn} = \langle s_m s_n \rangle$$

$$= \frac{1}{Z} \sum_{\{s_i = \pm 1\}} s_m s_n \, e^{-\beta H}$$

$$= C(\vec{x}, \vec{y})$$

$$\vec{x} = a m \qquad \vec{y} = a n$$

THE CORRELATION LENGTH

FROM $C(\vec{x},\vec{y}) \sim e^{-|\vec{x}-\vec{y}|/\xi(k)}$

$[\xi(k)] = L$ LENGTH UNITS

DEPENDS ON ADIMENS. PARAM K

$$C(\vec{x},\vec{y}) = \frac{1}{Z(N,k)} \sum_{\{s_i=\pm 1\}} s_m s_n \, e^{+k \sum_{i=1}^{N} s_i s_{i+1}}$$

SAY THEY ARE ODD

- WE USE THE RELATION BETWEEN PARTITION SUMS BEFORE & AFTER DECIMATION
- WE DECIMATE THE EVEN LABELED SPINS.

$$= \frac{1}{(f(k))^{\frac{N-1}{2}} \, Z\left(\frac{N+1}{2}, k'\right)} \sum_{\{s_I=\pm 1\}_{I\,(ODD)}} s_m s_n \, (f(k))^{\frac{N-1}{2}} \, e^{k' \sum_{I\,(ODD)} s_I s_{I+2}}$$

eg. $I=1$ $s_1 s_3$

- WE RENAME $k = \frac{i+1}{2} = 1, \dots, \frac{N+1}{2}$

$$= \frac{1}{Z\left(\frac{N+1}{2}, k'\right)} \sum_{\substack{s_k = \pm 1 \\ k = 1, \dots, \frac{N+1}{2}}} s_{\frac{m+1}{2}} \, s_{\frac{n+1}{2}} \, e^{k' \sum_{k=1}^{\frac{N+1}{2}} s_k s_{k+1}}$$

THE CORR FCT CAN BE COMPUTED IN BOTH WAYS AND ONE SHOULD GET THE SAME RESULT

IT'S JUST TWO WAYS OF CALCULATING

- IN THE SYST WITH (k, N, a)
- IN THE SYST WITH $\left(k', \frac{N+1}{2}, 2a\right)$

$$C(\vec{x}, \vec{y})\Big|_{N, k, a} = C(\vec{x}, \vec{y})\Big|_{\frac{N+1}{2}, k', 2a}$$

SHOULD HAVE THE SAME FUNCTIONAL FORM

CLOSE TO THE CRIT POINT EXP. DECAY WITH A CORRELATION LENGTH ξ

$[\xi] = L$ LENGTH DIMENSION

$\bar{\xi}(k)\, a = \bar{\xi}(k')\, 2a$ ($\bar{\xi}$: ADIMENSIONAL)

SAY IT'S $\xi = 10\,\text{cm} = \bar{\xi}(k)\, 1\,\text{cm} = \bar{\xi}(k')\, 2\,\text{cm}$

$\Rightarrow \quad \bar{\xi}(k') = \bar{\xi}(k)/2$

$$\boxed{\bar{\xi}(k') = \frac{\bar{\xi}(k)}{2}}$$

HOW DOES ONE GET $\bar{\xi}(k)$ FOR $k \to \infty$?

NOTICE THAT $k_{n+1} = \frac{1}{2} \ln \operatorname{ch} 2k_n$

FOR LARGE k_n

$\to \frac{1}{2} \ln\left(e^{2k_n}/2\right) = \frac{1}{2}\left(2k_n - \ln 2\right)$

$k_{n+1} - k_n = \frac{-\ln 2}{2}$ DECREASE PER ITERATION STEP

RECALL $K = \beta J$ $\begin{cases} K^* = \infty \quad (T_c = 0) \\ K^* = 0 \quad (T_c \to \infty) \end{cases}$ FIXED POINTS

SAY ONE STARTS FROM $k_0 = k \gg 1$.
HOW MANY ITERATIONS n DOES IT TAKE TO GO TO $k_f \sim \Theta(1)$? CALL THAT LIMIT NUMBER $\bar{n}$:

$$k_f = \Theta(1) = K - \frac{\ln 2}{2} \cdot \bar{n} \Rightarrow$$

IN EACH STEP k DECREASES BY THE SAME AMOUNT

USING $K \gg 1$
$\bar{n} \gg 1$ AND $k_f = \Theta(1)$ $\Rightarrow$ $\bar{n} \sim \frac{2K}{\ln 2}$

THEN USING $\bar{\Sigma}(k_{n+1}) = \frac{\bar{\Sigma}(k_n)}{2}$ AFTER $\bar{n}$ STEPS

$$\Rightarrow \quad \Theta(1) = \bar{\Sigma}(k_f) = \frac{\bar{\Sigma}(k)}{2^{\bar{n}}}$$

$$\bar{\Sigma}(k) \sim 2^{\bar{n}} = 2^{\frac{2K}{\ln 2}} \sim e^{2K}$$

THE LAST IDENTITY IS A CONSEQ. OF

$$2 = e^{\ln 2} \Rightarrow$$

$$2^{\frac{2k}{\ln 2}} = \left(e^{\ln 2}\right)^{\frac{2k}{\ln 2}}$$

$$= e^{\ln 2 \, \frac{2k}{\ln 2}} = e^{2k}$$

$$\bar{\Sigma}(k) \sim e^{2k}$$

NOTE THAT THIS IMPLIES

$$\Sigma(k) \sim a\, e^{2k}$$

THE LENGTH UNITS ARE RECOVERED. WHAT'S IMPORTANT IS THE EXP DIVERGENCE FOR $T \to 0$

EXPON. DIVERGENCE OF $\Sigma(k)$ SPECIFIC OF $d=1$

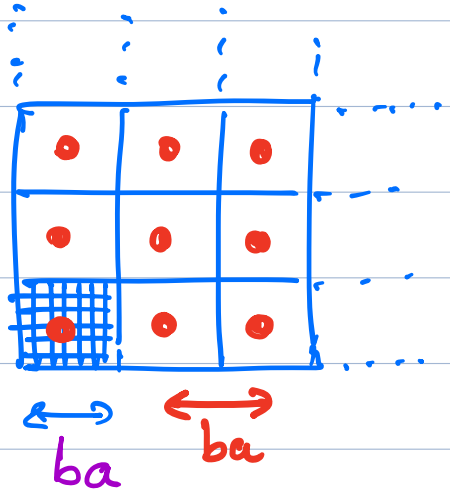
KADANOFF BLOCK SPINS & RENORMALIZATION Vd

AS LANDAU, BUILD BLOCK SPINS

$$S'_I = b^{-x} \sum_{i \in B_I} S_i \qquad [b] = 1$$

NORMALIZATION

SINCE SOME SPINS IN B_I ARE REVERSED THE SUM IS NOT EQUAL TO $N_I = \#$ SPINS IN B_I $\Big\}$ BUT $\Rightarrow b^{-x}$ WITH $x < d$



- S'_I ISING VARIABLES

NEW LATTICE CONST MAGNIFIED

$$\boxed{a' = ba}$$

(NB $b > 1$ A NUMBER)



$$\xi = \bar{\xi} \cdot a = \bar{\xi}' a' = \bar{\xi}' ba$$

ξ CORR. LENGTH IN ABSOLUTE UNITS (eg. CM)

$$\bar{\xi} = \bar{\xi}' b \Rightarrow \boxed{\bar{\xi}' = \bar{\xi}/b}$$

$\bar{\xi}'$ REDUCED ! IT'S AS SYST FURTHER AWAY FROM CRITICALITY

SAME AS WHAT WE SAW IN $d=1$

SCALE TRANSF. TO GET BACK TO ORIGINAL:

$$x'_\alpha = \frac{x_\alpha}{b} \qquad \alpha = 1, \ldots, d$$

Block spins: ISING VARIABLES OCCUPYING THE SITES OF A LATTICE W/ SAME LATTICE SPACING a AFTER DISTANCE RE-SCALING ($\times 1/b$)

BUT LESS SPINS: $N' = N/b^d$

FOR THE 1d ISING CHAIN WE CALCULATED BY DECIMATION THE NEW HAMILTONIAN

$$N \to \frac{N+1}{2}, \quad k \to k', \quad a \to a' = ba = 2a$$

IN GENERAL, WE DON'T KNOW HOW TO DO IT.

PROPOSE

$$H'(\{S'_I\}) = R_b \; H(\{S_i\})$$

FOR A TRANSFORMATION WITH SCALE FACTOR b

PROPERTIES

COMPOSITION

- APPLY TRANSF b & APPLY TRANSF b SHOULD BE EQUIV TO APPLYING TRANSF b^2. IT'S MULTIPLICATIVE.

$$R_b^2 = R_b R_b \Longrightarrow \boxed{R_{b^n} = (R_b)^n}$$

n TIMES

HYPOTHESIS:

THIS TRANSF. SHOULD HAVE A FIXED POINT AT CRITICALITY

$$\boxed{H'(3\xi'\S) = H(3\xi\S)}$$

FOR $\xi \gg r \gg ba \gg a$

WE CALL WITH A STAR THE FIXED POINT.

IN SIMPLE CASES, EXACT SAME FUNCT FORM AS ORIGINAL $\mathcal{H}$

IN OTHERS, CHANGES ARE POSSIBLE*.

THE NEW PARAMETERS AT THE FIXED POINT $\Rightarrow$ TO BE FOUND.

$$R_b \mathcal{H}^* = \mathcal{H}^*$$

$\mathcal{H}^* \Rightarrow$ LARGE SCALE PROPS.

COUPLING CONST FLOW

NB. FORM OF HAMLT. MAY CHANGE WRT. ORIGINAL ONE

$$\mathcal{H}(\{s_i\}) = -\sum_A J_A S_A$$

↳ eg ALL POSSIBLE GROUPS OF SPINS

(* NOTE THAT IN A GENERIC PROBLEM ONE COULD START WITH TWO-BODY INTERACTIONS AND GENERATE MANY-BODY ONES AFTER DECIMATION)

$$\beta J_A = K_A \quad \Rightarrow \quad [K] = (K_1, K_2 \dots K_k)$$

ADIM. ENSEMBLE OF PARAMETERS

$$[K'] = R_b [K]$$

b FINITE SCALE OF TRANSF.

TRANSFORMATION OF COUPLING CONSTANTS RG

- TAYLOR EXPAND

$$K'_\alpha = \underbrace{\sum_\beta R_{\alpha\beta}(b)\, K_\beta}_{\text{LINEAR}} + \overbrace{\cdots}^{\text{NON LINEAR}}$$

WHERE WE'RE MEASURING K_α FROM CRIT. POINT

MEANING $K_\alpha - K_\alpha^c$ EVENTUALLY SMALL

eg. $\frac{T-T_c}{T_c}$, βh IN FM-PM PROBLEM

THAT'S WHY THERE'S NO ZERO-th ORDER

eg. FM PROBLEM

$$K' = A(b)\,K + B(b)\,H + \text{h.o.t}$$

$$H' = C(b)\,K + D(b)\,H' + \text{h.o.t.}$$

IDEA: DROP h.o.t. & KEEP JUST LINEAR.

(WORKING CLOSE TO CRIT. POINT)

FOR $b=1 \Rightarrow$ NO TRANSFORMATION $\Rightarrow$ IDENTITY.

$$A(1)=1 \qquad B(1)=0 \qquad C(1)=0 \qquad D(1)=1$$

USE SYMM TO KILL SOME TERMS

eg. ISING $\{s_i \to -s_i\}$ $\{H \to -H\}$ $\{K \to K\}$

$$B(b)=0 \qquad C(b)=0$$

$$K' = A(b)\,K$$

$$H' = D(b)\,H$$

h.o.t. ALREADY DROPPED

FINALLY, APPLY TWO TRANSF. IN A ROW BOTH WITH SCALE b. IT SHOULD BE THE SAME AS APPLYING ONE WITH SCALE b^2

$$A(b)\,A(b) = A(b^2)$$

$$D(b)\,D(b) = D(b^2)$$

THEREFORE, $A(b) = b^{y_K}$ $\qquad D(b) = b^{y_H}$

$y_K > 0$ $\qquad y_H > 0$

TO MOVE AWAY FROM CRIT W/TRANSF, eg.

$$k' = b^{y_k} k$$

$y_k > 1$

TO MOVE AWAY FROM CRIT W/ TRANSF. PERFORMED

IN SUMMARY

$$k' = b^{y_k} k$$

$$H' = b^{y_H} H$$

EX. WITH TWO CONTROL PARAM. (TERM LINEAR IN S, 2 TERM QUAD. IN S)

NOW, IN FREE ENERGY

$$F(k', H') = F(k, H)$$ AT CRITICALITY

$$N' f(k', H') = N f(k, H)$$

NB. THIS f IS THE FREE ENERGY DENSITY. ALSO CALLED f ABOVE

$$f(k, H) = \left(\frac{N'}{N}\right) f(k', H')$$

$$f(k,H) = b^{-d} f(b^{y_k} k, b^{y_H} H)$$

HOMOGENEITY PROP. PROVEN !

Choose now $b = k^{-1/y_k}$

$$f(k,H) = k^{d/y_k} f\left(1, \frac{H}{k^{y_H/y_k}}\right)$$

$$\Delta \equiv \frac{y_H}{y_k} \qquad \frac{d}{y_k} = 2-\alpha$$

$$f(k,H) = k^{2-\alpha} g_f\left(\frac{H}{|k|^{\Delta}}\right)$$

WE FOUND THE SCALING FORM.

RELATION BETWEEN PHYSICAL EXPONENTS AND THE POWERS IN PARAM TRANSF

$$\frac{\bar{\xi}(R_b(k))}{\bar{\xi}(k)} = \frac{1}{b}$$

RECALL 1d IM CALCULATION & ARGUMENT

THE ADIMENSIONAL ONE

ASSUMING $\bar{\xi} \sim (k - k_c)^{-\nu}$ POWER LAW DIVERGENCE

$$\frac{1}{b} = \left(\frac{R_b(k) - R_b(k^*)}{k - k^*} \right)^{-\nu} = \left(\frac{R_b(k) - k^*}{k - k^*} \right)^{-\nu}$$

$$= \left(\frac{dR_b(k)}{dk} \Big|_{k=k^*} \right)^{-\nu}$$

FROM HERE,

$$-\ln b = -\nu \ln \frac{dR_b}{dk} \Big|_{k=k^*}$$

$$\nu = \frac{\ln b}{\ln \frac{dR_b}{dk} \Big|_{k=k^*}}$$

FOR THE POWER LAW $R_b \sim b^{y_k} k$

$$\frac{dR_b(k)}{dk} = \frac{d}{dk}\left(b^{y_k} k\right) = b^{y_k}$$

$$\ln \frac{dR_b}{dk} = y_k \ln b$$

$$\Rightarrow \quad \boxed{\nu = 1/y_k}$$

NOTE THAT $y_k = 1/\nu$ $\qquad y_k = \dfrac{d}{2-\alpha}$

$\Rightarrow d\nu = 2-\alpha$ WHICH IS THE JOSEPHSON IDENTITY DERIVED ABOVE

WE ALSO HAD $\Delta \equiv \dfrac{y_H}{y_k}$, $\qquad \dfrac{d}{y_k} = 2-\alpha$

WHICH IMPLIES

$$y_H = \Delta \frac{d}{2-\alpha}$$

THE TRANSF OF ξ $\Rightarrow$

POWER LAW DIVERGENCE OF ξ

AGAIN USING NOT. k FOR DIST FROM CRIT POINT

$$\bar{\xi}(k_n) = \bar{\xi}(b^{n y_k} k)$$ REPLACING k_n

$$= b^{-n} \bar{\xi}(k)$$ USING FLOW AWAY FROM CRITICALITY N-TRANSFORMS

MULTIPLYING BY b^n

$$b^n \bar{\xi}(b^{n y_k} k) = \bar{\xi}(k)$$

CHOOSING $b^n = \left(\frac{b}{k}\right)^{\frac{1}{y_k}} = \left(b k^{-1}\right)^{\frac{1}{y_k}}$

ONE HAS $b^{n y_k} k = \left(b k^{-1}\right)^{y_k / y_k} k = b$

& REPLACING ABOVE

$$(bk^{-1})^{\frac{1}{y_k}}\,\overline{\xi}(b) = \overline{\xi}(k)$$

ISOLATING THE k DEPENDECE

$$\overline{\xi}(k) \sim k^{-1/y_k}$$

(THE REST IS A NUMBER $b^{1/y_k}\,\overline{\xi}(b)$)

Σ SHOWED POWER LAW DIVERGENCE

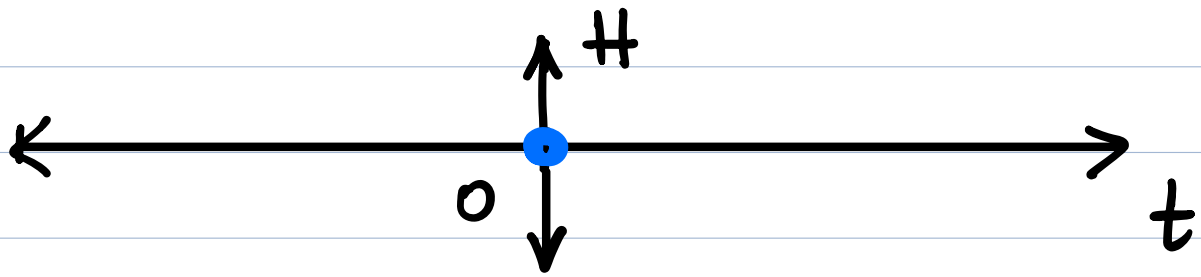
WITH

$$\boxed{\nu = 1/y_k}$$

WHAT HAPPENS IF THERE ARE MORE THAN ONE CONTROL PARAM?

eg. $t = \frac{T - T_c}{T_c}$ $\quad H = \beta h$

FOR MAGN. SYST.



DEPENDING ON WHICH DIR IS TAKEN THE CRIT EXP. MEAS. COULD BE DIFF.

eg. ν FROM $\xi = |t|^{-\nu}$

WHILE ν_H FROM $\xi \sim |H|^{-\nu_H}$

HOW CAN ONE SEE THIS ?

FROM SCALING FORM of ξ:

$$\xi \sim |t|^{-\nu} \; g_\xi\left(\frac{H}{|t|^{\beta\delta}}\right)$$

(1) $g_\xi(y \to 0) = \text{ct.}$ $\qquad H = 0$

(2) $g_\xi(y \to \infty) = \left(\frac{H}{|t|^{\beta\delta}}\right)^{-\frac{\nu}{\beta\delta}}$ $\qquad t = 0$

THEN (1) $\Rightarrow$ $\xi \sim |t|^{-\nu}$ AT $H = 0$

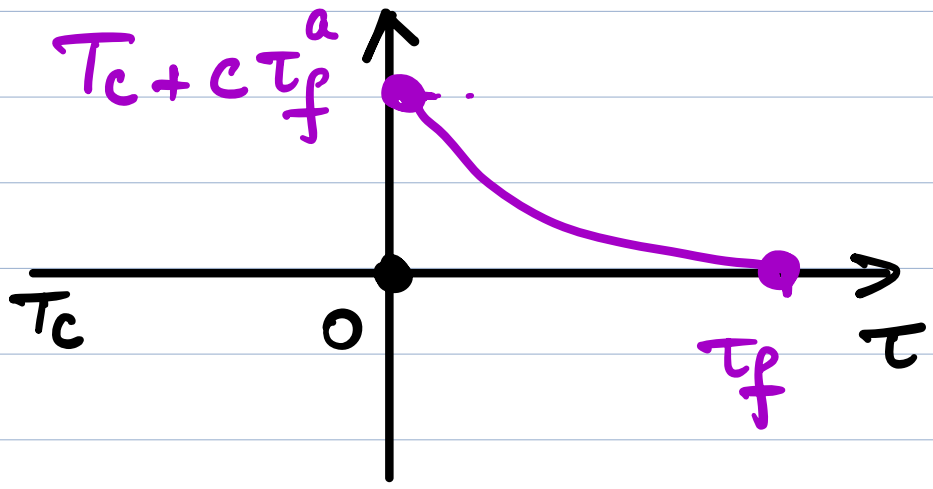
WHILE (2) $\Rightarrow$ $\xi \sim H^{-\nu/(\beta\delta)}$ AT $t = 0$

DEPENDING ON DIRECTION IN PARAM. SPACE WE GET $\neq$ EXP.

ALSO: IF WE CHANGE THE WAY IN WHICH PARAM IS CHANGED

ANNEALING OF TEMP.

$$T(\tau) = T_c + c \cdot (\tau_f - \tau)^a$$



τ : TIME

TENDS TO 0 $\tau = \tau_f$

SAY $\lambda = \tau_f - \tau$ A PARAM.

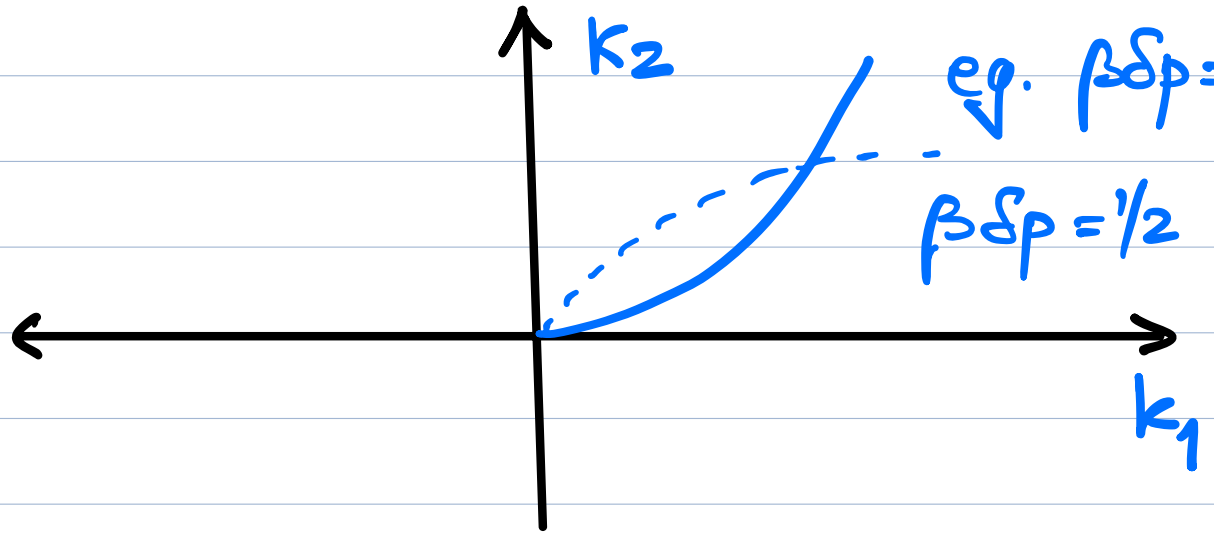
IF WE MEASURE AS A FUNCTION OF λ WHICH CONTROLS DIST FROM CRIT POINT

$$\xi \sim \lambda^{-\nu_\lambda}$$

ν_λ DIFFERENT VALUE.

SAY ONE TAKES A PATH

$$k_2 = c \cdot k_1^{\beta\delta p} \qquad p \geq 1$$



$$\xi(k_1, k_2) \sim k_1^{-\nu} \, g_\xi\left(\frac{k_2}{k_1^{\beta\delta}}\right)$$

$$\sim k_1^{-\nu} \, g_\xi\left(c k_1^{\beta\delta(p-1)}\right)$$

- $p \geq 1$, THE ARGUMENT $y \to 0$ FOR $k_1 \to 0$

$$g_\xi(y \to 0) = \text{ct.} \implies \xi \sim k_1^{-\nu}$$

- $p < 1$, THE ARGUMENT $y \to \infty$ FOR $k_1 \to \infty$

WE HAVE TO USE

$$g_\xi(y) \sim y^{-\frac{\nu}{\beta\delta}}$$

THEN

$$\xi \sim k_1^{-\nu} \, k_1^{\frac{-\beta\delta(p-1)}{\beta\delta}\nu}$$

$$\sim k_1^{-p\nu} \qquad k_1 \sim k_2^{\frac{1}{\beta\delta p}}$$

$$\sim k_2^{-\frac{\nu}{\beta\delta}}$$

$$\Rightarrow \quad \xi \sim k_2^{-\frac{\nu}{\beta\delta}}$$

So, FOR $p > 1$, k_1 DOMINATES

$p < 1$, k_2 DOMINATES

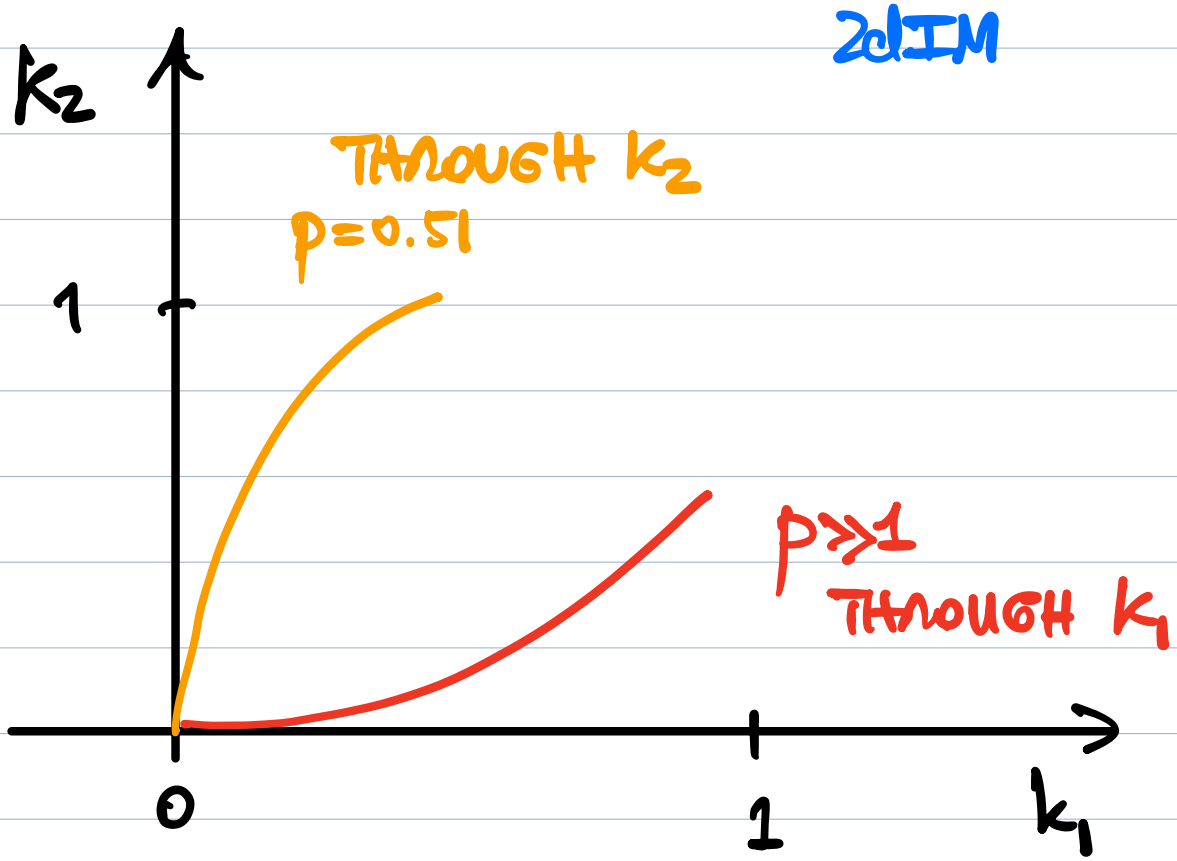
2dIM $\beta = 1/8$ $\delta = 1/15$

$$\beta\delta = \frac{1}{8 \cdot 15} \sim \frac{1}{100} \sim 0.01$$

very small!

$$k_2 = k_1^{\beta\delta p}$$

2dIM



CAREFUL

WHEN WE DISCUSSED WIDOM, JOSEPHSON, ETC RELATIONS. BETWEEN EXP.

WERE DERIVED FOR USUAL WAY OF MEASURING DIST FROM CRIT POINT

$$\frac{T - T_c}{T_c} \qquad \beta H$$

GENERAL ANY NUMBER of COUPLINGS.

$$[k'] = R_b [k]$$ TRANSF PARAM. (ABSOLUTE VALUE HERE)

$$\bar{\xi}(k') = \frac{\bar{\xi}(k)}{b}$$ TRANSF CORR LENGTH

FIXED POINT FROM CORR LENGTH

$\xi^* \to 0$ or $\xi^* \to \infty$

TRIVIAL INTERESTING

STABLE UNSTABLE UNDER FLOW

ATTRACTIVE REPULSIVE

CLOSE TO THE FIXED POINT

$[k'] = R_b [k]$ $\Rightarrow$ $\alpha = 1, \ldots, \#$ PARAM.

$$K'_\alpha(b) - k^*_\alpha = \sum_\beta R_{\alpha\beta}(b)\,(k_\beta - k^*_\beta)$$

LINEAR FORM CLOSE TO $[k^*]$

DROPPING h.o.t.

$$\delta k'_\alpha(b) = R_{\alpha\beta}(b)\,\underbrace{\delta k_\beta}$$

DIST FROM CRIT. POINT

$$R_{\alpha\beta}(b) = \left.\frac{\partial k'_\alpha(b)}{\partial k_\beta}\right|_{[k^*]}$$

AT FIXED POINT

MATRICIAL FORM

$$\delta k' = R\,\delta k$$

PARAM x # PARAM MATRIX

DIAGONALIZE WITH $UU^T = U^T U = \mathbb{1}$

$$\underbrace{U\delta k'}_{\downarrow} = \underbrace{URU^T}_{D}\ \underbrace{U\delta k}_{\downarrow}$$

$\delta\kappa'(b) = D(b)\ \delta\kappa$ $\delta\kappa$ ALPHA IS $\delta\kappa$ ROTATED BY U

↓

DIAGONAL MATRIX WITH EIGENVALUES $\{\lambda_\alpha\}$ IN DIAGONAL

FROM TRANSF. COMPOSITION RULE (MULTIPLICATIVE)

$$\lambda_\alpha(b) = b^{y_\alpha} \Rightarrow y_\alpha = \frac{d\ln\lambda_\alpha}{d\ln b}$$

DERIVATIVE JUSTIFIED SINCE b CONT. & CLOSE TO 1 ASSUMED.

THE TRANSF OF PARAM. YIELDS $\lambda_\alpha(b)$ $\Rightarrow$ y_α

$\Rightarrow$ DIVERGENCE OF OBSERV. eg

$$\bar{\xi}(k) \sim k^{-1/y_k}$$

FOLLOWING THE DIRECTION OF k.

CLASSIFICATION of PARAMETERS

- $y_\alpha > 0 \Rightarrow$ RELEVANT (INCREASES)
- $y_\alpha < 0 \Rightarrow$ IRRELEVANT (VANISHES)
- $y_\alpha = 0 \Rightarrow$ MARGINAL (h.o.t.)

IN VIEW OF WHAT WAS DISCUSSED FOR THE TWO PARAM. CASE, THE EXP. TO MEASURE AFTER THESE TRANSF HAVE TO BE STUDIED CAREFULLY.

WHICH PARAM. CONTROLS ?

BETTER CHOOSE THEM FIRST AND THEN APPROACH CRIT POINT FOLLOWING ITS DIRECTION.

COMMENTS

- RG TRANSF MOVES WITHIN A PHASE, DOESN'T GO ACROSS PHASE TRANS.

- REPULSIVE FIXED POINTS $\Rightarrow$ CRIT. POINTS (ξ DECREASED UNDER RG)

  - IN DISORDERED PHASE RG TO $T \to \infty$
  - IN ORDERED PHASE RG TO $T \to 0$

THE EXAMPLE WE SOLVED 1dIM $T_c = 0$
SO THE FLOW WENT TO $T \to \infty$
FOR 2dIM $0 < T_c < \infty$ $\Rightarrow$ SO FLOW AWAY FROM IT IN THE TWO PHASES.