Advanced Statistical Physics

Lectures by **Leticia F. Cugliandolo**[1]

Exercise sessions **Ada Altieri**[2] & **Marco Tarzia**[1]

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2025 – Lectures Monday, 08:30 - 10:30

Exercises Monday, 10:45 - 12:45

Focus

Classical and Quantum Phase transitions

Phenomenology, concepts & formalism

Critical phenomena & universality, topology, renormalization, nucleation

Current research, *e.g.*

Non-Equilibrium Universality : From Classical to Quantum and Back, KITP UC Santa Barbara 2021

Quantum localization and glassy physics, Summer School at Cargèse 2023

Out-of-equilibrium Dynamics and Quantum Information of Many-body Systems with Long-range Interactions
KITP UC Santa Barbara 2023

All material (program, lectures, TDs & exams from previous years) downloadable from the webpage www.lpthe.jussieu.fr/˜leticia/enseignement.html

Plan

Mathematical preliminaries (download the file from the webpage)

1. One introductory lecture
2. Six lectures on classical statistical physics
3. Six lectures on quantum statistical physics
4. Thirteen TD sessions in half group (even/odd ending last number student ID)
5. One homework for the mid-term holidays (in pairs/binôme)
6. Final written exam - concepts and exercises (see past years examples from the webpage)

Final Mark 20% homework and 80% written exam

Plan

1. Interest and background, principles and formalism, e.g.
   — (In)Equivalence of ensembles for (long) short-range interactions
   — Systems' reduction (role of environments)
2. Classical phase transitions
   — Important concepts (phase diagrams, order parameters, spontaneous symmetry breaking, pinning fields, etc.)
   — Uncommon mechanisms (e.g. topological phases, condensation)
   — Renormalization group ideas
   — Effects of quenched randomness
3. Quantum statistical physics
   — Generics : Linear response, Kubo formula, FDT, etc.
   — Quantum – classical equivalence, path-integrals, imaginary time
   — Quantum spin models - chains and mean-field - phase transitions

Plan

A very typical program

Absolutely necessary basic knowledge, needed to carry out theoretical or experimental research in

condensed matter - atomic physics - statistical physics

Two examples of syllabus :

Vlad Dobrosavljevic (Florida State University)

Experimental systems showing classical and quantum critical phenomena.

Thermodynamic potentials. Heat capacity. Magnetic susceptibility.

Phases. Phenomenology of 1st order phase transitions. Continuous transitions.

Landau theory. Order parameters. Spontaneous symmetry breaking.

Critical behavior. Scaling. Critical exponents. Relations between critical exponents.

Kadanoff scaling. Universality conjecture.

Calculation of critical exponents : Real space RG methods.

RG of Wilson and Fisher, ϕ^4 theory, 4-ϵ expansion.

Continuous symmetry : Mermin-Wagner theorem.

Non-linear sigma-model ; $2 + \epsilon$ expansion.

Scaling theory of localization. Quark confinement in QCD.

Topological order. Kosterlitz-Thouless phase transition.

Quantum critical phenomena. Hertz-Millis theory. Dissipative quantum tunneling.

Ben Simons (the University of Cambridge)

All mathematical methods in the Math Support file will be assumed to be mastered by the students

It is not enough to listen to the oral lectures to understand/learn the content of this course

You have to read the lectures notes or any book of your choice on Phase Transitions and Critical Phenomena (see the list below)

The final exam will evaluate the comprehension of the concepts presented and discussed and not only the ability to solve guided exercises (see examples of previous years)

You have to study and learn these concepts in between the Monday sessions during the semester. Do not wait until the Xmas holidays to do it

Bibliography

– Berlinsky & Harris - Statistical Mechanics - An Introductory Graduate Course

– Castiglione, Falcioni, Lesne & Vulpiani - Chaos and coarse-graining in statistical physics

– Khinchin - Mathematical Foundations of Statistical Mechanics

– Lesne & Lagües - Scale Invariance from Phase Transitions to Turbulence

– Parisi - Statistical Field Theory

– Goldenfeld - Phase Transitions and the Renormalization Group

– Altland & Simons - Condensed Matter Field Theory

There are many other excellent books that you can use as support to the course

Ask Questions

Introduction

What are you going to learn ?

Phase transitions

Definition

Sharp changes in the behaviour of macroscopic systems at points (or curves) in parameter space.

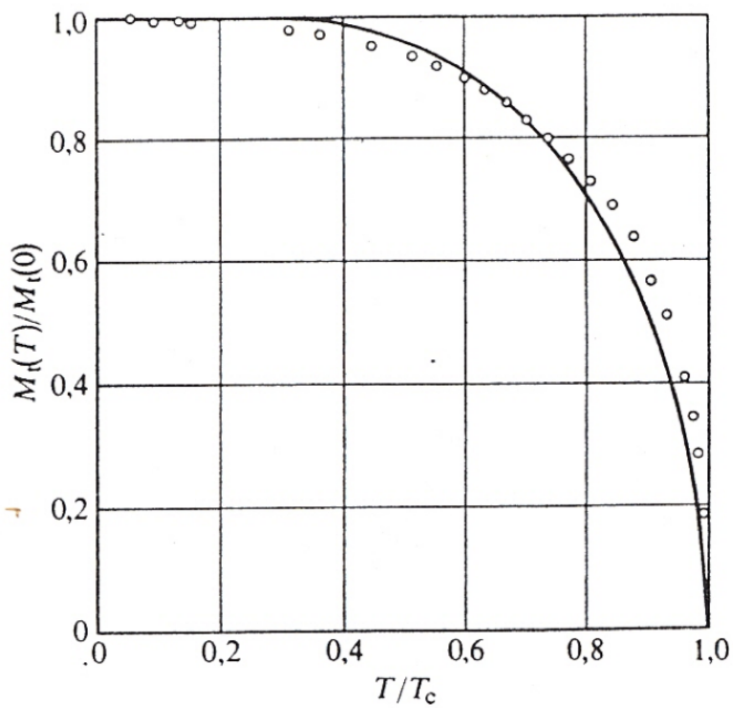
Non-trivial collective phenomena in the thermodynamic limit.

Historical development : experimental observation $\longrightarrow$ phenomenological description $\longrightarrow$ mathematical modelling $\longrightarrow$ full understanding with the development of a new theoretical framework.

Magnetic phase transition

Nickel (classical)

The magnetisation sharply vanishes at T_c *under no applied field* $H = 0$



Does it ?

Weiss & Forrer, Ann. Phys. 5, 153 (1926)

Magnetic phase transition

An exact solution

PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

LARS ONSAGER
Sterling Chemistry Laboratory, Yale University, New Haven, Connecticut

(Received October 4, 1943)

The partition function of a two-dimensional "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of vanishing field. The eigenwert problem involved in the corresponding computation for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is solved by direct product decomposition; in the special case $n=\infty$ an integral replaces a sum. The choice of different interaction energies $(\pm J, \pm J')$ in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

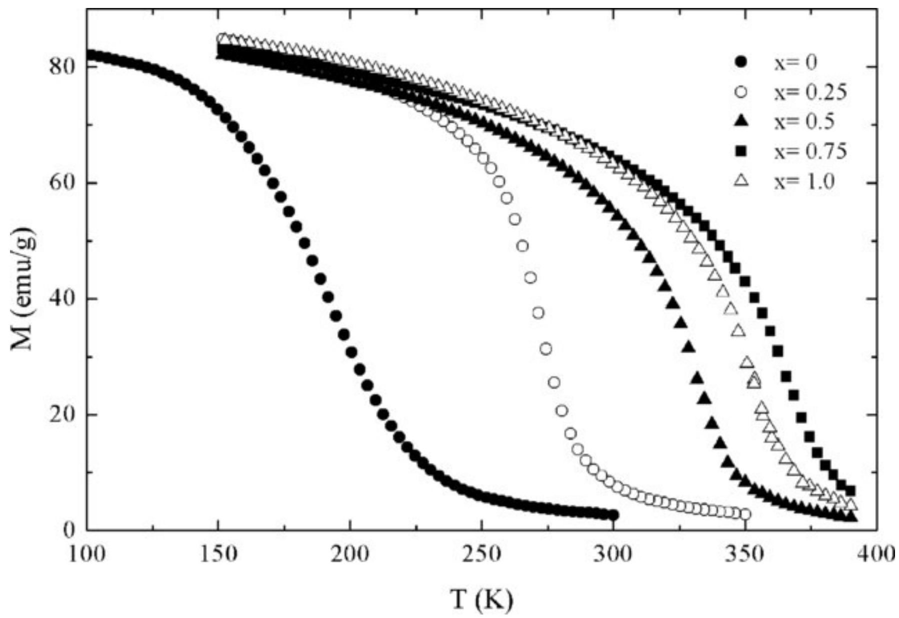
Lars Onsager, *Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition*, Physical Review 65, 117 (1944).

The two-dimensional Ising model, with the transfer matrix method

Magnetic phase transition

Manganites

It continuously decays to zero at $T \to \infty$ *under a magnetic field* $H \neq 0$



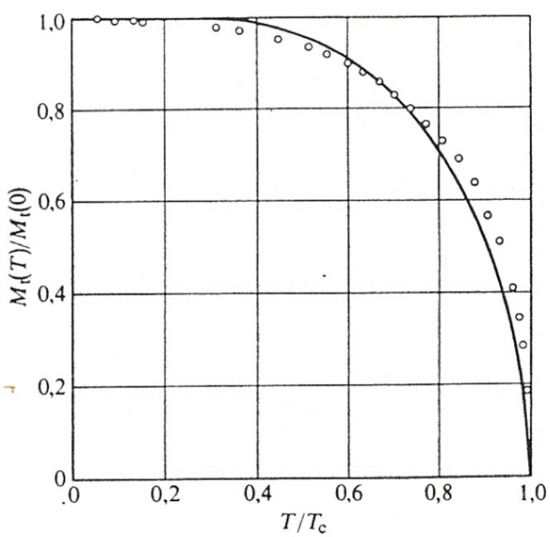
Applied magnetic field of $1T$ for different samples x

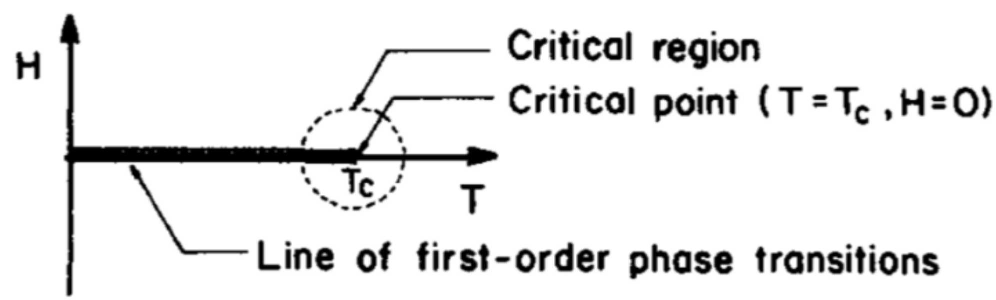
Gutiérrez, Olivares, Betancourt & Morales, J. Mat. Res. 24, 1585 (2009)

Magnetic phase diagram

Nickel (classical)

The magnetisation sharply vanishes at T_c *under no magnetic field* $H = 0$





Discontinuous jump from

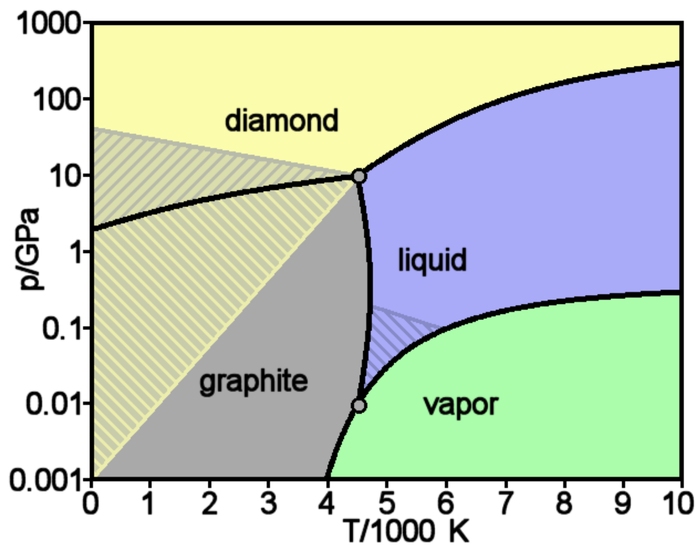
positive to negative magnetization

at $T < T_c$ *when* H *changes sign*

Simplest model for these features ? $\boxed{\textbf{Mean-field}}$

Structure phase diagram

Carbon (classical)



Theoretical phase diagram of carbon, which shows the state of matter for varying temperatures and pressures. The hatched regions indicate conditions under which one phase is metastable, so that two phases can coexist.

Wikipedia

Measurements

Magnetization^3 vs. temperature $m^3 \sim (T - T_c)^{3\beta} \Rightarrow \beta \sim 0.3$

Nuclear magnetic resonance in MnF4 near a critical point

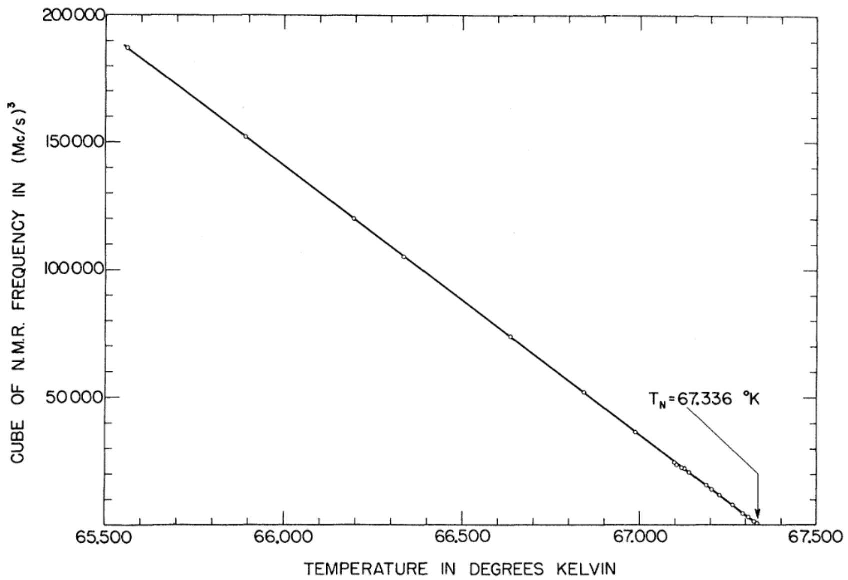


FIG. 1. Temperature dependence of the cube of the F^{19} nuclear resonance frequency for the first 1.8 degrees below T_N. The points lie on the straight line shown to within the experimental uncertainty of about 5 millidegrees.

Double linear plot

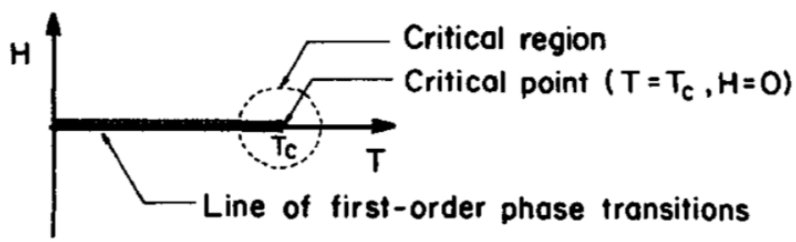
$$m^3 \sim c\,(T - T_c)$$

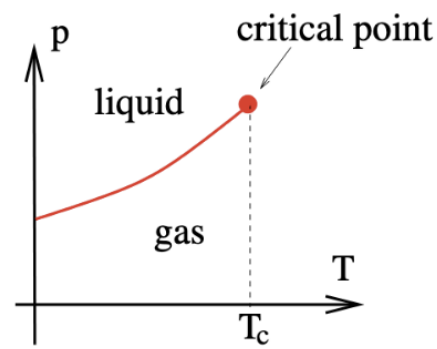
Mean-Field yields $\beta = 1/2$ **PROBLEM !**

P. Heller and G. B. Benedek, Phys. Rev. Lett. 8, 428 (1962)

Universality

Para-Ferro magnetism & Liquid-Gas





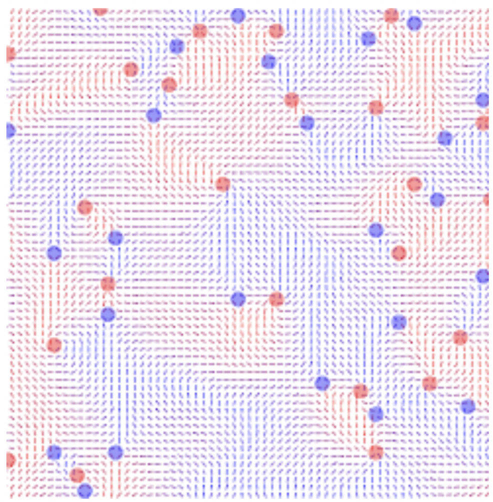
Similar phase diagrams

Very different variables and control parameters and, still, **same critical properties**

Renormalization Group

Topological phase transitions

Planar magnets



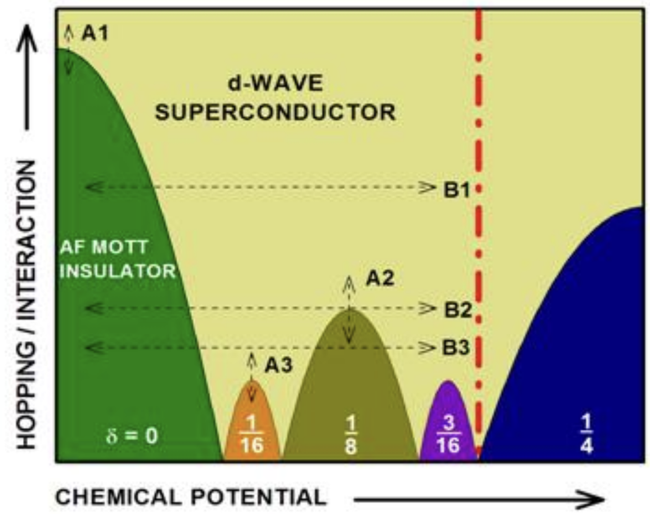
The Berezinskii-Kosterlitz-Thouless (BKT) transition is a phase transition from bound vortex-antivortex pairs at low temperatures to unpaired vortices and anti-vortices at some critical temperature realized by the two-dimensional $2d$XY model.
Also in Josephson junction arrays, thin superconducting films, ultracold atomic gases in 2d

Kosterlitz & Thouless, J. Phys. C : Solid State Phys. 5, L124 (1972)

Nobel 2016

Quantum Phase Diagram

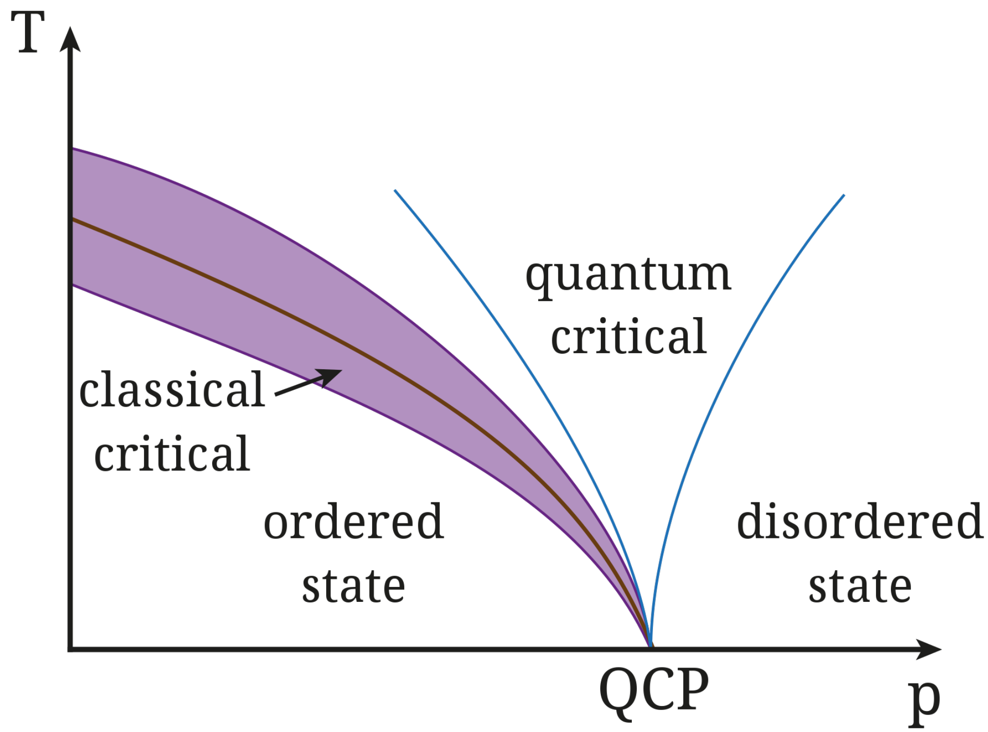
High T_c superconductors



... to propose a bosonic effective **quantum** Hamiltonian based on the projected SO(5) model with extended interactions, which can be derived from the microscopic models of the cuprates. The global phase diagram of this model is obtained using mean-field theory and Quantum Monte Carlo simulations ...

Quantum Phase Transitions

Occur at zero temperature

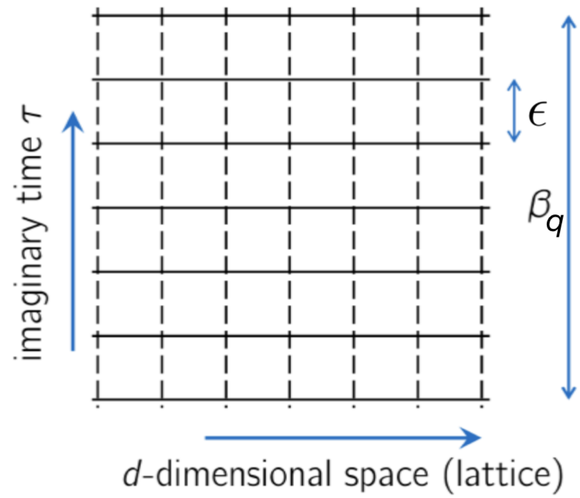


S. Sachdev, ***Quantum Phase Transitions*****, Harvard Univ. Press, 1999**

Quantum Phase Transitions

Methods

Functional methods **Quantum – classical mapping** **Spin chains**



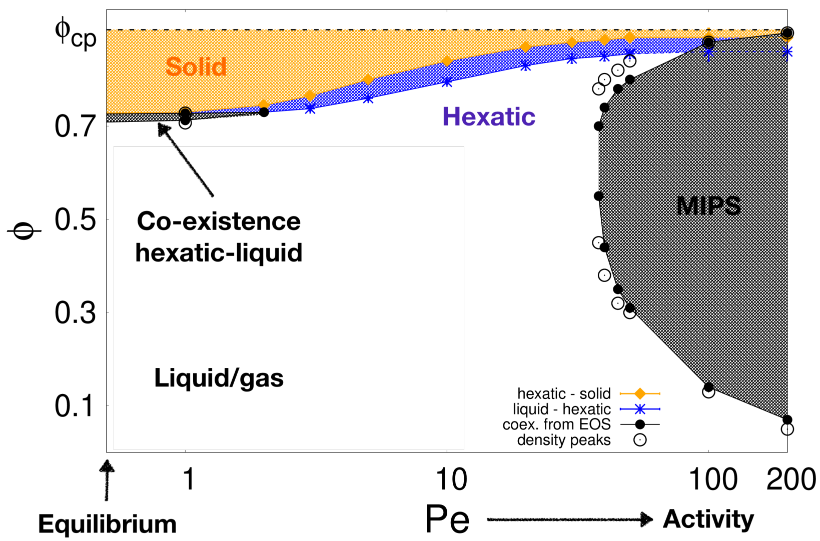
Generic properties

Transfer matrix

Beyond physics

Active Matter

Phase diagram with solid, hexatic, liquid, co-existence and MIPS



1st order **hexatic-liquid** close to Pe = 0

KT-HNY **solid-hexatic**

- universal dislocation unbinding

Breakdown of KT-HNY **hexatic-liquid** picture

- disclination unbinding within the **liquid** phase

- **percolation** of defect clusters in the liquid

Pressure $P(\phi, \mathrm{Pe})$ (EoS), correlations $G_T(r)$, $G_6(r)$, distributions of ϕ_i, and $|\psi_{6i}|$

defect identification & counting

Digregorio, Levis, Suma, LFC, Gonnella & Pagonabarraga, PRL 121, 098003 (2018)

Digregorio, Levis, LFC, Gonnella & Pagonabarraga, Soft Matter 18, 566 (2022)

Computer science

Algorithms & hard problems

Cristopher Moore (Santa Fe Institute, USA)

August 30th, 10:45AM EDT (Zoom meeting starts at 10:30 EDT)

The physics of inference, phase transitions, and networks

Finding patterns in data is a lot like finding ground states in physics. Each "state" corresponds to a hypothesis about the data, and the most-likely state is the one with the lowest energy. More generally, the Boltzmann distribution corresponds to the posterior distribution in Bayesian statistics. But reaching equilibrium can be hard, especially in glassy systems. We can get stuck for exponential time at local optima that have nothing to do with the true pattern, and are separated from the "correct" state by energy barriers. In many problems, this creates **phase transitions** where **finding patterns in noisy data suddenly becomes computationally hard or impossible**. These transitions occur when the amount of noise in the data –which is analogous to the temperature– crosses a critical threshold. I'll discuss these phase transitions using an example from the study of social networks, where we try to classify nodes according to which community they belong to.

Transition between **solvable** and **unsolvable**

Framework

In the course, focus on magnetic models

Why ?

Because

— they are simpler to define and work with

— thanks to universality, what you learn from them is applicable to other problems

— you will learn about bosons & fermions in other courses

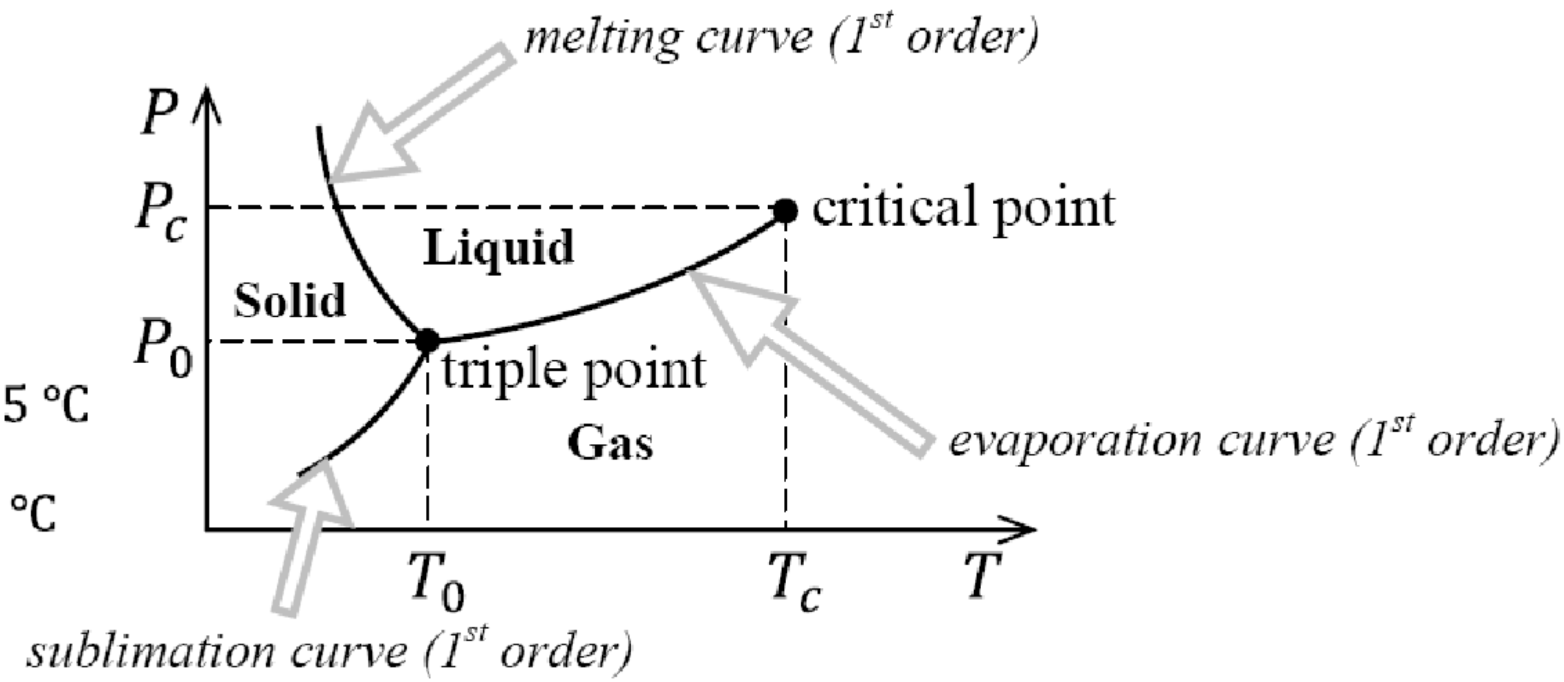
— we have only thirteen lectures & your background is heterogeneous

End of introductory part

Phase transitions

$T_0 = 0.0075\ °C$

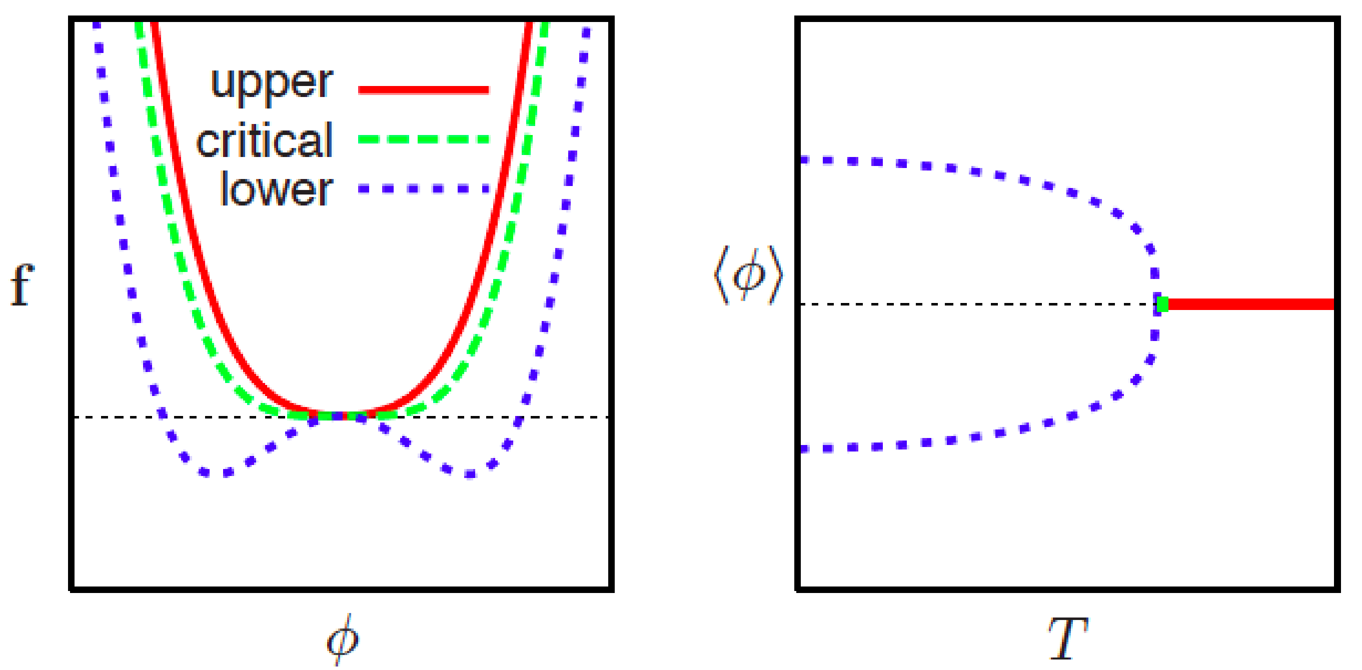
$T_c = 374.2\ °C$



Continuous phase-transition

Bi-valued equilibrium states related by symmetry

$\mathcal{F}$



Ginzburg-Landau free-energy

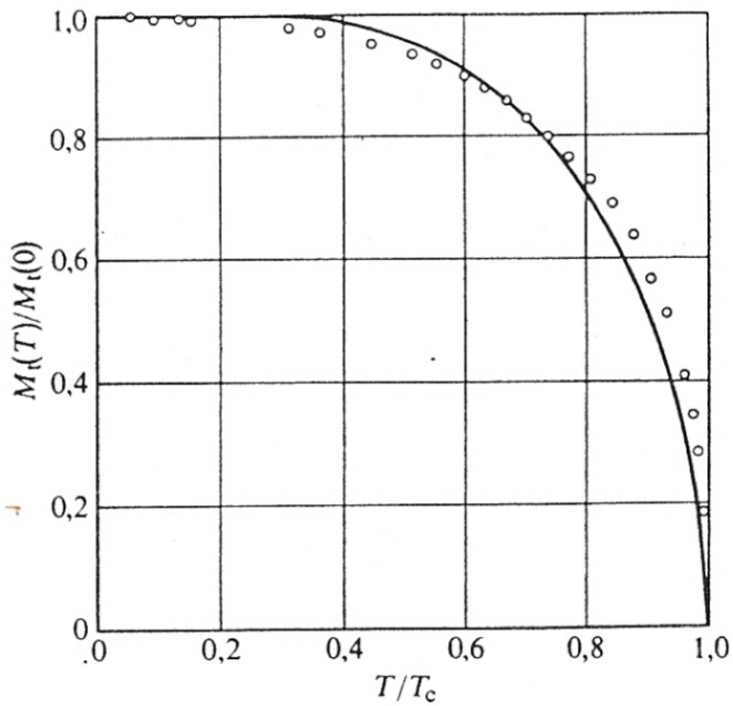
Scalar order parameter

e.g. Ising magnets at $h = 0$**,** $m = \langle s_i \rangle = \frac{1}{N}\sum_{k=1}^{N}\langle s_k\rangle$

Continuous phase-transition

Bi-valued equilibrium states related by symmetry

Aimantation et phénomène magnétocalorique du nickel



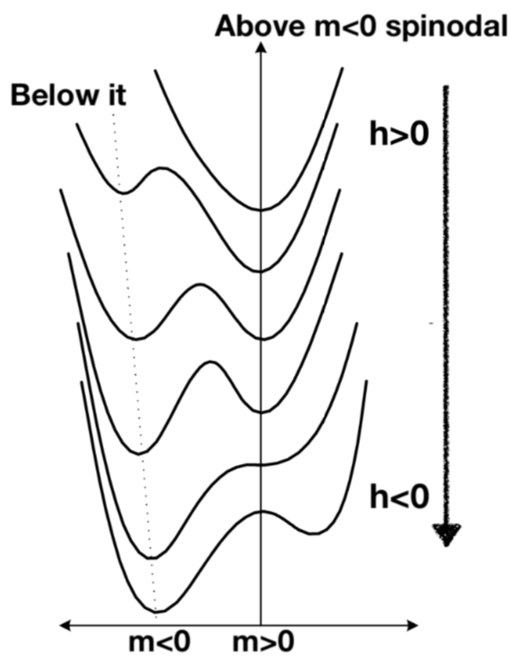
Nickel data vs. mean-field $m(T/T_c)$

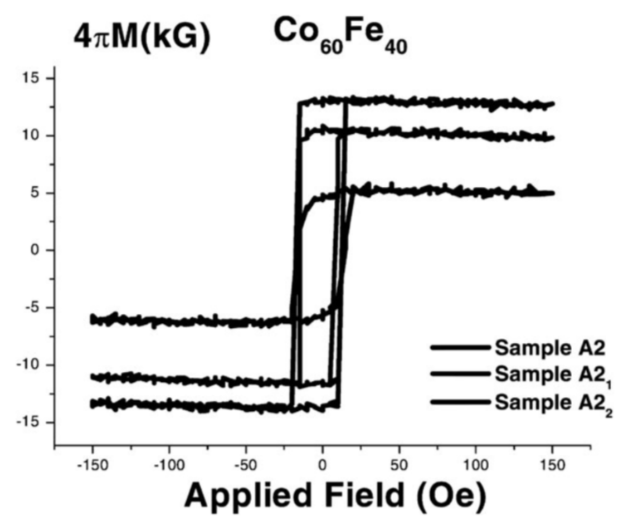
Weiss & Forrer, Ann. Phys. 5, 153 (1926)

Discontinuous phase transition

Hysteresis loops

Growth Rate Effects in Soft CoFe Films





M. Vopsaroiu, K. O'Grady, M. T. Georgieva, P. J. Grundy, and M. J. Thwaites,

IEEE Transactions on magnetics 41, 3253 (2005)

Curie-Weiss Mean-Field

Exponents for a continuous ($\mathbb{Z}_2$ spont symm broken) transition

$$
\begin{aligned}
-\beta f &= \beta J z \frac{m^2}{2} + \beta h m - \left(\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right) \\
&= -\beta J z \frac{m^2}{2} + \ln \left[2 \cosh \left(\beta J z m + \beta h \right) \right]
\end{aligned}
$$

$$m = \tanh \left(\beta J z m + \beta h \right)$$

$$t = (T - T_c)/T_c$$

	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto \lvert t \rvert^{-\alpha}$	$t \to 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^{\beta}$	$t \to 0^-, \quad h = 0$	$1/2$
Susceptibility	γ	$\chi \propto \lvert t \rvert^{-\gamma}$	$t \to 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto \lvert m \rvert^{\delta} \mathrm{sign}(m)$	$h \to 0, \quad t = 0$	3

Measurements

Magnetization^3 vs. temperature $m^3 \sim (T - T_c)^{3\beta} \Rightarrow \beta \sim 0.3$

Nuclear magnetic resonance in MnF4 near a critical point

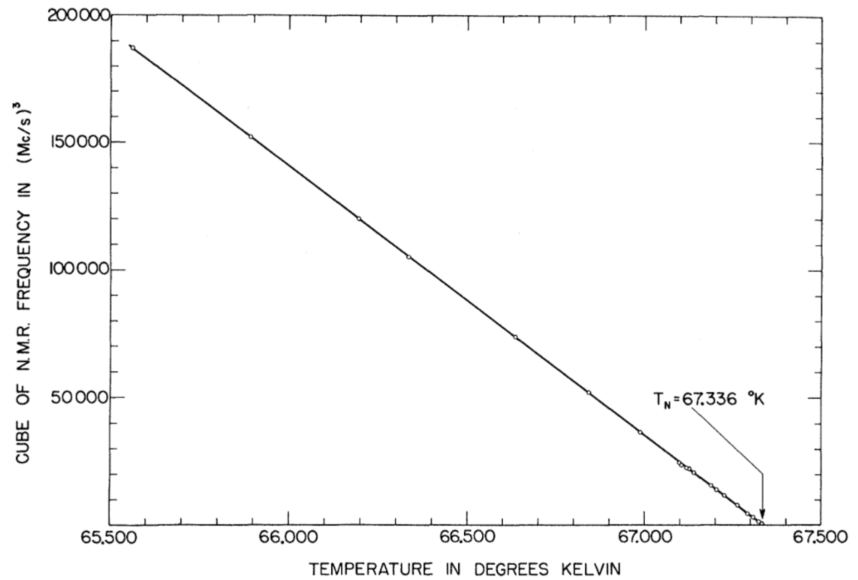


FIG. 1. Temperature dependence of the cube of the F^{19} nuclear resonance frequency for the first 1.8 degrees below T_N. The points lie on the straight line shown to within the experimental uncertainty of about 5 millidegrees.

Double linear plot

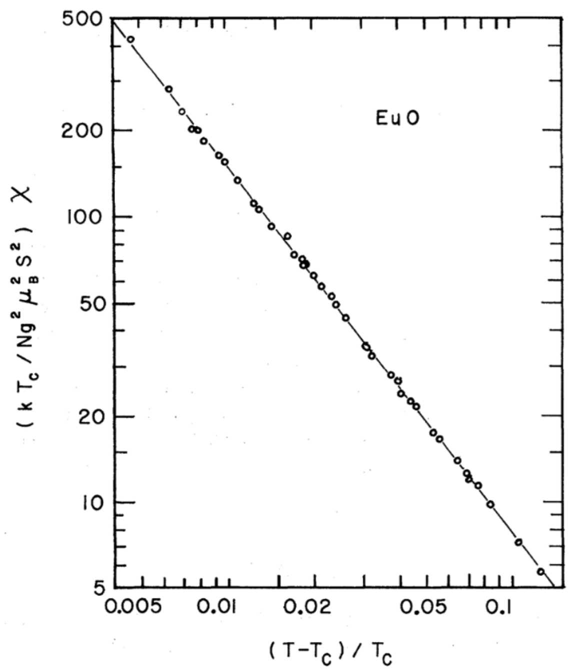
$$m^3 \sim c\,(T - T_c)$$

Mean-Field yields $\beta = 1/2$ **PROBLEM !**

P. Heller and G. B. Benedek, Phys. Rev. Lett. 8, 428 (1962)

Measurements

Magnetic susceptibility $\chi \sim (T_c - T)^{-1.3}$



Double log scale

$$\chi \sim \chi_0 \, (T - T_c)^{-\gamma}$$

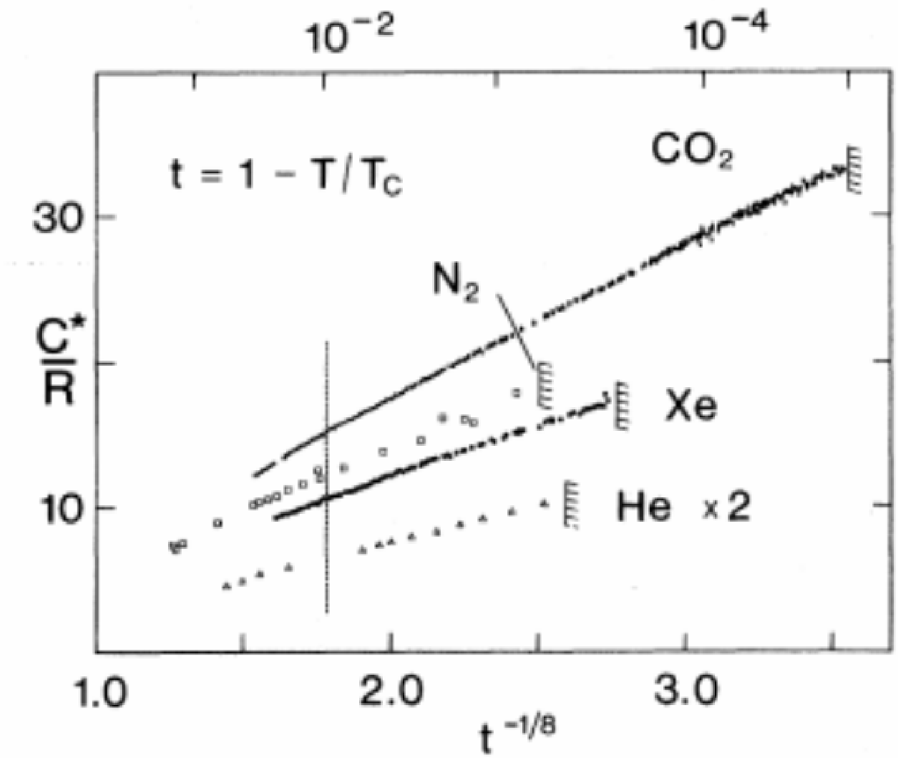
$$\underbrace{\ln \chi}_{y} \sim -\gamma \underbrace{\ln(T - T_c)}_{x} + \ln \chi_0$$

Mean-Field yields $\gamma = 1$

Measurements

Heat capacity vs reduced temperature$^{-1/8}$

Precision Measurement of the Specific Heat of CO_2 Near the Critical Point

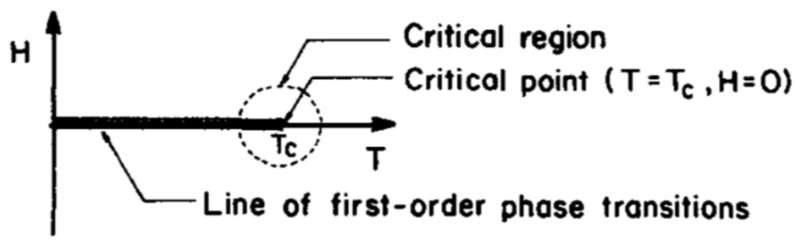


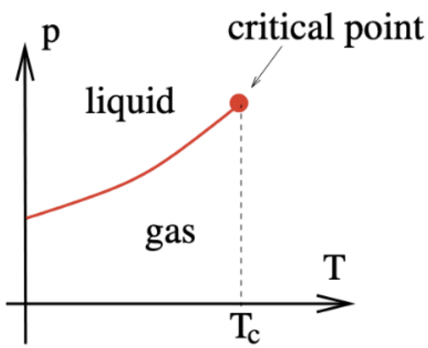
Mean-Field yields $\alpha = 0$

J. Lipa, C. Edwards, and M. Buckingham, Phys. Rev. Lett. 25, 1086 (1970)

Universality

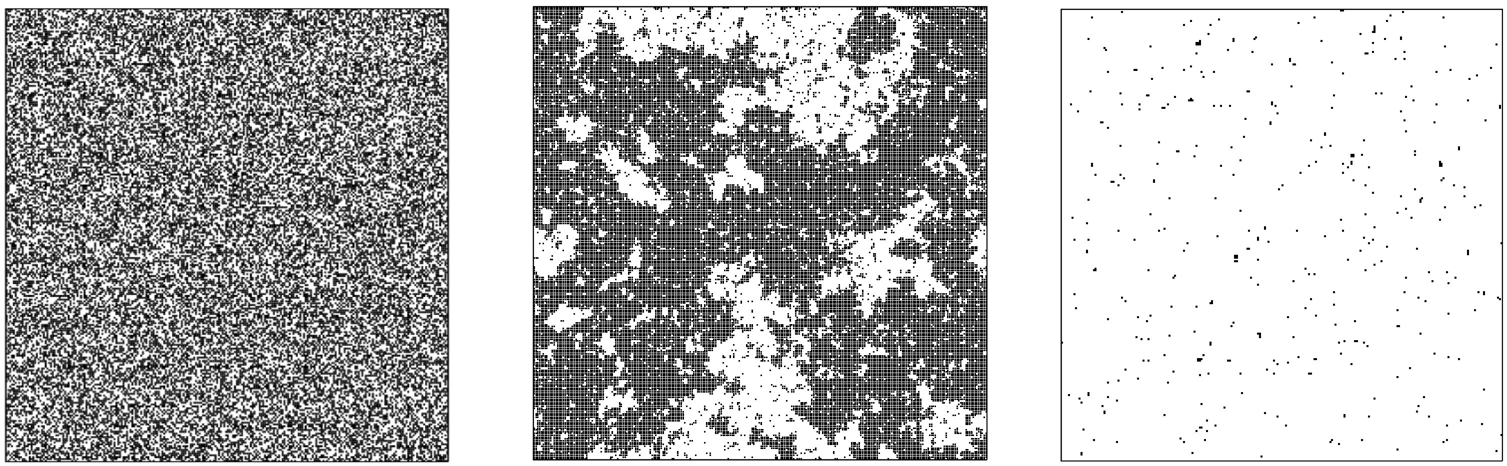
Para-Ferro magnetism & Liquid-Gas





Typical configurations

Up & down spins in a $2d$ Ising model



$T \to \infty$ $\qquad$ $T = T_c$ $\qquad$ $0 < T < T_c$

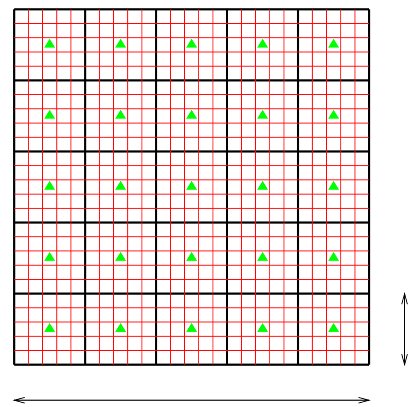
Real space viewpoint

Ginzburg-Landau

Continuous scalar statistical field theory

Coarse-grain the spin

$$\phi(\boldsymbol{r}) = n_{\boldsymbol{r}}^{-1} \sum_{i \in V_{\boldsymbol{r}}} s_i$$



The partition function is $\mathcal{Z} = \int \mathcal{D}\phi \; e^{-\beta \mathcal{F}(\phi)}$ with

$$\boxed{\mathcal{F}(\phi) = \int d^d r \; \left\{ \frac{1}{2} [\nabla \phi(\boldsymbol{r})]^2 + \frac{T - T_c}{2} \phi^2(\boldsymbol{r}) + \frac{\lambda}{4} \phi^4(\boldsymbol{r}) \right\}}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments)

Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\boldsymbol{r}) = \langle \phi(\boldsymbol{r}) \rangle = \phi_0$

The free-energy density is $\lim_{V \to \infty} f_V(\beta, J, g) = \lim_{V \to \infty} V^{-1} \mathcal{F}(\phi_0)$

Ginzburg-Landau

Exponents from the saddle-point analysis *vs.* finite d ones

	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto \|t\|^{-\alpha}$	$t \to 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^{\beta}$	$t \to 0^-, \quad h = 0$	$1/2$
Susceptibility	γ	$\chi \propto \|t\|^{-\gamma}$	$t \to 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto \|m\|^{\delta} \mathrm{sign}(m)$	$h \to 0, \quad t = 0$	3
Correlation length	ν	$\xi \propto \|t\|^{-\nu}$	$t \to 0, \quad h = 0$	$1/2$
Correlation function	η	$G(\mathbf{r}) \propto \|\mathbf{r}\|^{-d+2-\eta}$	$r = 0, \quad h = 0$	0

d	β	$\alpha = \alpha'$	$\gamma = \gamma'$	δ	ν	η	
2	$1/8$	0	$7/4$	15	1	$1/4$	exact
3	0.325	0.11	1.24	4.82	0.63	0.032	approx
MF	$1/2$	0	1	3	$1/2$	0	exact

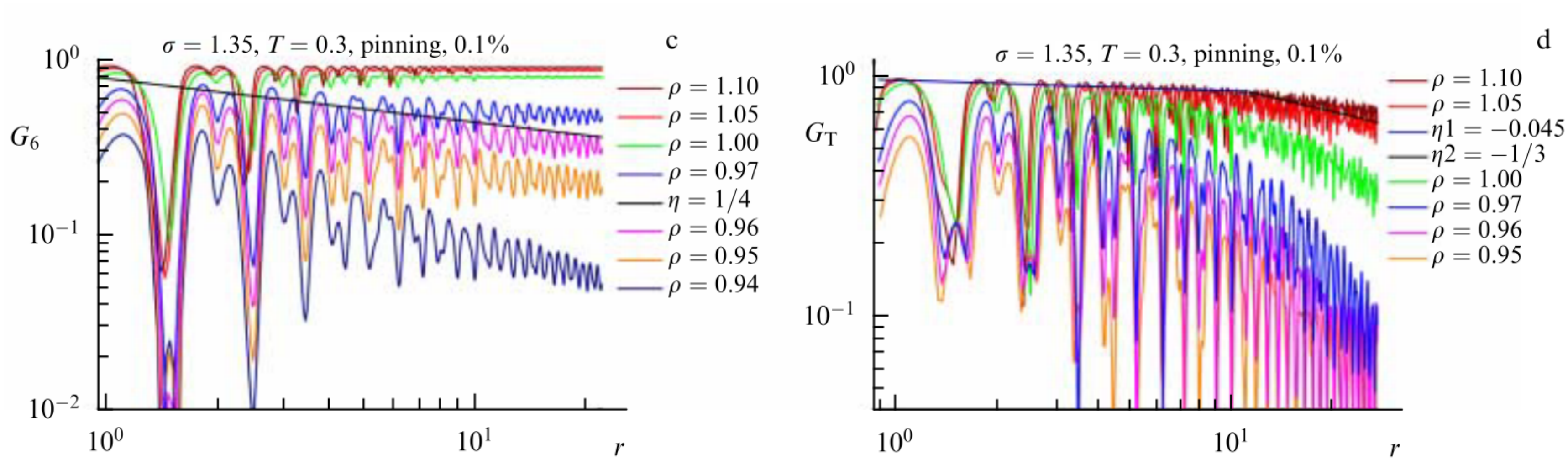
NB The mean-field exponents are independent of d, which is incorrect for $d < d_u$

Ginzburg-Landau does not detect the absence of a $T > 0$ phase transition in $d = 1$

(remember Peierls) but yields the suspicious result $C(r) \sim r^{2-d} = r$ at T_c

Correlation functions

Melting: from solid to liquid in $2d$



straight lines　　　curved

Translational order correlation functions G_T: exponential decay $e^{-r/\xi}$

Orientational order correlation functions G_6: power law decay $r^{2-d-\eta}$ - critical

Correlation functions

Atomic gases in $2d$ - Stochastic Gross-Pitaevskii equation

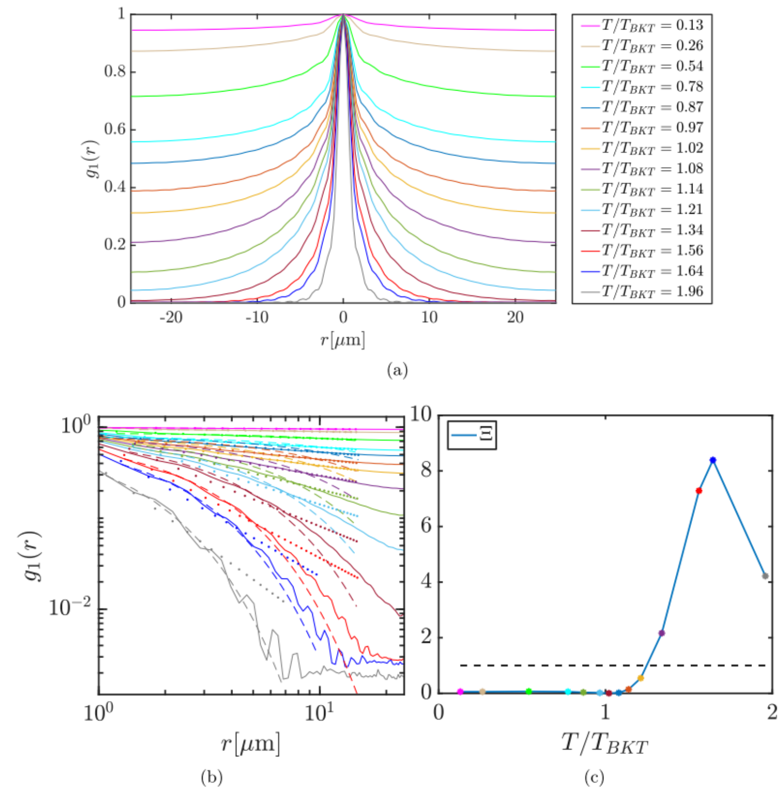
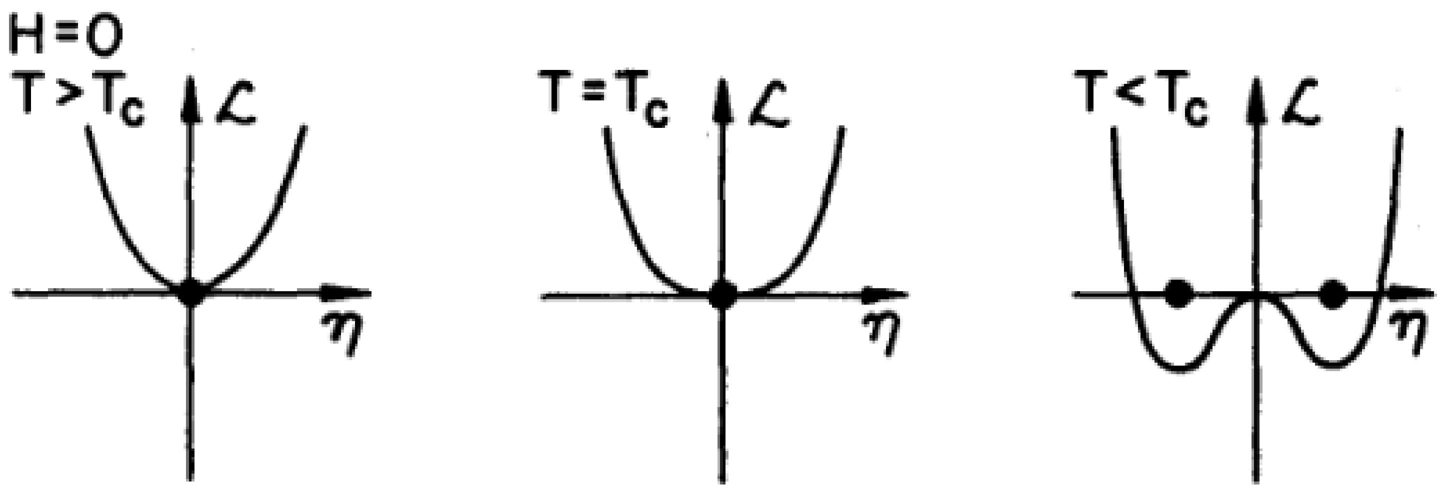


Figure 4.7: (a) Equilibrium profiles for the first-order correlation function $g_1(r)$ for different temperatures. (b) Correlation function profiles as in the previous figure, in logarithmic scale. The fitted algebraic (dotted lines) and exponential (dashed lines) functions are defined as in (4.56). (c) Ratio Ξ between the mean squared errors for the fits in (b), as defined in (4.57). The dashed line expresses the value of Ξ for which the two fits apply equally well to the data. The resulting critical point presents a shift of about ~ 0.25 with respect to the theoretical relation for T_{BKT}, equation (4.41).

F. Larcher, *Dynamical excitations in low-dimensional condensates: sound, vortices and quenched dynamics*, PhD thesis, Newcastle University & Università di Trento (2018)

Landau theory

Second order phase transition

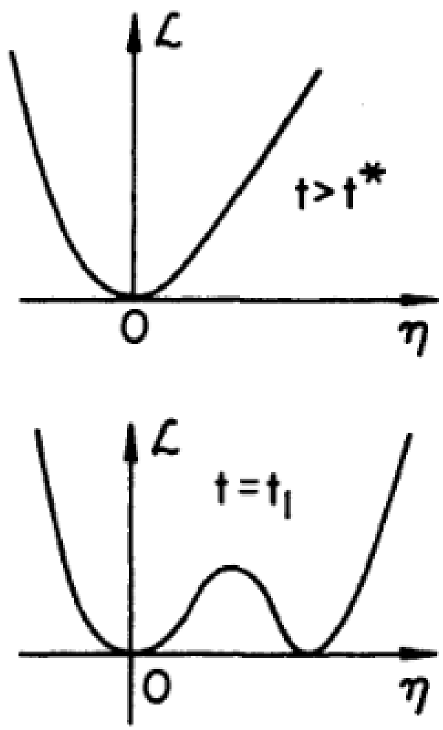


Notation : $\mathcal{L}$ is the “potential” in the Landau free-energy density

Notation : η is the “field”

Landau theory

First order phase transitions



Notation : $\mathcal{L}$ is the "potential" in the Landau free-energy density, η is the "field", t_I is the transition

Landau theory

Important points

- The actual values of the parameters are not important

  they are material/model dependent

- The dimension of the order parameter (scalar, vectorial) and the potential

  decide whether the transition is second, first order or infinite order

- In 2nd order phase transitions the correlation length diverges & the linear

  susceptibility as well.

- In continuous phase transitions the exponents are universal

- The strength of the Gaussian fluctuations limit the validity of the saddle-point

  treatment of the Landau theory.

  upper critical dimension, critical region $\xi^{4-d} < 1$ for the scalar $\lambda\phi^4$ theory

Critical Scaling

Singular part of the free-energy density

Singular part of the free-energy density

$$f_{\text{sing}}(|t|, h) \sim |t|^{2-\alpha}\, g_f\left(\frac{h}{|t|^{\Delta}}\right)$$

Limits of the scaling function

$$g_f(y=0) = \text{ct} \quad \Longrightarrow \quad f_{\text{sing}}(|t|, 0) \sim |t|^{2-\alpha}$$

$$g_f(y \to \infty) = y^x \quad \Longrightarrow \quad f_{\text{sing}}(|t| = 0, h) \sim h^x\, |t|^{2-\alpha-\Delta x}$$

$$f_{\text{sing}}(|t| = 0, h) \sim h^x$$

$$2 - \alpha - \Delta x = 0 \;\Longrightarrow$$

$$\Delta = (2-\alpha)/x$$

Example Curie-Weiss

$$x = 4/3 \qquad \alpha = 0 \qquad \Delta = 3/2$$

Critical Scaling

Relations between exponents

Widom's identity	$\delta - 1 = \gamma/\beta$
Rushbrooke's identity	$\alpha + 2\beta + \gamma = 2$
Josephson's identity	$2 - \alpha = d\nu$
From susceptibility	$\gamma = \nu(2 - \eta)$

Only two are independent

Critical Scaling

Ferromagnetic transition

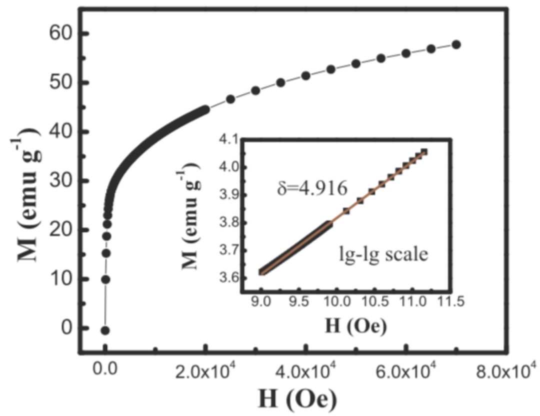


Fig. 4. Isothermal M vs H plot of $Nd_{0.55}Sr_{0.45}Mn_{0.98}Ga_{0.02}O_3$ at T_C = 247 K; the inset shows the same plot in log–log scale and the solid line is the linear fit following Eq. (3).

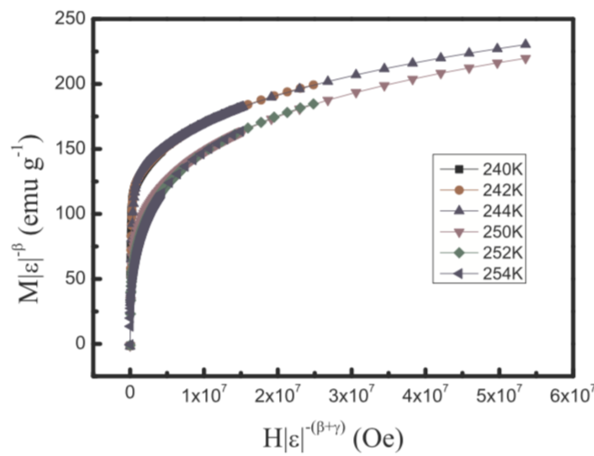


Fig. 5. Scaling plots of renormalized magnetization M vs. renormalized field H above and below T_C.

Bo Yua *et al*, *Scaling study of magnetic phase transition and critical behavior in* $Nd_{0.55}Sr_{0.45}Mn_{0.98}Ga_{0.02}O_3$ *manganite*, Materials Research Bulletin 99, 393 (2018)

Critical Scaling

Jamming transition

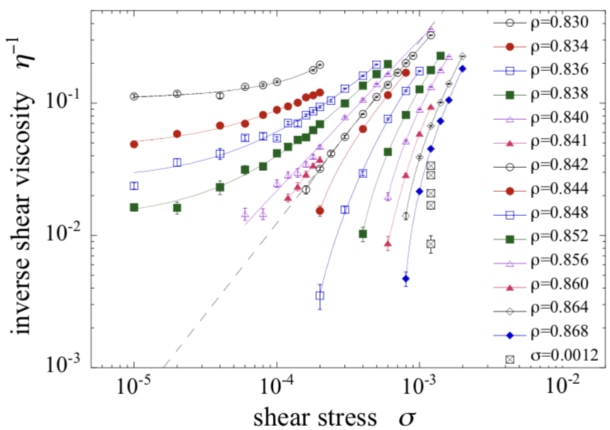


FIG. 2: (color online) Plot of inverse shear viscosity η^{-1} vs applied shear stress σ for several different values of the volume density ρ. The dashed line represents the power law dependence expected exactly at $\rho = \rho_c$ and has a slope $\beta/\Delta = 1.375$. Solid lines are guides to the eye. Points labeled $\sigma = 0.0012$ correspond to densities $\rho = 0.870$, 0.872, 0.874, 0.876, and 0.878.

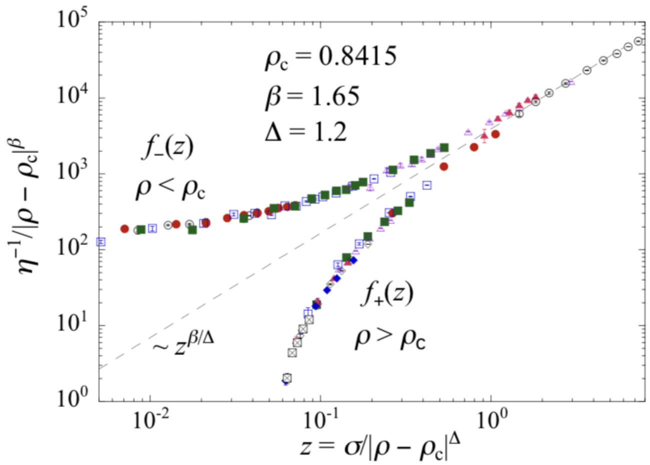


FIG. 3: (color online) Plot of scaled inverse viscosity $\eta^{-1}/|\rho - \rho_c|^{\beta}$ vs scaled shear stress $z \equiv \sigma/|\rho - \rho_c|^{\Delta}$ for the data of Fig. 2. We find an excellent collapse to the scaling form of Eq. (6) using values $\rho_c = 0.8415$, $\beta = 1.65$ and $\Delta = 1.2$. The dashed line represents the large z asymptotic dependence, $\sim z^{\beta/\Delta}$. Data point symbols correspond to those used in Fig. 2.

P. Olsson and S. Teitel, *Critical Scaling of Shear Viscosity at the Jamming Transition*, Phys. Rev. Lett. 99, 178001 (2007).

Critical Scaling

Jamming transition

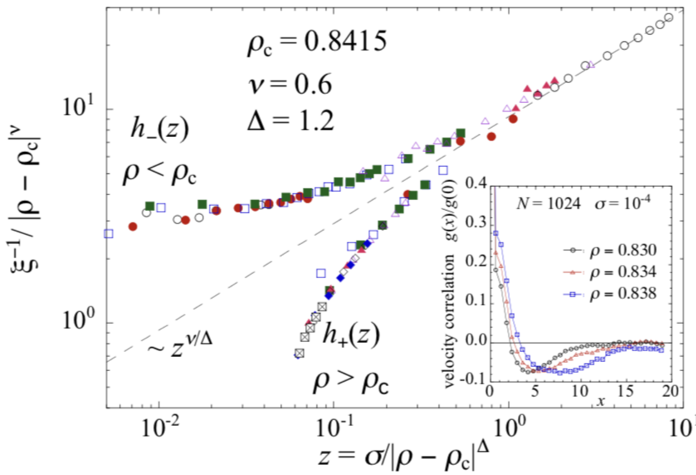


FIG. 4: (color online) Inset: Normalized transverse velocity correlation function $g(x)/g(0)$ vs longitudinal position x for $N = 1024$ particles, applied shear stress $\sigma = 10^{-4}$, and volume densities $\rho = 0.830$, 0.834 and 0.838. The position of the minimum determines the correlation length ξ. Main figure: Plot of scaled inverse correlation length $\xi^{-1}/|\rho-\rho_c|^\nu$ vs scaled shear stress $z \equiv \sigma/|\rho-\rho_c|^\Delta$ for the data of Fig. 2. We find a good scaling collapse using values $\rho_c = 0.8415$, $\Delta = 1.2$ (the same as in Fig. 3) and $\nu = 0.6$. Data point symbols correspond to those used in Fig. 2.

P. Olsson and S. Teitel, *Critical Scaling of Shear Viscosity at the Jamming Transition*, Phys. Rev. Lett. 99, 178001 (2007).

Finite size effects

Rounding of magnetization curves

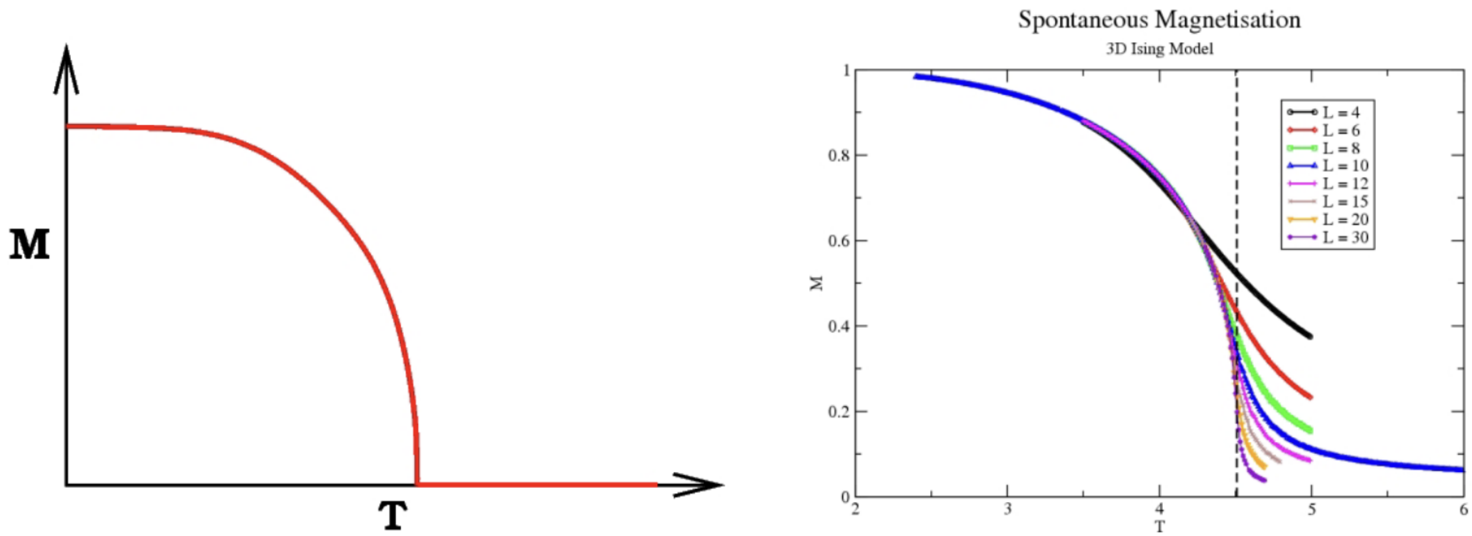
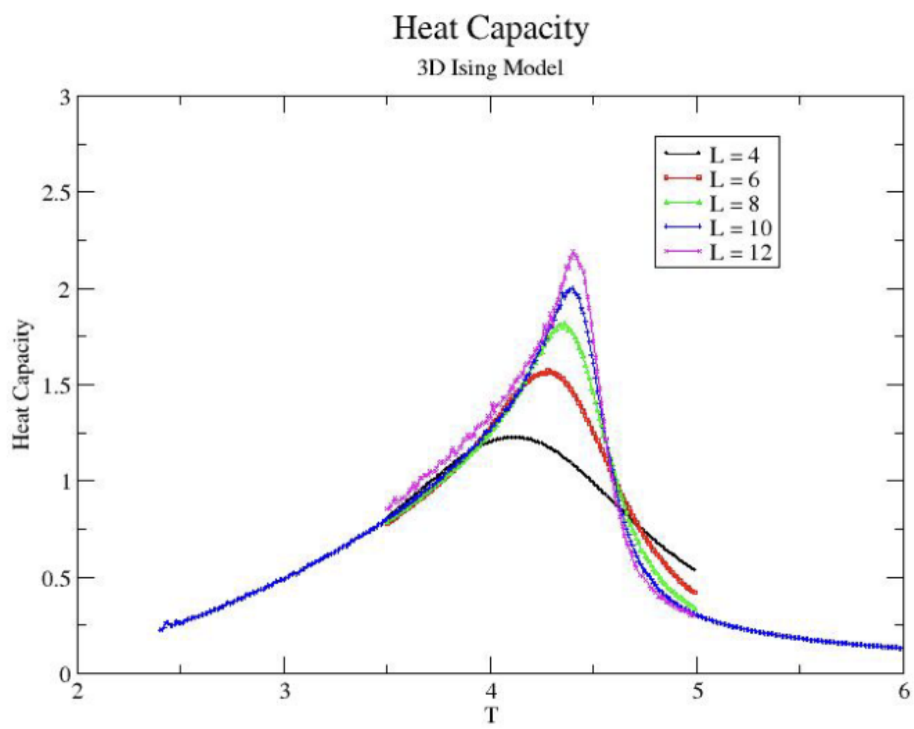


Figure: In TD limit the order parameter is 0 for $T > T_c$ but in FS the transition is smeared out.

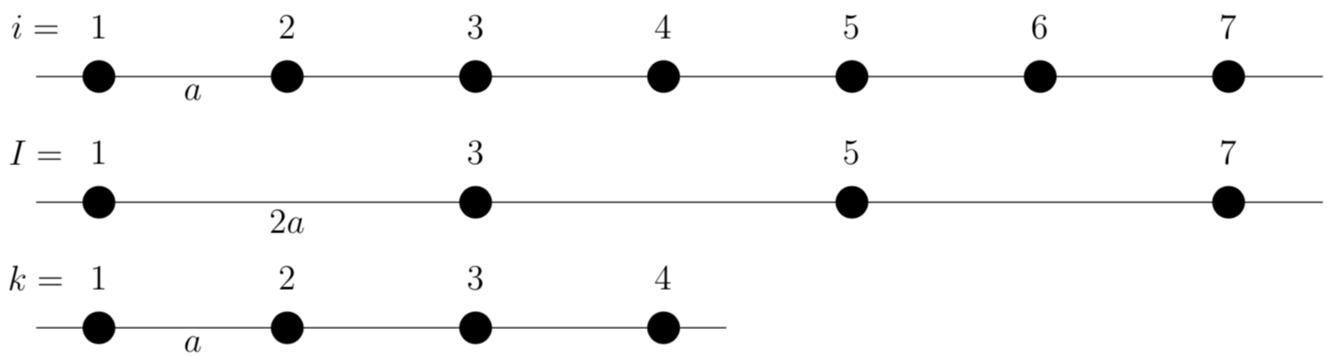
Finite size effects

Rounding of the heat capacity



Decimation

1d Ising chain $K = \beta J, \;\; \zeta = \ln \mathcal{Z}$



$$K' = \tfrac{1}{2}\ln\cosh(2K) \to 0 \quad \text{that is} \quad (T/J)^* \to \infty$$

$$\boxed{\zeta(K') = 2\zeta(K) - \ln[2(\cosh(2K))^{1/2}]}$$

Decimation : the killing of one in every ten of a group of people as a punishment for the whole group (originally with reference to a mutinous Roman legion)

Decimation

1d Ising chain $K = \beta J, \quad \zeta = \ln \mathcal{Z}$ **inverse relations**

$$\boxed{K = \frac{1}{2}\cosh^{-1}(\exp(2K'))\ , \qquad \zeta(K) = \frac{1}{2}\ln 2 + \frac{1}{2}K' + \frac{1}{2}\zeta(K')\ ,}$$

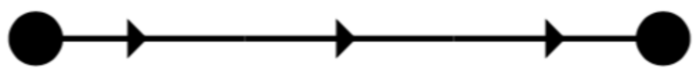
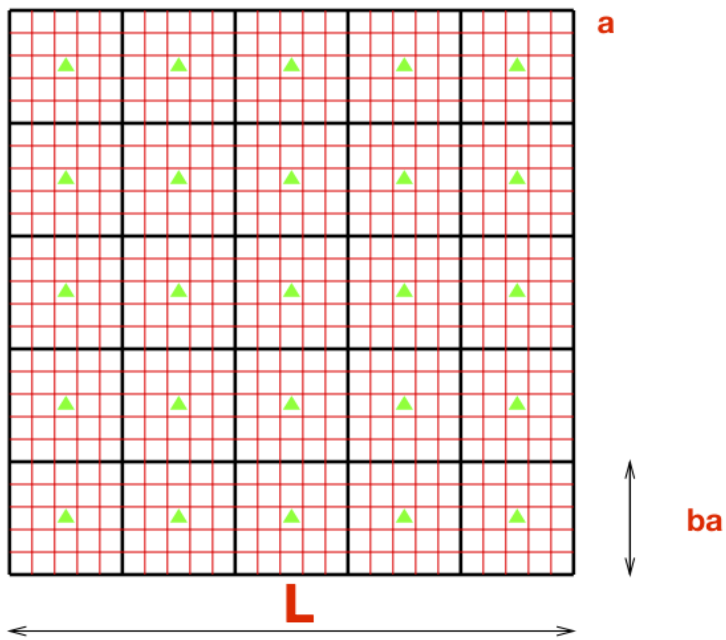
$K' = 0$  $K \to \infty$ $\quad T \to 0$

Table I. Values of ζ for the 1D Ising model calculated from the recursion formulas Eqs. (31) and (32) of the renormalization group and from the exact formula derived from Eq. (9). ζ is related to the partition function Z through Eq. (28).

K	$\zeta(K)$ Renormalization group	Exact
0.01	ln 2	0.693 197
0.100 334	0.698 147	0.698 172
0.327 447	0.745 814	0.745 827
0.636 247	0.883 204	0.883 210
0.972 710	1.106 299	1.106 302
1.316 710	1.386 078	1.386 080
1.662 637	1.697 968	1.697 968
2.009 049	2.026 876	2.026 877
2.355 582	2.364 536	2.364 537
2.702 146	2.706 633	2.706 634

RG

From block spins to coupling renormalization



degrees of freedom

$b^d \mapsto 1$

lattice spacing

$a \mapsto ba$

re-scaling of lengths

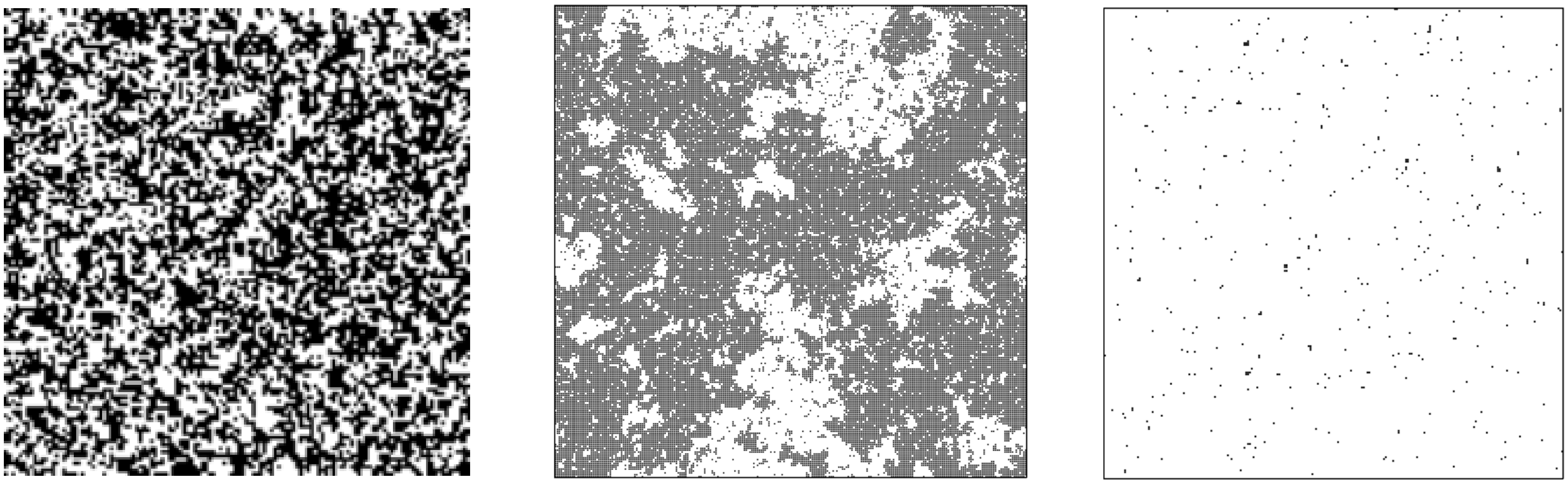
$\vec{x} \mapsto \vec{x}/b$

From $\mathcal{H}'_{[K']}(\{s'_I\}) = R_b \mathcal{H}_{[K]}(\{s_i\})$ to

$$K'_\alpha = \sum_\beta (R_b)_{\alpha\beta} K_\beta \quad \text{close to} \quad [K_c] = [0]$$

Typical configurations

Up & down spins in a $2d$ Ising model



$T = 2T_c$ $\qquad$ $T = T_c$ $\qquad$ $0 < T < T_c$

Real space viewpoint

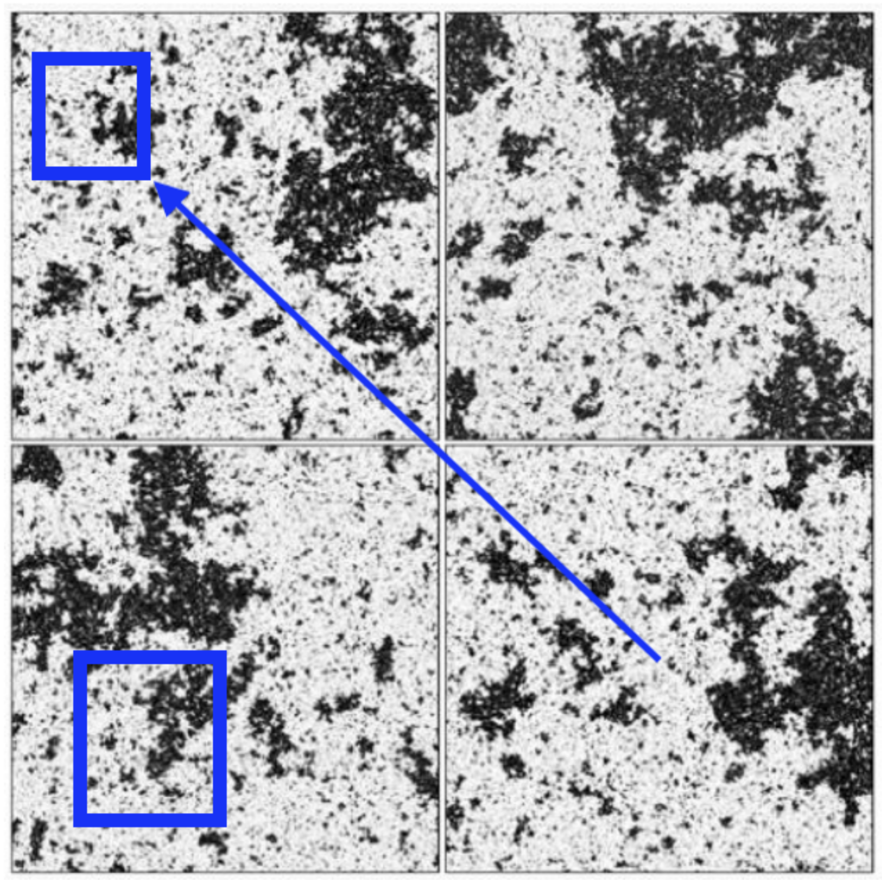
Zoom out by observing at larger scales

move away from criticality if $T \neq T_c$

$[K_c]$ repulsive fixed points

Scale invariance

2d Ising model at T_c



D. Ashton, *Scale invariance in the critical Ising model*,

https://www.youtube.com/watch?v=fi-g2ET97W8

RG

finite d Ising model

Two adimensional parameters in $\mathcal{H}_{[K]}(\{s_i\})$

$$K_1 = \beta J = K \qquad\qquad K_2 = \beta h = K$$

Composition $R(b_1 b_2) = R(b_1) R(b_2)$ and symmetries

$$K' = b^{y_K} K \qquad\qquad H' = b^{y_H} H$$

Repulsive fixed point

$$y_K > 0 \qquad\qquad y_H > 0$$

RG

finite d Ising model

Identity of total free-energy $\Longrightarrow$ homogeneity

$$F(K', H') = F(K, H) \qquad \Longrightarrow \qquad b^{-d} f(b^{y_K} H, b^{y_H} H)$$

choosing $b = |K|^{-1/y_K}$ $\Longrightarrow$ scaling

$$\begin{aligned} f(K, H) &= |K|^{d/y_K} \; f(1, |K|^{-y_H/y_K} H) \\ &= |K|^{d/y_K} \; g_f \left(\frac{H}{|K|^{y_H/y_K}} \right) \end{aligned}$$

with $\Delta = y_H / y_K$ and $2 - \alpha = d/y_K$

RG

Generic

Transformation

$$K'_\alpha - K^*_\alpha = \sum_\beta R(b)_{\alpha\beta}\left(K_\beta - K^*_\beta\right)$$

in other terms $\quad R(b)_{\alpha\beta} = \left.\dfrac{\partial K'_\alpha}{\partial K_\beta}\right|_{[K^*]}$

To solve one needs to diagonalize the linear (vectorial) relation above

Use the

eigenvalues λ^i and left orthonormal eigenvectors u^i_α of the matrix $\mathbb{R}(b)$

$$\sum_j u^j_\beta u^j_\gamma = \delta_{\beta\gamma}$$

RG

Generic

Transformation

$$\delta K_\alpha = K'_\alpha - K^*_\alpha = \sum_\beta R(b)_{\alpha\beta}\,(K_\beta - K^*_\beta) = \sum_\beta R(b)_{\alpha\beta}\,\delta K_\beta$$

multiply by the left eigenvector u^i_α and sum over α

$$\begin{aligned}
\delta\kappa'^{\,i} \equiv \sum_\alpha u^i_\alpha\,\delta K'_\alpha \quad &= \quad \underbrace{\sum_\alpha u^i_\alpha \sum_\beta R(b)_{\alpha\beta}}_{\lambda^i \sum_\beta u^i_\beta} \sum_\gamma \underbrace{(\sum_j u^j_\beta u^j_\gamma)}_{\delta_{\beta\gamma}}\,\delta K_\gamma \\
&= \quad \lambda^i \sum_j \underbrace{\sum_\beta u^i_\beta u^j_\beta}_{\delta^{ij}} \sum_\gamma u^j_\gamma \delta K_\gamma \\
&= \quad \lambda^i \sum_\gamma u^i_\gamma\,\delta K_\gamma = \lambda^i\,\delta\kappa^i
\end{aligned}$$

RG

Generic

Diagonalized transformation

$$\delta\kappa'^{\,i} \equiv \sum_{\alpha} u_{\alpha}^{i}\, \delta K'_{\alpha} = \lambda^{i}\, \delta\kappa^{i}$$

thus

$$\lambda^{i} = \left.\frac{\partial \delta\kappa'^{\,i}}{\partial \delta\kappa^{\,i}}\right|_{[K^*]} = \left.\frac{\partial \kappa'^{\,i}}{\partial \kappa^{\,i}}\right|_{[K^*]}$$

and

$$\lambda^{i} = b^{y_i} \qquad \Longrightarrow \qquad y_i = \frac{d \ln \lambda^{i}}{db}$$

RG

Generic

Parameters

- $y_i > 0$ – *relevant*, with the corresponding $\kappa_i(b)$ coupling growing under coarse-graining, i.e., getting away from the fixed point
- $y_i < 0$ – *irrelevant*, with the $\kappa_i(b)$ coupling vanishing under coarse- graining, i.e., approaching the fixed point
- $y_i = 0$ – *marginal*, with the flow of the $\kappa_i(b)$ coupling determined by the higher order terms in the couplings of the RG transformation.

Concepts

The scale of observation does not modify the properties of the critical state

nor the disordered or order character of the phases

Kadanoff blocks modify the physical state by coarse-graining but,

the modern RG (Wilson) transforms the form of the model until reaching the one that describes the critical state

At criticality and close to it the microscopic details are not important

The critical points are repulsive fixed points of the RG

Above the critical temperature, evolution to the free fixed point, $T \to \infty$

Below the critical temperature, evolution to $T = 0$

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At criticality and close to it the microscopic details are not important

A renormalisation transformation is a scale transformation that leaves the partition function invariant. Since the thermodynamic properties of a system are governed by the partition function, the physics is preserved.

Concepts

The critical behaviour does not depend on the type of subcritical order but on the number n of components of the order parameter and the dimension of space d.

Renormalisation enables the classification in universality classes with the same critical properties, depending on (n, d)

Scaling laws apply to global, macroscopic properties of systems containing a large number of elementary microscopic units.

They are found in other domains, *e.g.* finance, biology, etc.

Exponents again

Comparison

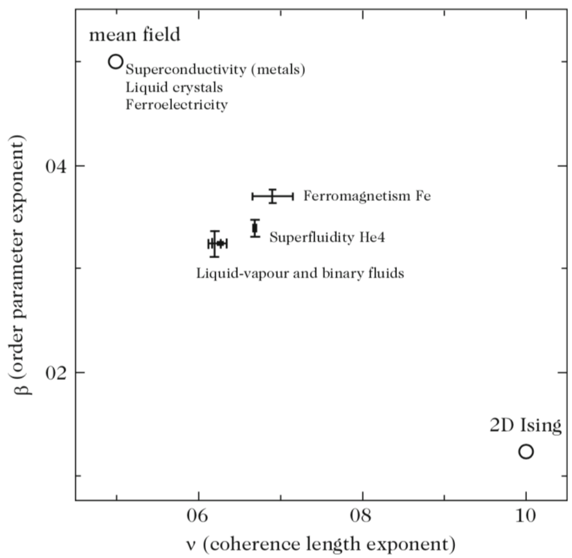

Fig. 1.22 Critical exponents β and ν: experimental results obtained for seven difference families of transitions, compared to values predicted by the mean field and 2D Ising models

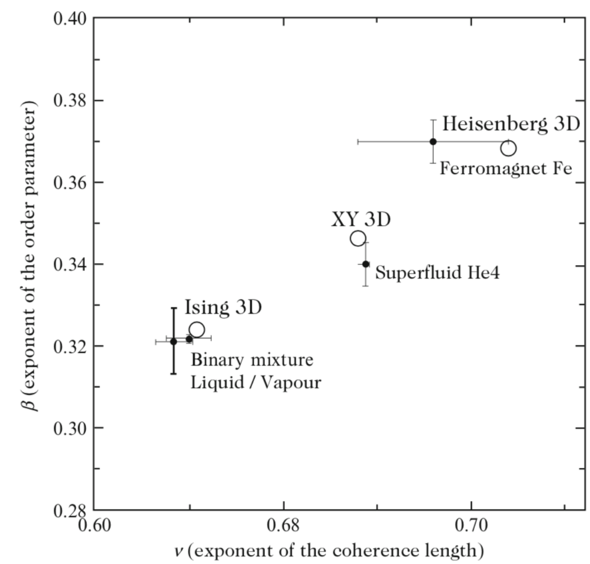

Fig. 1.23 Critical exponents measured for four families of transitions, compared to values predicted by the three corresponding models that take into account the scale invariance of the critical state

Images from

A. Lesne & M. Lagües, *Scale invariance from phase transitions to turbulence* (Springer-Verlag, 2012)