

Advanced Statistical Physics

Exam

Duration: 3h

January 2026

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English or French, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but also the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The answers must be written neatly within the boxes.

The problems roughly follow the order of the chapters in the Lecture Notes but are not necessarily of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

1 Ergodic Hypothesis 2 pt

Figure 1 shows the relative sunshine duration – defined as the fraction of the possible daylight time during which the Sun actually shines and is not covered by clouds – on the island of Lastovo (Croatia) during the month of March for five different years. The light coloured data-points correspond to the daily measurements for each of the five years indicated in the legend. The dark blue points represent the daily values averaged over these five years, and the thick blue segments simply connect these averaged points. Ignore the black horizontal line.

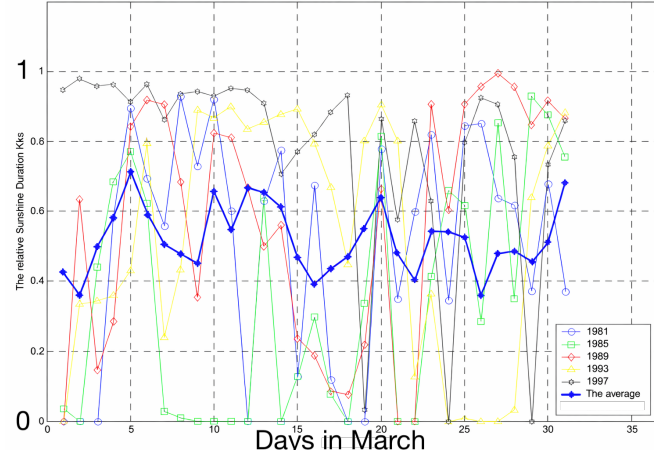


Figure 1: Relative sunshine duration per day in March. See the text for an explanation. Figure adapted from M. Perić et al., Stochastic Model of Irradiation for the Analysis of Solar System, CD Proceedings of the 11th Mediterranean Conference on Control and Automation (MED'03).

1. How would you check whether the process is ergodic? 1 pt

The data are very noisy and not that numerous (31 days for each time series and only 5 time series).

It is hard to make good averages.

What we want to check is whether the time average (for one typical year, say the α one here) is equal to the ensemble average (at any typical day, say the t_j th one here) over the five years.

$$\overline{K_{\alpha}} \equiv \frac{1}{31} \sum_{i=1}^{31} K_{\alpha}(t_i) \stackrel{?}{=} \frac{1}{5} \sum_{\gamma=1}^5 K_{\gamma}(t_j) \equiv \langle K \rangle$$

Here i label the times, and α labels the years. α is a typical year and t_j is a typical day. The result should be independent of α and j .

2. Would you consider the process generating these time series to be ergodic? Justify your answer. **1 pt**

If I look at the green data, for 1985, and I average the data at each day of the month (over time) I get a result which is very close to the blue data points which are the ensemble averages at each day. So, although very noisy and hard to be sure, I could consider the process to be ergodic.

2 Conceptual questions

Answer the questions in this Section conceptually, without developing technical details.

1. Define and distinguish the following terms: **10pt**
- (a) Explain what is the difference of the behaviour of the order parameter at a first and a second order phase transition. **1pt**

It is discontinuous (jumps) at a 1st order phase transitions and it is continuous in a 2nd order one.

(b) Spontaneous symmetry breaking. **2pt**

Spontaneous symmetry breaking is the phenomenon in which the underlying laws of a system are symmetric, but the system's equilibrium state selects one of several asymmetric configurations.

It can be triggered by a fluctuation or an external perturbation

(c) Correlation length. **2pt**

The correlation length is the characteristic distance over which fluctuations in a system remain significantly correlated before becoming effectively independent.

(d) Universality class. **2pt**

A universality class is a category of systems that share the same critical behavior – such as critical exponents and scaling functions – despite having different microscopic details.

They classify 2nd order transitions

- (e) Give one example of a classical and one example of a quantum phase transition, stating the simplest order parameter and the control parameter in each case. **3pt**

The classical ferromagnetic Ising model in $d \geq 2$ has a finite T/J phase transition with the magnetization density as the order parameter and, in canonical conditions, the temperature of the bath as the control parameter.

The quantum, transverse field (Γ) ferromagnetic ($J > 0$) Ising model, at $d \geq 1$ has a zero temperature phase transition controlled by Γ/J . The order parameter is the also the global magnetization in the direction of the ferromagnetic coupling.

2. Kosterlitz-Thouless Transition **10pt**

- (a) Define the $2d$ XY model. **1pt**

$H = -\frac{J}{2} \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$, with $\vec{s}_i = (\cos \phi_i, \sin \phi_i)$. In the FM case $J > 0$. Typically the spins are placed on the vertices of a square lattice but other geometries are possible.

- (b) Which is the value of $\langle \vec{s}_i \rangle$ at equilibrium in the thermodynamic limit at all temperatures? **1pt**

0 at any $T > 0$. At $T = 0$ the ground state can be fully ordered.

- (c) Which are the relevant topological defects for this model? **1pt**

Vortices.

- (d) Which argument can be used to argue that there should be a finite temperature phase transition in this model? Explain it with words (without doing all the calculations). **3pt**

Peierls argument, applied to the vortices. Estimate the energy and entropy of creating an isolated vortex and see that at a finite T the free energy difference between a state with and without such a vortex becomes negative.

Otherwise, low T spin-wave solution yielding algebraic decay of correlation vs high T expansion yielding exponential decay $\rightarrow$ suggests finite T transition in between.

- (e) How does the spin-spin correlation function $C(r) = \langle \vec{s}_0 \cdot \vec{s}_r \rangle$ behave at low temperatures $T < T_{KT}$? What type of order is this typically called? **2pt**

It decays as a power law. Quasi long-range order.

- (f) How does the correlation length ξ behave as the critical temperature T_{KT} is approached from above in the disordered phase? **2pt**

It diverges exponentially with the distance from the critical point,

$$\xi \sim e^{b/|T-T_{KT}|^\nu} \text{ with } \nu = 1/2.$$

3. Beyond mean-field **10pt**

- (a) What is the physical origin of the upper critical dimension d_u in the context of phase transitions? **3pt**

As the spatial dimension increases, interactions are effectively averaged over many neighbours and critical fluctuations are suppressed; beyond d_u , the correlation length still diverges at criticality, but fluctuations of the order parameter are too weak to modify the mean-field scaling (irrelevant).

The Ginzburg criterium gives an operational way of estimating d_u by comparing the fluctuations of the order parameter to its value.

- (b) Explain what does d_u imply about the validity of mean-field theory in dimensions $d > d_u$. **1pt**

Above d_u mean-field yields the correct critical exponents.

- (c) List the main steps of the Renormalization Group (RG) procedure via decimation and/or coarse-graining. **2pt**

- Pick the microscopic Hamiltonian
- Decimate or coarse-grain some selected degrees of freedom
- Find the new Hamiltonian
- Identify the new parameters as functions of the old ones.
- Rescale space
- Find the fixed points on the RG iteration
- Linearize the transformation around the fixed point

(d) Explain what are the fixed points. **2pt**

Special values of the adimensional control parameters which leave the iteration unchanged

Repulsive ones represent phase transitions

Attractive ones represent the phases

(e) Consider two models with slightly different microscopic Hamiltonians but the same symmetries. What does the RG do with their microscopic details? What does this imply on universality? **2pt**

Eliminates them. The exponents are just determined by the RG transformation close to the critical points, microscopic details do not matter as long as the symmetries are not different.

4. Quantum phase transitions **10pt**

(a) Why do quantum phase transitions strictly exist only at zero temperature? **2pt**

At any finite temperature, thermal fluctuations introduce a finite energy scale that masks the ground-state level crossing or nonanalyticity in the free energy, turning the sharp quantum critical point into a crossover rather than a true phase transition.

Compare the energy scales $\hbar\omega$ vs $k_B T$. For $\hbar\omega > k_B T$ quantum; otherwise classical.

- (b) Briefly describe the quantum-to-classical mapping for d -dimensional quantum systems discussed in the course and TDs. **3pt**

The classical-to-quantum mapping that we discussed relates a d -dimensional quantum system like the quantum Ising chain at zero temperature to a $(d + 1)$ -dimensional classical statistical system (like the 2d Ising model) by interpreting imaginary time as an extra dimension whose extent is set by inverse temperature.

- (c) Define the dynamical critical exponent z . **2pt**

z is the critical exponent characterising the relation between the divergence of the correlation length in the spatial direction ξ and the divergence of the characteristic (imaginary) time, $\xi \sim \tau^{1/z}$.

- (d) Explain the origin of the KMS relation and what it implies **3pt**

The periodicity of the trace and one obtains the FDT relation between linear response and correlation functions, which reads easy in the frequency domain.

3 Complex Ginzburg Landau 20pt

Consider a system described by a complex scalar order parameter $\psi(\mathbf{x}) = \rho(\mathbf{x}) e^{i\phi(\mathbf{x})} \in \mathbb{C}$, with the following Ginzburg–Landau free energy functional:

$$F[\psi] = \int d^d x \left[a|\psi|^2 + \frac{b}{2}|\psi|^4 + c|\nabla\psi|^2 + \frac{u}{2}(\psi^2 + \psi^{*2}) \right],$$

where a, b, c, u are real constants, with $b > 0$, $c > 0$, and ψ^* the complex conjugate. This model is a variation on a model for superfluidity, in which the amplitude $|\psi(\mathbf{x})|^2$ is proportional to the superfluid density $n_s(\mathbf{x})$ and the phase $\phi(\mathbf{x})$ is related to the superfluid velocity $\mathbf{v}_s \mathbf{x} = \hbar/m \nabla \phi(\mathbf{x})$.

1. Determine the symmetry of F for $u = 0$ and for $u \neq 0$. 2pt

If $u = 0$, the free-energy is invariant under the continuous transformation $\psi(\mathbf{x}) \mapsto \psi'(\mathbf{x}) = e^{i\alpha}\psi(\mathbf{x})$, with α independent of position in space. This is any global change of the phase of the complex field ψ . Global U(1) rotations, a continuous symmetry.

If $u \neq 0$ the symmetry is reduced to just $\psi(\mathbf{x}) \mapsto \psi'(\mathbf{x}) = e^{ik\pi}\psi(\mathbf{x}) = -\psi(\mathbf{x})$, with k any odd integer. This is a global change of sign, a discrete $\mathbb{Z}_2$ symmetry.

2. Assume that ψ is spatially uniform and write $\psi = \rho e^{i\phi}$ with $\rho \geq 0$. 13pt

- (a) Express the free energy density $f(\rho, \phi)$ in terms of the amplitude ρ and the phase ϕ . 2pt

For a spatially uniform $\psi = \rho e^{i\phi}$ the term proportional to c vanishes, and the integral over space yields a volume factor that cancels out when we work with the free-energy density (by dividing by volume).

$$\begin{aligned} f(\rho, \phi) &= a\rho^2 + \frac{b}{2}\rho^4 + \frac{u}{2}(\rho^2 e^{2i\phi} + \rho^2 e^{-2i\phi}) \\ &= a\rho^2 + \frac{b}{2}\rho^4 + u\rho^2 \cos(2\phi) \end{aligned}$$

- (b) For $a < 0$, determine the equilibrium phase ϕ_0 when $u \neq 0$ (considering ρ fixed). Distinguish $u > 0$ and $u < 0$ and distinguish the stable and unstable configurations. **2pt**

One has to be careful with the signs of the parameters. In the figure below we plot $\pm \cos(2\phi)$, the relevant term and factor in $f(\rho, \phi)$ where the minima are visible. Taking the derivative $\partial f(\rho, \phi)/\partial \phi = 0$ and evaluating at $\phi = \phi_0$:

$$0 = -2u \rho^2 \sin(2\phi_0) \implies 2\phi_0 = k\pi \text{ with } k \in \mathbb{Z}$$

This gives four possible values for ϕ_0 in the interval $[0, 2\pi)$:

$$\begin{array}{ll} k = 0 \implies \phi_0 = 0 & k = 2 \implies \phi_0 = \pi \\ k = 1 \implies \phi_0 = \frac{\pi}{2} & k = 3 \implies \phi_0 = \frac{3\pi}{2} \end{array}$$

To know which are minima and maxima one needs to calculate the second derivative of $f(\rho, \phi)$:

$$\left. \partial^2 f(\rho, \phi) / \partial \phi^2 \right|_{\phi_0} = -4u \rho^2 \cos(2\phi_0)$$

and evaluate its sign (positive $\implies$ minimum, negative $\implies$ maximum). The outcome depends on the sign of u .

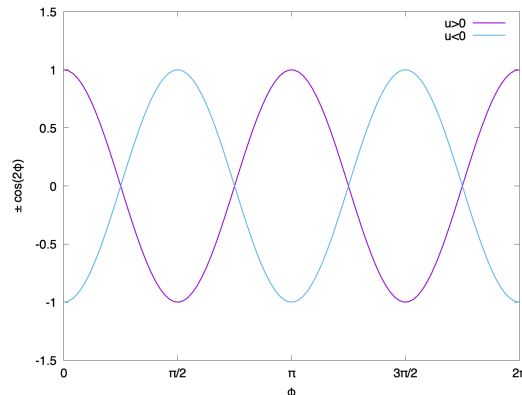
For $u > 0$:

$$\begin{array}{lll} k = 0 \implies \phi_0 = 0, \pi & \implies & \text{max} \\ k = 1 \implies \phi_0 = \frac{\pi}{2}, \frac{3\pi}{2} & \implies & \text{min, equilibrium} \end{array}$$

Instead, for $u < 0$:

$$\begin{array}{lll} k = 0 \implies \phi_0 = 0, \pi & \implies & \text{min, equilibrium} \\ k = 1 \implies \phi_0 = \frac{\pi}{2}, \frac{3\pi}{2} & \implies & \text{max} \end{array}$$

In conclusion, the equilibrium values are $\phi_0 = \pi/2$ and $\phi_0 = 3\pi/2$ for $u > 0$ and $\phi_0 = 0$ and $\phi_0 = \pi$ for $u < 0$



- (c) Fix $\phi = \phi_0$, one of the equilibrium ϕ values found in the previous item, and consider $f(\rho, \phi_0)$ as a function of ρ now. Which is the combination of parameters a and u that determines the critical point? **2pt**

In general, $a + u \cos(2\phi_0) = 0$, the coefficient of the ρ^2 term vanishing.

Take $u > 0$, the minima are $2\phi_0 = \pi, 3\pi$ and then the condition is $a - u = 0$.

Take $u < 0$, the minima are $2\phi_0 = 0, \pi$ and then the condition is $a + u = 0$.

Putting them together, $a - |u| = 0$.

- (d) What happens with the previous analysis if $u = 0$? **2pt**

The continuous symmetry is restored, there is no longer a dependence of the free-energy density on ϕ and all the angles are equally admissible in equilibrium.

The critical point is $a = 0$.

- (e) Consider $u \neq 0$. Find the equilibrium amplitude ρ_0 , for generic ϕ . **2pt**

$$0 = 2a\rho_0 + 2b\rho_0^3 + 2u\rho_0 \cos(2\phi) \quad \implies \quad \rho_0^2 = -\frac{a + u \cos(2\phi)}{b}$$

$$\text{or } \rho_0 = 0$$

- (f) Fix now ϕ to the equilibrium values obtained in (b). Which are the expressions of the equilibrium amplitudes ρ_0 ? Distinguish $u > 0$ and $u < 0$. **3pt**

We should distinguish $u > 0$ and $u < 0$.

For $u > 0$, $\phi_0 = \pi/2$ and $\phi_0 = 3\pi/2$. Then:

$$\phi_0 = \pi/2 \implies 0 = a\rho_0 + b\rho_0^3 - u\rho_0$$

with two solutions $\rho_0 = 0$ and $0 = a + b\rho_0^2 - u$ which yields

$$\rho_0^2 = (u - a)/b$$

This should be positive. Here, $u > 0$ and $b > 0$, so to be OK one needs $a < 0$.

In the case $\phi_0 = 3\pi/2$, $2\phi_0 = 3\pi$, $\cos(2\phi_0) = -1$ as well, and the ρ_0 that minimize $f(\rho, \phi)$ are the same.

For $u < 0$, $\phi_0 = 0$ and $\phi_0 = \pi$. Then.

$$\phi_0 = 0 \implies 0 = a\rho_0 + b\rho_0^3 + u\rho_0$$

with two solutions $\rho_0 = 0$ and $0 = a + b\rho_0^2 + u$ which yields

$$\rho_0^2 = (-u - a)/b$$

Again, the signs of the parameters are such that $\rho_0^2 > 0$.

The two cases together are

$$\rho_0^2 = (|u| - a)/b$$

3. Discuss how the term $\frac{u}{2}(\psi^2 + \psi^{*2})$ affects the degeneracy of the equilibrium state. **2pt**

The u term selects specific angles, reducing continuous degeneracy to two discrete minima. For each sign of u , these are the two angles ϕ_0 determined above. Recall that there remains the $\psi \mapsto -\psi$ symmetry for $u \neq 0$.

4. Let $\psi(\mathbf{x}) = (\rho_0 + \eta(\mathbf{x}))e^{i(\phi_0 + \varphi(\mathbf{x}))}$, with $\rho_0 \neq 0$ and $\phi_0 = 0$ the equilibrium configuration for $u < 0$. We consider η and φ very small and we Taylor expand f up to quadratic order. The “potential” part of the expansion (not depending on space derivatives) is

$$\Delta f_p(\eta, \varphi) = 2V^{-1} \int d^d x \left[b\rho_0^2 \eta^2(\mathbf{x}) + \rho_0^2 (b\rho_0^2 + a) \varphi^2(\mathbf{x}) \right]$$

What controls the fluctuations of the amplitude and phase of the complex field? Discuss the cases $u < 0$ and $u = 0$. **3pt**

The coefficients are different and so are the fluctuations.

The “mass” of the term proportional to η^2 is $\propto b\rho_0^2 > 0$.

The term proportional to φ^2 gets a “mass” proportional to $\rho_0^2 > 0$ and $(b\rho_0^2 + a)$. This last factor is also positive, because $b\rho_0^2 = -(u + a) > 0$ and then the factor is just $-u$.

We note that for $u = 0$, $\rho_0^2 = -a/b > 0$ and then the coefficient of the φ^2 term vanishes, as it should because of the rotational symmetry.

4 Harmonic chain 18pt

Consider a classical one-dimensional harmonic chain (lattice) with a real displacement field $x_i \in \mathbb{R}$ at each lattice site i . The energy (Hamiltonian) is

$$H[\{x_i\}] = \sum_i \left[\frac{1}{2} \mu x_i^2 + \frac{1}{2} k (x_i - x_{i+1})^2 \right],$$

where $\mu > 0$ is an onsite stiffness (mass term) and $k > 0$ is the nearest-neighbour spring constant. Consider the statistical weight $e^{-H[x]}$ (i.e. set $k_B T = 1$ for simplicity).

We perform a real-space RG by decimation: integrate out the displacements on the odd sites x_{2n+1} , to obtain an effective Hamiltonian for the even sites $X_n \equiv x_{2n}$. After decimation, rescale the lattice spacing by $b = 2$ to bring the effective chain back to the original lattice spacing.

Hint: Use the Gaussian integral identity

$$\int_{-\infty}^{\infty} dy \exp\left(-\frac{1}{2}ay^2 + Jy\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{J^2}{2a}\right) \quad (a > 0).$$

Answer the following:

1. For a single odd site x_o with left neighbour x_L and right neighbour x_R , write the part of the Hamiltonian involving x_o , expand it in quadratic form, and perform the Gaussian integration over x_o to show that the resulting effective contribution to the Hamiltonian of the even sites is:

$$H_{\text{eff}}(x_L, x_R) = \left(\frac{1}{2}k - \frac{k^2}{2a}\right)(x_L^2 + x_R^2) - \frac{k^2}{a}x_L x_R + \text{const}$$

with const a constant (independent of x_L and x_R) which we will neglect and $a = \mu + 2k$. 5pt

Pick one odd site x_o with neighbours x_L and x_R . The portion of the Hamiltonian involving x_o is

$$H_o = \frac{1}{2} \mu x_o^2 + \frac{1}{2} k (x_L - x_o)^2 + \frac{1}{2} k (x_o - x_R)^2.$$

Expand the spring terms:

$$H_o = \frac{1}{2} \mu x_o^2 + k x_o^2 - k x_o(x_L + x_R) + \frac{1}{2} k (x_L^2 + x_R^2).$$

Collect the quadratic coefficient of x_o :

$$a \equiv \mu + 2k > 0, \quad H_o = \frac{1}{2} a x_o^2 - k(x_L + x_R) x_o + \frac{1}{2} k (x_L^2 + x_R^2).$$

Integrating out x_o (Gaussian integral) contributes (up to an additive constant)

$$H_{\text{eff}}(x_L, x_R) = -\frac{[k(x_L + x_R)]^2}{2a} + \frac{1}{2} k (x_L^2 + x_R^2).$$

Equivalently,

$$H_{\text{eff}}(x_L, x_R) = \underbrace{\left(\frac{1}{2}k - \frac{k^2}{2a} \right)}_{\text{onsite piece per neighbour}} (x_L^2 + x_R^2) - \frac{k^2}{a} x_L x_R.$$

(One finds the cross term $-\frac{k^2}{a}x_L x_R$ which generates an effective coupling between the even sites.)

- Each even site has an odd neighbour on its left and another odd neighbour on its right. Summing all contributions from all odd sites and identifying the renormalized parameters μ' and k' (onsite and spring) of the effective Hamiltonian for the even sites, one finds

$$H'[\{X_n\}] = \sum_n \left[\frac{1}{2} \mu' X_n^2 + \frac{1}{2} k' (X_n - X_{n+1})^2 \right].$$

Prove that μ' and k' are given in terms of μ and k by **5pt**

$$k' = \frac{k^2}{\mu + 2k}, \quad \mu' = \mu + 2k - \frac{4k^2}{\mu + 2k}$$

Every even site X_n has two neighbouring odd sites (to its left and right). Therefore:

- The original onsite contribution for X_n is $\frac{1}{2}\mu X_n^2$.
- From the two adjacent spring halves (before integrating) one had contributions $+\frac{1}{2}k X_n^2$ from each, hence $+k X_n^2$ total.

- From integrating the two neighbouring odd sites one gets two contributions $-\frac{k^2}{2a}X_n^2$, hence a total $-\frac{k^2}{a}X_n^2$.
- Effective nearest-neighbour coupling between adjacent even sites X_n and X_{n+1} arises from the odd site sitting between them; each such odd produces $-\frac{k^2}{a}X_nX_{n+1}$.

Collecting terms, the effective Hamiltonian for the even sites reads

$$H_{\text{eff}}[\{X_n\}] = \sum_n \left[\frac{1}{2}\mu X_n^2 + kX_n^2 - \frac{k^2}{a}X_n^2 \right] + \sum_n \left[-\frac{k^2}{a}X_nX_{n+1} \right] + \text{const.}$$

We wish to put this in the standard form

$$H'[\{X_n\}] = \sum_n \left[\frac{1}{2}\mu' X_n^2 + \frac{1}{2}k'(X_n - X_{n+1})^2 \right] = \sum_n \left[\frac{1}{2}\mu' X_n^2 + k'X_n^2 - k'X_nX_{n+1} \right].$$

Comparing coefficients:

$$\text{cross term:} \quad -k' = -\frac{k^2}{a} \implies k' = \frac{k^2}{a} = \frac{k^2}{\mu + 2k},$$

$$\text{onsite:} \quad \frac{1}{2}\mu' + k' = \frac{1}{2}\mu + k - \frac{k^2}{a}.$$

Solving for μ' gives

$$\frac{1}{2}\mu' = \frac{1}{2}\mu + k - \frac{k^2}{a} - k' = \frac{1}{2}\mu + k - \frac{k^2}{\mu + 2k} - \frac{k^2}{\mu + 2k},$$

hence

$$\boxed{k' = \frac{k^2}{\mu + 2k}, \quad \mu' = \mu + 2k - \frac{4k^2}{\mu + 2k}}.$$

A compact alternative form for μ' is

$$\mu' = \frac{\mu(\mu + 4k)}{\mu + 2k}.$$

This is an *exact* result for the Gaussian (quadratic) model under decimation of every other site.

3. Define the dimensionless ratio $r \equiv \mu/k$. Show that under the RG step

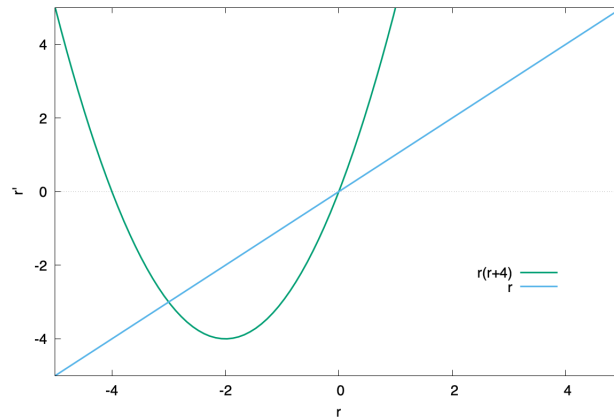
$$r \mapsto r' = r(r + 4).$$

Find the fixed points of this map and discuss their physical meaning. **4pt**

Define the dimensionless ratio $r \equiv \mu/k$. Compute $r' = \mu'/k'$. Using the closed forms above:

$$r' = \frac{\mu'}{k'} = \frac{\frac{\mu(\mu+4k)}{\mu+2k}}{\frac{\mu+2k}{k^2}} = \frac{\mu(\mu+4k)}{k^2} = \frac{\mu}{k} \left(\frac{\mu}{k} + 4 \right) = r(r+4).$$

Thus the RG map for r is $r \mapsto r' = r(r+4)$ and it is graphically shown in the figure



Fixed points satisfy $r^* = r^*(r^* + 4)$. For real finite r this gives:

$$r^* = 0 \quad \text{or} \quad r^* = -3.$$

The negative fixed point $r^* = -3$ is unphysical (it would require negative μ or k), so the only physically relevant fixed point is

$$r^* = 0$$

which corresponds to $\mu = 0$, i.e. a massless (gapless) chain. This is the “Gaussian” critical point separating a “massive” regime ($\mu > 0$) from the critical massless case.

4. Linearize the RG map near the (non-trivial) critical fixed point and extract the correlation length exponent ν . (Recall that for an RG step by factor b , a relevant eigenvalue λ obeys $\lambda = b^{1/\nu}$.) **4pt**

Linearize the map near the critical fixed point $r^* = 0$. For small r ,

$$r' = r(r + 4) = 4r + \mathcal{O}(r^2).$$

Hence the linear eigenvalue is $\lambda_r = 4$, indeed, $dr'/dr|_{r^*=0} = 4$.

Under an RG step with length rescaling factor $b = 2$, an operator with eigenvalue λ scales as $\lambda = b^y$ where y is the scaling exponent of that perturbation. The mass ratio r is proportional to the squared inverse correlation length ξ^{-2} (i.e. $r \sim \mu/k \sim \xi^{-2}$), so the RG eigenvalue of r is related to the correlation length exponent ν by

$$\lambda_r = b^{1/\nu_r} \implies \nu = \frac{\ln b}{\ln \lambda_r}.$$

Here $b = 2$ and $\lambda_r = 4$, therefore

$$\nu = \frac{\ln 2}{\ln 4} = \frac{\ln 2}{2 \ln 2} = \frac{1}{2}.$$

Thus the RG analysis reproduces the well-known Gaussian (mean-field) result $\nu = \frac{1}{2}$, consistent with the fact that for a free (quadratic) theory the mass gap scales with exponent $1/2$.