Advanced Statistical Physics

TD2: The XY model and the spin-wave regime

September 2025

The Kosterlitz–Thouless (KT) transition is a peculiar phase transition occurring in two-dimensional systems, in which topological defects play a crucial role. We will study it in the framework of the XY model, which is defined by two-dimensional (classical) vector spins located at the vertices $\mathbf{r}$ of a square lattice of N sites and linear size L ($N = (L/a)^2$, where a is the lattice spacing). The spins interact ferromagnetically according to the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} \,.$$

The model is considered in canonical equilibrium with a thermal bath at temperature T.

At low temperatures, the spin–spin correlation function decays algebraically as a power law, whereas above a certain temperature it becomes short-ranged. Although the nature of correlations changes dramatically between the low- and high-temperature regimes, there is no conventional symmetry-breaking phase transition.

The goal of this exercise is to analyze the XY model using both low- and high-temperature expansions.

A) Phenomenological analysis

1. Which kind of order is favored by the exchange J?
2. At which temperatures do you expect to see this kind of order?
3. Which kind of configurations do you expect to find at high temperatures?

B) Low temperature expansion: The spin-wave regime

Each spin $\boldsymbol{S}_\mathbf{r}$ can be simply characterized by an orientation $\theta_\mathbf{r} \in [0,2\pi)$ with respect to any chosen axis.

1. What is the ground state of $\mathcal{H}$ in terms of the angles?
2. Why is

$$\mathcal{H}_{\text{sw}} = \frac{J}{2} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} (\theta_\mathbf{r} - \theta_{\mathbf{r}'})^2$$

a good approximation of $\mathcal{H}$ at low temperature?

3. We define the discretized derivative along x as

$$\frac{\partial f}{\partial x} = \frac{f(x+a/2) - f(x-a/2)}{a}.$$

Show that the discretized Laplacian in 2d is

$$\nabla^2 f(x,y) = \frac{f(r+a\mathbf{e}_x) + f(r-a\mathbf{e}_x) + f(r+a\mathbf{e}_y) + f(r-a\mathbf{e}_y) - 4f(r)}{a^2}.$$

4. We introduce the Green's function of ($-a^2$ times) the 2d Laplacian on the square lattice:

$$-a^2\nabla^2 G_{\mathbf{r}} = \delta_{\mathbf{r},\mathbf{0}}.$$

The properties of G are given in the Appendix. Show that the partition function under this approximation is

$$Z_{\text{sw}} = \int \mathcal{D}\theta \, \exp\left[-\frac{K}{2}\sum_{\mathbf{r}} \theta_{\mathbf{r}}(-a^2\nabla^2)\theta_{\mathbf{r}}\right],$$

with $K = \beta J$ and $\mathcal{D}\theta = \prod_{\mathbf{r}} \mathrm{d}\theta_{\mathbf{r}}$, and express the correlation $\langle \theta_{\mathbf{r}}\theta_{\mathbf{r}'}\rangle$ in terms of in terms of the Green's function of the discrete Laplacian operator.

5. What is the average angle $\langle \theta_{\mathbf{r}}\rangle$? Is there any spontaneous magnetization $\langle \boldsymbol{S}_{\mathbf{r}}\rangle \neq 0$?
6. How does the spin-spin correlation $C(\mathbf{r},\mathbf{r}') = \langle \boldsymbol{S}_{\mathbf{r}} \cdot \boldsymbol{S}_{\mathbf{r}'}\rangle$ behave?
7. What is the correlation length ξ?
8. What is the linear magnetic susceptibility?

C) The high temperature expansion

1. Let $\mathcal{N}(\mathbf{r})$ be the number of shortest paths connecting $\mathbf{r} = (x,y)$ to the origin. Express $\mathcal{N}(\mathbf{r})$ as a function of $|x|,|y|$. Argue that $\mathcal{N}(\mathbf{r}) \leq 2^{\|\mathbf{r}\|_1}$ with $\|\mathbf{r}\|_1 = |x| + |y|$ (called the "Manhattan distance").
2. Show that

$$\int d\theta_2 \cos(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3) = \pi\cos(\theta_1 - \theta_3).$$

3. Justify that in the high-T regime, to leading order in K,

$$C(|\mathbf{r} - \mathbf{r}'|) \sim N(\mathbf{r} - \mathbf{r}')(\pi K)^{\|\mathbf{r}-\mathbf{r}'\|_1}.$$

Give an estimate of ξ in terms of K.

Appendix: Green's function of the 2d Laplacian on a square lattice

We define the Fourier transform as

$$\hat{G}_{\mathbf{q}} = \sum_{r} e^{i\mathbf{q}\cdot\mathbf{r}} G_{\mathbf{r}}, \quad G_{\mathbf{r}} = \frac{1}{N}\sum_{\mathbf{q}\neq\mathbf{0}} e^{-i\mathbf{q}\cdot\mathbf{r}}\hat{G}_{\mathbf{q}},$$

with $\mathbf{q} = \frac{2\pi}{L}(n_x,n_y)$ and integers $n_x,n_y \in [-L/(2a),L/(2a)]$.

Inserting into the definition gives

$$\hat{G}_{\mathbf{q}} = \frac{1}{4 - 2\cos(aq_x) - 2\cos(aq_y)}.$$

Hence

$$G_{\mathbf{r}} = \frac{1}{N}\sum_{\mathbf{q}\neq\mathbf{0}} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{4 - 2\cos(aq_x) - 2\cos(aq_y)}.$$

Useful properties:

$$G_{\mathbf{0}} \simeq \frac{1}{2\pi}\log\frac{L}{a}, \qquad G_{|\mathbf{r}|\gg a} - G_{\mathbf{0}} \simeq -\frac{1}{2\pi}\log\frac{|\mathbf{r}|}{a} - c + o(1),$$

with $c = \frac{1}{2\pi}(\gamma + \frac{3}{2}\log 2) \approx \frac{1}{4}$.