QUANTUM STATISTICAL PHYSICS

REFS

SIMONS & ALTMAN

SHANKAR, KADANOFF

FRADKIN

FIRST : CLASS $\Leftrightarrow$ QUANTUM

LATER : PATH INTEGRALS

INSTANTONS

GENERIC PROPS OF CORR. FUNCTIONS

KUBO, FDT, ETC.

MAHAN, SIMONS ALTLAND

FINALLY

QUANTUM PHASE TRANSITIONS SACHDEY

(AT STRICT ZERO TEMP.)

CLASSICAL – QUANTUM CORRESPONDENCE

CANON. EQUILIBRIUM OF CLASS. SYSTEM H
IN $d+1$ DIMENSIONS AT
INVERSE TEMP β

$$\Vert$$

CANONICAL EQUIL. QUANTUM SYST $\hat{\mathcal{H}}$
IN d DIMENSIONS AT
INVERSE TEMP β_q

NON-TRIVIAL RELATIONS BTW PARAMETERS IN THE TWO MODELS H AND $\hat{\mathcal{H}}$ (INCLUDING β AND β_q)

EVERYTHING WE SAID ABOUT CLASSICAL EQUIL. PHASE TRANS.
APPLIES TO THE QUANTUM PROBLEMS

METHODS

- TRANSFER MATRIX   CLASSICAL → QUANTUM

  eg. 1D ISING CHAIN → 1 QUANTUM SPIN

- GENERAL $Z_q^{(d)}(\beta_q, p_q) \to Z_{cl}^{(d+1)}(\beta, p)$

  INVERSE TEMP   QUANTUM   INV. TEMP   CLASS.

  PARAM. IN QUANTUM MODEL   PARAM. CLASS. MODEL

  THE HAMILTONIANS ARE DIFFERENT

- EVOL. OPERATORS

  QUANTUM REAL TIME DYNAMICS t

  → CLASSICAL EQUIL $-i\beta\hbar$

- PATH INTEGRALS

SEC. 3.1

THE CLASSICAL ONE DIM ISING CHAIN

- TRANSFER MATRIX

$$Z = \sum_{\{s_i\}} e^{-\beta H} = \sum_{\{s_i\}} e^{-\beta W(\{s_i\})}$$

$$= \sum_{\{s_i\}} e^{k \sum_i s_i s_{i+1} + H \sum_i s_i}$$

$k = \beta J \qquad H = \beta h \qquad$ PBC $\quad s_{N+1} = s_1$

CONVENIENCE: SYMMETRIZE THE WEIGHTS ON LINKS

$e^{-\beta w(s_i, s_{i+1})}$ WITH $w(s_i, s_{i+1}) = w(s_{i+1}, s_i)$

1st TERM IS ALREADY SYMM

2nd TERM $\quad \frac{h}{2} \sum_i (s_i + s_{i+1}) \quad$ SYMMETRIZED

$$-\beta w(s_i, s_{i+1}) = \underbrace{+\beta J}_{k} s_i s_{i+1} + \overbrace{\beta \frac{h}{2}}^{H} (s_i + s_{i+1})$$

$$-\beta W(s_i, s_{i+1}) = K s_i s_{i+1} + \frac{H}{2}(s_i + s_{i+1})$$

WHY CAN ONE WRITE THE Z AS A PRODUCT OF MATRICES ?

LOOK AT THREE SUBSEQ SPINS $i-1, i, i+1$:

$$\sum_{\{s_i = \pm 1\}} \underbrace{e^{-\beta W(s_{i-1}, s_i)}}_{\substack{M_l \\ 2\times 2}} \underbrace{e^{-\beta W(s_i, s_{i+1})}}_{\substack{M_r \\ 2\times 2}}$$

TWO TERMS CONTRIB.

1st VARIABLE s_{i-1} IN M_l, ITS VALUES ORGANIZED IN THE ROWS
s_i IN M_r

2nd VARIABLE s_i IN M_l, ITS VALUES ORG. IN COLUMNS
s_{i+1} IN M_r

$$M_l = \begin{pmatrix} M(s_{i-1} = 1, s_i = 1) & M(s_{i-1} = 1, s_i = -1) \\ M(s_{i-1} = -1, s_i = 1) & M(s_{i-1} = -1, s_i = -1) \end{pmatrix}$$

$$M_r = \begin{pmatrix} M(s_i = 1, s_{i+1} = 1) & M(s_i = 1, s_{i+1} = -1) \\ M(s_i = -1, s_{i+1} = 1) & M(s_i = -1, s_{i+1} = -1) \end{pmatrix}$$

IN MORE DETAIL

$$\sum_{\{s_i=\pm 1\}} \underbrace{e^{-\beta W(s_{i-1},s_i)}}_{M^{\ell}_{i-1,i}} \underbrace{e^{-\beta W(s_i,s_{i+1})}}_{M^{r}_{i,i+1}}$$

$$= e^{-\beta W(s_{i-1},-1)}\, e^{-\beta W(-1,s_{i+1})} +$$

$$e^{-\beta W(s_{i-1},1)}\, e^{-\beta W(1,s_{i+1})}$$

ORGANIZE IN A 2X2 MATRIX AGAIN

$$= M_{\ell\, i-1,i}\, M_{R\, i\, i+1}$$

SOMME SUR LES POSSIBLES VALEURS QUE LA VAR. EN i PEUT PRENDRE

$-\beta W(s_i, s_{i+1})$ CAN TAKE 4 VALUES $\{s_i, s_{i+1} = \pm 1\}$

$\Rightarrow$ ORGANIZE THEM IN A 2x2 MATRIX

$$-\beta \begin{array}{cc} s_{i+1}=1 & s_{i+1}=-1 \\ \downarrow & \downarrow \end{array}$$

$$-\beta \begin{pmatrix} W(1,1) & W(1,-1) \\ W(-1,1) & W(-1,-1) \end{pmatrix} = \begin{array}{l} \leftarrow s_i = 1 \text{ ROW} \\ \leftarrow s_i = -1 \text{ ROW} \end{array}$$

$$= \begin{pmatrix} +K+H & -K \\ -K & K-H \end{pmatrix}$$

BUT WE REALLY HAVE $e^{-\beta W(s_i, s_{i+1})}$ IN THE MATRICES M

$$\Rightarrow \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix} = M$$

THIS IS THE <u>TRANSFER MATRIX</u>

KEEP THIS ONE IN BLACKBOARD

COMMENT: WE NOTE IT'S THE SAME FOR ALL LINK PAIRS (s_i, s_{i+1})

INDEP OF i

IN OTHER HETEROGENEOUS MODELS

COULD BE M_i

FOCUS ON HOMOGENEOUS ONES

$$Z = \sum_{\{s_i\}} e^{K\sum_i s_i s_{i+1} + H \sum_i s_i}$$

ROWS COLUMNS

$$= \sum_{\{s_i\}} \prod_i e^{-\beta w(s_i, s_{i+1})}$$

PRODUCT OF TRANSFER MATRICES

$$= \mathrm{Tr} \prod_{i=1}^{N} M$$

Tr FROM PBC

$$= \mathrm{Tr}\, M^N$$

$$= \sum_{a=1}^{n} \lambda_a^N$$

EIGENVALUES OF M

$$= \lambda_1^N + \lambda_2^N$$

FOR THE $n=2$ CASE ("INTERNAL DIM" OF VARIABLE)

$$Z = \lambda_1^N \left[1 + \left(\frac{\lambda_2}{\lambda_1} \right)^N \right] \underset{N \gg 1}{\approx} \lambda_1^N$$

$\lambda_1 > \lambda_2$

$$-\beta f = \frac{1}{N} \ln Z = \ln \lambda_1$$

THE LARGEST EIGENVALUE OF M DOMINATES

$$\boxed{-\beta f = \ln \lambda_1}$$

CALCULATION OF f BOILS DOWN TO GETTING λ_1.

PERRON-FROBENIUS –

SYMMETRIC MATRIX M w/ $\mathbb{R}$ ELEM

$\lambda_1 > 0$, NON DEGENERATE

THE TRANSFER MATRIX M IS $\begin{pmatrix} e^{k+H} & e^{-k} \\ e^{-k} & e^{k-H} \end{pmatrix}$

CHARACTERISTIC POLYN.

$$P(\lambda) = (e^{k+H} - \lambda)(e^{k-H} - \lambda) - e^{-2k}$$

ROOTS $\Rightarrow$ EIGENVALUES

$$e^{2k} - \lambda e^{k+H} - \lambda e^{k-H} + \lambda^2 - e^{-2k} = 0$$

$$\lambda^2 - \lambda e^k \, 2\operatorname{ch} H + 2\operatorname{sh} 2k = 0$$

$$2\lambda_{1,2} = e^k \, 2\operatorname{ch} H \pm \left[e^{2k} 4\operatorname{ch}^2 H - 4 \, 2\operatorname{sh} 2k \right]^{1/2}$$

$$\boxed{\lambda_{1,2}} = e^k \operatorname{ch} H \pm \left[e^{2k} \operatorname{ch}^2 H - 2\operatorname{sh} 2k \right]^{1/2}$$

$$= e^k \operatorname{ch} H \pm e^k \left[\operatorname{ch}^2 H - (e^{2k} - e^{-2k}) e^{-2k} \right]^{1/2}$$

$$= e^k \operatorname{ch} H \pm e^k \left[\cancel{1} + \operatorname{sh}^2 H - \cancel{1} + e^{-4k} \right]^{1/2}$$

$$\boxed{= e^k \operatorname{ch} H \pm e^k \left[\operatorname{sh}^2 H + e^{-4k} \right]^{1/2}}$$

$$\lambda_1 = e^{K}\, ch H + e^{K} \left[sh^2 H + e^{-4K} \right]^{1/2}$$

$$H=0: \quad \lambda_1 = e^{k} + e^{-k} = 2 ch K$$

$$-\beta f = \ln \lambda_1 = \ln(2 ch K)$$

AS WE ALREADY KNEW

OK

IN TD $\Rightarrow$ $\xi = \left(\ln \frac{\lambda_1}{\lambda_2} \right)^{-1}$

EXERCISE COMPLETE THIS CALC.

$$m(K, H) \begin{cases} \xrightarrow[H \to \infty]{} 1 \\ \xrightarrow[H \to 0]{} 0 \end{cases}$$

END SOLUTION OF 1D ISING CHAIN WITH TRANSFER MATRIX.

QUANTUM : PAULI MATRICES

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

BASIS OF 2x2 HERMITIAN MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \in \mathbb{C}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PROPERTIES

HERMITIAN $\quad \sigma_a^{\dagger} = \sigma_a \qquad a = 1, 2, 3$

$\dagger : (t, *)$

$$\sigma_a^2 = \mathbb{I}$$ IDEMPOTENT

$$[\sigma_a, \sigma_b] = 2i\, \varepsilon_{abc}\, \sigma_c$$ COMM. RELATIONS

$$\{\sigma_a, \sigma_b\} = 0$$ ANTI-COMMUTE

IDEA

THINK OF $e^{-\beta W(s_i, s_{i+1})}$ AS THE EXPECTATION VALUE OF A QUANTUM OPERATOR

$$e^{-\beta W(s_i, s_{i+1})} = \langle s_{i+1} | \hat{T} | s_i \rangle$$

$$= \langle s_{i+1} | e^{-\varepsilon \hat{\mathcal{H}}_0} | s_i \rangle$$

DIRAC NOTATION BRA-KETS

$\hat{\mathcal{H}}_0$ QUANTUM HAMILT (HERMITIAN OPERATOR)

TASK: FIND $\hat{T}$ AND HENCE $\hat{\mathcal{H}}_0$

STRATEGY FROM CLASSICAL TO QUANTUM

- WE KNOW M FROM THE CLASSICAL TRANSFER MATRIX

- WE ASK $M = \langle S_{i+1} | \hat{T} | S_i \rangle$ WITH

$$\hat{T} = a_0 + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z$$

AND FIND $\{a_0, a_1, a_3\}(K, H)$

- WE ASK $\hat{T} = e^{-\varepsilon \hat{\mathcal{H}}}$ WITH

$$\hat{\mathcal{H}} = b_0 + b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z$$

AND FIND $\{b_0, b_1, b_3\}(K, H)$

- ONCE FOUND THE INFINITESIMAL TRANSFER

GO TO THE MACROSCOPIC LIMIT

ε MICROSC $\qquad \beta_q = N\varepsilon$ FINITE

$$e^{-\beta_q \hat{\mathcal{H}}}$$

MODEL OF A SINGLE QUANTUM SPIN AT $\beta_q = N\varepsilon$

SYST SIZE IN β_q, PARAM. (K, H) IN (Δ_x, Δ_z)

NON-TRIVIAL $\varepsilon \to 0$, $N \to \infty$ LIMIT

FIRST IDENTIFY $\hat{T}$ FROM

$$\begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix} = M = \langle S_{i+1} | \hat{T} | S_i \rangle$$

REWRITE $\hat{T}$ AS A LINEAR COMBINATION OF PAULI MATRICES OPERATORS ACTING ON A VECT HILBERT SPACE

$$\hat{T} = a_0 \hat{I} + a_1 \hat{\sigma}_x + \underset{\substack{\| \\ 0}}{a_2} \hat{\sigma}_y + a_3 \hat{\sigma}_z$$

SINCE $M \in \mathbb{R}$

$\Rightarrow$ a_0, a_1, a_3 ?

4 EQS $\rightarrow$ 3 EQS (SYMM) & 3 UNKNOWNS a_0, a_1, a_3

EXPLICIT CALCULATION

$\langle S_{i+1} | \hat{T} | S_i \rangle$

$$\hat{T} = a_0 \mathbb{I} + a_1 \hat{\sigma}_x + \underbrace{a_2}_{=0} \hat{\sigma}_y + a_3 \hat{\sigma}_z$$

NO NEED SINCE $T_{ab} \in \mathbb{R}$

SANDWICH $\hat{T}$ WITH $\langle + | \cdots | + \rangle$:

$$a_0 + a_1 \underbrace{\langle + | \hat{\sigma}_x | + \rangle}_{\langle + | - \rangle = 0} + a_3 \underbrace{\langle + | \hat{\sigma}_z | + \rangle}_{1}$$

$$= a_0 + a_3$$

SIMILARLY, OTHER SANDW.

$$\langle + | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | - \rangle = a_1$$

$$\langle - | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | + \rangle = a_1$$

$$\langle - | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | - \rangle = a_0 - a_3$$

ASKING EQUALITY WITH ELEMENTS IN M :

$$a_0 + a_3 = e^{k+H} \quad (1)$$

$$a_1 = e^{-k} \implies \boxed{a_1 = e^{-k}}$$

$$a_0 - a_3 = e^{k-H} \quad (2)$$

ADD (1)+(2) : $2a_0 = e^{k+H} + e^{k-H}$

$$\boxed{a_0 = e^{k} \, ch \, H}$$

SUBTRACT (1)−(2) :

$$\boxed{a_3 = e^{k} \, sh \, H}$$

$$\hat{T} = a_0 \, \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z$$

$$\boxed{\hat{T} = e^{k} \, ch H \, \mathbb{I} + e^{-k} \hat{\sigma}_x + e^{k} \, sh H \, \hat{\sigma}_z}$$

TRANSFER OPERATOR IN PAULI MATRIX NOTATION

TRANSFER OPERATOR AS THE EXP OF A QUANTUM HAMILTONIAN

$$\hat{T} = e^{k} \,\mathrm{ch}H\, \mathbb{I} + e^{-k}\hat{\sigma}_x + e^{k}\,\mathrm{sh}H\,\hat{\sigma}_z$$

$$= e^{-\varepsilon\hat{\mathcal{H}}}$$ DEFINITION OF A QUANTUM HAMILT

$$\varepsilon\hat{\mathcal{H}} = \left[-b_0\mathbb{I} - b_1\hat{\sigma}_x - \underbrace{b_2\hat{\sigma}_y}_{=0} - b_3\hat{\sigma}_z\right]\varepsilon$$

WHICH IS $\hat{\mathcal{H}}$?

SAME REASON

$$e^{-\varepsilon\hat{\mathcal{H}}} = e^{\varepsilon b_0 \mathbb{I}}\; e^{\varepsilon[b_1\hat{\sigma}_x + b_3\hat{\sigma}_z]}$$

$$e^{\varepsilon b_0}\mathbb{I}$$

$\hat{\sigma}_x, \hat{\sigma}_z$ DON'T COMMUTE

TAYLOR EXPAND

$$\sum_{n=0} \frac{(b_1\hat{\sigma}_x + b_3\hat{\sigma}_z)^n}{n!}\varepsilon^n$$

WE NEED EXPRESSIONS FOR EVEN/ODD POWERS

WE WILL SEE THE SERIES OF $\operatorname{ch} y, \operatorname{sh} y$ APPEARING

$$\operatorname{ch} y = \frac{e^{y}+e^{-y}}{2}$$

$$= \frac{1}{2}\sum_n \frac{y^n}{n!} + \frac{1}{2}\sum_n \frac{(-y)^n}{n!}$$

$$= \sum_{n\,\text{even}} \frac{y^n}{n!}$$ (ODD CANCEL)

$$\operatorname{sh} y = \frac{e^{y}-e^{-y}}{2}$$

$$= \frac{1}{2}\sum_n \frac{y^n}{n!} - \frac{1}{2}\sum_n \frac{(-y)^n}{n!}$$

$$= \sum_{n\,\text{odd}} \frac{y^n}{n!}$$ (EVEN CANCEL)

$$(b_1\hat{\sigma}_x + b_3\hat{\sigma}_z)^2 = b_1^2 \overset{\overset{I}{\shortparallel}}{\hat{\sigma}_x^2} + b_3^2 \overset{\overset{I}{\shortparallel}}{\hat{\sigma}_z^2} +$$

$$+\, b_1 b_3 \underbrace{(\hat{\sigma}_x\hat{\sigma}_z + \hat{\sigma}_z\hat{\sigma}_x)}_{=0}$$

$$= (b_1^2 + b_3^2)\, \mathbb{I}$$

$\Rightarrow$ ALL EVEN POWERS ARE PROP TO $\mathbb{I}$

$$\mathbb{I}\left\{ 1 + \frac{(b_1^2 + b_3^2)}{2!} + \frac{(b_1^2+b_3^2)^2}{4!} + \dots \right\}$$

$$\boxed{= \mathbb{I}\ ch\left[(b_1^2+b_3^2)^{1/2}\varepsilon\right]}$$

RESUMMATION OF EVEN POWERS: PROPORTIONAL TO $\mathbb{I}$

ODD POWERS

FIRST ONE IS $b_1\vec{\sigma}_x + b_3\vec{\sigma}_z$

NEXT

$$(b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^3 = (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)\cdot (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^2$$

SO THEN

$$(b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^5 = (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)\cdot (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^4$$

SO WE CAN USE RESULTS FROM THE EVEN CASE TO WRITE

$$(b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)\left[\mathbb{I} + (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^2\frac{1}{3!} + (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)^4\frac{1}{5!} + \dots\right]$$

$$= (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)\,\mathbb{I}\left[1 + (b_1^2 + b_3^2)\frac{1}{3!} + (b_1^2 + b_3^2)^2\frac{1}{5!} + \dots\right]$$

$$\Rightarrow (b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)\,\mathbb{I}\,\frac{1}{\sqrt{b_1^2 + b_3^2}}\,\mathrm{sh}\left[(b_1^2 + b_3^2)^{1/2}\varepsilon\right]$$

$$= \underbrace{\frac{\mathrm{sh}\left[(b_1^2 + b_3^2)^{1/2}\varepsilon\right]}{\sqrt{b_1^2 + b_3^2}}(b_1\vec{\sigma}_x + b_3\vec{\sigma}_z)}$$

RESUMMATION OF ODD POWERS PROP. TO PAULI OPER.

$$\boxed{e^{-\varepsilon\hat{\mathcal{H}}} = e^{\varepsilon b_0}\left\{ \mathbb{I}\, ch\left[(b_1^2+b_3^2)^{1/2}\varepsilon\right] + \frac{sh\left[(b_1^2+b_3^2)^{1/2}\varepsilon\right]}{\sqrt{b_1^2+b_3^2}}\left(b_1\hat{\sigma}_x + b_3\hat{\sigma}_z\right)\right\}}$$

THIS IS THE EXPRESSION OF $\hat{T}$ IN TERMS OF THE THREE PARAMETERS IN $\hat{\mathcal{H}}$

NOW, COMPARE TO $\hat{T}$

$$\hat{T} = e^{k}\, ch H\, \mathbb{I} + e^{-k}\hat{\sigma}_x + e^{k}\, sh H\, \hat{\sigma}_z$$

$\underline{\mathbb{I}}$:

$$e^{\varepsilon b_0}\, ch\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right] = e^{k}\, ch H \qquad (1)$$

$\underline{\hat{\sigma}_x}$:

$$e^{\varepsilon b_0}\,\frac{sh\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]}{\sqrt{b_1^2+b_3^2}}\, b_1 = e^{-k} \qquad (2)$$

$\underline{\sigma_2^1}$: $$e^{\varepsilon b_0} \frac{sh\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]}{\sqrt{b_1^2+b_3^2}} b_3 = e^k \, sh H \qquad (3)$$

THREE EQS & THREE UNKNOWNS b_0, b_1, b_3

$\frac{(2)}{(3)} \Longrightarrow$ $$\frac{b_1}{b_3} = e^{-2k} \frac{1}{sh H} \qquad (4)$$

TAKING $(1)^2 - (2)(b/b_1\varepsilon)^2$ $\quad b \equiv \sqrt{b_1^2+b_3^2}\,\varepsilon$

$$e^{2b_0\varepsilon}\left(ch^2 B - sh^2 B\right) =$$
$$= e^{2k} ch^2 H - \frac{(b_1^2+b_3^2)}{b_1^2} e^{-2k}$$

SINCE $ch^2 B - sh^2 B = 1 \Longrightarrow$

$$e^{2b_0\varepsilon} = e^{2k} ch^2 H - \left(1+\frac{b_3^2}{b_1^2}\right) e^{-2k}$$
$$= e^{2k} ch^2 H - \left(1+e^{4k} sh^2 H\right) e^{-2k}$$

USING (4)

$$= e^{2k}\underbrace{(ch^2 H - sh^2 H)}_{1} - e^{-2k}$$

$$\boxed{e^{2b_0\varepsilon} = 2\,sh\,2k}$$

FIXES $\varepsilon b_0 = \frac{1}{2}\ln(2\,sh\,2k)$

WE CAN NOW GO BACK TO EQ. (2) AND REPLACE $e^{2b_0\varepsilon}$ FROM THIS RESULT

$$\sqrt{2\,sh\,2k}\;\frac{sh\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]b_1}{\sqrt{b_1^2+b_3^2}} = e^{-k}$$

$$\Rightarrow \boxed{b_1 = \frac{e^{-k}\sqrt{b_1^2+b_3^2}}{\sqrt{2\,sh\,2k}\;sh\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]}} \quad (*)_1$$

IMPLICIT EXPRESSION IN TERMS OF $b = b_1^2 + b_3^2$ IN rhs

AND NOW IN (3)

$$\sqrt{2\,\mathrm{sh}\,2k}\;\frac{\mathrm{sh}\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]}{\sqrt{b_1^2+b_3^2}}\,b_3 = e^{k}\,\mathrm{sh}\,H$$

ALSO IMPLICIT IN TERMS OF b IN rhs

$$\Rightarrow \quad \boxed{b_3 = \frac{e^{k}\,\mathrm{sh}\,H\,\sqrt{b_1^2+b_3^2}}{\sqrt{2\,\mathrm{sh}\,2k}\;\mathrm{sh}\left[\sqrt{b_1^2+b_3^2}\,\varepsilon\right]}} \quad (*_2)$$

CAN BUILD AN EQ FOR $b=\sqrt{b_1^2+b_3^2}$ FROM $(*)_1$ AND $(*)_2$

SOLVE FOR b AND THE REPLACE IN $(*)_1$ AND $(*)_2$ TO GET b_1 AND b_3. FULL EXPRESSIONS GIVEN BELOW

TAKE $H \to 0$ AS A SIMPLER CASE $\quad \boxed{b_3 = 0}$

AND TAKE $\frac{(2)}{(1)}$:

$$\operatorname{th} b_1 \varepsilon = e^{-2k}$$

$$\frac{e^{b_1\varepsilon} - e^{-b_1\varepsilon}}{e^{b_1\varepsilon} + e^{-b_1\varepsilon}} = \frac{1 - e^{-2b_1\varepsilon}}{1 + e^{-2b_1\varepsilon}} = e^{-2k}$$

$$1 - e^{-2b_1\varepsilon} = e^{-2k}\left(1 + e^{-2b_1\varepsilon}\right)$$

$$1 - e^{-2k} = e^{-2b_1\varepsilon}\left(1 + e^{-2k}\right)$$

$$\frac{1 - e^{-2k}}{1 + e^{-2k}} = e^{-2b_1\varepsilon}$$

$$\operatorname{th} k = e^{-2b_1\varepsilon}$$

$$-2b_1\varepsilon = \ln \operatorname{th} k$$

$$\boxed{b_1\varepsilon = -\frac{1}{2} \ln \operatorname{th} k}$$

CASE $H=0$

$b_3 = 0$

IF I FOLLOW THIS ROUTE FOR $H \neq 0$
(I SET $\varepsilon = 1$ IN THIS PART.
IT'S RECOVERED BELOW)

$$\operatorname{ch}\sqrt{b_1^2+b_3^2} = e^{k}\operatorname{ch}H$$

$$\frac{\operatorname{sh}\sqrt{b_1^2+b_3^2}}{\sqrt{b_1^2+b_3^2}}\, b_1 = e^{-k}$$

$$\frac{\operatorname{th}\sqrt{b_1^2+b_3^2}}{\sqrt{b_1^2+b_3^2}}\, b_1 = e^{-2k}\operatorname{ch}H$$

USE NOW $\quad b_3 = b_1 e^{2k}\operatorname{sh}H$

$$\frac{\operatorname{th} b_1\sqrt{1+e^{4k}\operatorname{sh}^2 H}}{\cancel{b_1}\sqrt{1+e^{4k}\operatorname{sh}^2 H}}\,\cancel{b_1} = e^{-2k}\operatorname{ch}H$$

CALL $\quad y = b_1\sqrt{1+e^{4k}\operatorname{sh}^2 H}$

$$\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} = e^{-2k} \operatorname{ch} H \sqrt{\cdots}$$

$$1-e^{-2y} = \left(1+e^{-2y}\right) e^{-2k} \operatorname{ch} H \sqrt{\cdots}$$

$$\left(1-e^{-2k} \operatorname{ch} H \sqrt{}\right) =$$

$$= \left(1+ e^{-2k} \operatorname{ch} H \sqrt{\cdots}\right) e^{-2y}$$

$$e^{-2y} = \frac{1-e^{-2k} \operatorname{ch} H \sqrt{\cdots}}{1+e^{-2k} \operatorname{ch} H \sqrt{\cdots}}$$

$$y = -\frac{1}{2} \ln \frac{1-e^{-2k} \operatorname{ch} H \sqrt{\cdots}}{1+e^{-2k} \operatorname{ch} H \sqrt{\cdots}}$$

$$= b_1 \sqrt{1+e^{4k}\, sh^2 H}$$

ε RECOVERED HERE

$$\boxed{\varepsilon b_1 = -\frac{1}{2} \frac{1}{\sqrt{1+e^{4k}\, sh^2 H}} \cdot \ln\left(\frac{1-e^{-2k}\, ch H \sqrt{1+e^{4k}\, sh^2 H}}{1+e^{-2k}\, ch H \sqrt{1+e^{4k}\, sh^2 H}} \right)}$$

CHECK $H = 0$

$$b_1 = -\frac{1}{2\varepsilon} \ln \frac{1-e^{-2k}}{1+e^{-2k}} = -\frac{1}{2\varepsilon} \ln \underline{\frac{th\, k}{}}$$

OK

AND WE KNOW ALSO THE OTHER TWO PARAMETERS

$$\boxed{\begin{aligned} \varepsilon b_3 &= \varepsilon b_1\, e^{2k}\, sh H \\ \varepsilon b_0 &= \frac{1}{2} \ln(2\, sh\, 2k) \end{aligned}}$$

$H=0$ SIMPLER EXPRESSIONS

SHOW THESE ONES ON BOARD.
MAYBE ALSO SHOW THE GENERIC ONES.

$$\begin{cases} b_0 = \frac{1}{2\varepsilon} \ln\left[2\sinh 2u\right] \\ b_1 = -\frac{1}{2\varepsilon} \ln \operatorname{th} k \\ b_3 = 0 \end{cases}$$

QUANTUM HAMILTONIAN IN THE CASE $H=0$

$$\boxed{-\varepsilon \hat{\mathcal{H}}_q = \frac{1}{2}\ln\left[2\sinh(2u)\right]\mathbb{I} - \frac{1}{2}\underbrace{\ln \operatorname{th} k}_{\delta}\, \hat{\sigma}_x}$$

"TRANSVERSE FIELD"

THIS IS THE QUANTUM HAMILTONIAN FOR THE "ELEMENTARY" TRANSFER OPERATOR TAKING FROM SITE i TO ITS NEIGHBOUR i+1 ON THE CLASSICAL CHAIN, WRITTEN IN QM LANGUAGE

TO GO TO THE "MACROSCOPIC" LIMIT OF N SITES AND M^N OR $\hat{T}^N$ WE NEED

$$e^{-\beta_q \hat{\mathcal{H}}_q}$$

$$\beta_q = N\varepsilon$$

CLASSICAL CHAIN LENGTH

$$-\beta_q \hat{\mathcal{H}}_q = -N\varepsilon \hat{\mathcal{H}}_q = A\,\mathbb{I} - \Delta\,\hat{\sigma}_x$$

CASE $H=0$

$$\boxed{\begin{aligned} \beta_q &= N\varepsilon \\ \Delta &= \frac{N}{2} \ln \operatorname{th} K \end{aligned}}$$

MAPPING FOR CLASSICAL CHAIN $\updownarrow$ QUANTUM SPIN

CLASSICAL SYSTEM SIZE PROP. TO β_q

$K = \beta J$ CLASSICAL GOES INSIDE A QUANT.

ε IS PLAYING THE ROLE OF A "LATTICE SPACING"

εN IS LIKE A "LENGTH."

BUT ACTUALLY $[\varepsilon N] = [\beta_q]$ INVERSE ENERGY

SO THAT $[\beta_q \beta_\phi^{-1}] = 1$

WE WANT β_q FINITE

$$\varepsilon \to 0 \ \& \ N \to \infty$$

WHAT HAPPENS WITH Δ FOR $N \to \infty$?

TO KEEP $\Delta = \frac{N}{2} \ln \operatorname{th} k$ FINITE $\Longleftrightarrow$

$$\ln \operatorname{th} k \sim \frac{2\Delta}{N} \Longleftrightarrow$$

$$\operatorname{th} k \sim e^{2\Delta/N} \sim 1 + \frac{2\Delta}{N} + \dots$$

$\operatorname{th} k$ CLOSE TO 1 MEANS $k \to \infty$

BUT SMALLER THAN 1 $\Longrightarrow$

NB $\ln \operatorname{th} k$ IS THEN NEGATIVE $\Rightarrow \Delta < 0$

$$\operatorname{th} k = \frac{e^{k} - e^{-k}}{e^{k} + e^{-k}} = \frac{1 - e^{-2k}}{1 + e^{-2k}}$$

$$\operatorname{th} k \simeq (1 - e^{-2k})(1 - e^{-2k} + \dots) \simeq 1 - 2e^{-2k} + \dots$$

$$\frac{\cancel{2}\Delta}{N} \simeq -\cancel{2} e^{-2k} \Rightarrow -\frac{1}{2} \ln\left(\frac{-\Delta}{N}\right) = k$$

$$k = \frac{1}{2} \ln\left(\frac{-N}{\Delta}\right)$$

OK, SINCE $\Delta < 0$; $k \gg 1$

LIKE THIS WE HAVE

$N \to \infty \qquad \varepsilon \to 0 \qquad \beta_q$ FINITE

$\Delta < 0$ FINITE $\Longleftrightarrow$

$$\underbrace{K = \beta J}_{\text{CLASSICAL MODEL}} = \frac{1}{2} \ln\left(-\frac{N}{\Delta}\right)$$

QUANTUM SPIN

$T \to 0 \qquad\qquad \to \infty$

ACCESS VERY LOW T OF CLASSICAL MODEL THIS WAY.

A GENERAL WAY OF SEEING THE EQUIVALENCE FROM QUANTUM TO CLASSICAL

$$Z_q = \mathrm{Tr}\, e^{-\beta_q \hat{\mathcal{H}}_q}$$

q ← FOR QUANTUM

GENERAL MODEL

$\vec{S}$ { SPACE & INTERNAL INDICES

SPIN NOTATION JUST TO CHOOSE A NOTATION

$$= \sum_{\vec{S}_0} \langle \vec{S}_0 | e^{-\beta_q \hat{\mathcal{H}}_q} | \vec{S}_0 \rangle$$

← SUM OVER A COMPLETE SET, BASIS OF HILBERT SPACE

$\hat{\mathcal{H}}_q$ COMMUTES WITH ITSELF

(CAREFUL IF ONE WANTS TO SPLIT IT IN TWO TERMS WHICH TYPICALLY DON'T COMMUTE $\hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2$)

WRITE THE FULL EXPONENTIAL AS

$$e^{-\beta_q \mathcal{H}_q} = \underbrace{e^{-\varepsilon \mathcal{H}_q} \cdots e^{-\varepsilon \mathcal{H}_q}}_{N\varepsilon = \beta_q}$$

$N \to$ NUMBER OF FACTORS

INTRODUCE RESOLUTIONS OF UNITY

$$1 = \sum_{\vec{S}_n} |\vec{S}_n\rangle\langle\vec{S}_n|$$

IN BETWEEN THOSE FACTORS

$$= \sum_{\vec{S}_0} \cdots \sum_{\vec{S}_{N-1}} \langle\vec{S}_0|e^{-\varepsilon \mathcal{H}_q}|\vec{S}_{N-1}\rangle \cdots \langle\vec{S}_1|e^{-\varepsilon \mathcal{H}_q}|\vec{S}_0\rangle$$

RECALL $|\vec{S}_N\rangle = |\vec{S}_0\rangle$ PBC

TRACE IN Z_q

NOTATION

eg. $$1 = \sum_a |\psi_a\rangle\langle\psi_a|$$

$|\psi_a\rangle$ BASIS ELEM OF HILBERT SPACE

if $|\vec{S}_n\rangle$ ARE BASIS ELEM OF HILBERT SPACE,

I WRITE $$\sum_{\{\vec{S}_n\}} |\vec{S}_n\rangle\langle\vec{S}_n|$$

THE SUM OVER THOSE PARAM. SIMPLY LIKE THIS

CALL

NB: COULD BE $\mathbb{C}$ BUT, IF $\mathbb{R}$ AND POSITIVE $\Rightarrow$ REWRITING OK

$$\underbrace{e^{-\beta\mathcal{E}(\vec{S}_n,\vec{S}_{n-1})}}_{\text{ADIMENSIONAL}} = \langle \vec{S}_n | e^{-\varepsilon \vec{\mathcal{H}}} | \vec{S}_{n-1}\rangle \Rightarrow$$

ANOTHER β

$$Z_q = \underbrace{\sum_{\{\{\vec{S}_n\}\}_{n=1,\dots,N}} e^{-\beta\sum_{n=1}^{N}\mathcal{E}(\vec{S}_n,\vec{S}_{n-1})}}_{\text{CLASSICAL PART. FUNCT } Z \text{ OF A MODEL IN } (d+1) \text{ DIM.}}$$

AND RECALL PBC

EXTRA DIM $\Rightarrow$ THE ONE USED TO SANDWICH $n=1,\dots,N$

(IN THE $\vec{\cdot}$ IN $\vec{S}$ THERE'S THE DEP. ON THE d-DIM. AS WELL AS THE POSS. INTERNAL DIM OF S)

HERE WE MAPPED A QUANTUM Z_q ON A CLASSICAL ONE (in $d+1$ dim)

IN NOTES $\Rightarrow$ QUANTUM ISING SPIN $\Leftrightarrow$ CHAIN AGAIN.

THE RELATION BETWEEN PARAM. HAS TO BE FOUND FOR EACH SPECIFIC PROBLEM

THIS IS JUST A GENERALIZATION OF THE MAPPING OF CLASSICAL ISING CHAIN ON SINGLE QUANTUM SPIN.

- NO NEED TO USE TRANSFER MATRIX
- NOT NECESSARY $d=1$ CLASS $\Longleftrightarrow$ $d=0$ QUANTUM

BUT d QUANTUM $\Longleftrightarrow$ $d+1$ CLASSICAL

WE ARE MAPPING

$$Z^{\text{CLASSICAL}}(\beta, \text{PARAM.}) =$$

$$Z_q^{\text{QUANTUM}}(\beta_q, \text{PARAM}')$$

Sec. 3.1.2 THE EVOLUTION OPERATOR

FROM REAL TO IMAGINARY TIME

SCHRÖDINGER'S EQUATION

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

START FROM QUANTUM & GO CLASSICAL

$$|\psi(t)\rangle = \hat{U}(t,t')\,|\psi(t')\rangle$$

$$\boxed{\hat{U}(t,t') = e^{-i\frac{\hat{H}(t-t')}{\hbar}}}$$

EVOLUTION OPERATOR

NOTE $\frac{-i(t-t')}{\hbar}$ LOOKS LIKE $e^{-\beta H}$ ...

$$\hat{U}\hat{U}^{+} = \mathbb{I} \quad \text{if} \quad \hat{H} = \hat{H}^{+}$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(t')|\psi(t')\rangle$$

WAVE FUNCTION

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|\hat{U}(t,t')\,|\psi(t')\rangle$$

$$= \int dx'\,\langle x|\hat{U}(t,t')|x'\rangle\langle x'|\psi(t')\rangle$$

$$= \int dx'\,\underbrace{U(x,t;x',t')}\,\psi(x',t')$$

WAVE FUNCTION EVOLUTION - PROPAGATION.

EVOLUTION KERNEL (IT'S A FUNCTION)

QUANTUM PARTITION FCT - CANONICAL EQ.

$$Z_q = \mathrm{Tr}\, e^{-\beta_q \hat{H}}$$

$$= \int dx \, \langle x | e^{-\beta_q \hat{H}} | x \rangle$$

SUM OVER STATES OF AN ORTH. BASIS OF HILBERT SPACE

eg. EIGENVECTOR OF $\hat{x}$

IT'S LIKE A PERIODIC EVOLUTION IN IMAGINARY TIME

$$t - t' = -i\beta_q \hbar$$

FROM $|x\rangle$ BACK TO $\langle x|$

$$Z_q = \int dx \; U(x, \overbrace{-i\beta_q \hbar}^{t}; x, 0)$$

AND COMING BACK TO SAME X BECAUSE OF TRACE PBC

$$\boxed{\text{QUANTUM STAT PHYS. IN CAN. EQ. IS LIKE IMAGINARY TIME EVOLUTION}}$$

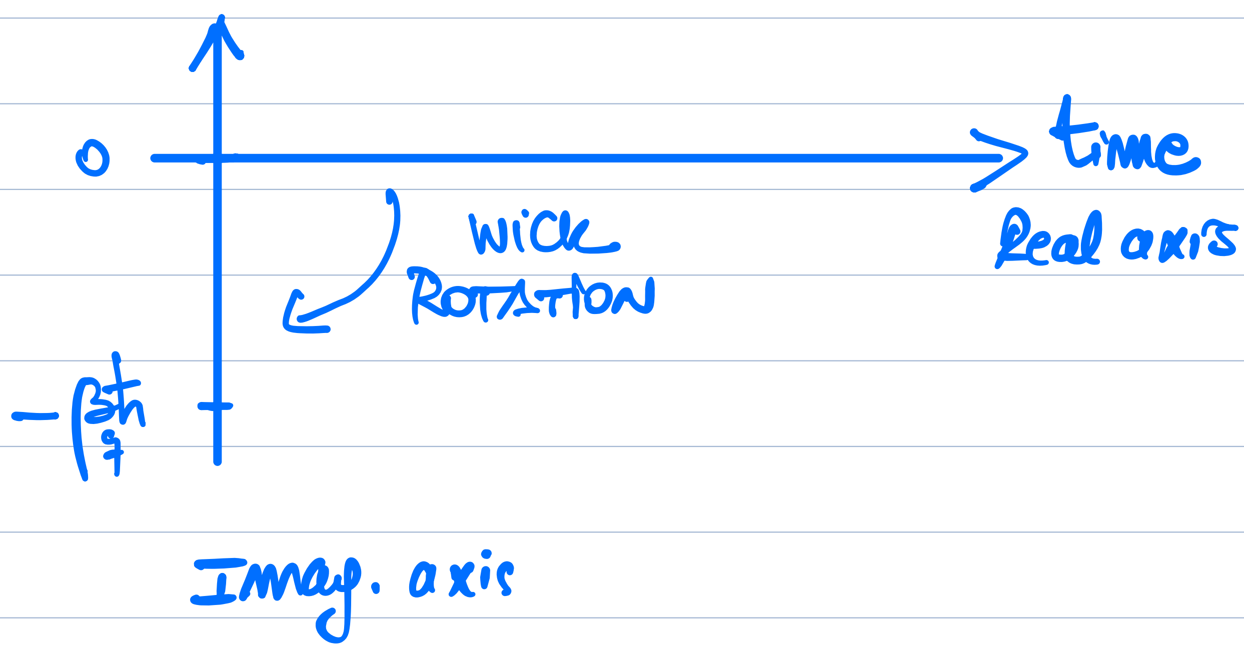
$$Z_q = \int dx \; \underbrace{U(x, \overbrace{-i\beta_q\hbar}^{t}; x, 0)}$$

EVOLUTION OPERATOR
DIAGONAL MATRIX
ELEMENT, WITH
REAL TIME REPLACED BY

GOING FROM $|x\rangle$ TO $|x\rangle$

THE IMAGINARY TIME: $-i\underbrace{\beta_q\hbar}_{\text{REAL}} = t \in \mathbb{I}$

$$t = -i\tau = -i(\beta_q\hbar)$$



3/ PATH INTEGRALS

THE PROPAGATOR FROM x' TO x IS

$$\boxed{\langle x | \hat{U}(t,t') | x' \rangle = U(x,t;x',t')}$$

GOAL WRITE $U(x,t;x',t')$

AS A SUM OVER CLASSICAL PATHS

LINKING (x',t') & (x,t)

TRAJECTORIES

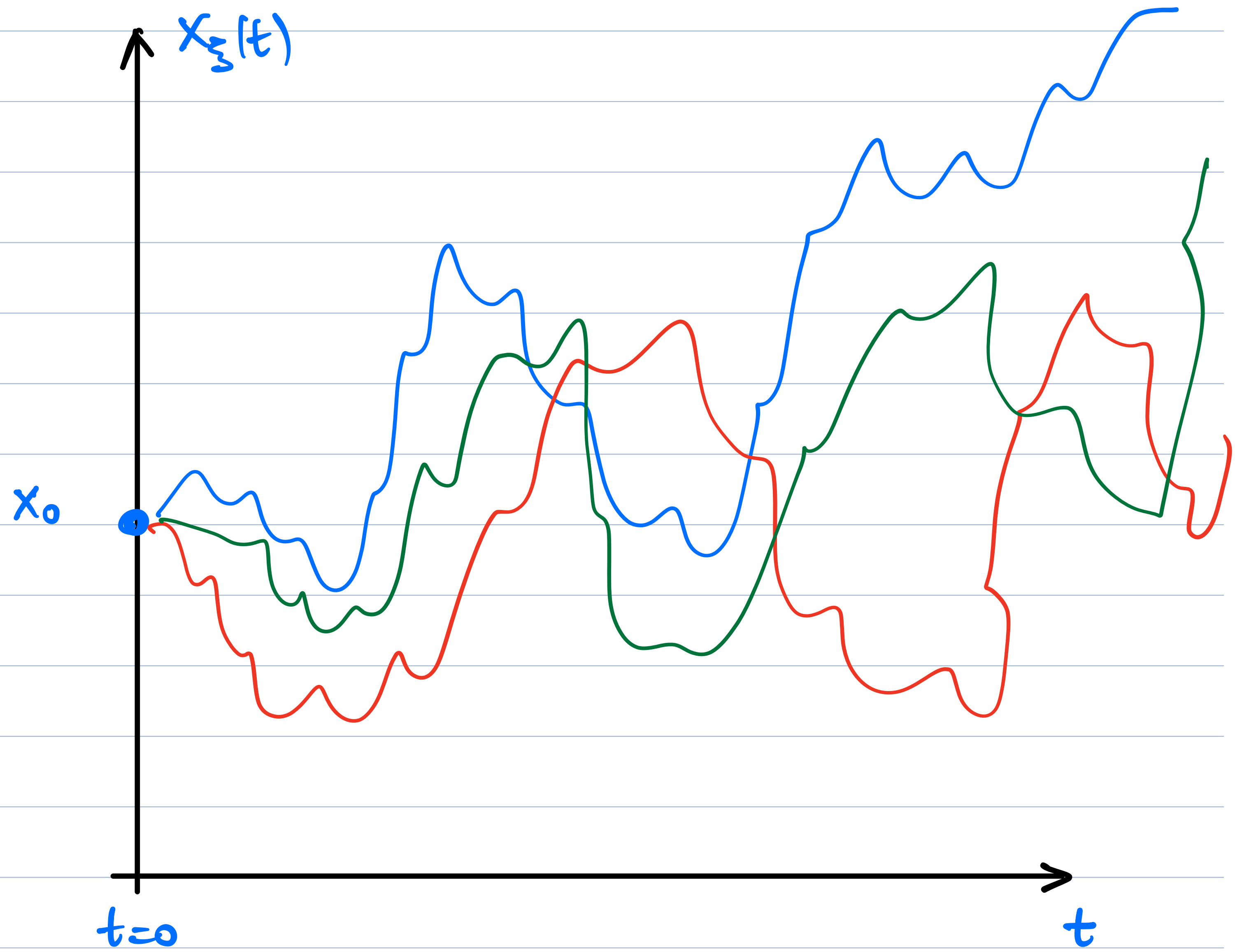
LET'S DISCUSS WIENER'S DESCRIPTION OF LANGEVIN PROCESSES

e.g. THE SIMPLEST PROCESS

$$\dot{x} = \xi \qquad \langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = 2 k_B T\, \delta(t-t')$$

BROWNIAN MOTION, LANGEVIN



TRAJECTORIES GENERATED BY DIFF NOISES $\xi(t)$

$$x(t) = x_0 + \int_0^t dt'' \, \xi(t'') \quad \Rightarrow \text{AVERAGES.}$$

$$\langle x(t) \rangle = x_0 + \int_0^t dt'' \, \langle \xi(t'') \rangle = x_0$$

$$\langle x^2(t) \rangle = x_0^2 + \int_0^t dt'' \int_0^t dt''' \underbrace{\langle \xi(t'') \, \xi(t''') \rangle}_{2k_BT \, \delta(t''-t''')}$$

$$\langle x^2(t) \rangle - x_0^2 = 2k_BT \, t \qquad \text{DIFFUSION}$$

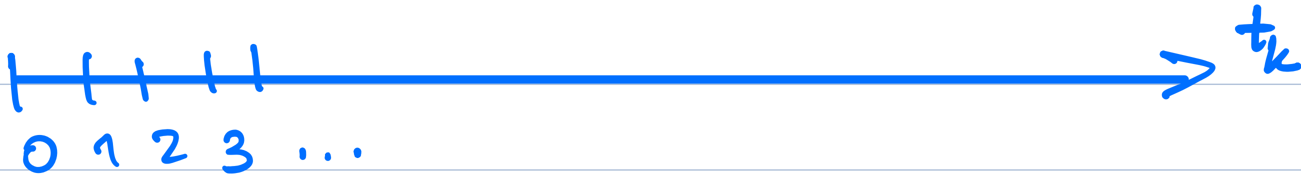
ALTERNATIVE DESCRIPTION

$P(x,t \mid x',t')$

FIND AN EXPRESSION FOR THE TRANSITION PROB. FROM $(x',t') \to (x,t)$

DISCRETIZE TIME

N- DISCRETE VALUES



$$t_k = t_0 + k\,\delta t \qquad k = 0, \dots, N$$

$$\delta t = \frac{t - t'}{N} \qquad t_0 = t' \qquad t_N = t$$

$$x(t') = x_0 = x' \qquad\qquad x(t) = x_N = x$$

$$P(x,t \mid x',t') = \int dx_1 \dots \int dx_{N-1}\; p(x,t \mid x_{N-1}, t_{N-1}) \dots$$

$$\dots p(x_1,t_1 \mid x',t')$$

MARKOV CHARACTER OF PROCESS

$$P(\xi) = \prod_k p(\xi_k) = \prod_k \frac{1}{\sqrt{2\pi\, 2k_BT/\delta t}}\, e^{-\frac{\xi_k^2}{2\,2k_BT/\delta t}}$$

GAUSSIAN pdf

JOINT pdf OF MEMORY-LESS NOISE

$\Rightarrow$

$$\langle \xi_k \rangle = 0 \quad \forall k$$

AVERAGES.

$$\langle \xi_k \xi_{k'} \rangle = \frac{2k_BT}{\delta t}\, \delta_{kk'}$$

DISCRETE VERSION OF DIRAC'S DELTA

$$x_{k+1} - x_k = \delta t\, \xi_k$$ SOL. TO LANG EQ (ITO)

$$\xi_k = \Theta(\delta t^{-1/2}) \Rightarrow \delta x_k = \Theta(\delta t^{1/2})$$

INFINITESIMAL $p(x_n, t_n \mid x_{n-1}, t_{n-1})$:

$$p(x_n, t_n \mid x_{n-1}, t_{n-1}) = \int d\xi_{k-1}\, p(\xi_{n-1})\, \delta(x_n - x_{n-1} - \underbrace{\delta t\, \xi_{n-1}}_{\bar{\xi}_{n-1}})$$

$$= \int \frac{d\bar{\xi}_{n-1}}{\delta t}\, p(\bar{\xi}_{n-1}/\delta t)\, \delta(\bar{\xi}_{n-1} - x_n + x_{n-1})$$

$$= \frac{1}{\delta t}\, p\left(\frac{x_n - x_{n-1}}{\delta t}\right)$$

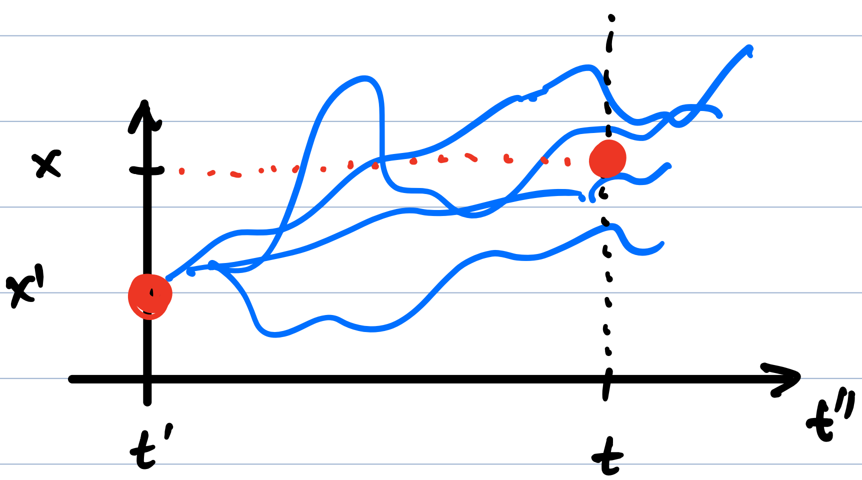
$$= \frac{1}{\sqrt{2\pi \;\, 2k_BT\,\delta t}}\; e^{-\frac{\delta t}{2\;2k_BT}\left(\frac{x_n - x_{n-1}}{\delta t}\right)}$$

NOW, BACK IN THE FULL TRANS. PROB TAKING A CONT TIME NOTATION $\delta t \to 0$ $N \to \infty$

$$p(x,t\,|\,x',t') = \underbrace{\int \mathcal{D}x}\; e^{-\frac{1}{4k_BT}\int_{t'}^{t} dt''\left(\dot{x}(t'')\right)^2}$$

$$\int_{-\infty}^{\infty}\frac{dx_1}{A}\cdots\int_{-\infty}^{\infty}\frac{dx_{N-1}}{A} \qquad \begin{cases} x(t) = x \\ x(t') = x' \end{cases}$$ BOUND COND.

$$\frac{1}{(2\pi\cdot 2k_BT\,\delta t)^{1/2}} = \frac{1}{A}$$



SUM OVER ALL TRAJECTORIES GENERATED BY LANG. GOING $x' \longrightarrow x$

QUANTUM PROBLEM

THE PROPAGATOR FROM x' TO x IS

$$\boxed{\langle x | \hat{U}(t,t') | x' \rangle = U(x,t;x',t')}$$

WE FOLLOW A SIMILAR ROUTE TO WHAT DONE ABOVE FOR THE STOCHASTIC EQ.

FIRST STEP

DISCRETIZE TIME

$$t_k = t_0 + k\,\delta t \qquad k = 0, \dots, N$$

$$\delta t = \frac{t - t'}{N} \qquad t_0 = t' \qquad t_N = t$$

ADD RESOLUTIONS OF THE IDENTITIES

$$\mathbb{1} = \int dx \; |x\rangle\langle x|$$

POSITION REPRESENTATION

$\langle x | \hat{U}(t,t') | x' \rangle =$

$$= \int dx_1 \dots \int dx_{N-1} \; \langle x | \hat{U}(t_N, t_{N-1}) | x_{N-1} \rangle$$

$$\langle x_{N-1} | \hat{U}(t_{N-1}, t_{N-2}) | x_{N-2} \rangle \cdots \langle x_1 | \hat{U}(t_1, t_0) | x' \rangle$$

VERY SIMILAR TO THE CLASSICAL PROCESS WITH TRANS. PROB. $p(x_{k+1}, t_{k+1} | x_k, t_k)$

WE NEED TO COMPUTE THE GENERIC FACTOR

$$\langle x_k | \hat{U}(t_k, t_{k-1}) | x_{k-1} \rangle =$$

$$= \langle x_k | e^{-\frac{i}{\hbar} \hat{\mathcal{H}}(t_k - t_{k-1})} | x_{k-1} \rangle$$

WITH $t_k - t_{k-1} = \delta t$ VERY SMALL

NOW WE USE $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

BUT WE RECALL THAT <u>1st & 2nd TERM DO NOT COMMUTE</u>. SO WE NEED TO EVALUATE THE AMPLITUDE IN THE LIMIT δt SMALL.

CAMPBELL - HAUSSDORF FORMULA

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B} + \frac{1}{2}[\hat{A},\hat{B}] + \dots}$$

WHICH, if $\left.\begin{array}{l}\hat{A} \propto \varepsilon \\ \hat{B} \propto \varepsilon\end{array}\right\} \Rightarrow$ 2nd TERM IN EXP

$\ll$ 1st TERM

& SO ON & SO FORTH

$$\Rightarrow \quad e^{\hat{A}} e^{\hat{B}} \approx e^{\hat{A}+\hat{B}}$$

USED BELOW

THANKS TO $\left\{\begin{array}{l}\hat{A} \propto \delta t \\ \hat{B} \propto \delta t\end{array}\right.$

APPLY IT TO THE FACTOR TO BE CALCULATED

$$\langle x_n | \, e^{-\frac{i}{\hbar}\hat{H}(t_n - t_{n-1})} \, | x_{n-1} \rangle$$

$$\sim \langle x_n | \, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t} \, \underbrace{e^{-\frac{i}{\hbar} V(\hat{x})\,\delta t} \, | x_{n-1} \rangle}_{\downarrow}$$

$$\sim \langle x_n | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t} | x_{n-1} \rangle \, e^{-\frac{i}{\hbar} V(x_{n-1})\,\delta t}$$

INSERT AN IDENTITY RESOLVED IN THE MOMENTUM REPRESENTATION

$$\mathbb{1} = \int dp_{k-1} \, | p_{k-1} \rangle \langle p_{k-1} |$$

$$\langle x_n | \, e^{-\frac{i}{\hbar}\hat{\mathcal{H}}(t_n - t_{n-1})} \, | x_{n-1} \rangle \simeq$$

$$e^{-\frac{i}{\hbar} V(x_{n-1})\,\delta t}$$

$$\int dp_{n-1} \, \underbrace{\langle x_n | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t} | p_{n-1} \rangle}_{\langle x_n | p_{n-1} \rangle \, e^{-\frac{i}{\hbar}\frac{p_{n-1}^2}{2m}\delta t}} \langle p_{n-1} | x_{n-1} \rangle$$

$$\langle x_n | \, e^{-\frac{i}{\hbar}\hat{H}(t_n - t_{n-1})} \, | x_{n-1} \rangle \simeq$$

$$e^{-\frac{i}{\hbar} V(x_{n-1}) \delta t}$$

$$\int dp_{n-1} \; e^{-\frac{i}{\hbar} \frac{p_{n-1}^2}{2m} \delta t} \; \langle x_n | p_{n-1} \rangle \langle p_{n-1} | x_{n-1} \rangle$$

THE FREE PARTICLE WAVE FUNCTIONS ARE

$$\langle p_{n-1} | x_{n-1} \rangle = \frac{1}{\sqrt{2\pi\hbar}} \; e^{-\frac{i}{\hbar} p_{n-1} x_{n-1}}$$

$$\langle x_n | p_{n-1} \rangle = \frac{1}{\sqrt{2\pi\hbar}} \; e^{\frac{i}{\hbar} p_{n-1} x_n}$$

THEN

$$\langle x_n | \, e^{-\frac{i}{\hbar}\hat{H}(t_n - t_{n-1})} \, | x_{n-1} \rangle \simeq e^{-\frac{i}{\hbar} V(x_{n-1}) \delta t}$$

$$\underbrace{\int \frac{dp_{n-1}}{2\pi\hbar} \; e^{-\frac{i}{\hbar} \frac{p_{n-1}^2}{2m} \delta t} \quad e^{\frac{i}{\hbar} p_{n-1} \frac{(x_n - x_{n-1})}{\delta t} \delta t}}_{*}$$

WHICH CAN BE REWRITTEN AS

$$\langle x_k | \, e^{-\frac{i}{\hbar}\hat{H}(t_k - t_{k-1})} \, | x_{k-1} \rangle \simeq$$

$$\int \frac{dp_{k-1}}{2\pi\hbar} \, e^{\frac{i}{\hbar}\left[p_{k-1}\frac{(x_k - x_{k-1})}{\delta t} \underbrace{- H(p_{k-1}, x_{k-1})}_{\text{NOT AN OPERATOR, A FUNCTION}} \right]\delta t}$$

*

THIS IS A GAUSSIAN INTEGRAL OVER THE MOMENTUM p_{k-1} WHICH READILY YIELDS

$$\boxed{\langle x_k | \, e^{-\frac{i}{\hbar}\hat{H}(t_k - t_{k-1})} \, | x_{k-1} \rangle} \simeq e^{-\frac{i}{\hbar}V(x_{k-1})\delta t}$$

$$\left(\frac{-im}{2\pi\hbar\,\delta t} \right)^{1/2} e^{\frac{im}{2}\left(\frac{x_k - x_{k-1}}{\delta t}\right)^2 \frac{\delta t}{\hbar}}$$

$$\propto \boxed{e^{\frac{i}{\hbar}\left[\frac{m\dot{x}_{k-1}^2}{2} - V(x_{k-1}) \right]\delta t}} \quad (+)$$

SHORT-HAND NOTATION

PUTTING ALL TOGETHER

PATH INTEGRAL REP. OF QUANTUM DYNAMICS

THE TRANSITION FROM $|x'\rangle$ TO $|x\rangle$ IN

THE INTERVAL $t' \to t$:

$$\langle x | \hat{U}(t,t') | x' \rangle =$$

IS GIVEN BY

$$U(x,t;\ x',t')$$

WITH

$$U(x,t;\,x',t') = \int_{x'}^{x} \mathcal{D}x \; e^{\frac{i}{\hbar} S}$$

RECALL THERE IS A PRODUCT OVER FACTORS LIKE ⊕ ⇒ SUM OVER

$\sum_k$ ⇒ INT OVER TIME

WITH THE ACTION

$$S = \int_{t'}^{t} dt'' \left[\frac{m}{2} \dot{x}^2(t'') - V(x(t'')) \right]$$

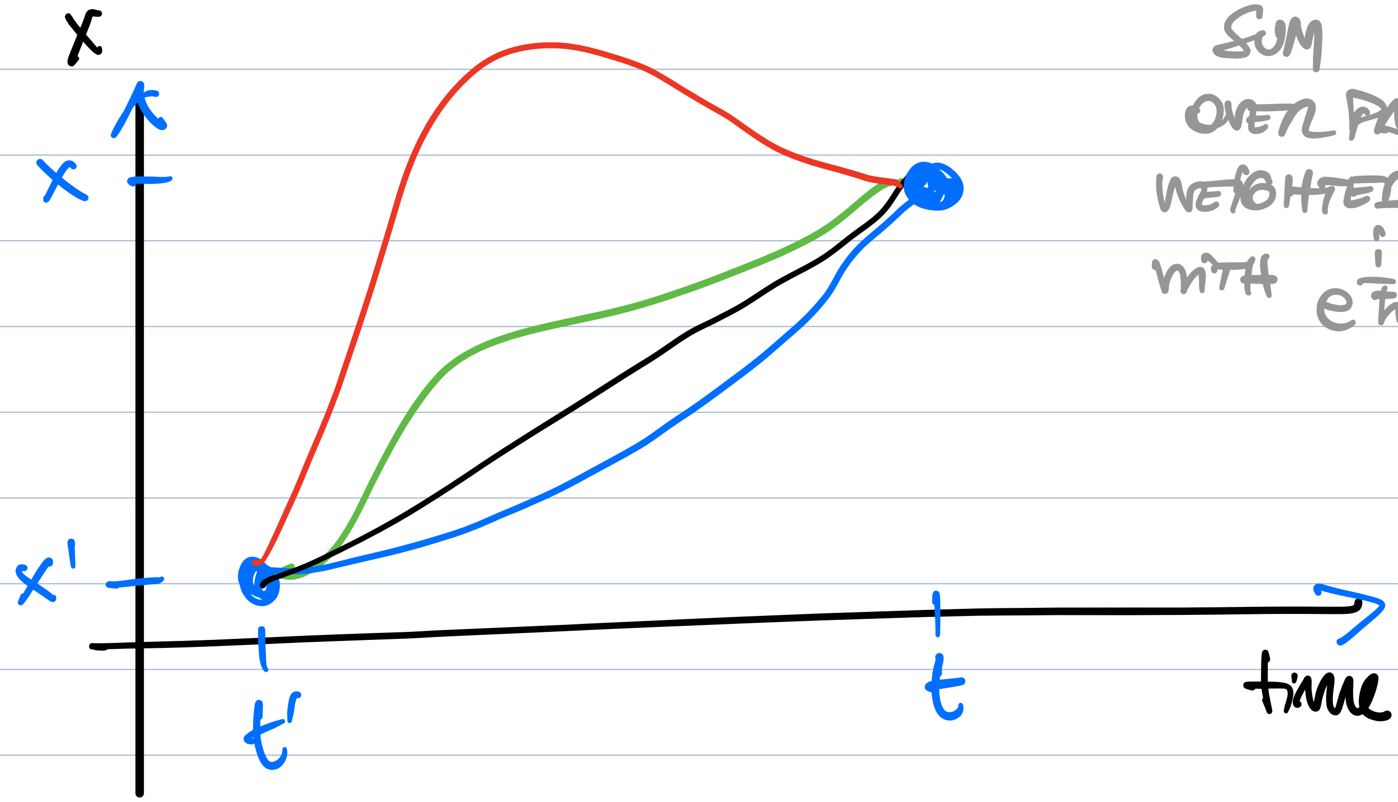
THE MULT FACTOR GOES IN $\mathcal{D}x$

$$S = \int_{t'}^{t} dt'' \; \mathcal{L}(x, \dot{x})$$

LAGRANGIAN

LIKE IN CLASSICAL MECHANICS.

FEYNMANN'S PATH INTEGRAL



SUM OVER PATHS WEIGHTED WITH $e^{\frac{i}{\hbar}S}$

SUM OVER ALL PATHS LINKING x' AND x FROM t' TO t

CONNECTION WITH QUANTUM STAT PHYS

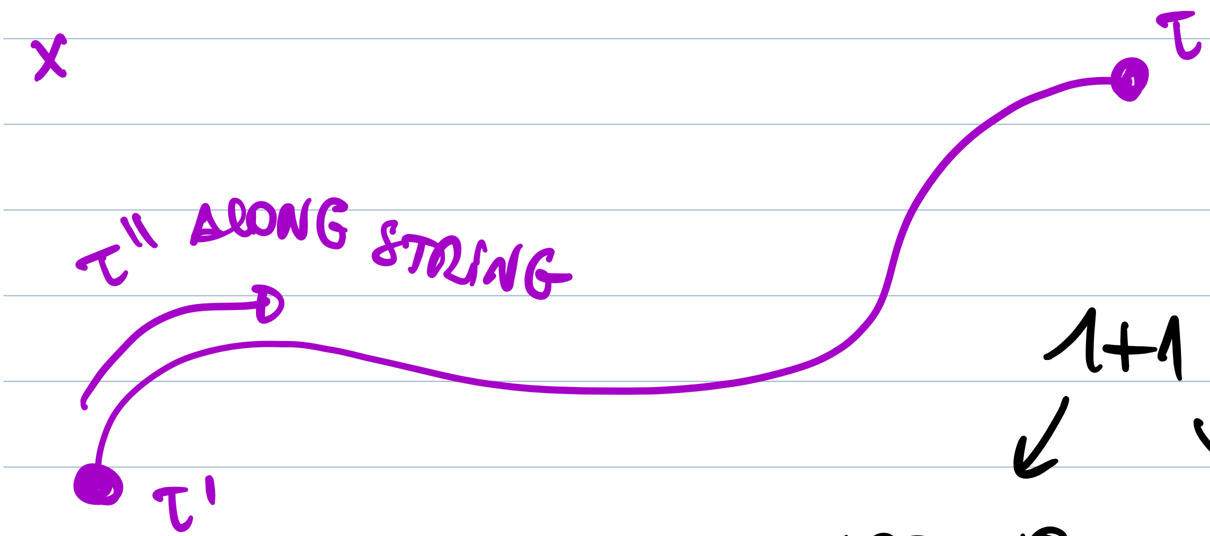
$$\boxed{t \to -i\tau}$$

WICK ROTATION

$$U(x, -i\tau; x', -i\tau') = \int \mathcal{D}x \, e^{-\frac{1}{\hbar}\int_{\tau'}^{\tau} d\tau'' \left[\frac{m}{2}\dot{x}^2(\tau'') + V(x(\tau''))\right]}$$

TO MAKE β APPEAR $\Longrightarrow$ τ'' FROM 0 UP TO $\beta\hbar$

IT'S THE CLASSICAL STATISTICAL MECHANICS PART SUM OF A STRING



1+1 PROBLEM

ON STRING

TRANSV TO STRING

THE BOUNDARY CONDITIONS ARE

$x(\tau') = x'$ $\quad$ $x(\tau) = x$ $\quad$ $x' \neq x$

$\tau' = 0$ $\quad$ $\tau = \beta\hbar$

PARTITION FUNCTION

$$Z = \mathrm{Tr}\, e^{-\beta \hat{\mathcal{H}}}$$

$$= \int dx \, \langle x | e^{-\beta \hat{\mathcal{H}}} | x \rangle$$

THIS IS THE EVOLUTION OPERATOR IN IMAGINARY TIME $t = -i\tau$ OVER AN INTERVAL OF DURATION $\beta\hbar$

$e^{-\frac{i}{\hbar}\hat{\mathcal{H}} t}$ WITH $t \to -i\tau$ ONE HAS

$$e^{-\frac{i}{\hbar}\hat{\mathcal{H}}(-i\tau)} = e^{-\hat{\mathcal{H}}\frac{\tau}{\hbar}}$$

NOW $\tau \to \beta\hbar \Rightarrow e^{-\beta\hat{\mathcal{H}}}$

THE OPERATOR IN Z

SO, IT'S THE EVOLUTION OPERATOR OVER $\tau = \beta\hbar$

WITH PERIODIC BOUNDARY CONDITIONS.

SO,

$$\boxed{Z = \int_{x(0)=x(\beta\hbar)} \mathcal{D}x \, e^{-\frac{1}{\hbar}\underbrace{\int_0^{\beta\hbar} d\tau'' \left[\frac{m}{2}\dot{x}^2(\tau'') + V(x(\tau''))\right]}_{S_E}}}$$

PARTITION FUNCTION IN PATH INT. REPRESENTATION

S_E EUCLIDEAN ACTION.

(TIME & SPACE IN FIELD TH. APPEAR WITH SAME sgn. cfr MINKOWSKY)

NOTE THAT $\mathcal{L}_E$ IS LIKE AN ENERGY. SUM OF KINETIC + POTENTIAL TERMS.

3.2.4 ZERO TEMP LIMIT & GROUND STATE E_0

PURE QUANTUM MECH. CALCULATION

$$\langle x | e^{-\beta \hat{\mathcal{H}}} | x' \rangle \qquad \hat{\mathcal{H}} |n\rangle = E_n |n\rangle$$

$$= \sum_n \langle x | e^{-\beta \hat{\mathcal{H}}} | n \rangle \langle n | x' \rangle$$

$$= \sum_n \langle x | n \rangle e^{-\beta E_n} \langle n | x' \rangle$$

$$= \sum_n e^{-\beta E_n} \psi_n^*(x) \psi_n(x')$$

$$\langle x | e^{-\beta \hat{\mathcal{H}}} | x \rangle = \sum_n e^{-\beta E_n} |\psi_n(x)|^2$$

$$\mathcal{Z} = \int dx \, \langle x | e^{-\beta \hat{\mathcal{H}}} | x \rangle$$

$$= \sum_n e^{-\beta E_n} \underbrace{\int dx \, |\psi_n(x)|^2}_{1 \text{ (NORM)}}$$

$$= \sum_n e^{-\beta E_n}$$

SELECTS THE MINIMAL E_n

THEN, $$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln \mathrm{Tr}\, e^{-\beta \mathcal{H}}$$

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln Z$$

IN PATH INT REP. BY TAKING $\beta\to\infty$; E_0 OBTAINED

$$\boxed{E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln \int_{x(0)=x(\beta\hbar)} \mathcal{D}x\, e^{-\frac{1}{\hbar} S_E}}$$

30'

3.2.5 CLASSICAL LIMIT

NOTE THAT BOTH IN THE $\mathbb{R}$-TIME REP. AS IN THE IMAGINARY TIME ONE, THERE'S A

$$\frac{1}{\hbar}$$

IN THE EXPONENTIAL

$\Longrightarrow$ SADDLE-POINT $\hbar \to 0$

$x_{cl}(t)$ OR $x_{cl}(\tau)$

$\mathbb{R}$ $\mathbb{I}$

STATIONARY PHASE APPROX SADDLE-POINT

DOMINATES THE SUM OVER PATHS.

$$\left.\frac{\delta S}{\delta x(t'')}\right|_{x_{cl}} = 0$$

STATIONARY TRAJECTORY

LIKE IN CLASSICAL MECHANICS

$$\frac{\partial L}{\partial x(t'')} - \frac{d}{dt''}\frac{\partial L}{\partial \dot{x}(t'')} = 0$$

EULER-LAGRANGE EQUATION

FIXES $x_{cl}(t'')$

WITH BOUNDARY VALUES

if GENERAL CASE

$$\begin{cases} x_{cl}(t') = x' \\ x_{cl}(t) = x \end{cases}$$

STABILITY

$$\left.\frac{\delta^2 S}{\delta x(t)\,\delta x(t')}\right|_{x_{cl}}$$

POS. DEF. IN IMAG TIME

3.2.6 QUANTUM CORRECTIONS

EXPAND

$$\frac{i}{\hbar} S[x] = \frac{i}{\hbar} S[x_{cl} + \delta x]$$

$$= \frac{i}{\hbar} S[x_{cl}]$$

$$+ \frac{i}{\hbar} \cancel{\int_{t'}^{t} dt'' \left. \frac{\delta S}{\delta x(t'')} \right|_{x_{cl}} \delta x(t'')} \to 0$$

$$\underbrace{+ \frac{i}{\hbar} \int_{t'}^{t} dt'' \int_{t'}^{t} dt''' \left. \frac{\delta S}{\delta x(t'') \delta x(t''')} \right|_{x_{cl}} \delta x(t'') \delta x(t''')}_{\text{GAUSSIAN CORRECTIONS}}$$

$$+ \dots$$

$$\Longrightarrow \left[\det \frac{i}{\hbar} \frac{\delta S}{\delta x(t'') \delta(t''')} \right]^{-1/2} \quad \text{TO CALCULATE}$$

CHECK SIGN OF POWER $-1/2$ IN $\left[\det\ldots\right]^{\text{POWER}}$

$$\int \frac{dz}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\frac{z^2}{\sigma^2}} = 1 \quad \Rightarrow$$

$$\int dz\, e^{-\frac{z^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}$$

FACTORS 2 AND 2π: DON'T WORRY ABOUT THEM

NOTE THAT $1/\sigma^2$ IN EXP GOES TO $(\sigma^2)^{1/2}$ IN RESULT

$\Rightarrow$ $-1/2$ IS OK ABOVE.

(FROM "NUM." IN GAUSSIAN WEIGHT TO 1/"NUM" IN RESULT OF GAUSS INT)

NB. FACTOR $(i/\hbar)$ WILL GIVE SOME PRE-FACTOR, ONE IGNORES IT WHEN CALC. CORR. FR.

3.2.7 QUANTUM HARMONIC OSC

CANONICAL EQUIL.

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}}$$

FROM PATH - INTEGRAL CALC.

WITHOUT DOING ALL CALCULATIONS
WOULD TAKE TOO LONG

FEYNMAN - HIBBS OR ANY OTHER
PATH - INT. BOOK

$$U(x,t \mid x';0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}}$$

$$\exp\left\{ \frac{im\omega}{2\hbar \sin\omega t} \left[(x^2 + x'^2)\cos\omega t - 2xx' \right] \right\}$$

IN REAL TIME

TAKE NOW $t = -i\tau$, $t' = 0$ & $x = x'$

$$U(x, -i\tau \mid x, 0) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh \omega\tau}}$$

$$\exp\left\{ -\frac{m\omega}{\hbar \sinh \omega\tau} \left[\cosh \omega\tau - 1 \right] x^2 \right\}$$

MOREOVER $\tau = \beta\hbar$ & INT OVER x

$$Z = \frac{1}{\sqrt{2\left[\mathrm{ch}\, \beta\hbar\omega - 1 \right]}}$$

INTEGRAL OF $U(x, -i\tau; x, 0)$ OVER x AND $\tau = \beta\hbar$

$$= \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}$$

$$= \sum_{n=0} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}$$

THE KNOWN RESULT
FOR THE PARTITION FUNCT OF
A QUANTUM HARM OSC.

PROOF $$e^{-\frac{\hbar\omega}{2}} \sum_{n=0} \left(e^{-\beta\hbar\omega}\right)^n$$ GEOM. SERIES

$$e^{-\frac{\hbar\omega}{2}} \cdot \frac{1}{1-e^{-\beta\hbar\omega}}$$

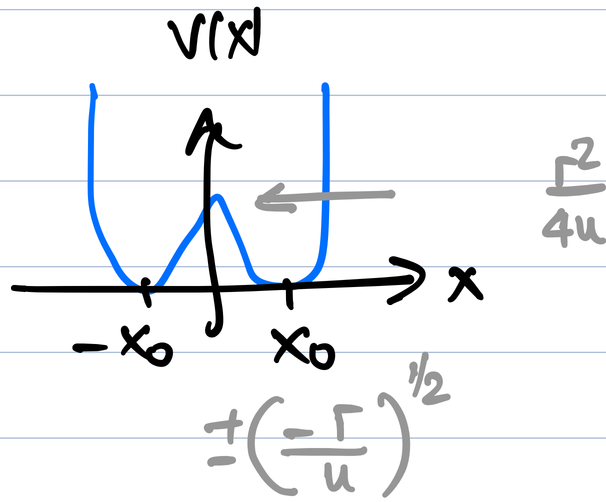
th

ONE OF THE MOST SPECTACULAR APPLIC OF PATH INTEGRALS

3.2.8 TUNNELING & INSTANTONS

PROBLEM

- DOUBLE WELL
- TUNNELING



QUARTIC POTENTIAL

$r<0$

$$\frac{r^2}{4u} + \frac{r}{2}x^2 + \frac{u}{4}x^4$$

$$x_0 = \pm \left(-\frac{\Gamma}{u}\right)^{1/2}$$

$$V(x_0) = 0 \qquad V(x_{max}) = V(0) = \frac{\Gamma^2}{4u}$$

CLASSICALLY $\quad E < V(x_{max})$

OSCILLATIONS IN A WELL

QUANTUM $\quad P(\text{TUNNELING}) > 0$

CALCULATE THE TRANSITION AMPLITUDE WITH THE PATH INTEGRAL METHOD.

IN REAL TIME TUNNELING IS CLASSICALLY FORBIDDEN

IF WE GO TO IMAGINARY TIME TUNNELING WILL BECOME POSSIBLE.

TRANSITION AMPLITUDE IN PATH INT

$$U(x,\tau;\, x',\tau') = \int_{x'}^{x} \mathcal{D}x \; e^{-\frac{1}{\hbar} S_E}$$

SUM OVER ALL TRAJECTORIES; BUT IN THE

SEMI-CLASSICAL LIMIT $\hbar \to 0$

SADDLE POINT EVALUATION $\frac{1}{\hbar} \to \infty$

(PLAYS THE ROLE OF N IN STAT PHYS)

$$\sim \underbrace{e^{-\frac{1}{\hbar} S_{E\,cl}}}_{U_{cl}} \; \underbrace{\left[\det \frac{\delta^2 S_E}{\delta x(\tau)\, \delta x(\tau')} \Big|_{x_{cl}} \right]^{-1/2}}_{U_q}$$

$$S_{E_{cl}} = S_E(x_{cl}(\tau))$$

"CLASSICAL TRAJECT"

MINIMIZER OF $S(x_{cl})$.

$$\frac{\delta S_E}{\delta x(\tau)} = 0 \quad \text{AT} \quad \underbrace{x_{cl}(\tau)}_{\text{THE INSTANTON}}$$

GOES FROM ONE WELL TO THE OTHER

2nd FACTOR U_q GAUSSIAN FUNCT.

WE NOTE THAT FIRST "CLASSICAL" FACTOR IS EXPONENTIAL WHILE U_q APPEARS IN NUMERATOR $\longrightarrow$ LESS IMPORTANT BUT TO BE CONSIDERED!

$$S_E^{cl} = \int_{\tau'}^{\tau} d\tau'' \left[m \frac{(\dot{x}(\tau''))^2}{2} + V(x(\tau'')) \right]$$

$$\frac{\delta S_E^{cl}}{\delta x(\tau)} = 0 = -m\ddot{x}(\tau) + V'(x(\tau))$$

NON TRIVIAL (NON CONST) SOLUTION

INSTANTON :

SAME EQ AS DOMAIN WALL IN GINZBURG LANDAU $\lambda\phi^4$ FIELD TH.

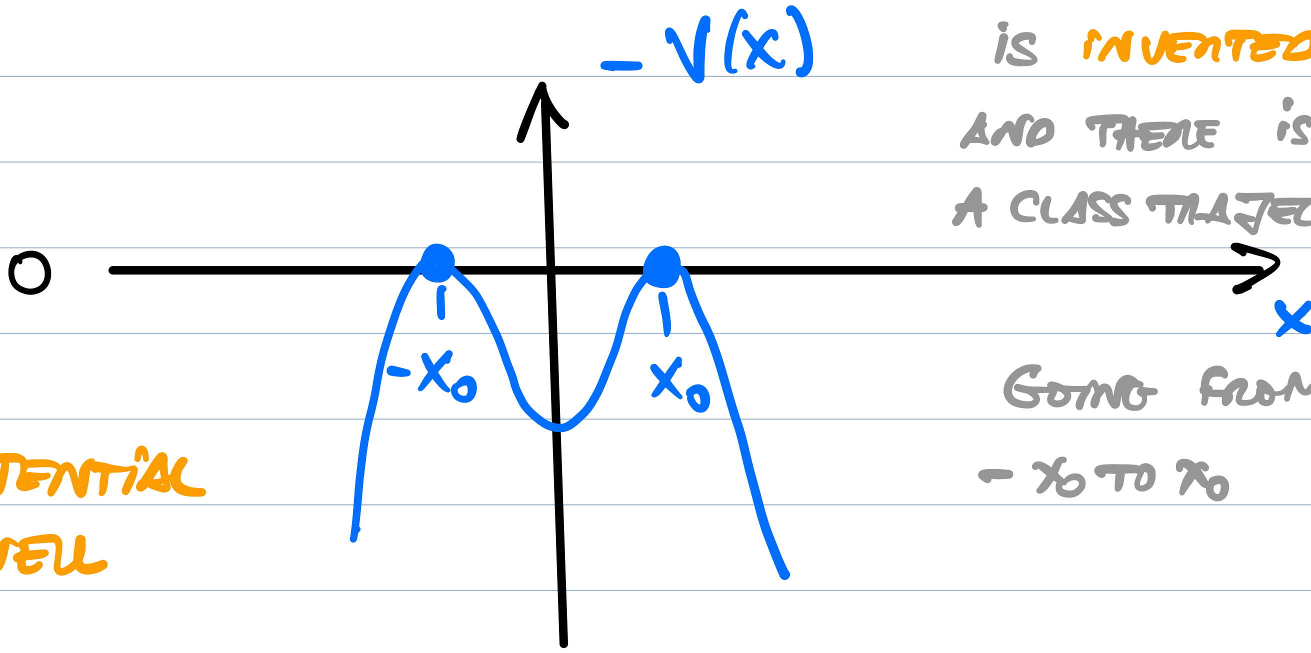
CLASSICAL TRAJECTORY IN IM. TIME

MOTION IN THE INVERTED POT $-V(x)$

REMARK : A BIT LIKE THE VORTICES IN 2d XY.
TOPOLOGICAL SOL. (FIXED BY BOUNDARY CONDITIONS) ARE IMPORTANT

NOTATION IN THE EQS BELOW I CALL τ THE TIME WHICH WILL RUN FROM τ' TO T_0. TO SIMPLIFY THE NOTATION

THE EQ. TO SOLVE : $m \dfrac{d^2 x(\tau)}{d\tau^2} = V'(x(\tau))$



THE POT IS INVERTED AND THERE IS A CLASS TRAJECT. GOING FROM $-x_0$ TO x_0

POTENTIAL WELL

PARTICLE ROLLING FROM $-x_0$ TO x_0
WITH $m = 1$

$x' = x(\tau') = -x_0$ TOWARDS

$x = x(\tau) = x_0$

$$m\ddot{x} = V'(x) = rx + ux^3$$

CLASSICALLY ALLOWED TRAJ. ROLLING FROM $-x_0$ TO x_0 $\qquad r < 0$

WKB RESULT

IN IMAGINARY TIME

$$m\ddot{x}\cdot\dot{x} = V'(x)\,\dot{x}$$

$$\frac{m}{2}\frac{d\dot{x}^2}{d\tau} = \frac{dV(x)}{d\tau}$$

$$\frac{d}{d\tau}\left(\frac{m\dot{x}^2}{2} - V(x)\right) = 0$$

$$\frac{m\dot{x}^2}{2}\underbrace{- V(x)}_{\text{REVERSE POT}} = E$$

WE SET $E=0$ SINCE $\dot{x}(+\infty) = \dot{x}(-\infty) = 0$

$V(x_0) = 0 = V(-x_0)$

$$\frac{m\dot{x}^2}{2} = V(x)$$

NOW IN ACTION

$$S_E^{cl} = \int_{-\infty}^{\infty} d\tau'' \left[\frac{m}{2} \left(\dot{x}(\tau'') \right)^2 + V\left(x(\tau'') \right) \right]$$

$=$

$$= m \int_{-\infty}^{\infty} d\tau'' \left(\dot{x}(\tau'') \right)^2$$

$$= m \int_{-\infty}^{\infty} \underbrace{d\tau \;\; \dot{x}(\tau'')}_{dx} \sqrt{\frac{2V(x(\tau''))}{m}}$$

$$= m \int_{-x_0}^{x_0} dx \sqrt{\frac{2V(x)}{m}}$$

$$\boxed{S_E^{cl} = \int_{-x_0}^{x_0} dx \sqrt{2m V(x)}}$$

WKB

SOLUTION TO THE EQ. MOTION $\forall \tau$

$$\boxed{x(\tau) = x_0 \; \mathrm{th}\left(\frac{\tau-\tau_0}{\tau_w}\right)}$$

RECALL DOMAIN WALL

CHECK IT'S A SOLUTION & FIX τ_w

$$\mathring{x} = \frac{x_0}{\tau_w}\left[\frac{\mathrm{ch}\,y\;\mathrm{ch}\,y - \mathrm{sh}\,y\;\mathrm{sh}\,y}{\mathrm{ch}^2 y}\right]$$

WHERE WE USED

$$\frac{d\,\mathrm{ch}\,y}{dy} = \frac{e^y - e^{-y}}{2} = \mathrm{sh}\,y$$

$$\frac{d\,\mathrm{sh}\,y}{dy} = \frac{e^y + e^{-y}}{2} = \mathrm{ch}\,y$$

$$\Longrightarrow \quad \mathring{x} = \frac{x_0}{\tau_w}\;\frac{1}{\mathrm{ch}^2\,\frac{\tau-\tau_0}{\tau_w}}$$

TAKING $\frac{d}{d\tau}$ AGAIN

$$\ddot{X} = \frac{x_0}{\tau_w} \frac{(-2)\cancel{ch \frac{\tau-\tau_0}{\tau_w}} \frac{1}{\tau_w}}{ch^{\cancel{4}3} \frac{\tau-\tau_0}{\tau_w}} sh \frac{\tau-\tau_0}{\tau_w}$$

$$= \frac{-2x_0}{\tau_w^2} th \frac{\tau-\tau_0}{\tau_w} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}}$$

BACK IN EQ :

$$-\frac{2\cancel{x_0}}{\tau_w^2} \cancel{th \frac{\tau-\tau_0}{\tau_w}} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}} \stackrel{?}{=}$$

$$r \cancel{x_0} \cancel{th \frac{\tau-\tau_0}{\tau_w}} + u x_0^{\cancel{3}2} th^{\cancel{3}2} \frac{\tau-\tau_0}{\tau_w}$$

$$\Rightarrow \frac{-2}{\tau_w^2} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}} \stackrel{?}{=} r + u x_0^2 th^2 \frac{\tau-\tau_0}{\tau_w}$$

$$\stackrel{?}{=} \frac{\Gamma \, ch^2 \frac{\tau-\tau_0}{\tau_W} + u x_0^2 \, sh^2 \frac{\tau-\tau_0}{\tau_W}}{ch^2 \frac{\tau-\tau_0}{\tau_W}}$$

if $\boxed{-\Gamma = u x_0^2} \Rightarrow$ WHICH IT IS, WE CHOSE THE INITIAL AND FINAL POSITION IN THIS WAY

RECALL $\Gamma < 0$

NUM $= \Gamma$

DEN CANCELS WITH THE ONE IN lhs

$\Rightarrow$

$$\frac{2}{\tau_W^2} = -\Gamma \Rightarrow$$

$$\boxed{\tau_W^2 = \frac{-2}{\Gamma}}$$

IF WE FURTHER REQUIRE

$$x(\tau') = -x_0 = x_0 \underbrace{\mathrm{th}\left(\frac{\tau'-\tau_0}{\tau_W}\right)}_{\text{WE NEED THIS FACTOR TO BE } -1}$$

$$x(\tau) = x_0 = x_0 \underbrace{\mathrm{th}\left(\frac{\tau-\tau_0}{\tau_W}\right)}_{\text{SHOULD BE } 1}$$

SO, WE NEED

$$\boxed{\begin{aligned} \tau' &\to -\infty \\ \tau &\to \infty \end{aligned}}$$

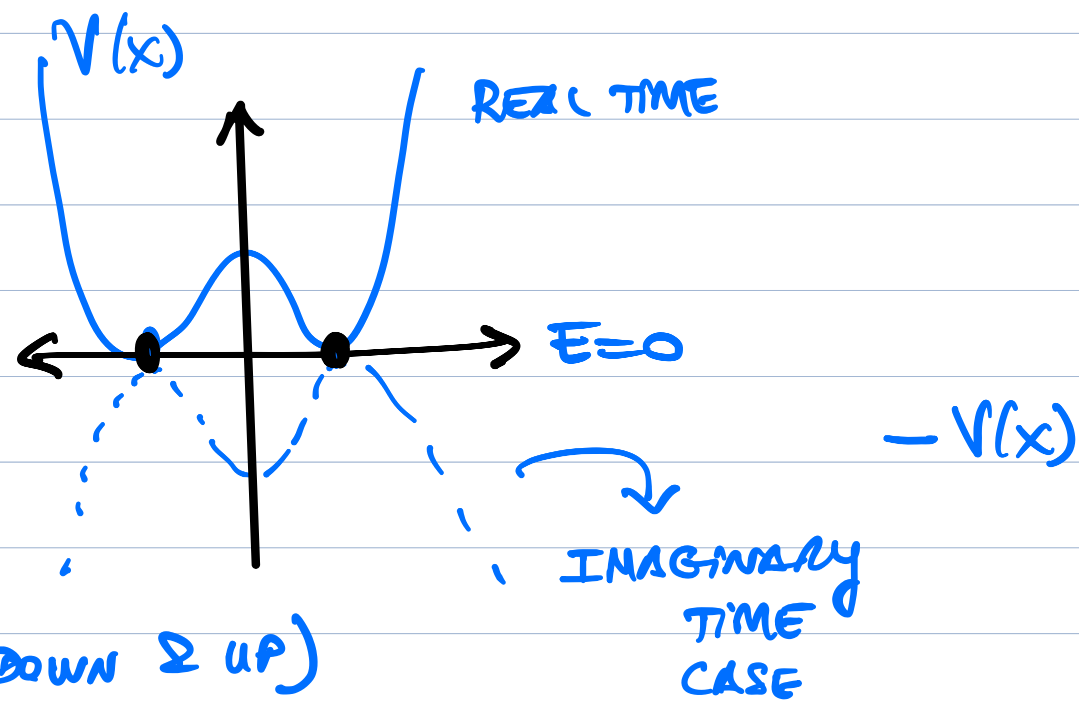
RECALL WHAT IS $x_0^2 = -\frac{r}{u}$

$$V(x) = \frac{r x^2}{2} + \frac{u}{4} x^4 + \text{CONST}$$

$$V'(x) = r x + u x^3 = 0 \implies$$

$$x = 0 \quad \text{OR} \quad x^2 = -\frac{r}{u}$$

$\Rightarrow$ THEY ARE THE MINIMA OF $V(x)$ OR THE MAXIMA OF THE INVERTED $-V(x)$

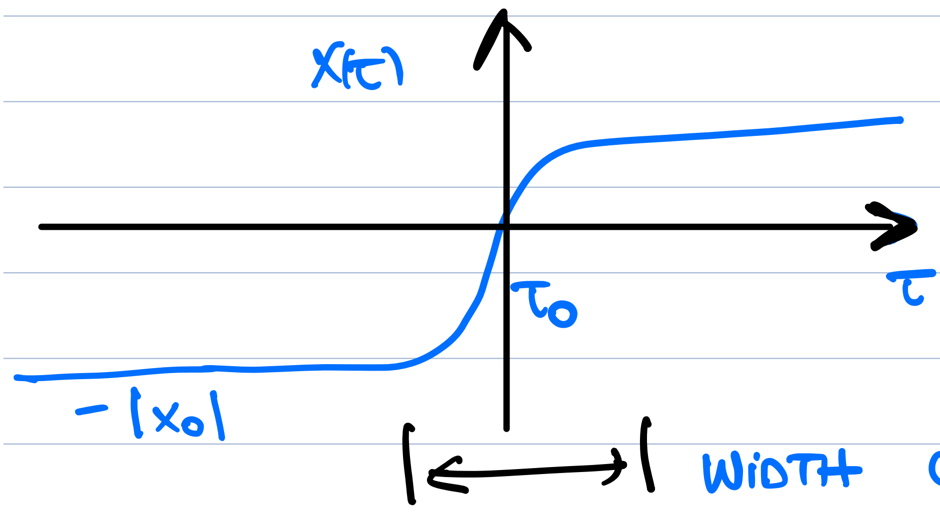


THE PART LEAVES $-|x_0|$ & GOES TO $+x_0$ (ROLLING DOWN & UP)

KEEPING E=0 ON TRAJECTORY

THE INSTANTON CONF

NB τ_0 is FREE
SYMM UNDER TRANSL OF INSTANTS



WIDTH CONTROLLED BY $\tau_W^2 = -\frac{2}{r}$

$\Rightarrow$ SHOULD INTEG OVER τ_0 & ALL # OF TUNNEL.

THE PARTICLE MOVES FROM LEFT TO RIGHT WELL IN AN "INSTANT" OF DURATION $\sqrt{-2/r}$ (TOP OF BARRIER)

THE CLASSICAL ACTION OF THE INSTANTON IS :

$$\boxed{S_E^{cl} = \frac{4x_0^2}{3\tau_W}}$$

JUST REPLACE x_{cl} IN $S(x_{cl})$

- INDEP OF τ_0
- FINITE

IMPORTANT PROPERTIES

CALCULATE S_E^{cl}

$$S_E^{cl} = \int_{-\infty}^{\infty} d\tau'' \left[\frac{1}{2}\left(\dot{x}_{cl}(\tau'')\right)^2 + V(x_{cl}(\tau'')) \right]$$

SHOW $\frac{1}{2}\dot{x}_{cl}^2(\tau) = V(x_{cl}(\tau))$

EQ MOTION

$$\ddot{x}_{cl}(\tau) = V'(x_{cl}(\tau))$$

MULT BY $\dot{x}_{cl}(\tau)$

$$\ddot{x}_{cl}\,\dot{x}_{cl} = V'(x_{cl})\,\dot{x}_{cl}$$

INTEG. OVER τ

$$\frac{1}{2}\dot{x}_{cl}^2 = V(x_{cl}) + \underbrace{\text{CONST}}_{\to\, 0}$$

SET $x_{cl} = \pm x_0$

$$S_{E_{cl}} = \int d\tau'' \left(\dot{x}_{cl}(\tau)\right)^2$$

THE TWO TERMS ARE =

NOW, FOR $x_{cl}(\tau) = x_0 \, th\left(\frac{\tau-\tau_0}{\tau_W}\right)$

$$\dot{x}_{cl}(\tau) = \frac{x_0}{\tau_W} \cdot \frac{ch^2 \frac{\tau-\tau_0}{\tau_W} - sh^2 \frac{\tau-\tau_0}{\tau_W}}{ch^2\left(\frac{\tau-\tau_0}{\tau_W}\right)}$$

$$= \frac{x_0}{\tau_W} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_W}}$$

REPLACE:

$$S_E^{cl} = \int_{\tau'}^{\tau} d\tau'' \left(\frac{x_0}{\tau_W}\right)^2 \frac{1}{ch^4 \frac{\tau-\tau_0}{\tau_W}}$$

$$= \left(\frac{x_0}{\tau_W}\right)^2 \int_{u'\to -\infty}^{u\to\infty} du'' \tau_W \frac{1}{ch^4 u''}$$

$u = \frac{\tau-\tau_0}{\tau_W}$

$$= \frac{x_0^2}{\tau_W} \cdot \frac{4}{3}$$

$du = \frac{d\tau}{\tau_W}$

THE INSTANTON HAS FINITE ACTION.

RECALL THAT IN A SADDLE-POINT APPROX IF WE HAVE MANY MINIMA,

WE HAVE TO SUM OVER ALL

$\Longrightarrow$ INTEGRAL OVER τ_0

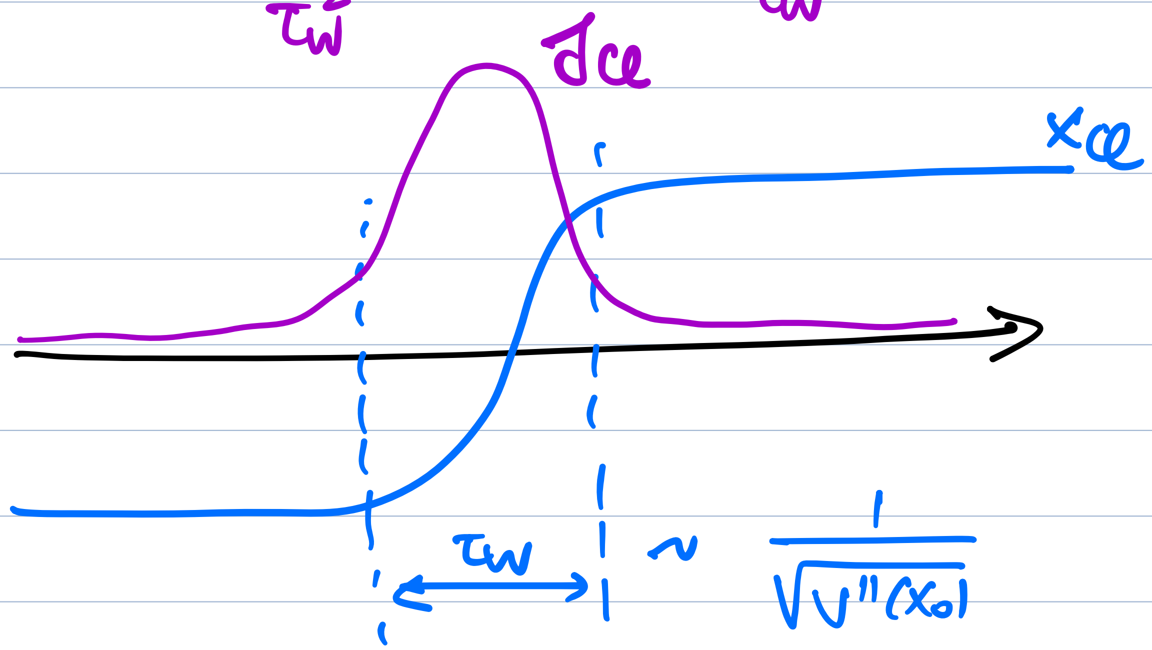
Summary

$$x_{cl}(\tau) = x_0 \tanh \frac{\tau-\tau_0}{\tau_w} \qquad \begin{array}{l} \tau' \to -\infty \\ \tau \to \infty \end{array}$$

$$\tau_w^2 = -\frac{2}{\Gamma}$$

$\mathcal{S}_{cl}$: DENSITY OF ACTION $= (\dot{x}_{cl})^2$

$$= \frac{x_0^2}{\tau_w^2} \operatorname{sech}^4 \frac{\tau-\tau_0}{\tau_w}$$



$$\tau_w \sim \frac{1}{\sqrt{V''(x_0)}}$$

$$S_E^{cl} = \frac{4}{3}\frac{x_0^2}{\tau_w} = \frac{4}{3} x_0^2 \sqrt{\frac{-\Gamma}{2}}$$

FOR QUARTIC POT.

THE TRANS. PROB WE WANT TO CALC. IS

$$U(x_0, \tau \to \infty ; -x_0, \tau' \to -\infty)$$

$$\propto \exp\left(-\frac{1}{\hbar} S_E^{cl}\right)$$

FOR THE MOMENT ONE INSTANTON

$$S_E^{cl} = \frac{4}{3} \frac{x_0^2}{\tau_W} \qquad \tau_W = \sqrt{\frac{-2}{r}}$$

WE IDENTIFIED A TOPOLOGICAL CONF.

THE INSTANTON

FINITE S_E^{cl}

CANNOT BE SMOOTHLY DEFORMED TO $x(\tau) = ct.$

LARGELY DETERMINED BY BORDER COND $x(-\infty) = -x_0$

$x(\infty) = x_0$

RECALL VORTICES IN 2d XY

ZERO MODE τ_0 IRRELEVANT

S_E^{cl} DOESN'T DEP. ON τ_0

WE CONSIDERED ONE INSTANTON. WHY NOT MANY ?

THEY ARE "LOCALIZED" IN THE SENSE OF SMALL WIDTH τ_W

HAVE TO SUM OVER ALL SADDLE POINTS WHICH COULD BE MADE OF MANY INSTANTONS

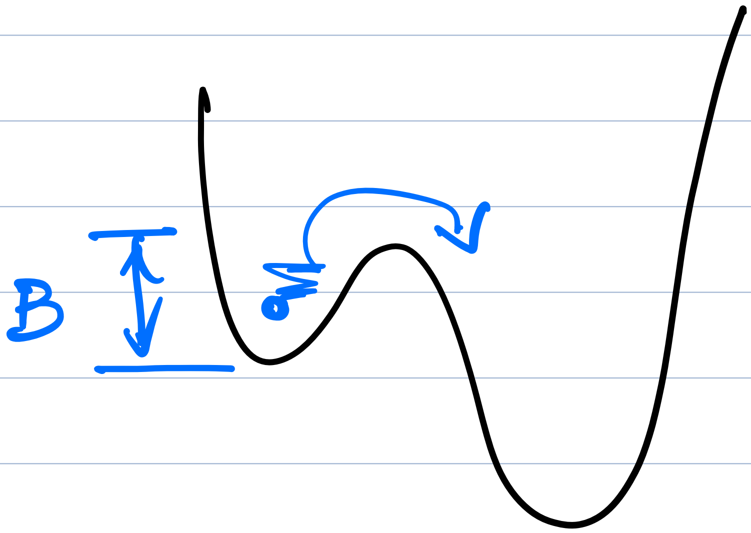
eg. A GAS OF INSTANTONS

THEY ALL CONTRIBUTE THE SAME S_E^{cl}

ALSO CONSIDER GAUSSIAN FLUCT AROUND EACH. S.P.

ARRHENIUS LAW

JUMP OVER A BARRIER WITH
CLASSICAL STOCHASTIC DYN.



ESCAPE FROM
METASTABLE
STATE

$$t_A \sim \tau_m \, e^{\frac{B}{k_B T}}$$

B = HEIGHT BARRIER

$k_B T$ = PLAYS ROLE OF $\hbar$

SEMICLASSICAL $\approx$ LOW T

τ_m = MICROSC ATTEMPT TIME

NB NOISE IS NEEDED TO KICK THE PART OVER THE BARRIER

PROBLEM SOLVED BY KRAMERS W/ ARGUMENTS FROM FP-LIKE EQ.

PATH INTEGRAL CALCULATION (À LA WIENER)

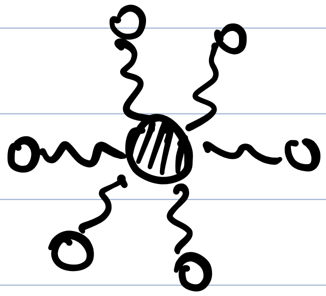
CAROLI, CAROLI, ROULET (81)

zh

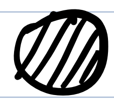
3.2.9. REDUCED SYSTEM

INFLUENCE FUNCTIONAL

RECALL DERIVATION OF LANGEVIN EQ IN 1st TD



PARTICLE m



POSITION x, MOMENTUM $p = mv$

COUPLED TO AN ENSEMBLE OF N HARMONIC OSCILLATORS

m_a, ω_a, q_a, π_a

ALL IN $d=1$ TO SIMPLIFY NOTATION

$H = H_{syst} + H_{BATH} + H_{INT} + H_{COUNTER}$

$H_{syst} \downarrow$ PART

$H_{BATH} \downarrow$ OSC

$H_{INT} \downarrow$ INTERACTION

- $H_{COUNTER}$ TO AVOID PART QUAD. TERM "RENORM."
  $-\mathbb{C}\, x^2$ WITH $\mathbb{C}(m_a, \omega_a^2, c_a)$

- WE SET NEWTON'S EQ FOR THE COUPLED SYSTEM

- WE SOLVED FOR OSC

- WE INTRODUCED SOL IN PARTICLE'S EQ.

ALL DETERMINISTIC 'TILL HERE

IMPORTANT

WE CLAIMED OSC'S INITIAL COND DRAWN FROM BOLTZMANN'S WEIGHT AT TEMP T IN PRESENCE OF PART'S INITIAL STATE (COUNTER)

$\Rightarrow$ STOCHASTICITY

WE OBTAINED

GENERALIZED LANGEVIN EQS.

WITH MEMORY - RETARDED FRICTION &
COLOURED NOISE

MULTIPLICATIVE NOISE - DEPENDING ON
COUPLING.

BI-LINEAR $x\, q_a \Rightarrow$ ADDITIVE

GENERAL $f(x)\, q_a \Rightarrow$ MULTIPLICATIVE

NB WE COULD HAVE BEEN INTERESTED IN

$$Z_{TOT} = \sum_{\{SYST\}} \sum_{\{BATH\}} e^{-\beta H_{TOT}}$$

$$Z_{RED} = \sum_{\{SYST\}} e^{-\beta H_{RED}}$$

"REDUCED" PART FUNCTION

QUANTUM MECHANICALLY ?

WE CAN ASK THE DYN. QUESTION

REAL TIME EVOL. OF SYST COUPLED TO BATH

OR THE EQUIL QUESTION

$$Z_{RED}$$

IN THESE LECTURES, MOSTLY EQ $\longrightarrow$ Z_{RED}

QUANTUM PARTICLE $\hat{x}$, $\hat{p}$ (m)

QUANTUM OSC. $\hat{q}_a$, $\hat{\pi}_a$ (m_a, ω_a)

$$[\hat{\pi}_a, \hat{q}_b] = -i\hbar\,\delta_{ab}$$

$$[\hat{x}, \hat{p}] = +i\hbar$$

DENSITY OPERATOR

$$\rho_{TOT}(x, q_a; x', q_a') =$$

$$= \frac{1}{Z_{TOT}} \langle x, q_a | e^{-\beta \hat{H}_{TOT}} | x', q_a' \rangle$$

$$Z_{TOT} = \mathrm{Tr}\; e^{-\beta \hat{H}_{TOT}}$$

USE THE PATH INTEGRAL LAGRANGIAN REPRESENTATION OF ρ_{TOT}

$$\rho_{TOT}\left(x, q_a ; x', q_a'\right) =$$

$$\frac{1}{Z_{TOT}} \int_{x(0)=x'}^{x(\beta\hbar)=x} \mathcal{D}x \int_{q_a(0)=q_a'}^{q_a(\beta\hbar)=q_a} \mathcal{D}q_a \; e^{-\frac{1}{\hbar} S_E}$$

$$S_E = S_E^{SYST} + S_E^{BATH} + S_E^{INT} + S_E^{COUNTER}$$

THE ACTION OF THE BATH IS

$$S_E^{BATH} = \int_0^{\beta\hbar} d\tau \left\{ \frac{m \dot{q}_a^2(\tau)}{2} + \frac{m_a \omega_a^2}{2} q_a^2(\tau) \right\}$$

THE INT. ONE

$$S_E^{INT} = \int^{\beta\hbar} d\tau \quad x(\tau) \quad 1 \sum^{N} C_a \, q_a(\tau)$$

$$N^{1/2} \quad a=1$$

BILINEAR COUP; COUPLING CONST SCALED W/ N SO AS TO HAVE AN INTERESTING RESULT

SIMPLEST CHOICE, $f(x)$ ALSO POSSIBLE

ONE CAN NOW SIMPLY INTEGRATE AWAY THE OSC. TO FIND

$$S_E^{RED} = S_E^{syst} + S_E^{\text{BATH EFFECT}}$$

RECALL THAT $q_a(\tau)$ ARE PERIODIC FUNCTIONS OF IMAGINARY TIME OVER $\beta\hbar$

$$q_a(0) = q_a(\beta\hbar) \quad \forall a$$

FOURIER REPRESENTATION - MATSUBARA FREQ.

$$q_a(\tau) = \sum_{n=-\infty}^{\infty} q_a^{(n)} e^{i\nu_n \tau}$$

$$\nu_n = \frac{2\pi n}{\beta\hbar}$$

PERIODICITY OVER FINITE τ- INTERVAL $\Rightarrow$ DISCRETE FREQ

AFTER SOME STRAIGHTFORWARD BUT LENGHTY STEPS

$$\rho_{RED}(x,x') = \mathrm{Tr}_{env.}\ \rho_{TOT}(x, q_a; x', q_a')$$

$$= \frac{1}{Z_{TOT}} \int_{x(0)=x'}^{x(\beta\hbar)=x} \mathcal{D}x\ e^{-\frac{1}{\hbar} S_E^{syst}}$$

$$\underbrace{e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'\ x(\tau)\, k(\tau-\tau')\, x(\tau')}_{\text{FROM INT OVER BATH-OSCILL. VAR.}}$$

WITH

$$k(\tau) = \frac{2}{\pi\beta\hbar} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\omega\ \frac{S(\omega)}{\omega}\ \frac{\nu_n^2}{\nu_n^2+\omega^2}\ e^{i\nu_n\tau}$$

$S(\omega)$ SPECTRAL DENSITY

$$S(\omega) = \frac{1}{N}\sum_{a=1}^{N} \frac{C_a^2}{m_a \omega_a^2}\,\delta(\omega-\omega_a)$$

PROPOSAL:

$$\frac{S(\omega)}{\omega} = 2\gamma_0 \left(\frac{|\omega|}{\tilde{\omega}}\right)^{\alpha-1} f_c\left(\frac{|\omega|}{\Lambda}\right)$$

$\alpha = 1$ OHMIC $\quad$ $\alpha < 1$ SUB OHMIC

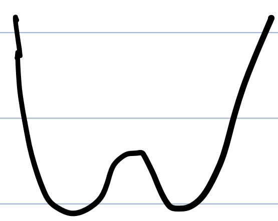
$\alpha > 1$ SUPER OHMIC

TYPICALLY $k(\tau)$ LONG-RANGE IN τ

STRONG EFFECTS ON PARTICLE'S BEHAVIOUR !

eg. LOCALIZATION

BRAY MOORE
CALDEIRA-
LEGGETT



$\alpha = 1$ $\quad$ γ_0

↑ SAY PART INITIALLY HERE

TUNNELING SUPPRESSED FOR SUFFICIENTLY STRONG COUPLING TO OHMIC BATH

3.3 GENERIC PROPERTIES OF QUANTUM OBSERV. CORRELATIONS & LINEAR RESPONSES

EXP. MEASUREMENTS

LINEAR RESP. ELECT CONDUCTIVITY
THERMAL CONDUCTIVITY
etc.

GENERIC PROPS. VALID $\forall$ SYSTEMS
VERY IMPORTANT TO KNOW THEM
& DON'T BREAK THEM
NUMERICALLY OR WITH APPROX.

FOCUS ON SYST WITH $\hat{H} \neq \hat{H}(t)$
NO EXPLICIT TIME DEP.

PICTURE: USE HEISENBERG'S

OPERATOR DEPEND ON TIME

EXPECTATION VALUES

FIRST, "ONE TIME" OBSERVABLES

$$\hat{A}(t) = e^{i\hat{H}t/\hbar}\,\hat{A}\,e^{-i\hat{H}t/\hbar}$$

NO EXPLICIT t DEP. IN $\hat{A}$

EXPECTATION OF $\hat{A}$ AT TIME t

$$\langle \hat{A}(t) \rangle \equiv \mathrm{Tr}\, \hat{A}(t)\, \hat{\rho}_0$$

$\mathrm{Tr}\,\hat{\rho}_0 = 1$ ASSUMED HENCEFORTH

$$\hat{\rho}_0 = \frac{e^{-\beta\hat{H}}}{Z}$$

TAKE BOLTZMANN EQ.

$$\langle \hat{A}(t) \rangle = \frac{1}{Z}\,\mathrm{Tr}\; e^{i\hat{H}t/\hbar}\,\hat{A}\,e^{-i\hat{H}t/\hbar}\;e^{-\beta\hat{H}}$$

COMMUTE

TRACE ALLOWS US TO MOVE IT HERE

$$\langle \hat{A}(t) \rangle = \frac{1}{Z}\,\mathrm{Tr}\,\hat{A}\,e^{-\beta\hat{H}}$$

INDEP. OF TIME

IF SYST IN EQ. $\Longrightarrow$ TIME INDEP. OBSERV. HAVE EXP. VALUES WHICH ARE INDEP OF t

ANY OTHER $\rho_0(\hat{H}) \Longrightarrow$ SAME t- INDEP OF $\langle \hat{A}(t) \rangle$

GREEN FUNCTIONS VS. CORRELATION FUNCTIONS

$\downarrow$

USED TO NAME
CORR FUNCTIONS OF
a, a^+ SAY OR
$\hat{\phi}, \hat{\phi}^+$

$\downarrow$

MORE GENERAL
CORR ANY OBSERV.

TWO TIME OBSERV.

$$C_{AB}(t,t') \equiv \frac{\mathrm{Tr}\, \mathcal{T}\, \hat{A}(t)\, \hat{A}(t')\, e^{-\beta \hat{H}}}{\mathrm{Tr}\, e^{-\beta \hat{H}}}$$

$\mathcal{T}$ TIME ORDERED OPERATOR

$$\mathcal{T}\, \hat{A}(t)\, \hat{A}(t') = \hat{A}(t)\, \hat{A}(t')\, \theta(t-t') + \hat{A}(t')\, \hat{A}(t)\, \theta(t-t')$$

CAREFUL WITH $t = t'$ $\quad \theta(0) = 1/2$

STATIONARITY

TAKE $t > t'$

$$C_{AB}(t,t') = \frac{1}{Z} \mathrm{Tr}\, e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{i\hat{H}t'/\hbar} \hat{B} e^{-i\hat{H}t'/\hbar} e^{-\beta\hat{H}}$$

AGAIN, USING CYCLIC PROPS OF Tr

$$C_{AB}(t,t') = \frac{1}{Z} \mathrm{Tr}\, e^{i\hat{H}(t-t')/\hbar} \hat{A} e^{-i\hat{H}(t-t')/\hbar} e^{-\beta\hat{H}}$$

ALL TIME DEP. IS OF THE FORM $t-t'$

$$= \bar{C}_{AB}(t-t')$$

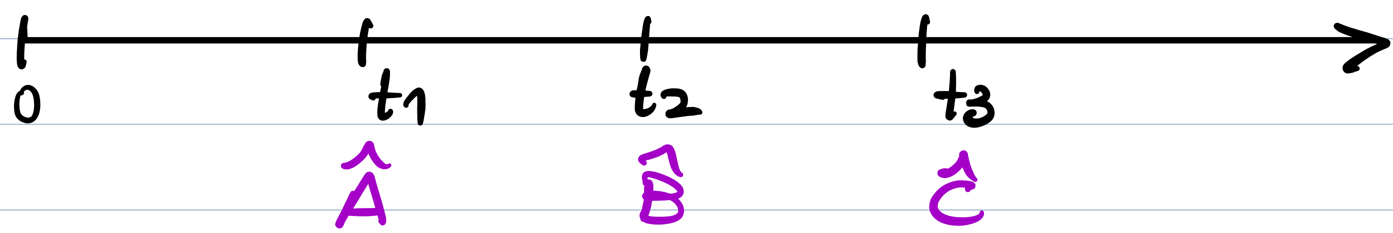
STATIONARITY OR

TIME TRANSLATION INVARIANCE

$$C_{AB}(t+\delta t,\ t'+\delta t) = C_{AB}(t, t')$$

TRANSLATE ALL TIMES BY THE SAME δt

THREE (OR MORE) TIMES



$$C_{ABC}(t_1, t_2, t_3) = \mathrm{Tr}\ \hat{C}(t_3)\,\hat{B}(t_2)\,\hat{A}(t_1)\,\hat{\rho}_0 =$$

$$= \mathrm{Tr}\ e^{i\hat{H}t_3/\hbar}\,\hat{C}\,e^{-i\hat{H}t_3/\hbar}$$
$$e^{i\hat{H}t_2/\hbar}\,\hat{B}\,e^{-i\hat{H}t_2/\hbar}$$
$$e^{i\hat{H}t_1/\hbar}\,\hat{A}\,e^{-i\hat{H}t_1/\hbar}\ \hat{\rho}_0$$

USE $\quad \hat{\rho}_0 = e^{-\beta \hat{H}} / Z$

$$= \mathrm{Tr} \; e^{i\hat{H} t_3/\hbar} \, \hat{C} \, e^{-i\hat{H}(t_3-t_2)/\hbar} \, \hat{B}$$

$$e^{-i\hat{H}(t_2-t_1)/\hbar} \, \hat{A} \, e^{-i\hat{H} t_1/\hbar} \; e^{-\beta \hat{H}}$$

$$= \mathrm{Tr} \; \hat{C} \, e^{-i\hat{H}(t_3-t_2)/\hbar} \, \hat{B} \, e^{-i\hat{H}(t_2-t_1)/\hbar}$$

$$\hat{A} \, e^{-i\hat{H}(t_1-t_3)/\hbar} \; e^{-\beta \hat{H}}$$

$$= \overline{C}_{ABC}(t_3-t_2, t_2-t_1, t_1-t_3)$$

$$= \overline{C}_{ABC}(\underbrace{t_3-t_2}_{\tau_{32}}, \underbrace{t_2-t_1}_{\tau_{21}}, t_1-t_2+t_2-t_3)$$

$$-\tau_{32}-\tau_{12}$$

$$= \bar{\bar{C}}_{ABC}(t_3 - t_2, t_2 - t_1)$$

TIME-TRANS INV.

$$t_3 \to t_3 + \delta t$$

$$t_2 \to t_2 + \delta t$$

$$t_1 \to t_1 + \delta t$$

NO CHANGE UNDER THESE TRANSLATIONS

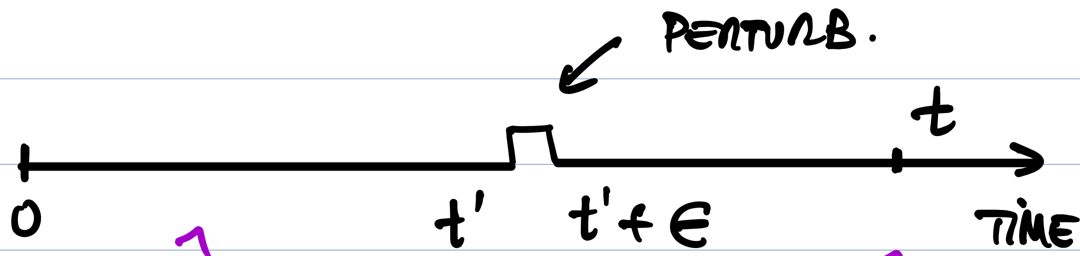
KUBO FORMULA

LINEAR RESPONSE THEORY

WRITING THE LINEAR RESPONSE AS A CORRELATION FUNCTION

LINEAR RESPONSE

VARIATION OF AN EXPECTATION OF AN OBSERVABLE DUE TO A PERTURBATION



$\hat H$ $\qquad\qquad$ $\hat H - \hat B h$ $\qquad\qquad$ $\hat H$

$$\hat U_h(t,0) = e^{-i\hat H(t-t'-\epsilon)/\hbar}\; e^{-i(\hat H - h\hat B)(\cancel{t'}+\epsilon-\cancel{t'})/\hbar}\; e^{-i\hat H t'/\hbar}$$

$$\hat U_h^\dagger(t,0) = e^{i\hat H t'/\hbar}\; e^{i(\hat H - h\hat B)\epsilon/\hbar}\; e^{i\hat H(t-t'-\epsilon)/\hbar}$$

$$\hat H = \hat H^\dagger \qquad \hat B = \hat B^\dagger$$

NO PROBLEM WITH NON-COMMUTATIVITY OF $\hat H$ AND $\hat B$ BELOW SINCE BOTH $h \to 0$

$\epsilon \to 0$

$$\frac{1}{\epsilon}\frac{\delta \hat{U}_h(t,0)}{\delta h}\bigg|_{h=0} = e^{-i\hat{H}(t-t')/\hbar}\,\frac{i}{\hbar}\hat{B}\,e^{-i\hat{H}t'/\hbar}$$

$$\frac{1}{\epsilon}\frac{\delta \hat{U}_h^{\dagger}(t,0)}{\delta h}\bigg|_{h=0} = -\,e^{i\hat{H}t'/\hbar}\,\frac{i}{\hbar}\hat{B}\,e^{i\hat{H}(t-t')/\hbar}$$

INSTANTANEOUS LINEAR RESPONSE

$$R_{AB}(t,t') = \frac{\delta\langle\hat{A}(t)\rangle}{\delta\epsilon h}\bigg|_{h=0} \qquad t>t'$$

$$= \frac{\delta}{\delta\epsilon h}\,\mathrm{Tr}\left(\hat{U}_h^{\dagger}(t,0)\,\hat{A}\,\hat{U}_h(t,0)\,\hat{\rho}_0\right)\bigg|_{h=0}$$

$$= -\frac{i}{\hbar}\,\langle\hat{B}(t')\,\hat{A}(t)\rangle + \frac{i}{\hbar}\,\langle\hat{A}(t)\,\hat{B}(t')\rangle$$

$$= \frac{i}{\hbar} \langle [\hat{A}(t), \hat{B}(t')] \rangle$$

$$\boxed{R_{AB}(t,t') = \frac{i}{\hbar} \langle [\hat{A}(t), \hat{B}(t')] \rangle \, \theta(t-t')}$$

KUBO FORMULA

NOTE THAT WE HAVE NOT USED $\hat{\rho}_0 = \frac{e^{-\beta \hat{H}}}{Z}$

$\hat{\rho}_0$ WAS KEPT GENERIC HERE

VALID EVEN OUT OF EQ.

JUST LINEAR RESPONSE USED

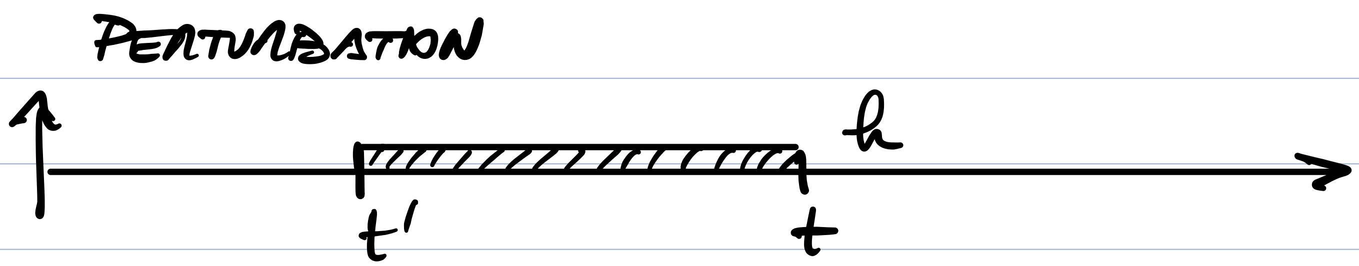
$$\underbrace{h \to 0 \qquad \varepsilon \to 0}_{h\varepsilon \to 0}$$

CAUSALITY THERE CANNOT BE A RESPONSE BEFORE THE PERTURBATION

INTEGRATED LINEAR RESPONSE

$$\chi(t,t') = \int_{t'}^{t} dt'' \; R(t,t'')$$

SUM OF INFINITESIMAL RESPONSES OVER A FINITE TIME INTERVAL



KMS (PROP of CYCLIC TRACE & EQUIL.)

$$C_{AB}(t,t') = \langle A(t)\, B(t') \rangle =$$

$$\frac{1}{\mathrm{Tr}\, e^{-\beta \hat H}}\, \mathrm{Tr}\Big[e^{i\hat H t/\hbar}\, \hat A\, e^{-i\hat H t/\hbar}\; e^{i\hat H t'/\hbar}\, \hat B\, e^{-i\hat H t'/\hbar}\; e^{-\beta \hat H} \Big]$$

WE INTRODUCE IDENTITIES $e^{-\beta \hat H}\; e^{\beta \hat H}$
AND USE CYCLIC PROPS
OF TRACE

• MOVE $\hat B(t')\, e^{-\beta \hat H}$

$$C_{AB}(t,t') = \frac{1}{\mathrm{Tr}\, e^{-\beta \hat H}}$$

$$\mathrm{Tr}\Big[\overbrace{e^{i\hat H t'/\hbar}\, \hat B\, e^{-i\hat H t'/\hbar}}^{\hat B(t')}\; e^{-\beta \hat H}\; e^{i\hat H t/\hbar}\, \hat A\, e^{-i\hat H t/\hbar}\; e^{\beta \hat H}\; e^{-\beta \hat H} \Big]$$

$$= \frac{1}{\mathrm{Tr}\, e^{-\beta\hat{H}}} \mathrm{Tr}\, \hat{B}(t')\, \hat{A}(t+i\beta\hbar)\, e^{-\beta\hat{H}}$$

USING $\frac{it}{\hbar} - \beta = \frac{i}{\hbar}(t + i\beta\hbar)$

$$-\frac{it}{\hbar} + \beta = \frac{-i}{\hbar}(t + i\beta)$$

$$\boxed{C_{AB}(t,t') = C_{BA}(t', t + i\beta\hbar)}$$

FOURIER TRANSFORM

FIRST, STATIONARY

$$C_{AB}(t,t') \longrightarrow C_{AB}(t-t')$$

$$C_{BA}(t', t+i\beta\hbar) \longrightarrow C_{BA}(t'-t-i\beta\hbar)$$

$$\tilde{C}_{AB}(\omega) = \int_{-\infty}^{\infty} d(t-t')\, e^{i\omega(t-t')}\, C_{AB}(t-t')$$

KMS

$$= \int_{-\infty}^{\infty} d(t-t')\, e^{i\omega(t-t')}\, C_{BA}(t'-t-i\beta\hbar)$$

$$= \int d(t-t'+i\beta\hbar)\, e^{i\omega(t-t'+i\beta\hbar)}$$

INSERT IDENTITY

TIME "TRANSL."

$$e^{-ii\beta\hbar\omega}\, C_{BA}(t'-t-i\beta\hbar)$$

CALL $\quad u = t-t'+i\beta\hbar$

$$e^{ii\beta\hbar} = e^{-\beta\hbar}$$

$$= e^{\beta\hbar\omega} \underbrace{\int du\, e^{i\omega u}\, C_{BA}(-u)}_{\tilde{C}_{BA}(-\omega)}$$

$u \to -u$

$$\boxed{\tilde{C}_{AB}(\omega) = e^{\beta\hbar\omega}\, \tilde{C}_{BA}(-\omega)}$$

KMS IN FOURIER

FDT

DEF $$C_{[A,B]}(t,t') \equiv \frac{1}{2}\langle [\hat{A}(t), \hat{B}(t')] \rangle$$

$$C_{\{A,B\}}(t,t') \equiv \frac{1}{2}\langle \{\hat{A}(t), \hat{B}(t')\} \rangle$$

FOURIER TRANSF. if STAT

$$2\tilde{C}_{[A,B]}(\omega) = \mathcal{F}\langle \hat{A}(t)\hat{B}(t')\rangle - \mathcal{F}\langle \hat{B}(t')\hat{A}(t)\rangle$$

$\mathcal{F}$: FOURIER TRANSFORM

$$\mathcal{F}\langle \hat{A}(t)\hat{B}(t')\rangle = \mathcal{F}\, C_{AB}(t-t') = \tilde{C}_{AB}(\omega)$$

$$\mathcal{F}\langle \hat{B}(t')\hat{A}(t)\rangle = \mathcal{F}\, C_{BA}(t'-t)$$

$$\mathcal{F}\, C_{BA}(t'-t) = \int d\underbrace{(t-t')}_{\tau}\, e^{i\omega\overbrace{(t-t')}^{(-\tau)}}\, C_{BA}\overbrace{(t'-t)}^{\tau}$$

$$= \tilde{C}_{BA}(-\omega)$$

$$2\,\tilde{C}_{[A,B]}(\omega) = \tilde{C}_{AB}(\omega) - \tilde{C}_{BA}(-\omega)$$

$$\underset{\text{KMS}}{=} \tilde{C}_{AB}(\omega) - e^{-\beta\hbar\omega}\,\tilde{C}_{AB}(\omega)$$

$$= \left(1 - e^{-\beta\hbar\omega}\right)\tilde{C}_{AB}(\omega)$$

$$2\,\tilde{C}_{\{A,B\}}(\omega) = \left(1 + e^{-\beta\hbar\omega}\right)\tilde{C}_{AB}(\omega)$$

TAKE THE RATIO BETWEEN THE TWO

$$\boxed{\tilde{C}_{[A,B]}(\omega) = \operatorname{th}\frac{\beta\hbar\omega}{2}\,\tilde{C}_{\{A,B\}}(\omega)}$$

THIS IS FDT IN FOURIER

USE KUBO NOW TO WRITE IT IN TERMS OF THE LINEAR RESPONSE

BUT CAREFULLY, BECAUSE OF θ !

WE PAUSE TO DISCUSS LINEAR RESPONSE IN FOURIER DOMAIN

$$R_{AB}(t,t') = \int \frac{d\omega}{2\pi} \, e^{-i\omega(t-t')} \, \tilde{R}_{AB}(\omega)$$

$\propto \theta(t-t')$ CAUSALITY

BECAUSE OF CAUSALITY $\Longrightarrow$ $z = z_R + i z_I$

$\tilde{R}_{AB}(z)$ ANALYTIC FOR $z_I > 0$

IN UPPER HALF COMPLEX PLANE

(NO POLES)

SKETCH OF A PROOF

- IMAGINE $t-t' < 0$
- EXTEND $\tilde{R}_{AB}(\omega) \longrightarrow \tilde{R}_{AB}(z)$
- $\int_C dz \; e^{-iz(t-t')} \; \tilde{R}_{AB}(z)$

  $\hookrightarrow$ CLOSE ON UPPER HALF PLANE

  $z = z_R + i z_I \,, \quad z_I > 0$

  $$e^{-iz(t-t')} = e^{-i z_R (t-t') + \underset{\vee\,0}{z_I} \underset{\wedge\,0}{(t-t')}}$$

  $$\xrightarrow[z_I \to \infty]{} 0$$

if NO POLES OF $\tilde{R}_{AB}(z) \Rightarrow$ INT. $=0$

$\Rightarrow R_{AB}(t,t') = 0$ if $t-t' < 0$

ANALYTICITY OF $\tilde{R}_{AB}(z)$ FOR $z_I > 0$ $\Rightarrow$ $R_{AB}(t,t') = 0$ FOR $t-t' < 0$

KRAMERS-KRÖNIG RELATIONS

$$\mathrm{Re}\,\tilde{R}_{AB}(\omega) = -\mathcal{P}\int \frac{d\omega'}{\pi} \frac{\mathrm{Im}\,\tilde{R}_{AB}(\omega')}{\omega-\omega'}$$

$$\mathrm{Im}\,\tilde{R}_{AB}(\omega) = \mathcal{P}\int \frac{d\omega'}{\pi} \frac{\mathrm{Re}\,\tilde{R}_{AB}(\omega')}{\omega-\omega'}$$

PROOF IN LECTURES NOTES 3.3.4

BACK TO FDT

$$\tilde{C}_{[A,B]}(\omega) = \operatorname{th}\frac{\beta\hbar\omega}{2}\,\tilde{C}_{\{A,B\}}(\omega)$$

HOW SHOULD ONE RELATE

$$\tilde{C}_{[A,B]}(\omega) \quad \text{TO} \quad \tilde{R}_{AB}(\omega) \; ?$$

$$R_{AB}(t,t') \overset{\text{KUBO}}{=} \frac{i}{\hbar}\theta(t-t') \left\langle \left[A(t), B(t')\right] \right\rangle$$

$$\overset{\text{DEF.}}{=} \frac{i}{\hbar}\theta(t-t')\, 2\, \underbrace{C_{[A,B]}(t,t')}_{\text{WRITE IN FOURIER}}$$

$$\overset{\text{FOURIER}}{=} \frac{i}{\hbar}\theta(t-t')\, 2 \int \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \tilde{C}_{[AB]}(\omega')$$

$$\overset{\text{KMS}}{=} \frac{i}{\hbar}\theta(t-t')\, 2 \int \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \operatorname{th}\frac{\beta\hbar\omega'}{2}\,\tilde{C}_{\{A,B\}}(\omega')$$

APPLY IDENTITY

$$\int_0^\infty dt\, e^{i\omega t} = \pi\,\delta(\omega) + i\,\frac{\mathcal{P}}{\omega}$$

BECAUSE 0 INSTEAD OF $-\infty$

TO FOURIER TRANSFORM $R_{AB}(t-t')$:

$$\underbrace{\int_0^\infty d(t-t')\, e^{i\omega(t-t')}\, R_{AB}(t-t')}_{\tilde{R}_{AB}(\omega)\,\in\,\mathbb{C}} =$$

$$= \int_0^\infty d(t-t')\, e^{i\omega(t-t')}\, \frac{2i}{\hbar} \times \int \frac{d\omega'}{2\pi}\, e^{-i\omega'(t-t')}\, \mathrm{th}\,\frac{\beta\hbar\omega'}{2}\, \tilde{C}_{\{A,B\}}(\omega')$$

THETA ALREADY TAKEN INTO ACCOUNT

$$= \frac{2i}{\hbar}\left\{ \frac{1}{\cancel{2\pi}}\cancel{\pi}\int d\omega'\, \delta(\omega-\omega')\, \mathrm{th}\,\frac{\beta\hbar\omega'}{2}\, \overbrace{\tilde{C}_{\{A,B\}}(\omega')}^{\in\,\mathbb{R}} + \frac{1}{2\pi}\, i\,\mathcal{P}\int d\omega'\, \frac{1}{\omega-\omega'}\, \mathrm{th}\,\frac{\beta\hbar\omega'}{2}\, \underbrace{\tilde{C}_{\{A,B\}}(\omega')}_{\in\,\mathbb{R}} \right\}$$

$\Longrightarrow$ TWO RELATIONS

CONDITION FOR $\tilde{C}_{\{A,B\}}(\omega) \in \mathbb{R}$

$$\tilde{C}_{\{A,B\}}(\omega) = \mathcal{F}\, \underbrace{C_{\{A,B\}}(t,0)}_{\text{STATIONARY}}$$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, C_{\{A,B\}}(t,0)$$

$$\tilde{C}^{*}_{\{A,B\}}(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, \underbrace{C^{*}_{\{A,B\}}(t,0)}_{\in \mathbb{R}}$$

$$= \int_{\infty}^{-\infty} (-dt)\, e^{i\omega t}\, C_{\{A,B\}}(-t,0)$$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, C_{\{A,B\}}(-t,0)$$

IF $C_{\{A,B\}}(t,0) = f(t-0) = f(t)$ $\Rightarrow$

$$C_{\{A,B\}}(-t-0) = f(-t)$$

$$\boxed{\tilde{C}_{\{A,B\}}(\omega) = \tilde{C}^{*}_{\{A,B\}}(\omega)}$$

$$\operatorname{Im} \tilde{R}_{AB}(\omega) = \frac{1}{\hbar} \operatorname{th} \frac{\beta \hbar \omega}{2} \tilde{C}_{\{A,B\}}(\omega)$$

$$\operatorname{Re} \tilde{R}_{AB}(\omega) = -\frac{2}{\hbar} \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{1}{\omega - \omega'} \operatorname{th} \frac{\beta \hbar \omega'}{2} \tilde{C}_{\{AB\}}(\omega')$$

$\boxed{\text{FDT}}$

BECAUSE $\tilde{R}_{AB}(\omega)$ HAS REAL & IM. PARTS, AND $\tilde{C}_{\{A,B\}}(\omega)$ WE TAKE IT REAL

LOOK AT $\operatorname{Im} \tilde{R}_{AB}(\omega)$ IN THE $\hbar \to 0$ LIMIT

$$\lim_{\hbar \to 0} \operatorname{Im} \tilde{R}_{AB}(\omega) = \frac{1}{\cancel{\hbar}} \frac{\beta \cancel{\hbar} \omega}{2} \tilde{C}_{AB}(\omega)$$

AND THIS IS FINE.

BACK IN TIME DOMAIN

$$R_{AB}(t-t') = -\beta \frac{d}{d(t-t')} C_{AB}(t-t') \, \Theta(t-t')$$

MODEL INDEP. RELATION VALID IN LINEAR RESPONSE & EQUILIBRIUM

GREEN FUNCTIONS

WE DEFINED

$$R_{AB}(t,t') = \frac{\delta \langle \hat{A}(t) \rangle}{\delta h_B(t')}\Bigg|_{h=0}$$

$$\hat{H} \rightarrow \hat{H} - h_B \hat{B}$$

$$\hat{B} = \hat{B}^{\dagger} \qquad \hat{H} = \hat{H}^{\dagger}$$

IF WE WANT TO WORK WITH FERMIONIC OPERATORS

$$G(t,t') = \langle T \, \hat{c}^{\dagger}(t) \, c(t') \rangle$$

etc. $\qquad$ TIME ORDER

WE CAN DEFINE 4 KINDS OF GREEN FUNCTIONS COMBINING

$\hat{C}$ AND C IN DIFFERENT WAYS

AND RELATE THEM

$\Longrightarrow$ EXTENSIONS OF FDTs TO FERM.
GREEN FUNCTIONS

BUT NOT REALLY LINEAR RESPONSES.
IN THE WAY INTRODUCED ABOVE

SEE e.g. KAMENEV'S
BOOKS.
MAHAN'S

3.4 QUANTUM PHASE TRANSITIONS

GENERIC

- STRICTLY AT $T=0$
- DRIVEN BY QUANTUM FLUCTUATIONS
- TUNE SOME PARAM IN HAMILTONIAN TO GO THROUGH A PHASE TRANS.

eg $$\hat{\mathcal{H}} = -J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Gamma \sum_i \hat{\sigma}_i^x$$

$g = \Gamma/J$ ADIM CONTROL PARAM.

- ZERO TEMP $\Longrightarrow$ GROUND STATE ENERGY PROPERTIES

NON-ANALYTICITY OF $e_{GS}(g)$

- COULD BE 2nd ORDER OR
  1st ORDER OR
  BKT.

FOCUS ON 2nd ORDER

- NB IN QUANTUM THERE IS REAL TIME

$$\hat{H} = \hat{K} + \hat{V}$$

DON'T COMMUTE

RECALL WHAT HAPPENS WITH TIME IN A CLASSICAL PHASE TRANS (THOUGH WE HAVEN'T DISCUSSED THIS WHEN DEALT W/ CLASS. PHASE TRANS.)

CRITICAL SLOWING DOWN

$$t_c \sim \xi^{z} \qquad \xi \to \infty \Rightarrow t_c \to \infty$$

$$C(t,t') = \langle \delta\phi(t)\, \delta\phi(t') \rangle$$

$$\longrightarrow |t-t'|^{-\text{POWER}} \quad e^{-\frac{|t-t'|}{t_c}}$$

z DYNAMIC CRITICAL EXPONENT

DIVERGING t_c $\Rightarrow$ VANISHING FREQ

$$t_c \to \infty \quad \omega_c \sim 1/t_c \sim \xi^{-z}$$

SUGGESTS THAT $\hbar\omega_c$ AN ENERGY GAP $\to 0$

- AT 2nd ORDER TRANSITIONS

AT CRITICAL POINT GAP CLOSES

$$\Delta \equiv e_{1st\,exc} - e_{G.S} \longrightarrow 0$$

$$\Delta \sim J\,|g-g_c|^{z\nu}$$

ν CRITICAL EXPONENT OF SPATIAL CORRELATION LENGTH

SPACE-TIME d+1 DIM. REPRESENTATION

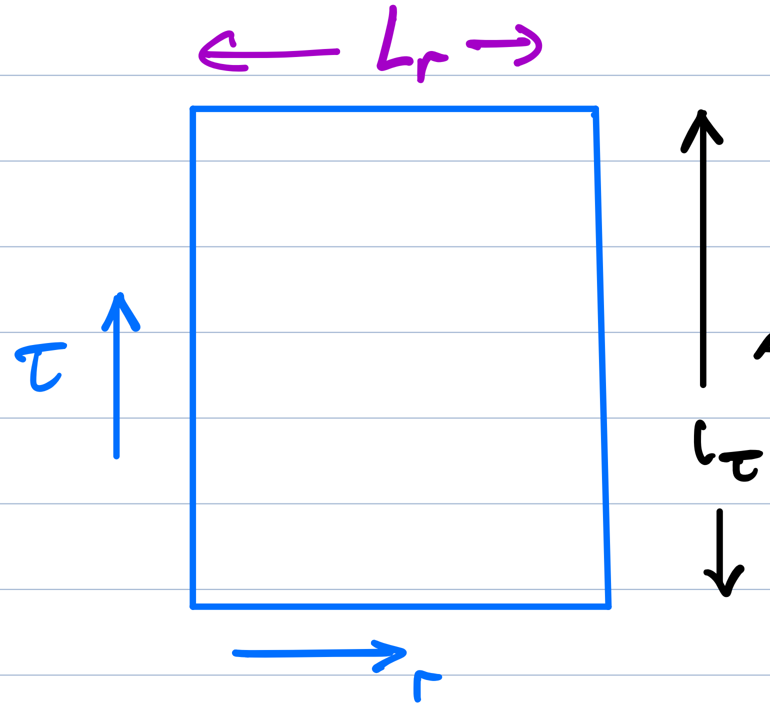
d- QUANTUM $\Longleftrightarrow$ d+1 CLASSICAL

$\downarrow$

IMAG. TIME

$\hat{A}(\vec{r})$ QUANTUM OBSERVABLE

$\Longrightarrow$ $A(\tau, r)$ CLASSICAL MAPPING



ANISOTROPIC RECT FORM

SPACE- IM. TIME

d+1 - DIM.

$$C_{\hat{A}\hat{A}}(\vec{r}) \Rightarrow C_{AA}(\vec{r},\tau)$$

QUANTUM FORMULATION

DIST ALONG SPATIAL DIRECTION

DIST ALONG TIME DIRECTION

AT CRITICALITY

$$C(r,\tau) \sim r^{-\text{POWER}} e^{-r/\xi_r}$$

τ FIXED

$$\xi_r \sim |g-g_c|^{-\nu}$$

z DYNAMIC CRIT EXPONENT

REMEMBER EXTRA DIM IN
CLASS-QUANTUM IS AN IMAGINARY
TIME SO CORR ALONG THIS
DIRECT. ARE TIME-CORR

$$C(r,\tau) \sim \tau^{-\text{POWER}} \, e^{-\tau/\xi_\tau}$$

FIXED

IT'S A TIME

$$\xi_\tau \sim |g-g_c|^{-\nu z}$$

$$\boxed{\xi_r \sim \xi_\tau^{1/z}}$$

A SIMPLE WAY TO SEE THAT Δ IS CONTROLLED BY ξ_τ DIVERGENCE

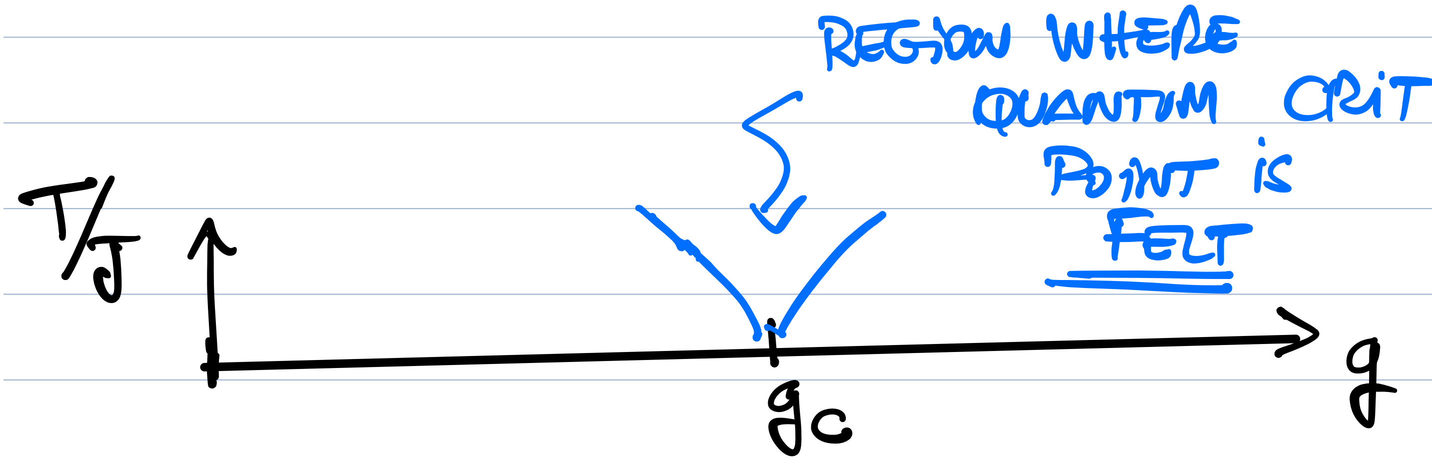
$$\begin{tikzcd} e^{-\beta H} \arrow[r] \arrow[d] & \text{ENERGY} \\ \tau/\hbar \arrow[ur, leftrightarrow] & \end{tikzcd}$$

TIME & ENERGY GO TOGETHER LIKE PARTNERS

$$\Rightarrow \boxed{\Delta \sim |g-g_c|^{\nu z}}$$

ENERGY GAP CLOSES AT g_c

LOW TEMP. EFFECTS



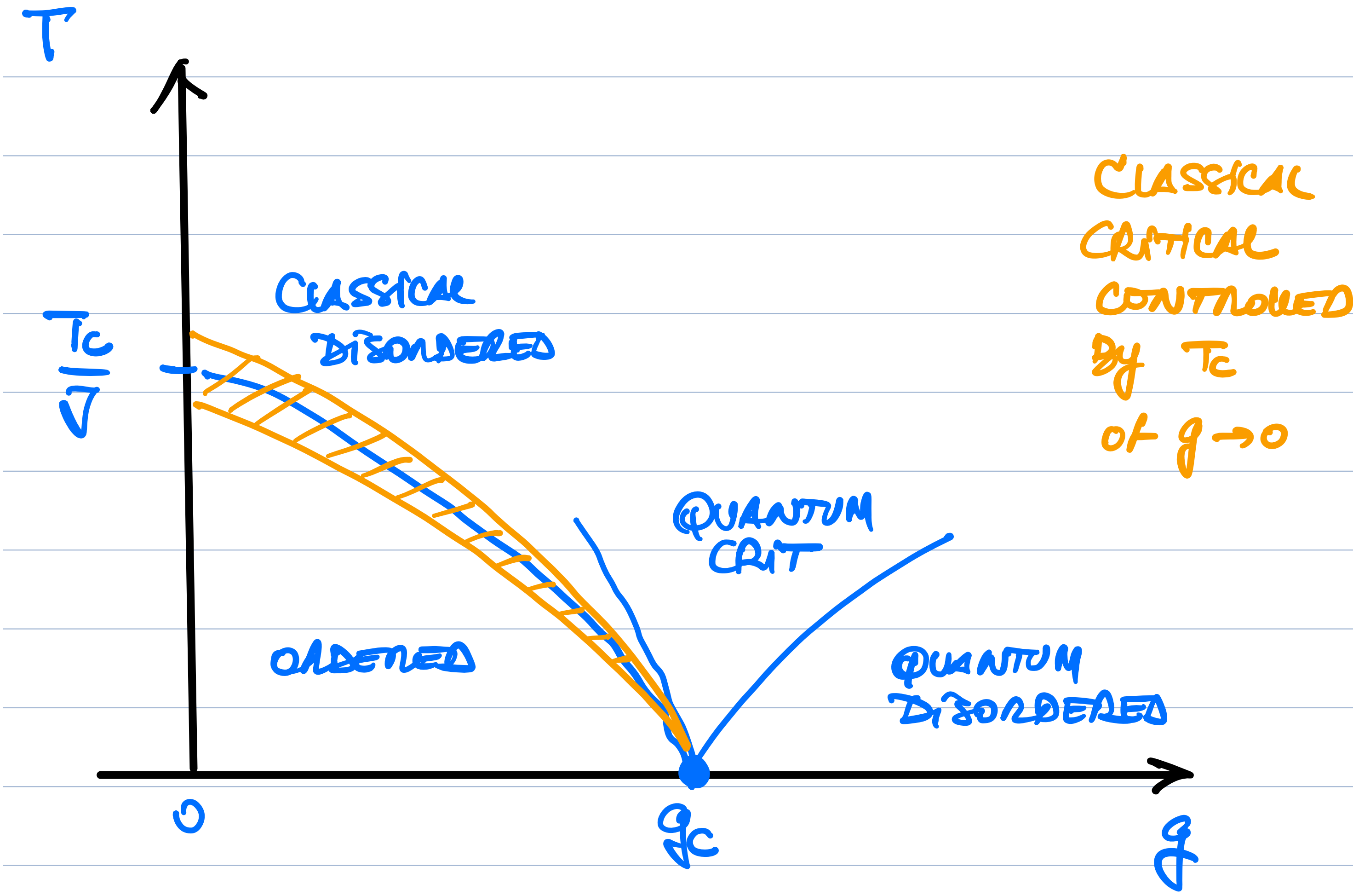
HAND WAVING ARGUMENT

$$\underbrace{\hbar\omega_c}_{\text{QUANTUM TYP ENERGY SCALE}} \sim \underbrace{k_B T}_{\text{THERMAL ENERGY}}$$

if lhs > rhs $\Longrightarrow$ QUANTUM

lhs < rhs $\Longrightarrow$ CLASSICAL

THIS IS THE ACTUAL TEMP AT WHICH THE QUANTUM MODEL IS SET



- CLASSICAL - QUANTUM

IMAGINARY TIME LENGTH

$\beta_q \to \infty$, $T_q \to 0$ EXTRA-DIMENSION DIVERGES $L_\tau \to \infty$

$\beta_q < +\infty$, $T_q > 0$ EXTRA-DIM. IS FINITE $L_\tau < \infty$

$L \to \infty$ REAL SPACE LENGTH

Relation btw Green's functions
Fermions

$$G^R(t,t') = -i\theta(t-t') \langle \{ \hat{c}(t), \hat{c}^\dagger(t') \} \rangle$$

$$G^A(t,t') = i\theta(t'-t) \langle \{ \hat{c}(t), \hat{c}^\dagger(t') \} \rangle$$

$$G^K(t,t') = -i \langle [\hat{c}(t), \hat{c}^\dagger(t')] \rangle$$

$\hat{H} - \mu \hat{N}$ in average

$$G^K(\omega) = \operatorname{th}\left(\frac{\beta\hbar\omega}{2}\right)\left(G^R(\omega) - G^A(\omega)\right)$$

See eg Kamenev's or Mahan's books

NB $\operatorname{th} \leftrightarrow \coth$

QUANTUM ISING CHAIN

RECALL PAULI MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

DON'T NEED σ_y

$$\hat{\sigma}_i = \underbrace{\mathbb{I} \otimes \cdots \otimes}_{\mathbb{I}} \underset{\substack{\uparrow \\ \text{SITE } i}}{\hat{\sigma}} \underbrace{\otimes \cdots \otimes \mathbb{I}}_{\mathbb{I}}$$

$$\left[\hat{\sigma}_i^a, \hat{\sigma}_j^b \right] = 0 \quad \forall \; i \neq j, \; a, b.$$

$J > 0$ ENERGY SCALE, g CONTROL PARAM.

THE MODEL

PBC OR FBC

$$\boxed{H = -J \sum_{i=1}^{L-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - Jg \sum_i \hat{\sigma}_i^x}$$

DON'T COMMUTE

- $g = 0 \Rightarrow$ CLASSICAL LIMIT
- $g \to \infty \Rightarrow$ JUST TRANSV. FIELD
  NO SPIN INTERACT

TWO LIMITS WILL BE FURTHER
DISCUSSED LATER

RECALL ACTION OF SPIN OPERATORS

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \sigma^z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ FLIPS

SYMMETRY

LIKE UP-DOWN IN CLASSICAL ISING MODEL

$$P = \prod_j \sigma_j^x$$

(MATRICIAL NOTATION. NO HATS)

PROPERTIES OF P

- HERMITIAN $\quad P = P^\dagger$

$$P^\dagger = \left(\prod_j \sigma_j^x\right)^\dagger = \prod_j (\sigma_j^x)^\dagger = \prod_j \sigma_j^x = P$$

- $P^2 = \mathbb{I}$

$$P^2 = \left(\prod_j \sigma_j^x\right)\left(\prod_k \sigma_k^x\right)$$

$j = 1, \dots, L$
$k = 1, \dots, L$

EXAMPLE $\quad L=3 \quad j,k = 1,2,3$

$$P^2 = (\sigma_1^x \sigma_2^x \sigma_3^x)(\sigma_1^x \sigma_2^x \sigma_3^x)$$

$$= (\sigma_1^x)^2 (\sigma_2^x)^2 (\sigma_3^x)^2 = \mathbb{I}$$ SINCE $\sigma_i^x \ \sigma_j^x$ COMMUTE $i \neq j$

$$P^2 = \prod_j (\sigma_j^x)^2 = \prod_j \mathbb{I} = \mathbb{I}$$

- UNITARY

$$PP^\dagger = P^\dagger P = \mathbb{1}$$

$$PP^\dagger = \prod_j \sigma_j^x \left(\prod_k \sigma_k^x\right)^\dagger =$$

$$= \sigma_1^x \sigma_2^x \dots \sigma_L^x \ \sigma_L^{x\dagger} \sigma_{L-1}^{x\dagger} \dots \sigma_1^{x\dagger}$$

$$= \sigma_1^x \sigma_2^x \dots \sigma_L^x \ \sigma_L^x \ \sigma_{L-1}^x \dots \sigma_1^x$$

$$= \sigma_L^x \sigma_{L-1}^x \dots \sigma_1^x \ \sigma_1^x \sigma_2^x \dots \sigma_L^x$$

$$= \sigma_L^{x\dagger} \sigma_{L-1}^{x\dagger} \dots \sigma_1^{x\dagger} \ \sigma_1^x \sigma_2^x \dots \sigma_L^x$$

$$= \left(\prod_k \sigma_k^x\right)^\dagger \prod_j \sigma_j^x = P^\dagger P$$

ACTION ON SPIN OPERATORS

$$P^\dagger \sigma_i^z P = -\sigma_i^z$$

$$P^\dagger \sigma_i^x P = \sigma_i^x$$

PROOFS

$$P^\dagger \sigma_i^z P = \left(\prod_j \sigma_j^x\right)^\dagger \sigma_i^z \left(\prod_k \sigma_k^x\right)$$

$$= \prod_{jk} \sigma_j^x \sigma_i^z \sigma_k^x$$

$$= \sigma_1^x \cdots \sigma_i^x \cdots \sigma_L^x \, \sigma_i^z \, \sigma_1^x \cdots \sigma_i^x \cdots \sigma_L^x$$

$$= \sigma_1^x \cdots \sigma_{i-1}^x \sigma_{i+1}^x \cdots \sigma_L^x \, \underbrace{\sigma_i^x \sigma_i^z \sigma_i^x}\, \sigma_1^x \cdots$$

$$\cdots \sigma_{i-1}^x \sigma_{i+1}^x \cdots \sigma_L^x$$

$$\sigma^x \sigma^z \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\sigma^z$$

USING

$$(\sigma^x)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbb{I}$$

$$\sigma^x \sigma^z \sigma^x = -(\sigma^x)^2 \sigma^z$$ THUS

$$P^\dagger \sigma_i^z P = \underbrace{\sigma_1^x \cdots \sigma_L^x}_{P} \underbrace{\sigma_1^x \cdots \sigma_L^x}_{P} (-\sigma_i^z)$$

$$= -\sigma_i^z \quad \checkmark$$

SIMILARLY FOR THE OTHER RELATION:

$$\hat{P}^\dagger \hat{\sigma}_i^x \hat{P} = \hat{\sigma}_i^x$$

$$\left(\prod_j \sigma_j^x\right)^\dagger \sigma_i^x \prod_k \sigma_k^x = \text{EVERYTHING COMMUTES} \Rightarrow$$

$$= P^\dagger P \sigma_i^x = I \sigma_i^x = \sigma_i^x$$

DONE ✓

ACTION ON STATES

eg $\hat{P}|\uparrow\rangle = |\downarrow\rangle$

$\hat{P}|\downarrow\rangle = |\uparrow\rangle$

REVERSES ALL SPINS

THIS IS CLEAR SINCE EACH FACTOR IN $\hat{P}$ DOES REVERSE ONE SPIN

COMMUTATION W/ $\hat{H}$

$$[\hat{H}, \hat{P}] = 0$$

PROOF $\hat{H} = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x$

$$[\hat{H}, \hat{P}] = \left(-J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x\right) \prod_k \hat{\sigma}_k^x$$
$$+ \prod_k \hat{\sigma}_k^x \left(-J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x\right)$$

TAKE ONE TERM IN $\sum_i \hat{\sigma}_i^x$

$$\hat{\sigma}_i^x \hat{\sigma}_1^x \ldots \hat{\sigma}_L^x = \hat{\sigma}_1^x \ldots \hat{\sigma}_L^x \hat{\sigma}_i^x$$

SO IT COMMUTES

TAKE ONE TERM IN $\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$

$$\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \quad \hat{\sigma}_1^x \ldots \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \ldots \hat{\sigma}_L^x =$$

$$= \hat{\sigma}_1^x \cdots \underbrace{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \cdots \hat{\sigma}_L^x =$$

$$= -\hat{\sigma}_1^x \cdots \underbrace{\hat{\sigma}_i^z \hat{\sigma}_i^x} \hat{\sigma}_{i+1}^x \underbrace{\hat{\sigma}_{i+1}^z} \hat{\sigma}_{i+2}^x \cdots \hat{\sigma}_L^x =$$

$$= \hat{\sigma}_1^x \cdots \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \cdots \hat{\sigma}_L^x \underbrace{\hat{\sigma}_i^z \sigma_{i+1}^z}$$

BASICALLY TWO (-1) SIGNS FROM ANTICOMMUTATION OF PAULI MATRICES

AND IT ALSO COMMUTES

ANTI-COMM. OF PAULI MATRICES

$$\sigma_x \sigma_z + \sigma_z \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 \quad \underline{\text{ok}}$$

$$\Longrightarrow \quad [\hat{H}, \hat{P}] = 0$$

IF TWO OPERATORS COMMUTE $\Longrightarrow$ THEY ARE

$\hat{H}$ & $\hat{P}$ DIAGON. IN SAME BASIS

$$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$$

$$\hat{P}\,|\psi_n\rangle = \pm\,|\psi_n\rangle$$

ENERGETIC CONSIDERATIONS

- **TRANSVERSE ALIGNMENT** $g \gg 1$

$$\hat{\mathcal{H}} \sim -Jg\sum_i \hat{\sigma}_i^x \equiv \hat{\mathcal{H}}_t$$

CALL IT TRANSVERSE TERM IN $\hat{\mathcal{H}}$

TWO EIGENSTATES TO FOCUS ON

GROUND STATE $\underbrace{|\rightarrow\rangle = \prod_i \otimes\, |\rightarrow\rangle_i}_{\text{QUANTUM PARAMAGNET}} = |0\rangle_{g\gg1}$

WITH $|\rightarrow\rangle_i = \dfrac{1}{\sqrt{2}}\left(|\uparrow\rangle_i \pm |\downarrow\rangle_i\right)$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\rightarrow$ + FLAT MIXTURE OF $|\uparrow\rangle_i$ & $|\downarrow\rangle_i$

$$\sigma_i^x |\rightarrow\rangle_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |\rightarrow\rangle_i$$

IT'S AN EIGENVECTOR WITH EIGENVALUE $(+1)$

$$\hat{\sigma}_i^x |\leftarrow\rangle_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -|\leftarrow\rangle_i$$

IT'S ALSO AN EIGENVECTOR BUT WITH EIGENVALUE (-1)

<u>GROUND STATE</u> $g \gg 1$ $\quad |0\rangle = \prod_i \otimes |\rightarrow\rangle_i = |\rightarrow\rangle$

$$\hat{\mathcal{H}} = -Jg \sum_i \hat{\sigma}_i^x$$

ENERGY:

$$E_0 = \langle \rightarrow | \hat{\mathcal{H}} | \rightarrow \rangle \Rightarrow$$

CONTRIB OF EACH i:

$$e_i = -Jg \frac{1}{\sqrt{2}} (1\ 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -Jg \frac{1}{2} (1\ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -Jg \frac{1}{2} \underbrace{(1+1)}_{2}$$

FOR EACH i

WHOLE SYST. $\Longrightarrow$

$$\boxed{E_0 = -Jg L}$$

MACROSC. ENERGY OF GROUND STATE

EXCITED STATE

JUST ONE SPIN FLIPPED (j)

$$|exc.\, j\rangle = \prod_{i (\neq j)} |\rightarrow\rangle_i \otimes |\leftarrow\rangle_j$$

$$e_1^{(j)} = {}_j\langle \leftarrow | \hat{h}_0 | \leftarrow \rangle_j$$

THIS IS THE CONTRIB OF THE REVERSED j SPIN

$$= -Jg \frac{1}{\sqrt{2}} (1\ -1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -\frac{Jg}{2} (1\ -1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{Jg}{2} (-2) = Jg$$

$$\boxed{e_1^{(j)} = Jg}$$

DIFFERENCE IN TOTAL ENERGY

$$\Delta \equiv E_1 - E_0 = \underbrace{-Jg(\cancel{L}-1) + Jg}_{E_1} \underbrace{+ \cancel{Jg L}}_{-E_0}$$

ONE REVERSED SPIN

$$= Jg + Jg = 2Jg$$

FOR EACH REVERSED SPIN (HERE JUST ONE) WE PAY $\Delta E = 2Jg$ FINITE

IF ONE REVERSES THEM ALL

$$|\leftarrow\rangle = \prod_i \otimes |\leftarrow\rangle_i$$

$\Rightarrow$ $\Delta E = 2JgL$ MACROSCOPIC

ONE NEEDS TO REVERSE A MACROSC. NUMBER TO GET $\Delta E = \Theta(L)$

CORRECTIONS FOR g LARGE BUT NOT $g \to \infty$

WHAT HAPPENS WITH THE 1st TERM IN $\hat{H}_\ell$? HOW DOES IT ACT ON $|0\rangle_{g\gg1}$?

SANDWICH $\quad {}_{g\gg1}\langle 0 | \hat{H}_e | 0 \rangle_{g\gg1}$

$$\langle \rightarrow | -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z | \rightarrow \rangle =$$

RECALL $\quad |\rightarrow\rangle = \prod_i \otimes |\rightarrow\rangle_i$

$$= \prod_j \otimes \frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}}$$

$$\hat{\sigma}^z_j \left(\frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}} \right) = \frac{|\uparrow\rangle_j - |\downarrow\rangle_j}{\sqrt{2}} = |\leftarrow\rangle_j$$

THUS, $\langle\rightarrow| -J\, \hat{\sigma}^z_i \hat{\sigma}^z_{i+1} |\rightarrow\rangle =$

$$= \prod_k \otimes \langle\rightarrow|_k \; -J \prod_{j(\neq i,i+1)} \otimes |\rightarrow\rangle_j \otimes |\leftarrow\rangle_i \otimes |\leftarrow\rangle_{i+1}$$

ALL SITES $\neq i, i+1$ YIELD 1; WHILE REMAINS

$$-J \, {}_i\langle\rightarrow|\leftarrow\rangle_i \; {}_{i+1}\langle\rightarrow|\leftarrow\rangle_{i+1} = 0$$

SINCE

$$(1 \quad 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

ON THE PERFECT GROUND STATE.

ON AN EXCITATION ?

CALL $|EXC\, i\rangle = \prod_{j(\neq i)} \otimes |\rightarrow\rangle_j \otimes \underbrace{|\leftarrow\rangle_i}_{\text{ONE SPIN FLIP}}$

$$\langle \text{exc } j | \hat{\mathcal{H}}_\ell | \text{exc } i \rangle = \langle \text{exc } j | (-J) \sum_k \hat{\sigma}^z_k \hat{\sigma}^z_{k+1} | \text{exc } i \rangle$$

RELEVANT

$$\hat{\sigma}^z_i \hat{\sigma}^z_{i+1} + \hat{\sigma}^z_{i-1} \hat{\sigma}^z_i \, | \text{exc } i \rangle$$

ACT 1st WITH THIS TERM:

FLIPS $|\leftarrow\rangle_i$ TO $|\rightarrow\rangle_i$

FLIPS $|\rightarrow\rangle_{i+1}$ TO $|\leftarrow\rangle_{i+1}$

MOVES FLIPPED SPIN IN $|\text{exc}_i\rangle$ FROM i TO $i+1$

ACT NEXT WITH THIS ONE

FLIPS $|\leftarrow\rangle_i$ TO $|\rightarrow\rangle_i$

FLIPS $|\rightarrow\rangle_{i-1}$ TO $|\leftarrow\rangle_{i-1}$

MOVES FLIPPED SPIN IN $|\text{exc}_i\rangle$ TO $i-1$

SO, THE "PERTURBATION" $\hat{\mathcal{H}}_\ell$ WITH REFR TO $\hat{\mathcal{H}}_t$ MOVES THE LOCAL EXCITATIONS TO NEARBY SITES.

WHEN SANDWICHED WITH $\langle \text{exc } j|$ $\Rightarrow$

$$-J \left(\delta_{j\,i-1} + \delta_{j\,i+1} \right)$$

LINEAR SUPERPOSITION OF LOCAL EXCITATIONS

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} e^{ikx_i} |\mathrm{exc}\ i\rangle$$ PLANE-WAVES

WHAT IS THEIR ENERGY IN FULL HAMILTONIAN?

$$\langle k | \hat{\mathcal{H}} | k \rangle = ?$$

$$\frac{1}{\sqrt{L}} \sum_i e^{-ikx_i} \frac{1}{\sqrt{L}} \sum_j e^{ikx_j} \underbrace{\langle \mathrm{exc}\ i | \hat{\mathcal{H}} | \mathrm{exc}\ j \rangle}_{\hat{\mathcal{H}}_\ell + \hat{\mathcal{H}}_t}$$

THE LONG. CONTRIBUTION

$\ell)$ $$\langle \mathrm{exc}\ i | \hat{\mathcal{H}}_\ell | \mathrm{exc}\ j \rangle = -J\left(\delta_{i\,j-1} + \delta_{i\,j+1}\right)$$

AND, IN THE SUM

$$\langle k | \hat{\mathcal{H}}_\ell | k \rangle = \frac{1}{L} \sum_{ij} e^{-ik(x_i - x_j)} (-J)\left(\delta_{i\,j-1} + \delta_{i\,j+1}\right)$$

$$= -\frac{J}{L}\sum_i \left[e^{-ik(x_i - x_{i+1})} + e^{-ik(x_i - x_{i-1})} \right]$$

$$= -\frac{J}{L}\sum_i \left[e^{+ika} + e^{-ika} \right]$$

$$= -\frac{J}{\cancel{L}} \cancel{L} \; 2\cos ka$$

$$= -2J \cos ka$$

THE TRANSVERSE CONTRIB

t) $\langle \text{EXC } i | \mathcal{H}_t | \text{EXC } j \rangle = \langle \text{EXC } i | \; Jg \sum_\ell \hat{\sigma}_\ell^x \; | \text{EXC } j \rangle$

ONE TERM IN SUM $\sum_\ell$

$$\langle \text{EXC } i | Jg \, \hat{\sigma}_\ell^x | \text{EXC } j \rangle = Jg \, \langle \text{EXC } i | \text{EXC } \ell, j \rangle$$

$$= Jg \, \delta_{\ell, j} \, \delta_{ij}$$

IN THE SUM $\sum_\ell$ THE $\delta_{\ell j}$ CUTS IT:

$$\langle k | \mathcal{H}_{\phi t} | k \rangle = Jg$$

$$\frac{1}{\sqrt{L}} \sum_i e^{-ikx_i} \frac{1}{\sqrt{L}} \sum_j e^{ikx_j} \delta_{ij}$$

$$= \frac{1}{\cancel{L}} Jg \cancel{\sum_i 1}$$

$$= Jg$$

THE TWO TERMS TOGETHER

$$\boxed{\langle k | \mathcal{H}_0 | k \rangle = Jg - 2J \cos ka = E_k^{EXC}}$$

MINIMIZED FOR $k=0 \implies Jg - 2J$

$$\Delta e_k = e_k^{EXC} - e_{G.S.} \qquad -Jg \quad \text{(PER UNIT LENGTH)}$$

$$= 2Jg \left[1 - \frac{1}{g} \cos ka \right] (+ \text{CORRECTIONS}$$

LOWEST EXCIT $k=0$: $\Delta e_{k=0} = 2J(g-1)$

$\Delta e_{k=0} \to 0$ AT $\boxed{g = g_c = 1}$

THUS COMING FROM $g \gg 1$ THE GAP CLOSED AT

$$\boxed{g = g_c = 1}$$

• LONGITUDINAL ALIGNMENT $g \ll 1$

$$|0\rangle_{g=0} = \prod_i \otimes |\uparrow\rangle_i \quad \text{AND} \quad |0\rangle_{g=0} = \prod_i \otimes |\downarrow\rangle_i$$

TWO GROUND STATES

$$ {}_{g=0}\langle 0|\hat{H}_{g=0}|0\rangle_{g=0} = $$

$$ = -J \prod_j {}_j\langle \uparrow | \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \prod_k |\uparrow\rangle_k $$

$$ = -JL $$

$$\hat{\sigma}_{i+1}^z |\uparrow\rangle_{i+1} = |\uparrow\rangle_{i+1}$$

JUST THE SAME CONTRIBUTION FROM EACH SITE

$$\boxed{E_0 = -JL}$$

GROUND STATE ENERGY

BASIC EXCITATIONS : DOMAIN WALLS

$$\uparrow \uparrow \uparrow \cdots \underset{i}{\uparrow} \underset{i+1}{\downarrow} \downarrow \downarrow \downarrow \cdots$$

LET'S CALL $|i+1\rangle$ THIS STATE W/DW $i \leftrightarrow i+1$

EFFECT OF $\hat{H}_\phi$ ON THESE DOMAIN WALL STATES

$$\langle i | \hat{H}_\phi | i+1 \rangle = \overbrace{\left(\prod_{j<i} \otimes \langle \uparrow | \prod_{k\geq i} \otimes \langle \downarrow | \right)}^{\text{DW } i-1 \leftrightarrow i} \hat{H}_\phi \underbrace{\left(\prod_{j<i+1} \otimes |\uparrow\rangle \prod_{k\geq i+1} \otimes |\downarrow\rangle \right)}_{|i+1\rangle \;\; \text{DW } i \leftrightarrow i+1} = -Jg$$

PLANE WAVES

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_i e^{ikx_i} |i\rangle$$

DOMAIN WALL AT $(i-1, i)$

ONE CAN MAKE AN ANALYSIS AS FOR $g \gg 1$ AND FIND

$$\Delta e_k = 2J \left[1 - g\cos(ka) + \mathcal{O}(g^2)\right]$$

AGAIN THIS IS MINIMIZED FOR $k=0$

$$\Delta e_{k=0} = 2J(1-g) + O(g^2)$$

GAP CLOSES AT

$$\boxed{g = g_c = 1}$$

WE IDENTIFIED THE GROUND STATES AND THEIR E_0 AT $g \gg 1$
$g = 0$

THEY ARE VERY ≠ STATES.

WE DEVELOPED ABOVE PERTURBATIVE ARGUMENTS TO SEE HOW THEY CHANGE FOR $g \gtrsim 0$ AND $1/g \gtrsim 0$

MORE DETAILS IN NOTES

DUALITY

DEFINE

$$\hat{\tau}_i^z = \prod_{j \le i} \hat{\sigma}_j^x \quad \& \quad \hat{\tau}_i^x = \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

STRING OPERATOR

ON A DUAL LATTICE
$\bar{i} = i + 1/2$

NEW SPIN OPERATORS

$$\{\hat{\tau}_i^\alpha, \hat{\tau}_i^\beta\} = 2\delta^{\alpha\beta} \quad \text{ON THE SAME } i$$

$$[\hat{\tau}_i^\alpha, \hat{\tau}_j^\beta] = 0 \quad \text{FOR } i \neq j$$

PROOFS

RECALL THE SPIN ALGEBRA

$$\{\hat{\sigma}_i^x, \hat{\sigma}_i^x\} = 2(\hat{\sigma}_i^x)^2 = 2$$

$$\{\hat{\sigma}_i^z, \hat{\sigma}_i^z\} = 2(\hat{\sigma}_i^z)^2 = 2$$

$$\{\hat{\sigma}_i^x, \hat{\sigma}_i^z\} = \hat{\sigma}_i^x \hat{\sigma}_i^z + \hat{\sigma}_i^z \hat{\sigma}_i^x$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

IF $i = j$ AND $\alpha = \beta = x$

$$\{\hat{\tau}_i^x, \hat{\tau}_i^x\} = \{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z\} =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$= \underbrace{\hat{\sigma}_i^z \left(\hat{\sigma}_{i+1}^z\right)^2}_{\mathbb{I}} \hat{\sigma}_i^z + \underbrace{\hat{\sigma}_i^z \left(\hat{\sigma}_{i+1}^z\right)^2}_{\mathbb{I}} \hat{\sigma}_i^z$$

$$= 2\left(\hat{\sigma}_i^z\right)^2 = 2$$

if $i = j$ AND $\alpha = x$ $\beta = z$

$$\{\hat{\tau}_i^x, \hat{\tau}_i^z\} = \{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \prod_{j\le i} \hat{\sigma}_j^x\}$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x \hat{\sigma}_i^x\right) +$$

$$+ \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x \hat{\sigma}_i^x\right) \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$= \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x\right) \underbrace{\left(\hat{\sigma}_i^z \hat{\sigma}_i^x + \hat{\sigma}_i^x \hat{\sigma}_i^z\right)}_{\{\hat{\sigma}_i^z, \hat{\sigma}_i^x\} = 0} \hat{\sigma}_{i+1}^z$$

$$= 0$$

if i $j = i+1$ AND $\alpha = \beta = x$

$$[\hat{\tau}_i^x, \hat{\tau}_{i+1}^x] =$$

$$= \hat{\sigma}_i^z \underbrace{\hat{\sigma}_{i+1}^z \hat{\sigma}_{i+1}^z}_{\mathrm{I}} \hat{\sigma}_{i+2}^z - \hat{\sigma}_{i+1}^z \hat{\sigma}_{i+2}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \quad \mathrm{II} \quad =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+2}^z - \hat{\sigma}_{i+2}^z \hat{\sigma}_i$$

$$= [\hat{\sigma}_i^z, \hat{\sigma}_{i+2}^z] = 0$$ COMMUT ok.

$$[\hat{\tau}_i^z, \hat{\tau}_{i+1}^z] =$$

$$\prod_{j \le i} \hat{\sigma}_j^x \prod_{k \le i+1} \hat{\sigma}_k^x - \prod_{k \le i+1} \hat{\sigma}_k^x \prod_{j \le i} \hat{\sigma}_j^x$$

$$= (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_i^x)(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i+1}^x) -$$

$$- (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i+1}^x)(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_i^x)$$

$$= \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - \hat{\sigma}_{i+1}^x \hat{\sigma}_i^x = 0$$ COMMUT ok

$$[\hat{\tau}_i^x, \hat{\tau}_{i+1}^z] =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \prod \hat{\sigma}_j^x - \prod \hat{\sigma}_j^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$= \left(\hat{\sigma}_1^x \cdots \hat{\sigma}_{i-1}^x\right) \left\{ \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \right\}$$

$$= \left(\hat{\sigma}_1^x \cdots \hat{\sigma}_{i-1}^x\right) \left\{ \hat{\sigma}_i^z \hat{\sigma}_i^x \hat{\sigma}_{i+1}^z \hat{\sigma}_{i+1}^x - \hat{\sigma}_i^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z \right\}$$

$$= \left(\hat{\sigma}_1^x \cdots \hat{\sigma}_{i-1}^x\right) \left\{ -\hat{\sigma}_i^x \hat{\sigma}_i^z \left(-\hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z\right) - \hat{\sigma}_i^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z \right\}$$

THEY CANCEL

$$= 0$$

THE HAMILTONIAN CAN BE RE-WRITTEN AS (WITH PERIODIC BOUND. CONDITIONS)

$$\hat{H} = -J \sum_{i=1}^{L} \hat{\tau}_i^x - Jg \sum_{i=1}^{L} \hat{\tau}_i^z \hat{\tau}_{i+1}^z$$

SAME HAMILTONIAN WITH

$$g \xrightarrow[1st]{} 1/g \quad \text{AND} \quad J \xrightarrow[Next]{} Jg$$

$$\Longrightarrow \quad \boxed{g_c = 1}$$

DUALITY

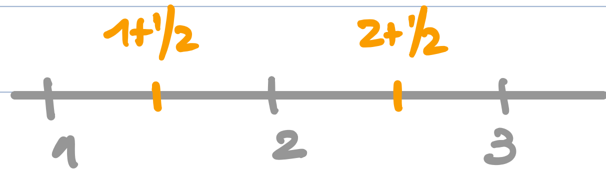
CHECK $\tilde{i} = i + 1/2$ REPLACING τ's BY THE σ's

$$\hat{H} = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$- gJ \sum_i \left(\prod_{j \leq i} \hat{\sigma}_j^x \right) \left(\prod_{k \leq i+1} \hat{\sigma}_k^x \right)$$

$$= -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$- gJ \sum_i \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \ldots \hat{\sigma}_i^x \right) \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \ldots \hat{\sigma}_{i+1}^x \right)$$



$$= -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$- Jg \sum_i \underbrace{(\hat{\sigma}_i^x)^2}_{\mathbb{I}} \underbrace{(\hat{\sigma}_2^x)}_{\mathbb{I}} \ldots \underbrace{(\hat{\sigma}_i^x)^2}_{\mathbb{I}} (\hat{\sigma}_{i+1}^x)$$

$$= -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - Jg \sum_i \hat{\sigma}_{i+1}^x$$

CALL $i+1 \to i$

THE FM PHASE OF ONE MODEL

$\Updownarrow$

THE PM OF THE OTHER

"FIXED POINT" $g_c = 1$

- THE FM PHASE OF THE τ MODEL $\Rightarrow$ DOMAIN WALL CONDENSATE

SOLVING THE FULL PROBLEM

1- JORDAN - WIGNER

$\Longrightarrow$ FERMIONS

2- BOGOLIUBOV

$\Longrightarrow$ FOURIER MODES

END WITH FREE FERMIONS

$$\varepsilon_k(g)$$

GAP AT $k=0$ CLOSES AT

$$g = g_c = 1.$$

PROBLEM WITH DEF OF FERMIONS NAIVELY

RECALL CLASSICAL ISING SPINS $\Longleftrightarrow$ OCCUP. #

$$n_i = \frac{S_i + 1}{2}$$

QUANTUM ? $\hat{n}_i = \hat{c}_i^{+} \hat{c}_i$ NUMBER OPER W/ FERMIONS

SO $\hat{c}_i^{+} \hat{c}_i = \frac{\hat{\sigma}_i^{z} + 1}{2}$?

OR $\boxed{\hat{c}_i^{+} \hat{c}_i = \frac{\hat{\sigma}_i^{+} + 1}{2}}$ WITH RAISING / LOWERING $\hat{\sigma}_i^{+}$ *

NOT SO EASY SINCE PROBLEMS TO SATISFY THE ANTI- COMMUTATION RELATIONS AT SAME & $\neq$ SITES. IN PARTICULAR,

$\{\hat{c}_i^{+}, \hat{c}_j\} = 0 \quad \forall i j$ WITH CHOISE ABOVE (SHOULDN'T FOR $i=j$)

TRUE FERMION OPERATORS CAN BE BUILT IN $d=1$ WITH STRING OPERATORS (NON-LOCAL IN SPIN OPERATORS)

JORDAN-WIGNER

BEFORE ENTERING DETAILS:

* RECALL RAISING & LOWERING OPERATORS

$$\hat{\sigma}_i^+ = \frac{\hat{\sigma}_i^x + i\hat{\sigma}_i^y}{2} \qquad \hat{\sigma}_i^- = \frac{\hat{\sigma}_i^x - i\hat{\sigma}_i^y}{2}$$

MATRIX

$$\sigma^{\pm} = \frac{\sigma^x \pm i\sigma^y}{2} = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

THEIR EFFECT

$$\sigma^- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow \text{REVERSES}$$

$$\sigma^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \quad ''$$

WITH NO sgn INVOLVED

ANTI-COMM $\quad \{\hat{\sigma}^+, \hat{\sigma}^-\} = ?$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

$$\{\hat{\sigma}_i^+, \hat{\sigma}_i^-\} = \mathbb{I}$$ LOCALLY SAME i

WHILE $[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = 0$ if $i \neq j$

COMMUTATOR AT SAME SITE

$$[\hat{\sigma}_i^+, \hat{\sigma}_i^-] = \hat{\sigma}_i^z$$ CHECK

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$$

DEFINE MODIFIED RAISING/LOW.

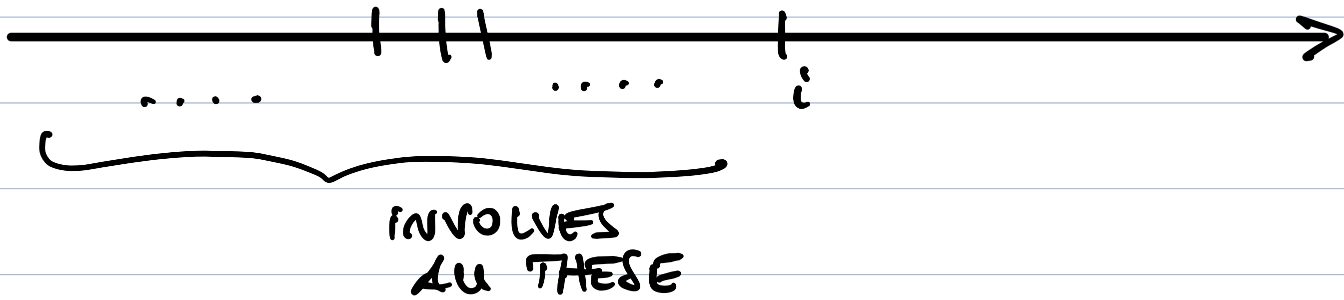
SHOW ONLY THE

EXPLAINING THAT THESE ARE "GOOD" FERMIONS

- $$\hat{\sigma}_i^{+} = \underbrace{\prod_{j<i}\left(1-2\hat{c}_j^{\dagger}\hat{c}_j\right)}_{\hat{S}\,\equiv\,\text{STRING}}\hat{c}_i$$

HERMITIAN OPERATOR

AND THE $\hat{\sigma}_i^{-}$ RELATES TO $\hat{c}_i^{+}$



$\hat{n}_j \equiv \hat{c}_j^{\dagger}\hat{c}_j$ ARE LOCAL # OPERATORS

$\Rightarrow$ THE $\hat{S}\,|\text{STATE}\rangle = \pm\,|\text{STATE}\rangle$

WHETHER EVEN / ODD # FERMIONS

TO THE LEFT OF SITE i IN $|\text{STATE}\rangle$

ONE CAN CHECK THAT

$$\hat{\sigma}_i^{+}\hat{\sigma}_k^{-} = \hat{\sigma}_k^{-}\hat{\sigma}_i^{+} \qquad k \neq i$$

INVERSE RELATIONS

FROM SPINS TO FERMIONS

- $$\hat{c}_i = \left(\prod_{j<i} \hat{\sigma}_j^z\right) \hat{\sigma}_i^+$$

SHOW THEM

- $$\hat{c}_i^+ = \left(\prod_{j<i} \hat{\sigma}_j^z\right) \hat{\sigma}_i^-$$

ONE CAN CHECK THAT

SHOW THEM

- $$\{\hat{c}_i, \hat{c}_j^+\} = \delta_{ij}$$
- $$\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^+, \hat{c}_j^+\} = 0$$

$\Longleftrightarrow$

$$\begin{cases} [\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij}\,\hat{\sigma}_i^z \\ [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\,\delta_{ij}\,\hat{\sigma}_i^\pm \end{cases}$$

COMMUTE AT $\neq$ SITES

- CONVENIENT: PERFORM A ROTATION AROUND THE y AXIS BY $90°$

$$\begin{cases} \hat{\sigma}_i^z \longrightarrow \hat{\sigma}_i^x \\ \hat{\sigma}_i^x \longrightarrow -\hat{\sigma}_i^z \end{cases}$$

BUT NOT NECESSARY FOR THE JORDAN-WIGNER TRANSF.

$$\hat{\mathcal{H}} = -J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ\sum_i \hat{\sigma}_i^x$$ OLD

$$\boxed{\hat{\mathcal{H}} = -J\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + gJ\sum_i \hat{\sigma}_i^z}$$ NEW

THE "DUAL" $\hat{H}$ WITH $g \to -g$

BUT CALL IT AGAIN g

- APPLY THE MAPPING PROPOSED ABOVE

$$\hat{\sigma}_i^x = \mathbb{I} - 2\hat{c}_i^\dagger \hat{c}_i$$

$$\hat{\sigma}_i^z = -\prod_{j<i}\left(1-2\hat{c}_j^\dagger \hat{c}_j\right)\left(\hat{c}_i + \hat{c}_i^\dagger\right)$$

AND IN TERMS OF $\{\hat{c}_i^\dagger, \hat{c}_i\}$ $\hat{\mathcal{H}}$ READS

- $$\hat{\mathcal{H}} = -J\sum_i\Big(\hat{c}_i^\dagger c_{i+1} + \underbrace{\hat{c}_i^\dagger \hat{c}_i^\dagger}_{\text{FERMION \# NON CONSERVED}} + h.c.\Big)$$

$$-gJ\sum_i\left(1-2\hat{c}_i^\dagger \hat{c}_i\right)$$ NON-INTERACTING FERMIONS

$$[\hat{\mathcal{H}}, \hat{P}] = 0$$

PARITY IS CONSERVED
BUT FERMION # NON CONS.

HOW TO PROVE THE CONNECTION WITH FERMIONIC $\hat{H}$:

e.g. $\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z = \prod_{j<i} (1-2\hat{c}_j^\dagger \hat{c}_j)(\hat{c}_i + \hat{c}_i^\dagger)$

$$\prod_{k<i+1} (1-2\hat{c}_k^\dagger \hat{c}_k)(\hat{c}_{i+1} + \hat{c}_{i+1}^\dagger)$$

expand and go ahead.

BOGOLIUBOV TRANSF

FOURIER TRANSFORM

(BUILD COLLECTIVE MODES)

$$\hat{c}_k = \sum_j \hat{c}_j e^{-ik \cdot \underbrace{ja}_{x_j}}$$

POSITION ON AXIS

AFTER MANY STEPS THAT YOU CAN FOLLOW ON YOUR OWN

$$\hat{\mathcal{H}}_\varphi = 2J \sum_{k>0} \left\{ (\hat{c}_k^\dagger \hat{c}_{-k}) A_k \begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix} \right.$$

$$- g + \left(g - \cos(ka) \right) \Big\}$$

$$+ 2J(g-1)\, \hat{c}_0^{\dagger} \hat{c}_0$$

$$A_k = \left(g - \cos(ka) \right) \sigma^z - \sin(ka)\, \sigma^y$$

NON-DIAGONAL.

DIAGONALIZABLE

$$D_k = U_k^{\dagger} A_k U_k$$

WITH ROTATION MATRIX

$$U_k = e^{-i\theta_k \sigma^x / 2} = \cos\left(\frac{\theta_k}{2}\right) \mathbb{I} - i \sin\frac{\theta_k}{2}\, \sigma^x$$

WITH
$$\operatorname{tg} \theta_k = \frac{\sin(ka)}{g - \cos(ka)}$$

$$D_k = \left[\left(g - \cos(ka) \right)^2 + \sin^2(ka) \right]^{1/2} \sigma^z$$

WHICH IS DIAGONAL

CONCERNING FERMIONS

$$\begin{pmatrix} \hat{c}_k \\ \hat{c}^{+}_{-k} \end{pmatrix} = U_k \begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}^{+}_{-k} \end{pmatrix}$$

$\hat{\gamma}_k$ STILL FERMION OPERATORS

$\{\gamma_k, \gamma^{+}_{k'}\} = \delta_{kk'}$

$\Rightarrow$ IN TERMS OF THE $\{\hat{\gamma}'s\}$:

$$\boxed{\begin{aligned} \hat{\mathcal{H}} &= \sum_k \epsilon_k \left(\hat{\gamma}_k \hat{\gamma}^{+}_k - \tfrac{1}{2} \right) + \text{CONST} \\ \epsilon_k &= 2J \left[\left(g - \cos(ka) \right)^2 + \sin^2(ka) \right]^{1/2} \end{aligned}}$$

ϵ_k ACHIEVES ITS MINIMUM AT $k=0$
SINCE EACH TERM MINIMIZES INDEP.

$$\epsilon_0 = 2J \, (g-1)^{2 \cdot 1/2}$$

$$\epsilon_0 = 2J \, |g-1|$$ VALID $\forall g$

$g \ll 1$ $\quad \epsilon_k \simeq 2J \left[1 - g \cos(ka) \right] + \mathcal{O}(g^2)$

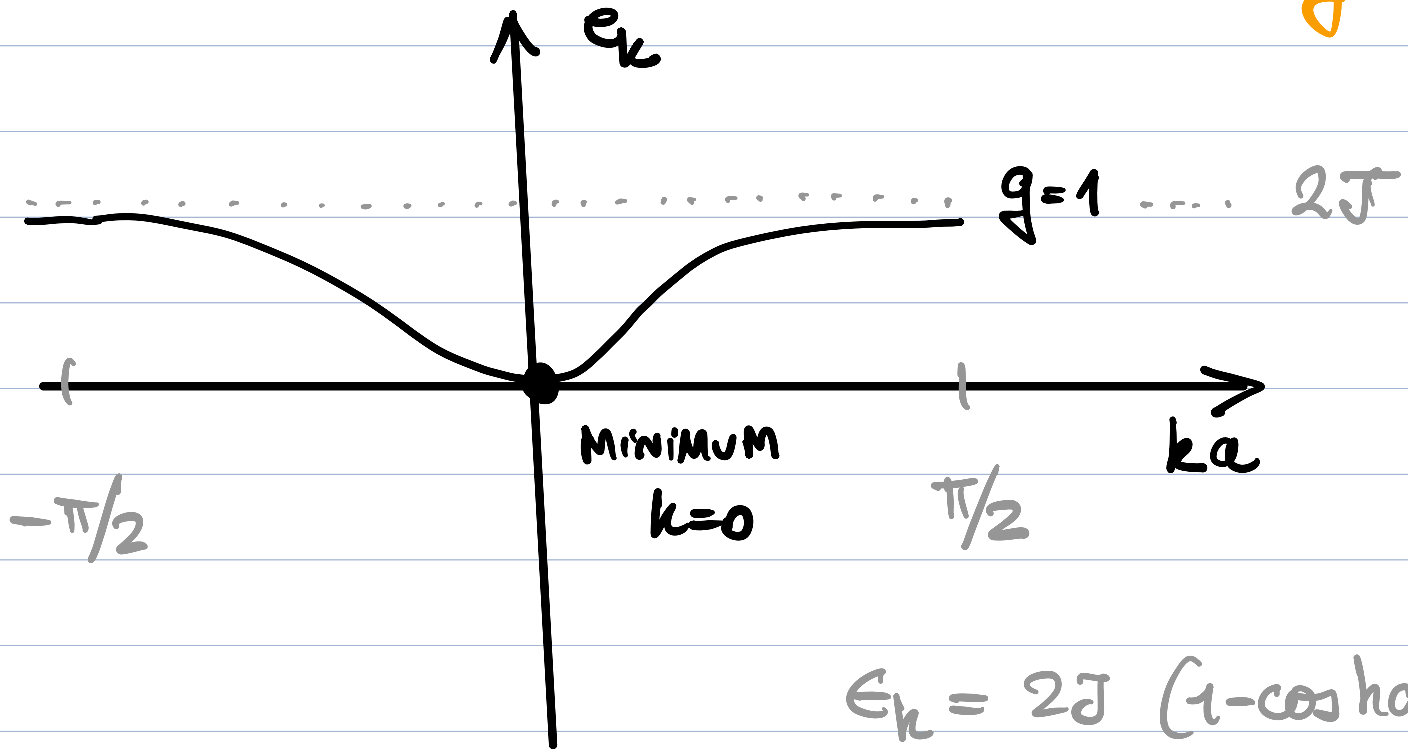
$g \gg 1 \qquad \epsilon_k \simeq 2Jg\left[1 - \frac{1}{g}\cos(ka)\right] + \mathcal{O}(1/g)$

AS WHEN WE STUDIED FLUCTUATIONS

NOTE

$g \leftrightarrow 1/g$

DUALITY



$\epsilon_k = 2J\,(1-\cos ka)$

MINIMUM OF ϵ_k

$$\epsilon_k = 2J\sqrt{\left(g - \cos(ka)\right)^2 + \sin^2 ka}$$

WITH RESPECT TO k:

ASK THAT EACH TERM MINIMIZES:

$\sin^2 ka \implies k = 0$

$g - \cos ka \implies +(\sin ka)\, a = 0$

$\implies k = 0$

SUMMARY

EQUILIBRIUM PROPERTIES

CLASSICAL $\longleftrightarrow$ QUANTUM

MAPPING

COLLECTIVE PHENOMENA

PHASE TRANSITIONS

(in) EQUIV. OF ENSEMBLES
(LONG) SHORT RANGE

1st ORDER	1d
2nd ORDER	MEAN-FIELD CURIE WEISS
TOPOLOGICAL	LANDAU - GINZBURG
	REAL SPACE RG METHODS

DOMAIN WALLS $\longleftrightarrow$ TUNNELING

COUPLING TO ENVIRONMENTS

LANGEVIN EQS

BATH $\rightarrow$ LONG-RANGE COUPL

IN IMAGINARY TIME

QUANTUM PHASE TRANSITIONS (T=0)

GROUND STATE PROPS.