Advanced Statistical Physics

Exam

January 2025

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English or French, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but also the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The answers must be written neatly within the boxes.

The problems roughly follow the order of the chapters in the Lecture Notes but are not necessarily of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

1 Ergodic Hypothesis

Consider a particle of mass m moving in a one-dimensional potential $V(x)$. Assume that the phase space dynamics follow the laws of classical mechanics and that the particle is isolated.

1. Which is the statistical ensemble that would describe the equilibrium properties of such a system?

2. State the ergodic hypothesis that we discussed in the lectures, as it would apply to the kinetic energy of the particle.

3. Do you expect the conditions of the measure proposed in item 1. to hold for a single harmonic oscillator with potential $V(x) = \frac{1}{2}kx^2$ and $k > 0$? Hint: think about the number of constants of motion and whether the dynamics of the oscillator covers the phase space available at the energy determined by the initial conditions.

4. Answer the same question for a harmonic oscillator in two dimensions with potential $V(\vec{x}) = \frac{1}{2}kx^2$ and $x = ||\vec{x}||$.

2 Linear response

Consider a classical particle of mass m in a harmonic potential $V(x) = \frac{1}{2}kx^2$ in canonical thermal equilibrium at temperature T.

1. Calculate the linear response of the particle's average position $\langle x \rangle_F$ to an external static and uniform force F (also called the static susceptibility) $\chi = \left.\frac{\partial \langle x \rangle_F}{\partial F}\right|_{F=0}$

2. Calculate the variance of the particle's position, $\sigma_x^2 = \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ in the absence of the applied force F.

3. Are these two quantities related? Is this relation only valid for this simple model or does it hold more generally? Discuss.

3 Phase transitions

1. Define the term “phase transition”.

2. Describe with words what happens to the free-energy of a system, and its derivatives, at the values of the parameters where the phase transition occurs.

3. Which condition on the system is necessary to allow for the existence of a phase transition?

4. What is an order parameter? How does it help distinguish different phases of matter?

5. What are the differences between a first-order and a second-order phase transition? Mention all the differences you know of.

6. Provide examples of each.

7. Explain the concept of spontaneous symmetry breaking in the context of the ferromagnetic phase transition.

8. Describe the behaviour of the order parameter across a second-order phase transition.

9. How does dimensionality influence the nature of phase transitions?

10. Do you know of a phase transition without order parameter? Give an example and explain.

11. Do the answer to the questions above apply to classical and quantum phase transitions or is there a difference when dealing with the quantum ones? Justify your answer.

12. In Fig. 1 a model for the propagation of an epidemy is studied. The model is defined on a graph and sites can be infected (active) or recovered (inactive). Say that at the initial time $t = 0$ one starts from a configuration in which the sites are active with probability ϕ or inactive with probability $1 - \phi$. A discrete time dynamics follows. In each time step, one random active site is picked and with probability p it infects a random neighbour whereas with probability $1-p$ the particle spontaneously recovers and is removed from the set of active sites. Time is then incremented by $1/N_{\rm act}$, where $N_{\rm act}$ is the number of active sites. If a state with only inactive sites is reached, the dynamics terminates.

    (a) What do you expect for $p = 1$?

    (b) What do you expect for $p = 0$?

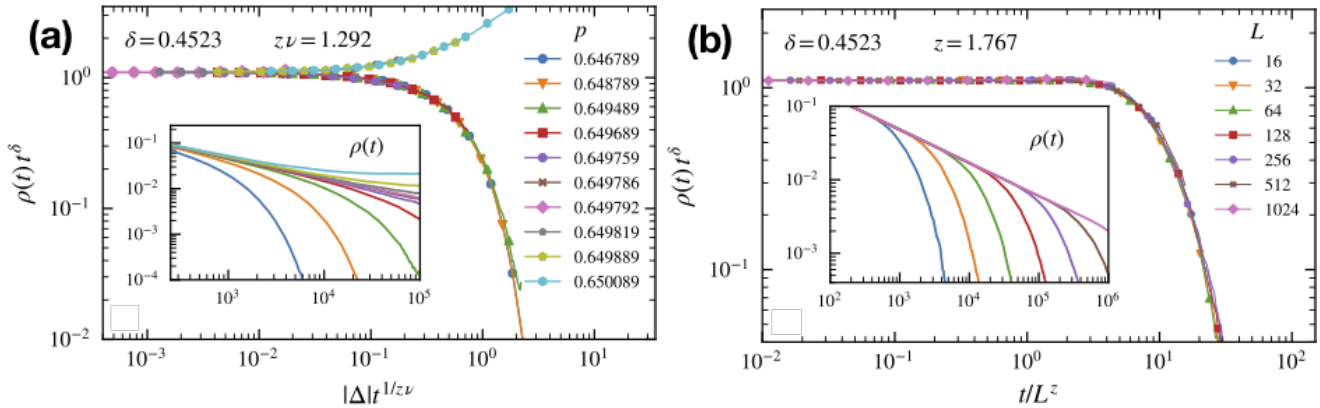

Figure 1: Images extracted from M. Schrauth and J. Portella, Phys. Rev. E **100**, 062118 (2019). $\rho(t)$ is the time dependent density of infected states at time t after the start of the dynamics at time $t = 0$. L is the size of the system.

(c) Explain what has been done in Fig. 1(a). Does it indicate the existence of a phase transition at $0 < p_c < 1$ in this problem?

(d) Explain what has been done in Fig. 1(b). Relate it to what has been discussed in the course. Explain.

4 Quantum Phase Transitions

1. Define a quantum phase transition. How does it differ from a classical phase transition in terms of temperature and critical behavior?

2. Discuss the phase diagram of a generic system with quantum and thermal phases. How does the quantum critical point extend into the finite-temperature regime?

5 Problem. The mean spherical one dimensional model

Instead of working with the usual Ising variables, we will consider here a real extension of them, that makes the statistical physics models easier to treat analytically.

The classical mean spherical model, introduced by Lewis and Wannier[1] in the 50s as a modification of the strict spherical model of Berlin and Kac,[2] is defined as follows. The Hamiltonian is the sum of the usual nearest-neighbour interactions one, and a new term:

$$H = -J \sum_i s_i s_{i+1} - h \sum_i s_i + \lambda \sum_i s_i^2 \ . \tag{1}$$

$J > 0$ is a parameter that tunes the nearest-neighbour coupling on the chain, h is a uniform applied magnetic field, and λ is a new real parameter. We consider the system in contact with a thermal bath at temperature T and in thermal equilibrium with it.

This last term in Eq. (1) would be a constant for Ising variables. In this new model, instead, *real valued* variables

$$-\infty < s_i < \infty \qquad\qquad i = 1, \dots, N$$

are placed on the i sites of a chain with lattice spacing a and periodic boundary conditions. The parameter λ is fixed *a posteriori* so that these variables obey a global mean constraint

$$\left\langle \sum_{i=1}^{N} s_i^2 \right\rangle = \left\langle \frac{\partial H}{\partial \lambda} \right\rangle = N \ , \tag{2}$$

where the angular brackets denote the average over the canonical distribution. With this requirement each s_i is bounded to vary, on average, in the interval $-\sqrt{N} < s_i < \sqrt{N}$. (If one interprets each s_i as a coordinate in an N dimensional space, the constraint above defines a hyper-sphere with radius $\sqrt{N}$ and thus the name "spherical" attached to these variables.)

5.1 Direct treatment

1. Are the Ising configurations included in all possible configurations of this system?

2. How many control parameters are there in this problem? Discuss.

[1]H. W. Lewis and G. H. Wannier, *Spherical model of a ferromagnet*, Phys. Rev. **88**, 682 (1952)

[2]T. H. Berlin and M. Kac, *The spherical model of a ferromagnet*, Phys. Rev. 86, 821 (1952)

3. Which would be the expected phases of this problem?

4. Do you expect a finite temperature phase transition? Think about what you know for 1d models on the one hand, and mean-field models on the other, to construct your answer. Note that at this stage, your answer can still be non-definitive.

5. Write down the partition function (and consider $\beta\Lambda$ as a free parameter at this stage).

6. Write the global mean spherical constraint (2) in terms of the free-energy

7. The explicit calculation of the Gaussian integrals in the partition function yields

$$Z = \prod_q \left(\frac{\pi}{\beta(\lambda - \mu_q)} \right) \, e^{\frac{Nh^2}{4\beta(\lambda-\mu_0)}} \tag{3}$$

with $\mu_q = J\cos(qa)$, $q = 2\pi n/N$ and $n = 0, 1, \ldots, N-1$. Explain how is this result obtained without doing the explicit calculation.

8. Using the expression (3) for Z, write the free energy.

9. Examine this expression and find a condition on the values that the parameter λ can take so as to keep the free-energy real valued.

10. Write the equation that determines λ.

11. Write the expression of the magnetisation density

12. Write the susceptibility.

13. Re-express the equation for λ obtained in the item 10. in terms of m.

14. The critical point is found studying which are the critical parameters $\beta J, \beta h$ which make λ approach its limiting value J. In the absence of an external field, the mathematical analysis of the sum over q in the large N limit yields

$$\lambda - J \sim 1/(\beta J)^2$$

Conclude and reconsider your answer to item 4.

5.2 Decimation

Let us focus on the $h = 0$ case, for simplicity. We add a constant K_0 to the Hamiltonian

$$-\beta H = K_0 + K \sum_i s_{i+1} s_i - \Lambda \sum_i s_i^2$$

but we keep in mind that $K_0 = 0$ in the original expression.

1. Explain, in words, what is the decimation technique about and where it leads to.

2. We will now put it in practice for this model. Consider N even and decimate the even sites of the lattice. How many variables remain and at which distance after the decimation?

3. Integrate away the variables sitting on the even sites. Can you set the new model in the form of the original one? Explain how you do it and show that the relation between the parameters K_0', K' and Λ' of the new model and the ones of the original one $K_0 = 0$, K and Λ are

$$K_0' = \frac{1}{\pi} \ln\left(\frac{\pi}{\Lambda}\right) + 2K_0 \,, \qquad K' = \frac{K^2}{2\Lambda} \,, \qquad \Lambda' = \Lambda - \frac{K^2}{2\Lambda} \,. \tag{4}$$

One can check (do not do it) that the mean spherical constraint is still satisfied after decimation, $\langle \sum_{j=1}^{N'} \sigma_j^2 \rangle = N'$.

4. Find the fixed points of the recurrence relations for K and Λ, and identify the interesting one.

5. Find the linearised form of the recurrence for K close to the interesting fixed point and deduce the critical exponent ν: