Advanced Statistical Physics

Lectures by **Leticia F. Cugliandolo**[1,2]

Exercise sessions **Ada Altieri**[3] & **Marco Tarzia**[1]

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2023 – Lectures Monday, 08 : 30 - 10 : 30

Exercises Monday, 10 : 45 - 12 : 45

Focus

Classical and Quantum Phase transitions

Phenomenology, concepts & formalism

Critical phenomena & universality, topology, renormalization, nucleation

Current research, *e.g.*

Non-Equilibrium Universality : From Classical to Quantum and Back, KITP UC Santa Barbara 2021

Quantum localization and glassy physics, Summer School at Cargèse 2023

Out-of-equilibrium Dynamics and Quantum Information of Many-body Systems with Long-range Interactions KITP UC Santa Barbara 2023

All material (program, lectures, TDs & exams from previous years) downloadable from the webpage www.lpthe.jussieu.fr/˜leticia/enseignement.html

Plan

Mathematical preliminaries (should have been done during the summer)

1. One introductory lecture
2. Six lectures on classical statistical physics
3. Six lectures on quantum statistical physics
4. Thirteen TD sessions in half group (even/odd ending last number student ID)
5. One homework for the mid-term holidays (in pairs/binôme)
6. Final written exam - concepts and exercises (see examples)

Final Mark 20% homework and 80% written exam

Plan

1. Interest and background, principles and formalism, e.g.
   — (In)Equivalence of ensembles for (long) short-range interactions
   — Systems' reduction (role of environments)
2. Classical phase transitions
   — Important concepts (phase diagrams, order parameters, spontaneous symmetry breaking, pinning fields, etc.)
   — Uncommon mechanisms (e.g. topological phases, condensation)
   — Renormalization group ideas
   — Effects of quenched randomness
3. Quantum statistical physics
   — Generics : Linear response, Kubo formula, FDT, etc.
   — Quantum – classical equivalence, path-integrals, imaginary time
   — Quantum spin chains

Plan

A very typical program

Absolutely necessary basic knowledge, needed to carry out theoretical or experimental research in

condensed matter - atomic physics - statistical physics

Two examples of syllabus :

Vlad Dobrosavljevic (Florida State University)

Experimental systems showing classical and quantum critical phenomena.

Thermodynamic potentials. Heat capacity. Magnetic susceptibility.

Phases. Phenomenology of 1st order phase transitions. Continuous transitions.

Landau theory. Order parameters. Spontaneous symmetry breaking.

Critical behavior. Scaling. Critical exponents. Relations between critical exponents.

Kadanoff scaling. Universality conjecture.

Calculation of critical exponents : Real space RG methods.

RG of Wilson and Fisher, ϕ^4 theory, 4-ϵ expansion.

Continuous symmetry : Mermin-Wagner theorem.

Non-linear sigma-model ; $2+\epsilon$ expansion.

Scaling theory of localization. Quark confinement in QCD.

Topological order. Kosterlitz-Thouless phase transition.

Quantum critical phenomena. Hertz-Millis theory. Dissipative quantum tunneling.

Ben Simons (the University of Cambridge)

All mathematical methods in the Math Support file are assumed to be mastered by the students

It is not enough to listen to the oral lectures to understand/learn the content of this course

You have to read the lectures notes or any book of your choice on Phase Transitions and Critical Phenomena (see list below)

The final exam will evaluate the comprehension of the concepts presented and discussed and not only the ability to solve guided exercises (see examples of previous years)

You have to study and learn these concepts in between the Monday sessions during the semester. Do not wait until the Xmas holidays to do it

Bibliography

– Berlinsky & Harris - Statistical Mechanics - An Introductory Graduate Course

– Castiglione, Falcioni, Lesne & Vulpiani - Chaos and coarse-graining in statistical physics

– Khinchin - Mathematical Foundations of Statistical Mechanics

– Lesne & Lagües - Scale Invariance from Phase Transitions to Turbulence

– Parisi - Statistical Field Theory

– Goldenfeld - Phase Transitions and the Renormalization Group

– Altland & Simons - Condensed Matter Field Theory

There are many other excellent books that you can use as support to the course

Phase transitions

Definition

Sharp changes in the behaviour of macroscopic systems at points (or curves) in parameter space.

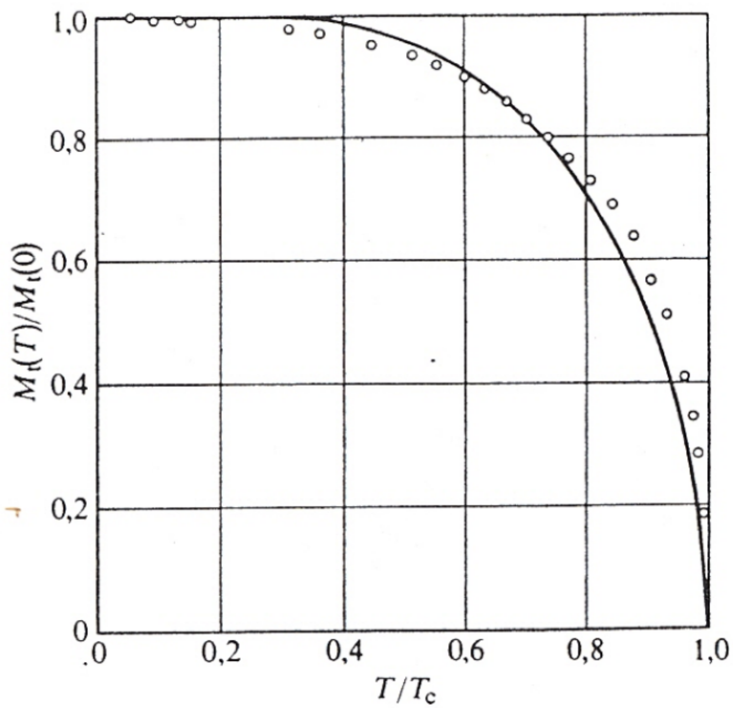
Non-trivial collective phenomena arising in the thermodynamic limit.

Historical development : experimental observation, phenomenological description, mathematical modelling, theoretical understanding.

Magnetic phase transition

Nickel (classical)

The magnetisation sharply vanishes at T_c *at* $H = 0$



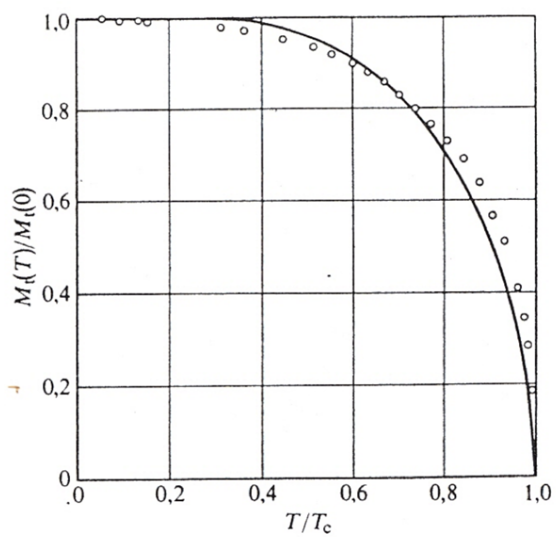
It continuously decays to zero at $T \to \infty$ *under a magnetic field* $H \neq 0$

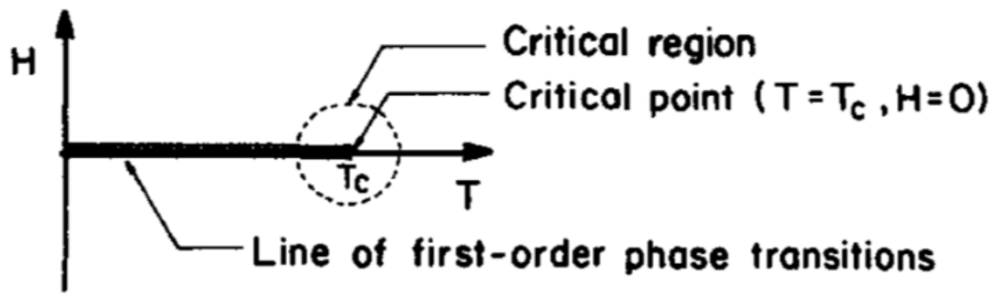
Weiss & Forrer, Ann. Phys. 5, 153 (1926)

Magnetic phase diagram

Nickel (classical)

The magnetisation sharply vanishes at T_c *for* $H = 0$





Discontinuous jump from

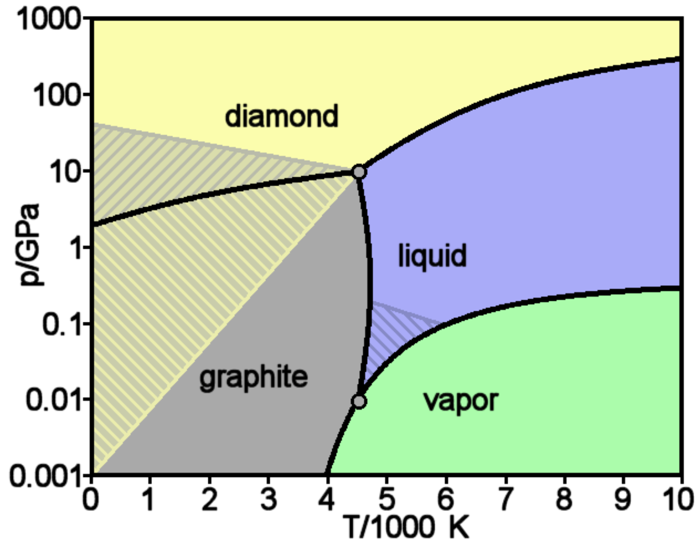
positive to negative magnetization

at $T < T_c$ *when* H *changes sign*

Simplest model for these features ? Mean-field

Structure phase diagram

Carbon (classical)

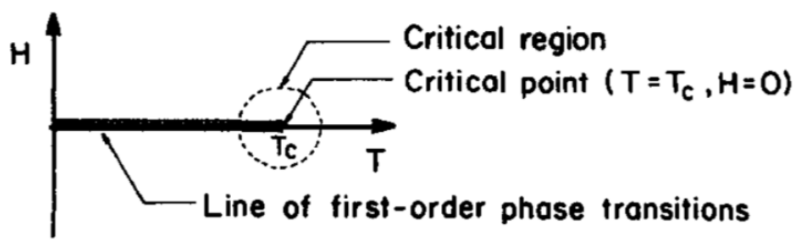


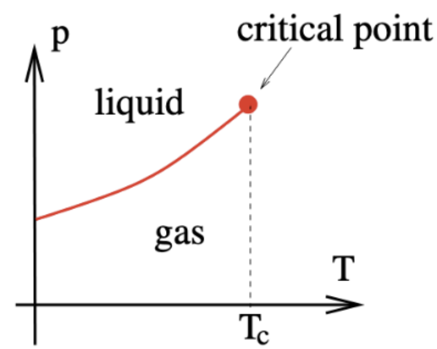
Theoretical phase diagram of carbon, which shows the state of matter for varying temperatures and pressures. The hatched regions indicate conditions under which one phase is metastable, so that two phases can coexist.

Wikipedia

Universality

Para-Ferro magnetism & Liquid-Gas





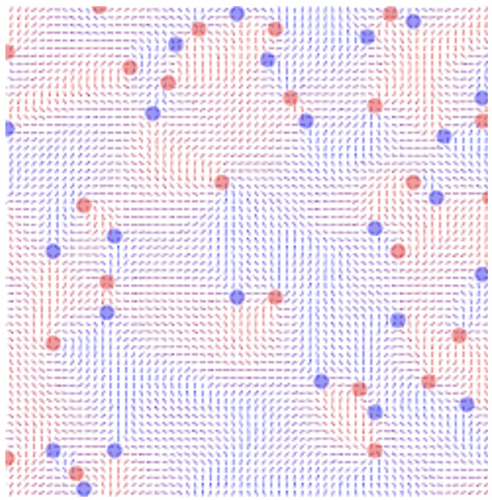
Similar phase diagrams

Very different variables and control parameters and, still, **same critical properties**

Why ? General approach : RG

Topological phase transitions

Planar magnets



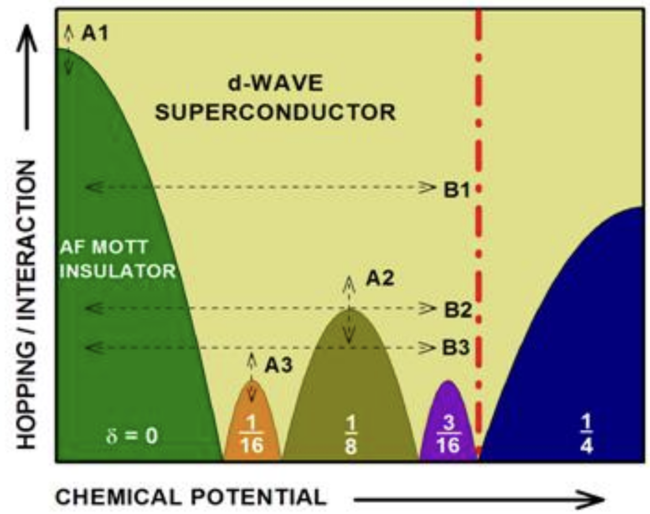
The Berezinskii-Kosterlitz-Thouless (BKT) transition is a phase transition from bound vortex-antivortex pairs at low temperatures to unpaired vortices and anti-vortices at some critical temperature realized by the two-dimensional $2d$XY model.
Also in Josephson junction arrays, thin superconducting films, ultracold atomic gases in 2d

Kosterlitz & Thouless, J. Phys. C : Solid State Phys. 5, L124 (1972)

Nobel 2016

Quantum Phase Diagram

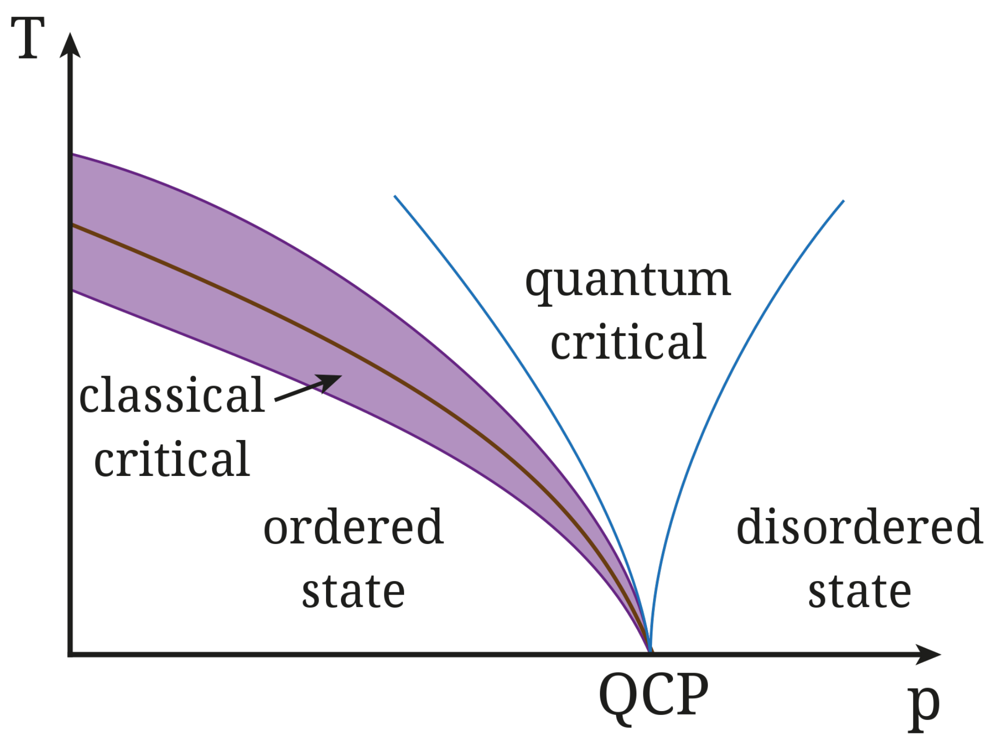
High T_c superconductors



... to propose a bosonic effective **quantum** Hamiltonian based on the projected SO(5) model with extended interactions, which can be derived from the microscopic models of the cuprates. The global phase diagram of this model is obtained using mean-field theory and Quantum Monte Carlo simulations ...

Quantum Phase Transitions

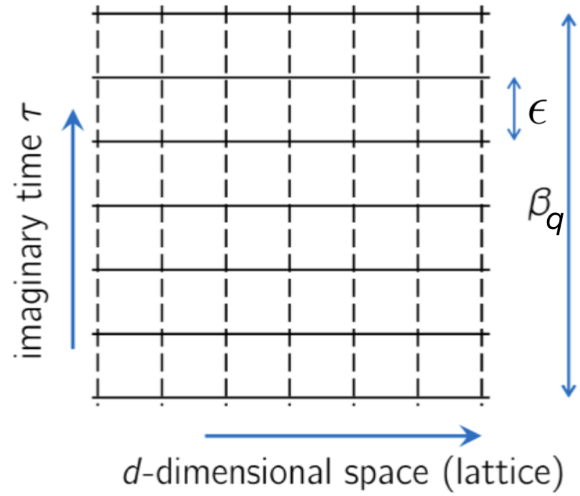
Occur at zero temperature



Quantum Phase Transitions

Methods

Functional methods **Quantum – classical mapping** **Spin chains**

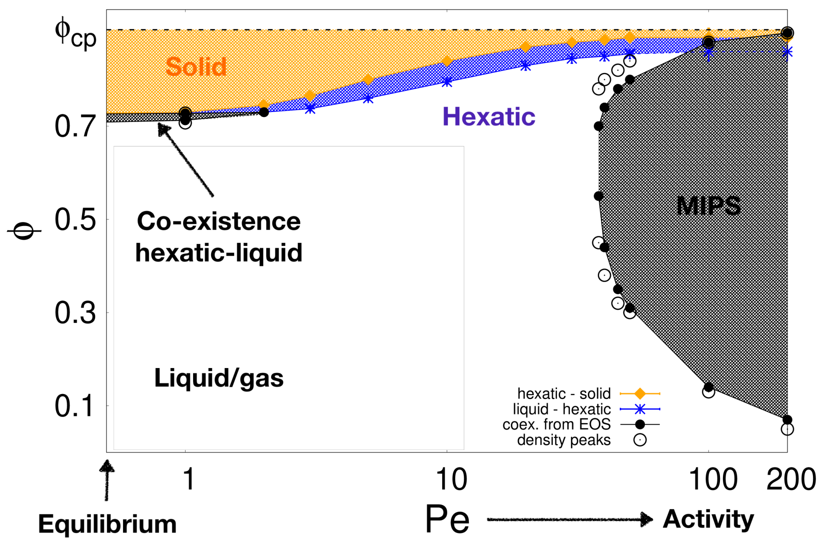


Generic properties

Beyond physics

Active Matter

Phase diagram with solid, hexatic, liquid, co-existence and MIPS



1st order **hexatic**-**liquid** close to Pe = 0

KT-HNY **solid**-**hexatic**

- universal dislocation unbinding

Breakdown of KT-HNY **hexatic**-**liquid** picture

- disclination unbinding within the **liquid** phase

- **percolation** of defect clusters in the liquid

Pressure $P(\phi, \text{Pe})$ (EoS), correlations $G_T(r)$, $G_6(r)$, distributions of ϕ_i, and $|\psi_{6i}|$

defect identification & counting

Digregorio, Levis, Suma, LFC, Gonnella & Pagonabarraga, PRL 121, 098003 (2018)

Digregorio, Levis, LFC, Gonnella & Pagonabarraga, Soft Matter 18, 566 (2022)

Computer science

Algorithms & hard problems

Cristopher Moore (Santa Fe Institute, USA)

August 30th, 10:45AM EDT (Zoom meeting starts at 10:30 EDT)

The physics of inference, phase transitions, and networks

Finding patterns in data is a lot like finding ground states in physics. Each "state" corresponds to a hypothesis about the data, and the most-likely state is the one with the lowest energy. More generally, the Boltzmann distribution corresponds to the posterior distribution in Bayesian statistics. But reaching equilibrium can be hard, especially in glassy systems. We can get stuck for exponential time at local optima that have nothing to do with the true pattern, and are separated from the "correct" state by energy barriers. In many problems, this creates **phase transitions** where **finding patterns in noisy data suddenly becomes computationally hard or impossible**. These transitions occur when the amount of noise in the data –which is analogous to the temperature– crosses a critical threshold. I'll discuss these phase transitions using an example from the study of social networks, where we try to classify nodes according to which community they belong to.

Transition between **solvable** and **unsolvable**

End of introductory part

Out of equilibrium

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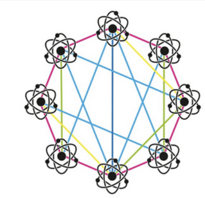
Out-of-equilibrium Dynamics and Quantum Information of Many-body Systems with Long-range Interactions

Coordinators: Nicolò Defenu, Alexey Gorshkov, Giovanna Morigi and Lea F. Santos

Scientific Advisors: Paola Cappellaro, Eugene Demler, David Mukamel, Manfred Salmhofer and Monika Schleier-Smith

The experimental progress in the field of quantum simulations has stimulated a strong theoretical interest in the quantum coherent dynamics of long-range interacting particles. The prominent collective character of these systems gives rise to novel dynamical phenomena, metastable phases of matter, and forms of dynamical criticality which do not have a counterpart in traditional quantum systems with local interactions. This program will give a broad scientific community an occasion to have systematic and organic discussions over several recently discovered non-equilibrium phenomena enabled by long-range interactions, bridging the insights coming from theory and experiment. The program will cover several aspects of long-range physics, including, but not limited to the dynamics of quantum correlations and entanglement, dynamical phase transitions, dynamically stabilized phases and metastable phases and anomalous slow dynamics and ergodicity breaking. The program will provide a framework for various research communities to discuss the core concepts and tools relevant to long-range interacting matter and will contribute to better bridging the gap between theorists and experimentalists.



DATES

Oct 16, 2023 - Dec 22, 2023

INFORMATION

Apply

Application deadline is:
Nov 13, 2022.
Primary deadline above

Long-Range Interactions

Statistical mechanics and dynamics of solvable models with long-range interactions

A. Campa (1), T. Dauxois (2), S. Ruffo (3) ((1) Complex Systems and Theoretical Physics Unit, ISS and INFN, Rome, Italy (2) Laboratoire de Physique and CNRS, ENS-Lyon, France (3) Dip. di Energetica, Univ. Firenze and INFN, Italy)

The two-body potential of systems with long-range interactions decays at large distances as $V(r) \sim 1/r^\alpha$, with $\alpha \leq d$, where d is the space dimension. Examples are: gravitational systems, two-dimensional hydrodynamics, two-dimensional elasticity, charged and dipolar systems. Although such systems can be made extensive, they are intrinsically non additive. Moreover, the space of accessible macroscopic thermodynamic parameters might be non convex. The violation of these two basic properties is at the origin of ensemble inequivalence, which implies that specific heat can be negative in the microcanonical ensemble and temperature jumps can appear at microcanonical first order phase transitions. The lack of convexity implies that ergodicity may be generically broken. We present here a comprehensive review of the recent advances on the statistical mechanics and out-of-equilibrium dynamics of systems with long-range interactions. The core of the review consists in the detailed presentation of the concept of ensemble inequivalence, as exemplified by the exact solution, in the microcanonical and canonical ensembles, of mean-field type models. Relaxation towards thermodynamic equilibrium can be extremely slow and quasi-stationary states may be present. The understanding of such unusual relaxation process is obtained by the introduction of an appropriate kinetic theory based on the Vlasov equation.

Comments: 118 pages, review paper, added references, slight change of content
Subjects: **Statistical Mechanics (cond-mat.stat-mech)**
Journal reference: Physics Reports 480 (2009), pp. 57-159

Long-Range Interactions

Notes on the Statistical Mechanics of Systems with Long-Range Interactions

David Mukamel

Thermodynamic and dynamical properties of systems with long range pairwise interactions (LRI) which decay as 1/r^{d+\sigma} at large distances r in d dimensions are reviewed in these Notes. Two broad classes of such systems are identified: (a) systems with a slow decay of the interactions, termed "strong" LRI, where the energy is super-extensive. These systems are characterized by unusual properties such as inequivalence of ensembles, negative specific heat, slow decay of correlations and ergodicity breaking. And (b) systems with faster decay of the interaction potential where the energy is additive, thus resulting in less dramatic effects. These interaction affect the thermodynamic behavior of systems near phase transitions, where long range correlations are naturally present. Long range correlations are often present in systems driven out of equilibrium when the dynamics involves conserved quantities. Steady state properties of driven systems are considered within the framework outlined above.

Comments: Notes of lectures given at a Summer School in Les Houches (France), 4-29 August 2008

Baths

Reservoir-controlled quantum materials

Responsable : Cristiano Ciuti (PR Université de Paris)

Enseignants : Cristiano Ciuti (PR Université de Paris)

nombre d'ECTS : 3

Langue d'enseignement : Anglais

Evaluation : examen écrit (questions sur le cours, tutoriels et articles donnés à l'avance)

Description :

In recent years, new concepts related to the coupling of quantum many-body systems to tailored reservoirs are providing elegant ways to modify the electronic, optical and mechanical properties of quantum materials in condensed matter, atomic, optical and quantum circuit platforms. Moreover, reservoir-based approaches are emerging in the context of machine learning and quantum information. This M2 course and its tutorials will provide in a pedagogical way the fundamental concepts for the reservoir-control of quantum systems and show the most recent developments, applications and experimental realizations. The interdisciplinary and self-contained character of this class should resonate with students belonging to the tracks of Condensed Matter, Quantum Physics, and Theoretical Physics.

Baths

Home > The Journal of Chemical Physics > Volume 148, Issue 11 > 10.1063/1.4991779

< PREV NEXT >

No Access . Submitted: 23 June 2017 . Accepted: 21 February 2018 . Published Online: 15 March 2018

Theories of quantum dissipation and nonlinear coupling bath descriptors

J. Chem. Phys. 148, 114103 (2018); https://doi.org/10.1063/1.4991779

Rui-Xue Xu, Yang Liu, Hou-Dao Zhang, *and* YiJing Yan[a)]

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PDF ABSTRACT FULL TEXT FIGURES CITED BY TOOLS SHARE METRICS

TOPICS

- Invariance properties
- Emission spectroscopy
- Nonequilibrium statistical mechanics
- Quantum dissipation
- Complex systems theory
- Stochastic processes
- Entropy
- Optical properties
- Oscillators
- Wick's theorem

ABSTRACT

The quest of an exact and nonperturbative treatment of quantum dissipation in nonlinear coupling environments remains in general an intractable task. In this work, we address the key issues toward the solutions to the lowest nonlinear environment, a harmonic bath coupled both linearly and quadratically with an arbitrary system. To determine the bath coupling descriptors, we propose a physical mapping scheme, together with the prescription reference invariance requirement. We then adopt a recently developed dissipaton equation of motion theory [R. X. Xu *et al.*, Chin. J. Chem. Phys. **30**, 395 (2017)], with the underlying statistical quasi-particle ("dissipaton") algebra being extended to the quadratic bath coupling. We report the numerical results on a two-level system dynamics and absorption and emission line shapes.

Baths

SCIENTIFIC REPORTS

OPEN

Decoherence and control of a qubit in spin baths: an exact master equation study

Received: 8 August 2017
Accepted: 8 January 2018
Published online: 24 January 2018

Jun Jing[1,2] & Lian-Ao Wu[2,3]

In spin-based nanosystems for quantum information processing, electron spin qubits are subject to decoherence due to their interactions with nuclear spin environments. In this paper, we present an exact master equation for a central spin-1/2 system in time-dependent external fields and coupled to a spin-half bath in terms of hyperfine interaction. The master equation provides a unified description for free and controlled dynamics of the central spin and is formally independent of the details and size of spin environments. Different from the previous approaches, the master equation remains exact even in the presence of external control fields. Using the parameters for realistic nanosystems with nonzero nuclear spins, such as GaAs, we investigate the Overhauser's effect on the decoherence dynamics of the central spin under different distributions of bath-spin frequencies and system-bath coupling strengths. Furthermore, we apply the leakage elimination operator, in a nonperturbative manner, to this system to suppress the decoherence induced by hyperfine interaction.

Baths

Bath-induced Zeno localization in driven many-body quantum systems

Markus Holzmann [1] , Thibaud Maimbourg [2] , Denis M. Basko [1] , Alberto Rosso [2] Details

1 LPM2C - Laboratoire de physique et modélisation des milieux condensés

2 LPTMS - Laboratoire de Physique Théorique et Modèles Statistiques

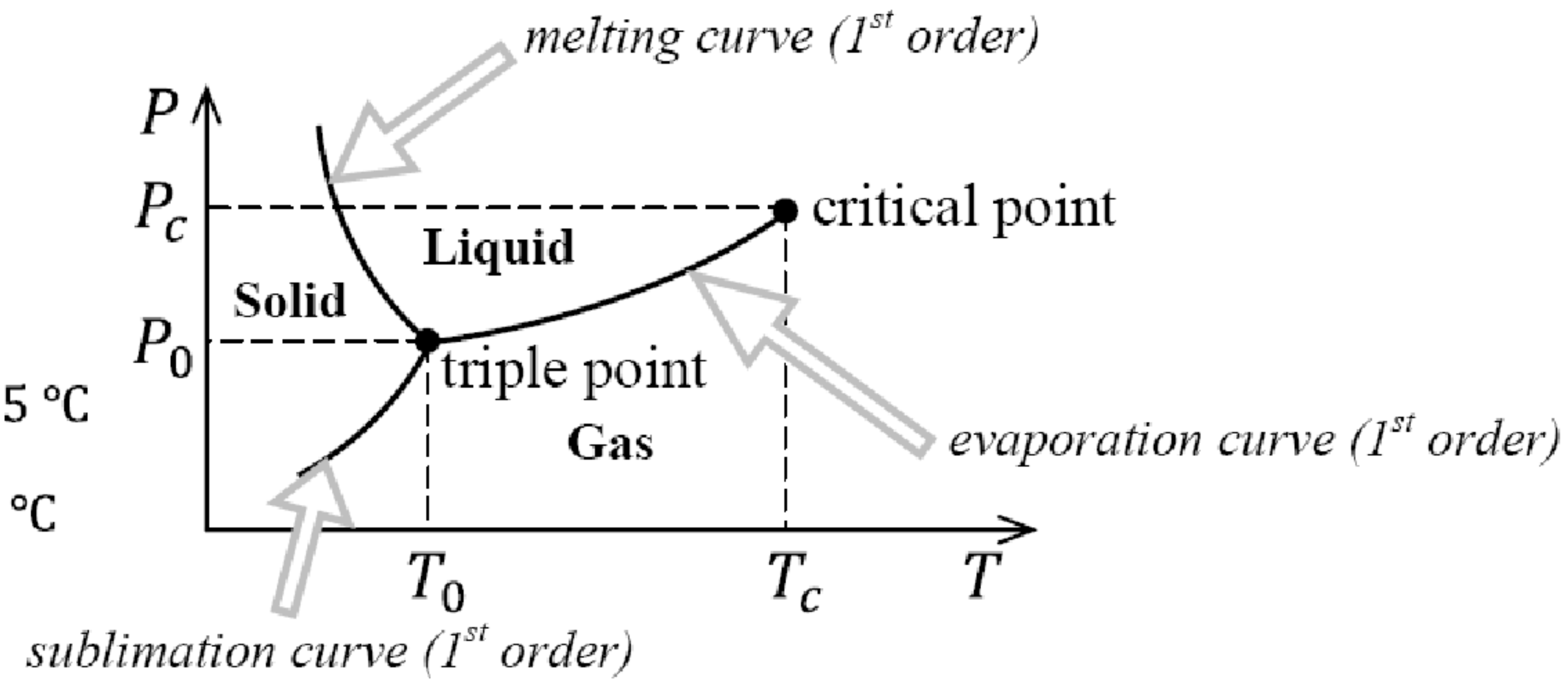
Abstract : We study a quantum interacting spin system subject to an external drive and coupled to a thermal bath of spatially localized vibrational modes, serving as a model of Dynamic Nuclear Polarization. We show that even when the many-body eigenstates of the system are ergodic, a sufficiently strong coupling to the bath may effectively localize the spins due to many-body quantum Zeno effect, as manifested by the hole-burning shape of the electron paramagnetic resonance spectrum. Our results provide an explanation of the breakdown of the thermal mixing regime experimentally observed above 4 - 5 Kelvin.

End of 1st Lecture

Phase transitions

$T_0 = 0.0075\ °C$

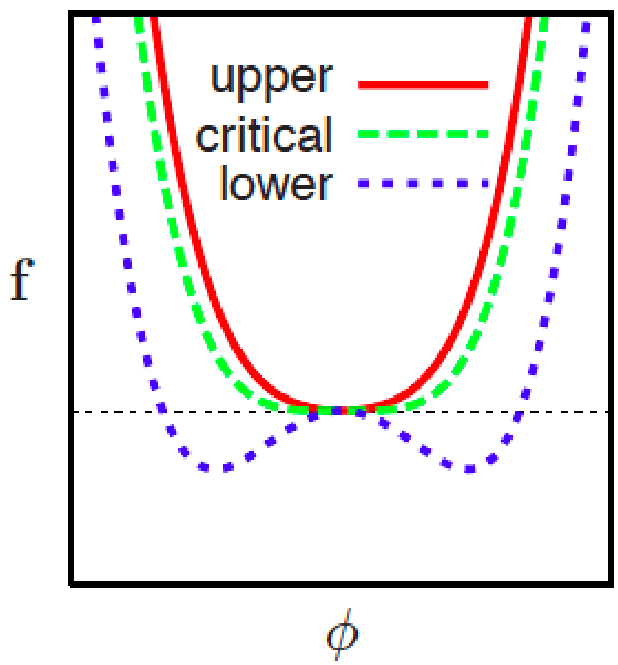
$T_c = 374.2\ °C$



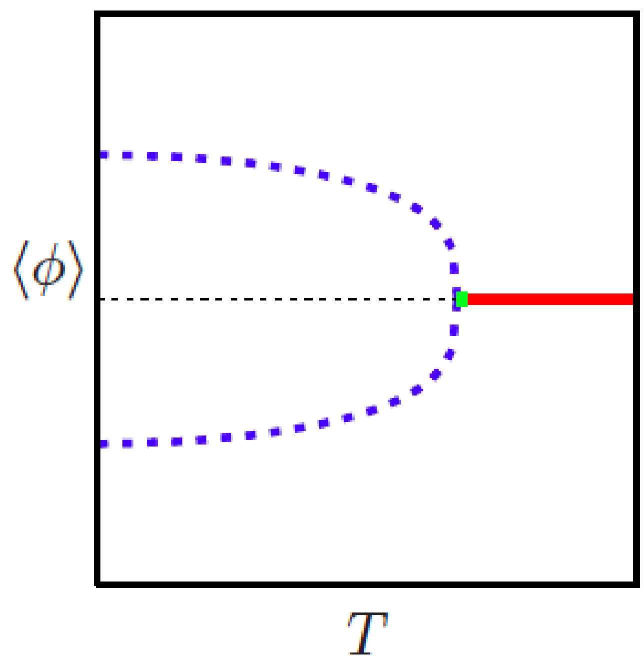
Continuous phase-transition

Bi-valued equilibrium states related by symmetry

$\mathcal{F}$



Ginzburg-Landau free-energy



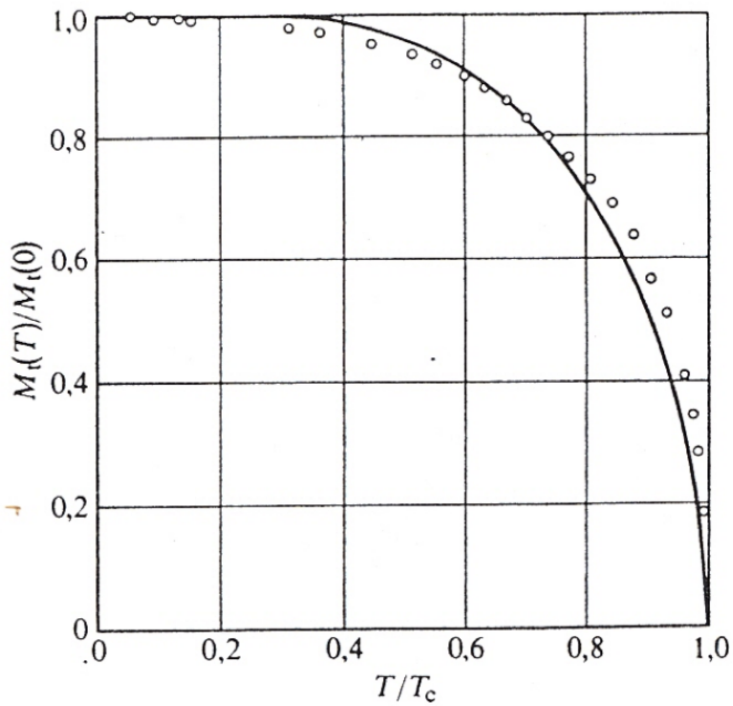
Scalar order parameter

e.g. Ising magnets at $h = 0$**,** $m = \langle s_i \rangle = \frac{1}{N}\sum_{k=1}^{N}\langle s_k \rangle$

Continuous phase-transition

Bi-valued equilibrium states related by symmetry

Aimantation et phénomène magnétocalorique du nickel



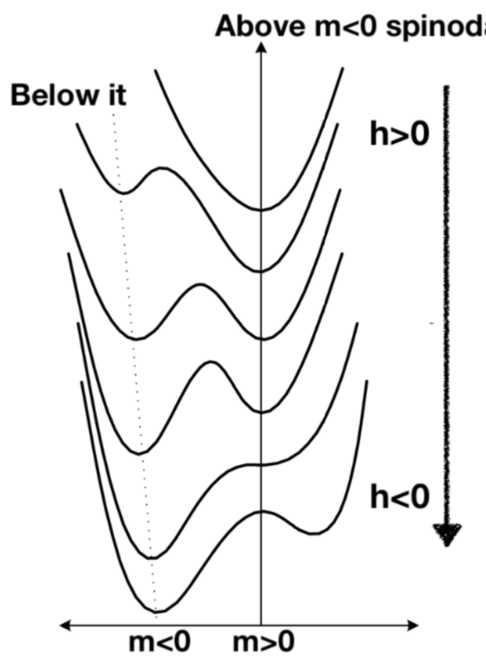
Nickel data vs. mean-field $m(T/T_c)$

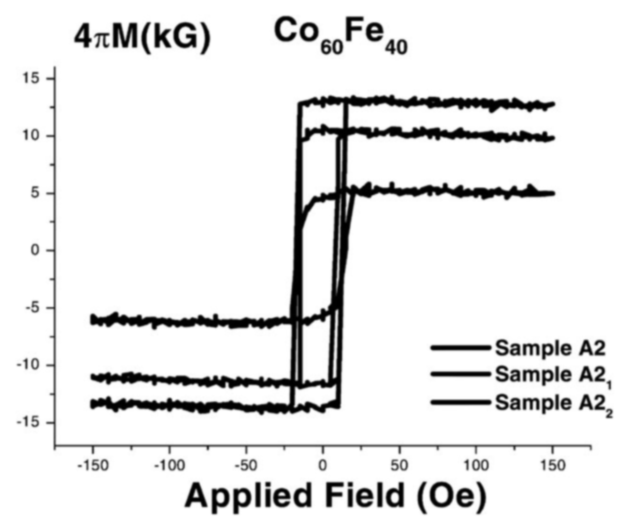
Weiss & Forrer, Ann. Phys. 5, 153 (1926)

Discontinuous phase transition

Hysteresis loops

Growth Rate Effects in Soft CoFe Films





M. Vopsaroiu, K. O'Grady, M. T. Georgieva, P. J. Grundy, and M. J. Thwaites,

IEEE Transactions on magnetics 41, 3253 (2005)

Curie-Weiss Mean-Field

Exponents for a continuous ($\mathbb{Z}_2$ spont symm broken) transition

$$
\begin{aligned}
-\beta f &= \beta J z \frac{m^2}{2} + \beta h m - \left(\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right) \\
&= -\beta J z \frac{m^2}{2} + \ln \left[2 \cosh \left(\beta J z m + \beta h \right) \right]
\end{aligned}
$$

$$m = \tanh \left(\beta J z m + \beta h \right)$$

$$t = (T - T_c)/T_c$$

	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto \lvert t \rvert^{-\alpha}$	$t \to 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^{\beta}$	$t \to 0^-, \quad h = 0$	$1/2$
Susceptibility	γ	$\chi \propto \lvert t \rvert^{-\gamma}$	$t \to 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto \lvert m \rvert^{\delta} \mathrm{sign}(m)$	$h \to 0, \quad t = 0$	3

Measurements

Magentisation^3 vs. temperature

Nuclear magnetic resonance in MnF4 near a critical point

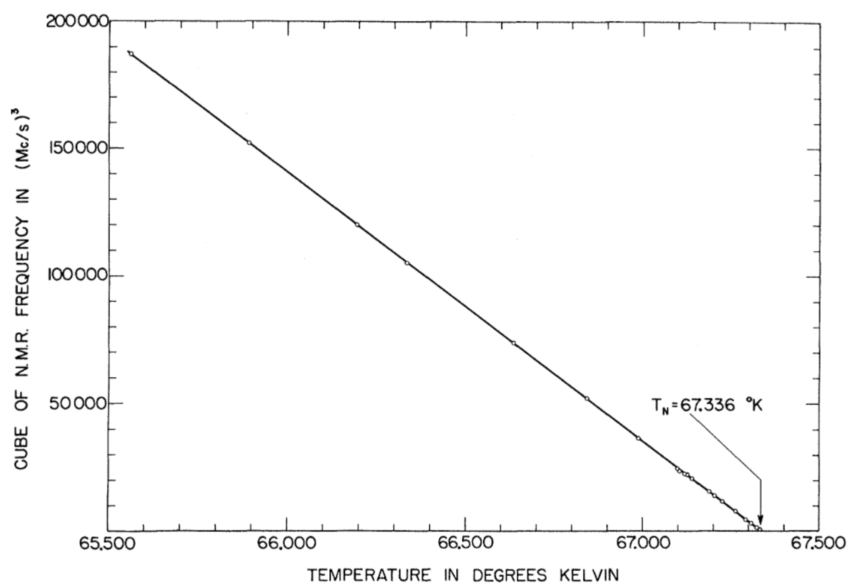


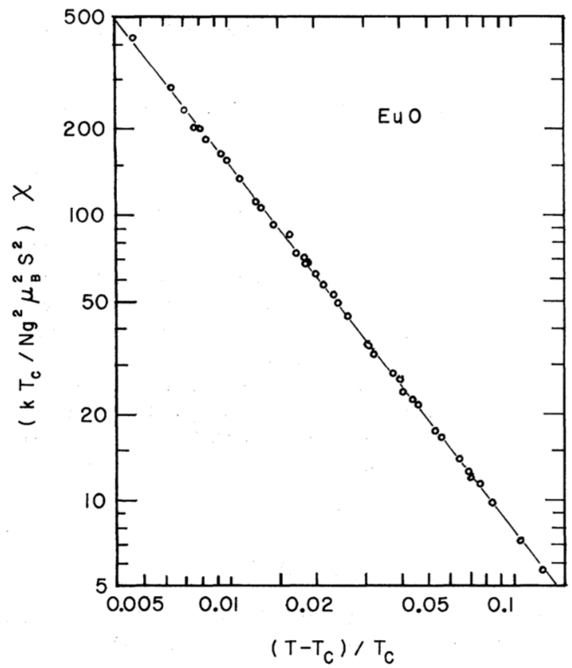
FIG. 1. Temperature dependence of the cube of the F^{19} nuclear resonance frequency for the first 1.8 degrees below T_N. The points lie on the straight line shown to within the experimental uncertainty of about 5 millidegrees.

Mean-Field yields $\beta = 1/2$

P. Heller and G. B. Benedek, Phys. Rev. Lett. 8, 428 (1962)

Measurements

Magnetic susceptibility $\chi \sim (T_c - T)^{-1.3}$

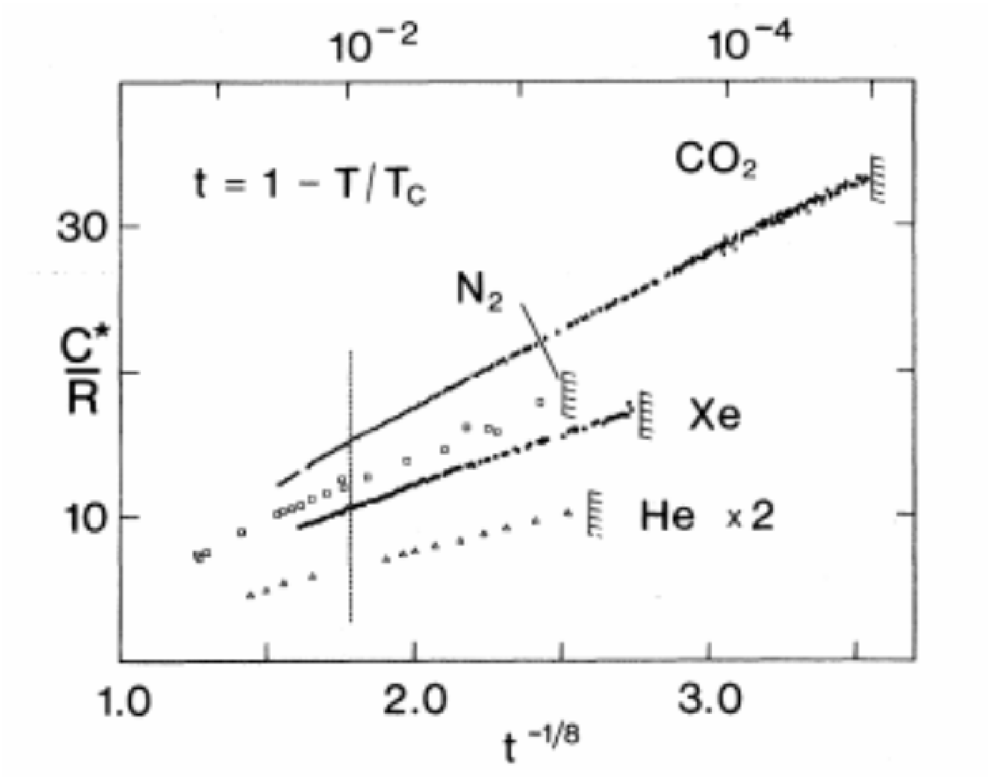


Mean-Field yields $\gamma = 1$

Measurements

Heat capacity vs reduced temperature$^{-1/8}$

Precision Measurement of the Specific Heat of CO_2 Near the Critical Point

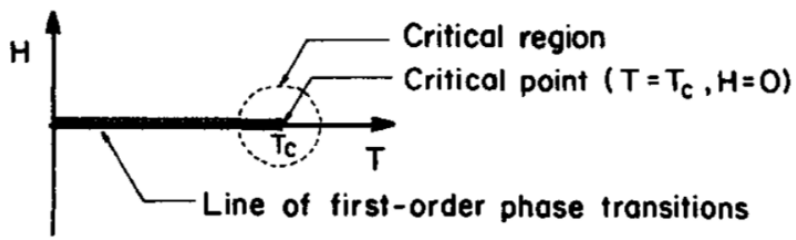


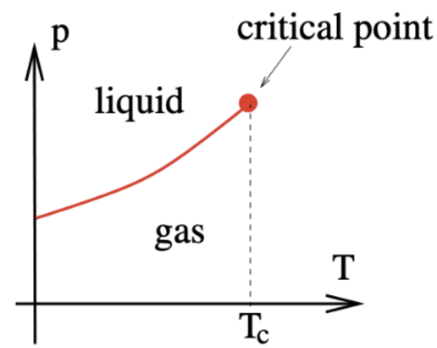
Mean-Field yields $\alpha = 0$

J. Lipa, C. Edwards, and M. Buckingham, Phys. Rev. Lett. 25, 1086 (1970)

Universality

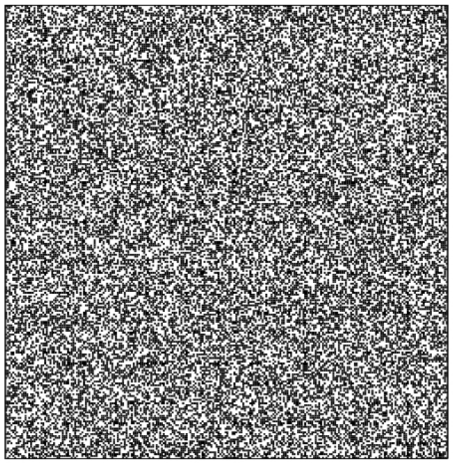
Para-Ferro magnetism & Liquid-Gas



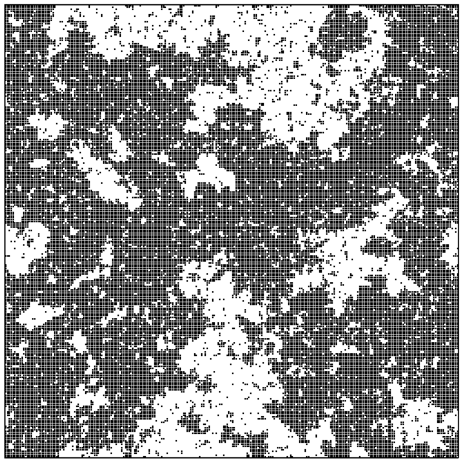


Typical configurations

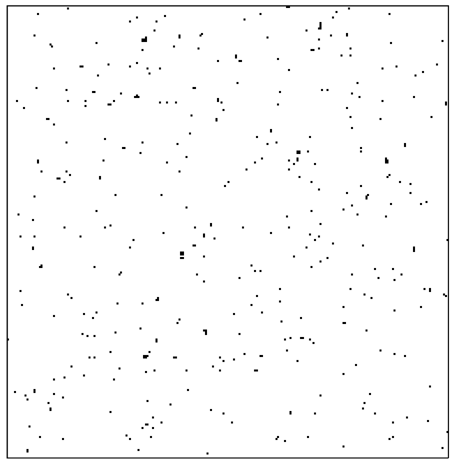
Up & down spins in a $2d$ Ising model



$T \to \infty$



$T = T_c$

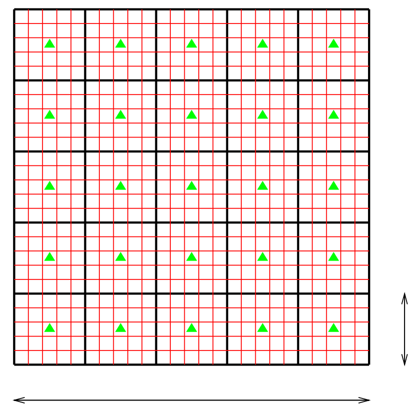


$0 < T < T_c$

Real space viewpoint

Ginzburg-Landau

Continuous scalar statistical field theory



Coarse-grain the spin

$\phi(\boldsymbol{r}) = V_{\boldsymbol{r}}^{-1} \sum_{i \in V_{\boldsymbol{r}}} s_i$.

The partition function is $\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\beta \mathcal{F}(\phi)}$ with

$$\mathcal{F}(\phi) = \int d^d r \;\left\{ \frac{1}{2} [\nabla \phi(\boldsymbol{r})]^2 + \frac{T-T_c}{2} \phi^2(\boldsymbol{r}) + \frac{\lambda}{4} \phi^4(\boldsymbol{r}) \right\}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments)

Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\boldsymbol{r}) = \langle \phi(\boldsymbol{r}) \rangle = \phi_0$

The free-energy density is $\lim_{V \to \infty} f_V(\beta, J, g) = \lim_{V \to \infty} V^{-1} \mathcal{F}(\phi_0)$

Ginzburg-Landau

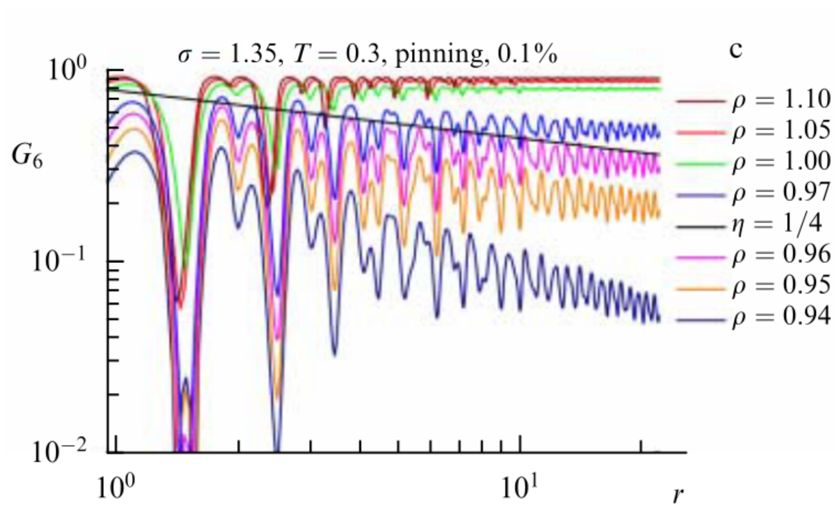
Exponents from the saddle-point analysis *vs.* finite d ones

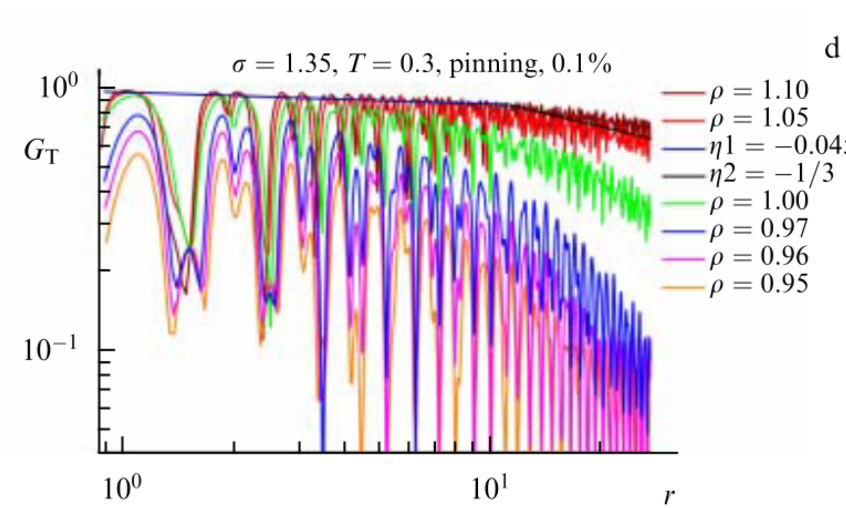
	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto \|t\|^{-\alpha}$	$t \to 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^{\beta}$	$t \to 0^-, \quad h = 0$	$1/2$
Susceptibility	γ	$\chi \propto \|t\|^{-\gamma}$	$t \to 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto \|m\|^{\delta} \mathrm{sign}(m)$	$h \to 0, \quad t = 0$	3
Correlation length	ν	$\xi \propto \|t\|^{-\nu}$	$t \to 0, \quad h = 0$	$1/2$
Correlation function	η	$G(\mathbf{r}) \propto \|\mathbf{r}\|^{-d+2-\eta}$	$r = 0, \quad h = 0$	0

d	β	$\alpha = \alpha'$	$\gamma = \gamma'$	δ	ν	η	
2	$1/8$	0	$7/4$	15	1	$1/4$	exact
3	0.325	0.11	1.24	4.82	0.63	0.032	approx
MF	$1/2$	0	1	3	$1/2$	0	exact

Correlation functions

Melting: from solid to liquid in $2d$





straight lines

curved

Translational order correlation functions G_T: exponential decay $e^{-r/\xi}$

Orientational order correlation functions G_6: power law decay $r^{2-d-\eta}$ - critical

Correlation functions

Atomic gases in $2d$ - Stochastic Gross-Pitaevskii equation

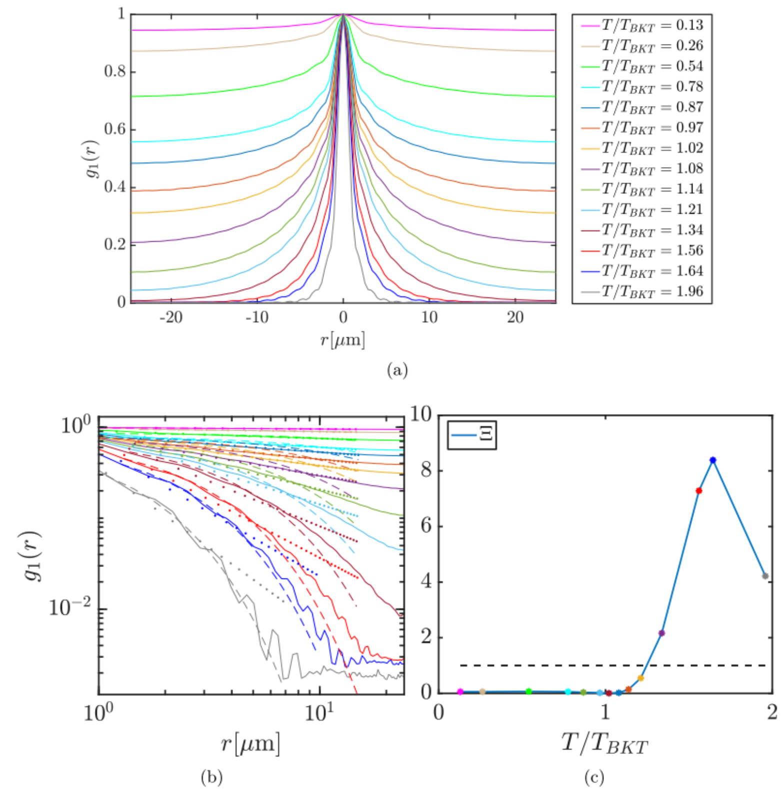
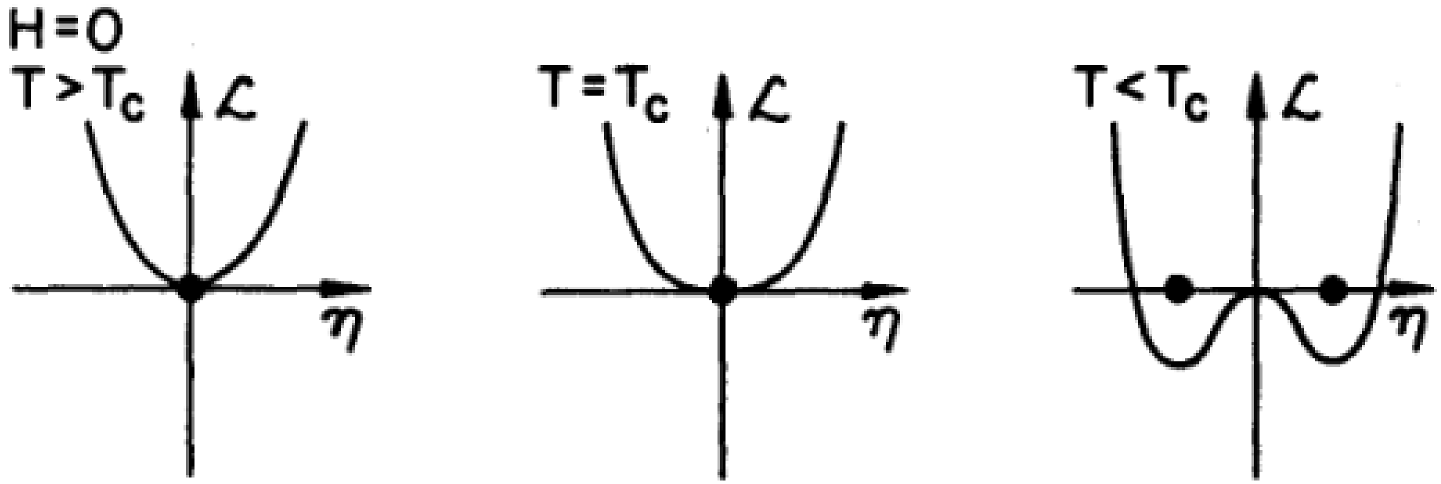


Figure 4.7: (a) Equilibrium profiles for the first-order correlation function $g_1(r)$ for different temperatures. (b) Correlation function profiles as in the previous figure, in logarithmic scale. The fitted algebraic (dotted lines) and exponential (dashed lines) functions are defined as in (4.56). (c) Ratio Ξ between the mean squared errors for the fits in (b), as defined in (4.57). The dashed line expresses the value of Ξ for which the two fits apply equally well to the data. The resulting critical point presents a shift of about ~ 0.25 with respect to the theoretical relation for T_{BKT}, equation (4.41).

F. Larcher, *Dynamical excitations in low-dimensional condensates: sound, vortices and quenched dynamics*, PhD thesis, Newcastle University & Università di Trento (2018)

Landau theory

Second order phase transition

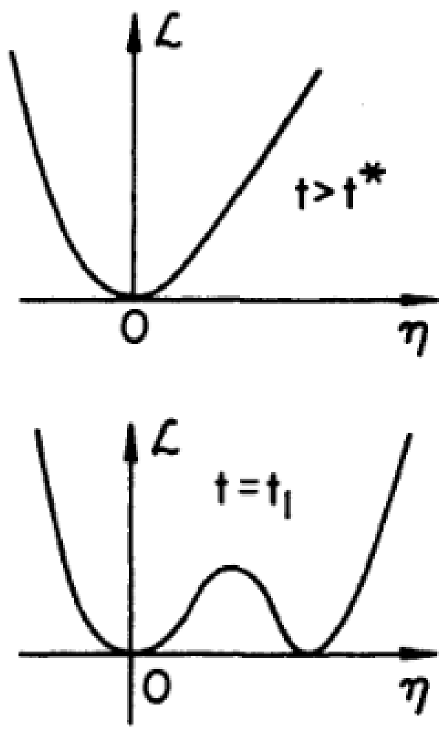


Notation : $\mathcal{L}$ is the “potential” in the Landau free-energy density

Notation : η is the “field”

Landau theory

First order phase transitions



Notation : $\mathcal{L}$ is the "potential" in the Landau free-energy density, η is the "field", t_I is the transition

Landau theory

Important points

- The actual values of the parameters are not important

  they are material/model dependent

- The dimension of the order parameter (scalar, vectorial) and the potential

  decide whether the transition is second, first order or infinite order

- In 2nd order phase transitions the correlation length diverges & the linear

  susceptibility as well.

- In continuous phase transitions the exponents are universal

- The strength of the Gaussian fluctuations limit the validity of the saddle-point

  treatment of the Landau theory.

  upper critical dimension, critical region $\xi^{4-d} < 1$ for the scalar $\lambda\phi^4$ theory

Critical Scaling

Ferromagnetic transition

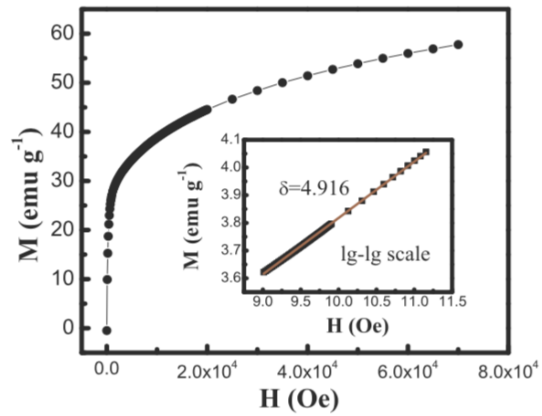


Fig. 4. Isothermal M vs H plot of $Nd_{0.55}Sr_{0.45}Mn_{0.98}Ga_{0.02}O_3$ at T_C = 247 K; the inset shows the same plot in log–log scale and the solid line is the linear fit following Eq. (3).

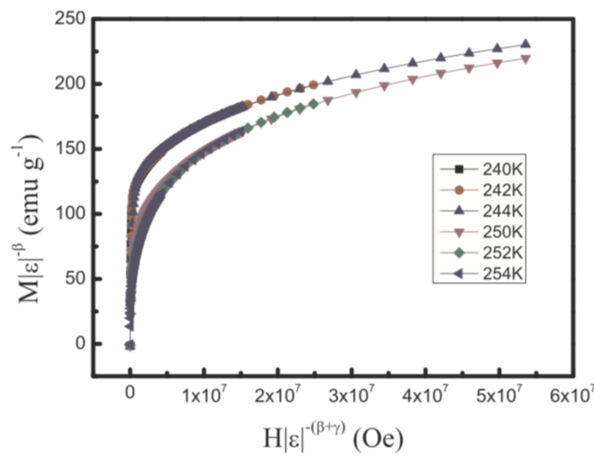


Fig. 5. Scaling plots of renormalized magnetization M vs. renormalized field H above and below T_C.

Bo Yua *et al*, *Scaling study of magnetic phase transition and critical behavior in* $Nd_{0.55}Sr_{0.45}Mn_{0.98}Ga_{0.02}O_3$ *manganite*, Materials Research Bulletin 99, 393 (2018)

Critical Scaling

Jamming transition

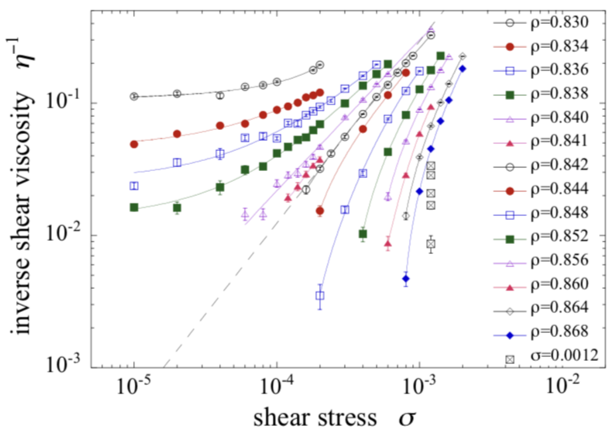


FIG. 2: (color online) Plot of inverse shear viscosity η^{-1} vs applied shear stress σ for several different values of the volume density ρ. The dashed line represents the power law dependence expected exactly at $\rho = \rho_c$ and has a slope $\beta/\Delta = 1.375$. Solid lines are guides to the eye. Points labeled $\sigma = 0.0012$ correspond to densities $\rho = 0.870$, 0.872, 0.874, 0.876, and 0.878.

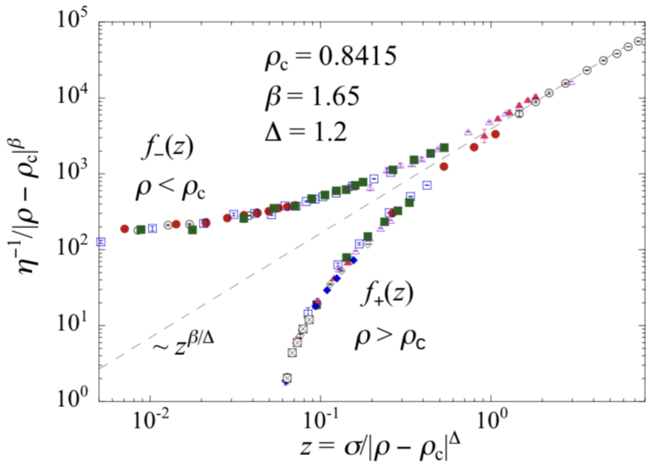


FIG. 3: (color online) Plot of scaled inverse viscosity $\eta^{-1}/|\rho - \rho_c|^\beta$ vs scaled shear stress $z \equiv \sigma/|\rho - \rho_c|^\Delta$ for the data of Fig. 2. We find an excellent collapse to the scaling form of Eq. (6) using values $\rho_c = 0.8415$, $\beta = 1.65$ and $\Delta = 1.2$. The dashed line represents the large z asymptotic dependence, $\sim z^{\beta/\Delta}$. Data point symbols correspond to those used in Fig. 2.

P. Olsson and S. Teitel, *Critical Scaling of Shear Viscosity at the Jamming Transition*, Phys. Rev. Lett. 99, 178001 (2007).

Critical Scaling

Jamming transition

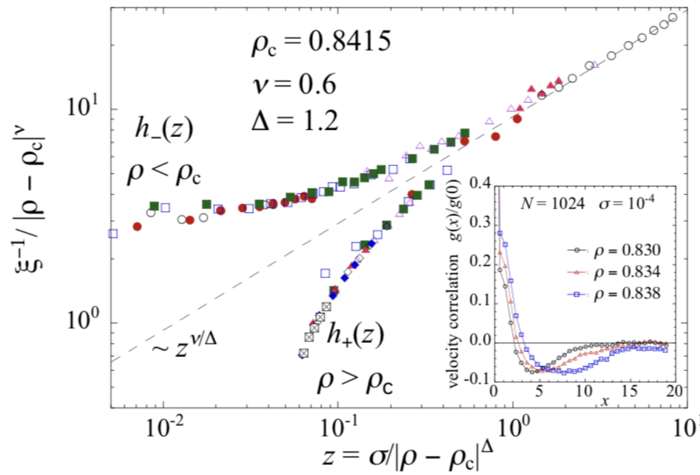
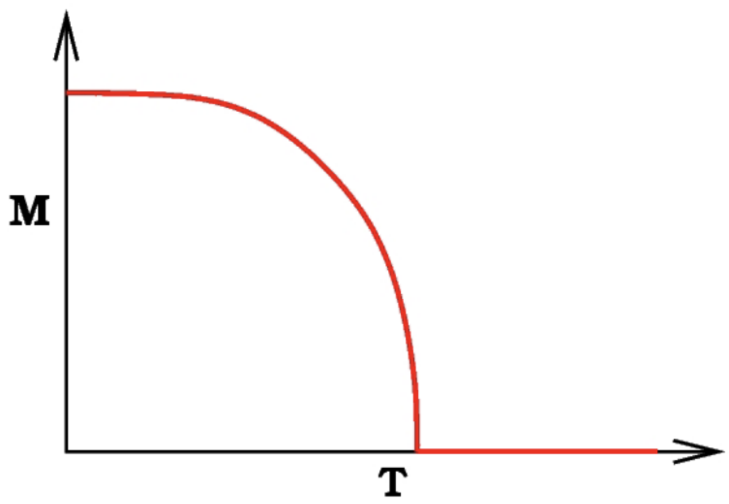


FIG. 4: (color online) Inset: Normalized transverse velocity correlation function $g(x)/g(0)$ vs longitudinal position x for $N = 1024$ particles, applied shear stress $\sigma = 10^{-4}$, and volume densities $\rho = 0.830$, 0.834 and 0.838. The position of the minimum determines the correlation length ξ. Main figure: Plot of scaled inverse correlation length $\xi^{-1}/|\rho - \rho_c|^\nu$ vs scaled shear stress $z \equiv \sigma/|\rho - \rho_c|^\Delta$ for the data of Fig. 2. We find a good scaling collapse using values $\rho_c = 0.8415$, $\Delta = 1.2$ (the same as in Fig. 3) and $\nu = 0.6$. Data point symbols correspond to those used in Fig. 2.

P. Olsson and S. Teitel, *Critical Scaling of Shear Viscosity at the Jamming Transition*, Phys. Rev. Lett. 99, 178001 (2007).

Finite size effects

Rounding of magnetization curves



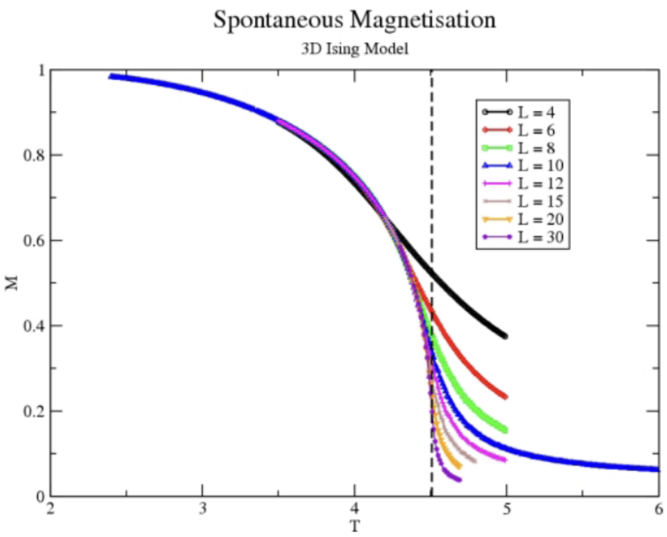
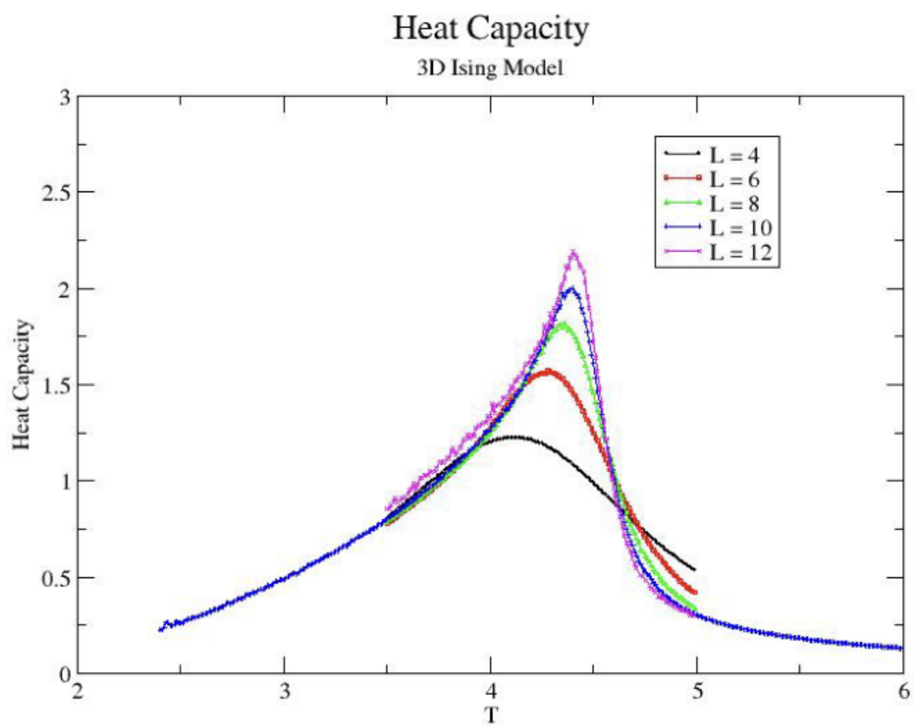


Figure: In TD limit the order parameter is 0 for $T > T_c$ but in FS the transition is smeared out.

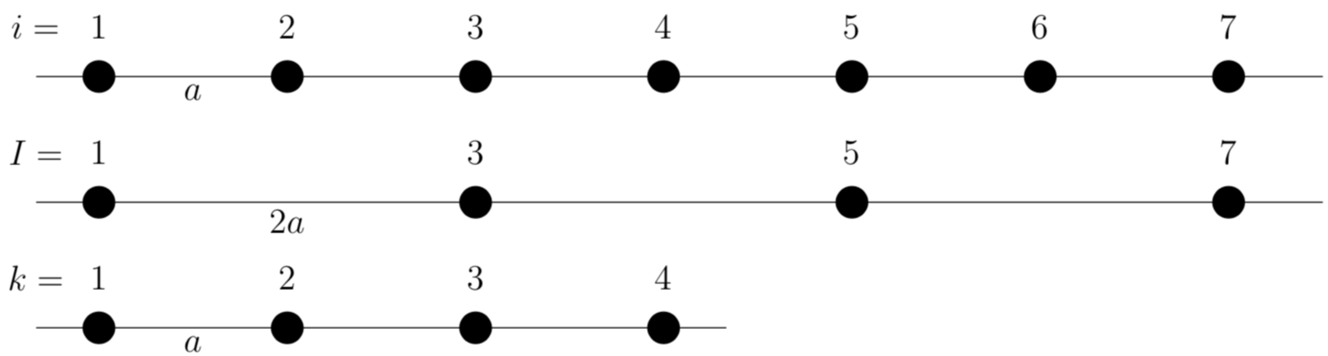
Finite size effects

Rounding of the heat capacity



Decimation

1d Ising chain $K = \beta J, \;\; \zeta = \ln \mathcal{Z}$



$$K' = \frac{1}{2} \ln \cosh(2K) \to 0 \quad \text{that is} \quad (T/J)^* \to \infty$$

$$\boxed{\zeta(K') = 2\zeta(K) - \ln[2(\cosh(2K))^{1/2}]}$$

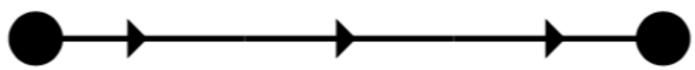
Decimation : the killing of one in every ten of a group of people as a punishment for the whole group (originally with reference to a mutinous Roman legion)

Decimation

1d Ising chain $K = \beta J, \;\; \zeta = \ln \mathcal{Z}$ **inverse relations**

$$K = \frac{1}{2}\cosh^{-1}(\exp(2K')) \, , \qquad \zeta(K) = \frac{1}{2}\ln 2 + \frac{1}{2}K' + \frac{1}{2}\zeta(K') \, ,$$

$K' = 0$



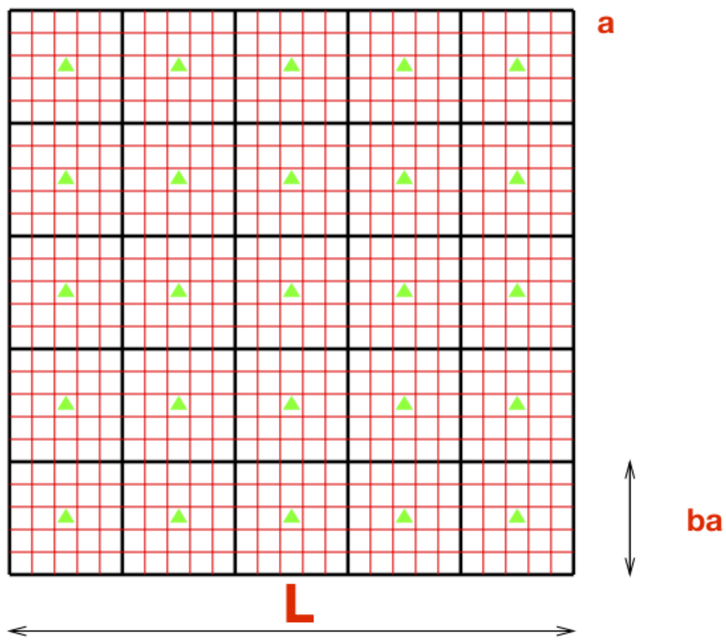
$K \to \infty$ $\qquad T \to 0$

Table I. Values of ζ for the 1D Ising model calculated from the recursion formulas Eqs. (31) and (32) of the renormalization group and from the exact formula derived from Eq. (9). ζ is related to the partition function Z through Eq. (28).

K	$\zeta(K)$ Renormalization group	Exact
0.01	ln 2	0.693 197
0.100 334	0.698 147	0.698 172
0.327 447	0.745 814	0.745 827
0.636 247	0.883 204	0.883 210
0.972 710	1.106 299	1.106 302
1.316 710	1.386 078	1.386 080
1.662 637	1.697 968	1.697 968
2.009 049	2.026 876	2.026 877
2.355 582	2.364 536	2.364 537
2.702 146	2.706 633	2.706 634

RG

From block spins to coupling renormalization



degrees of freedom

$b^d \mapsto 1$

lattice spacing

$a \mapsto ba$

re-scaling of lengths

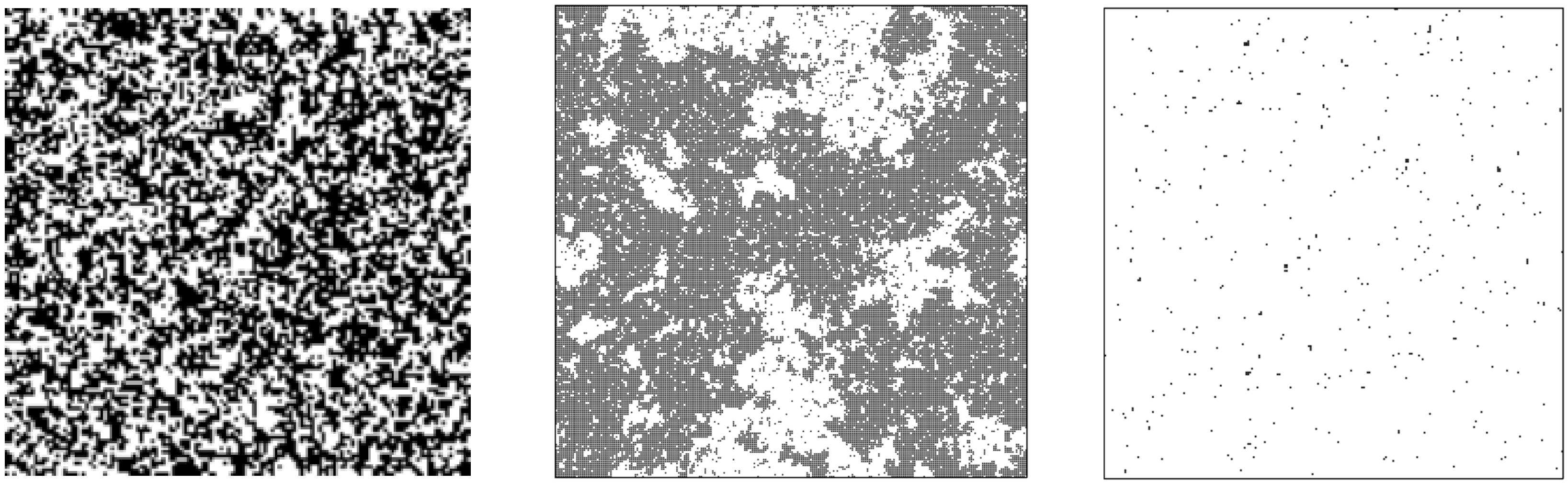
$\vec{x} \mapsto \vec{x}/b$

From $\mathcal{H}'_{[K']}(\{s'_I\}) = R_b \mathcal{H}_{[K]}(\{s_i\})$ to

$K'_\alpha = \sum_\beta (R_b)_{\alpha\beta} K_\beta$ close to $[K_c] = [0]$

Typical configurations

Up & down spins in a $2d$ Ising model



$T = 2T_c$ $\qquad$ $T = T_c$ $\qquad$ $0 < T < T_c$

Real space viewpoint

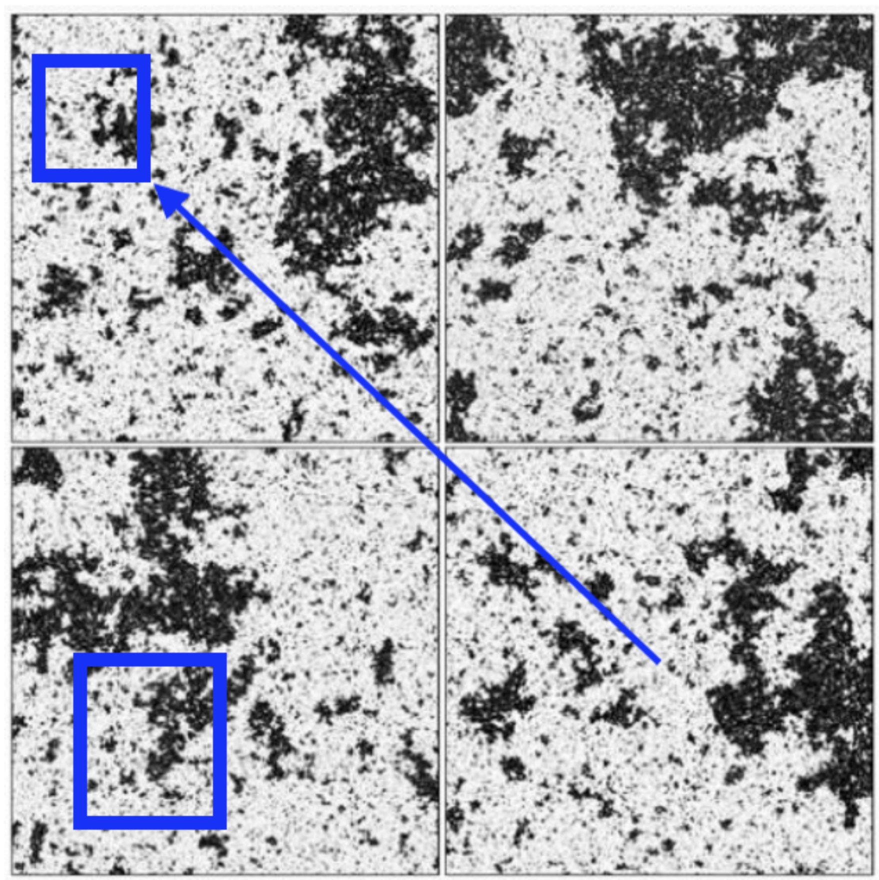
Zoom out by observing at larger scales

move away from criticality if $T \neq T_c$

$[K_c]$ repulsive fixed points

Scale invariance

2d Ising model at T_c



D. Ashton, *Scale invariance in the critical Ising model*,

https://www.youtube.com/watch?v=fi-g2ET97W8

RG

finite d Ising model

Two adimensional parameters in $\mathcal{H}_{[K]}(\{s_i\})$

$$K_1 = \beta J = K \qquad\qquad K_2 = \beta h = K$$

Composition $R(b_1 b_2) = R(b_1)R(b_2)$ and symmetries

$$K' = b^{y_K} K \qquad\qquad H' = b^{y_H} H$$

Repulsive fixed point

$$y_K > 0 \qquad\qquad y_H > 0$$

RG

finite d Ising model

Identity of total free-energy $\Longrightarrow$ homogeneity

$$F(K', H') = F(K, H) \qquad \Longrightarrow \qquad b^{-d} f(b^{y_K} H, b^{y_H} H)$$

choosing $b = |K|^{-1/y_K}$ $\Longrightarrow$ scaling

$$\begin{aligned} f(K, H) &= |K|^{d/y_K} \; f(1, |K|^{-y_H/y_K} H) \\ &= |K|^{d/y_K} \; g_f\left(\frac{H}{|K|^{y_H/y_K}}\right) \end{aligned}$$

with $\Delta = y_H/y_K$ and $2 - \alpha = d/y_K$

RG

Generic

Transformation

$$K'_\alpha - K^*_\alpha = \sum_\beta R(b)_{\alpha\beta} \left(K_\beta - K^*_\beta\right)$$

in other terms $$R(b)_{\alpha\beta} = \left.\frac{\partial K'_\alpha}{\partial K_\beta}\right|_{[K^*]}$$

To solve one needs to diagonalize the linear (vectorial) relation above

Use the

eigenvalues λ^i and left orthonormal eigenvectors u^i_α of the matrix $\mathbb{R}(b)$

$$\sum_j u^j_\beta u^j_\gamma = \delta_{\beta\gamma}$$

RG

Generic

Transformation

$$\delta K_\alpha = K'_\alpha - K^*_\alpha = \sum_\beta R(b)_{\alpha\beta}\,(K_\beta - K^*_\beta) = \sum_\beta R(b)_{\alpha\beta}\,\delta K_\beta$$

multiply by the left eigenvector u^i_α and sum over α

$$\begin{aligned}
\delta\kappa'^{\,i} \equiv \sum_\alpha u^i_\alpha\,\delta K'_\alpha \quad &= \quad \underbrace{\sum_\alpha u^i_\alpha \sum_\beta R(b)_{\alpha\beta}}_{\lambda^i \sum_\beta u^i_\beta} \sum_\gamma \underbrace{(\sum_j u^j_\beta u^j_\gamma)}_{\delta_{\beta\gamma}}\,\delta K_\gamma \\
&= \quad \lambda^i \sum_j \underbrace{\sum_\beta u^i_\beta u^j_\beta}_{\delta^{ij}} \sum_\gamma u^j_\gamma \delta K_\gamma \\
&= \quad \lambda^i \sum_\gamma u^i_\gamma\,\delta K_\gamma = \lambda^i\,\delta\kappa^i
\end{aligned}$$

RG

Generic

Diagonalized transformation

$$\delta\kappa'^{\,i} \equiv \sum_\alpha u^i_\alpha \, \delta K'_\alpha = \lambda^i \, \delta\kappa^i$$

thus

$$\lambda^i = \left.\frac{\partial \delta\kappa'^{\,i}}{\partial \delta\kappa^{\,i}}\right|_{[K^*]} = \left.\frac{\partial \kappa'^{\,i}}{\partial \kappa^{\,i}}\right|_{[K^*]}$$

and

$$\lambda^i = b^{y_i} \qquad \Longrightarrow \qquad y_i = \frac{d \ln \lambda^i}{db}$$

RG

Generic

Parameters

- $y_i > 0$ – *relevant*, with the corresponding $\kappa_i(b)$ coupling growing under coarse-graining, i.e., getting away from the fixed point

- $y_i < 0$ – *irrelevant*, with the $\kappa_i(b)$ coupling vanishing under coarse- graining, i.e., approaching the fixed point

- $y_i = 0$ – *marginal*, with the flow of the $\kappa_i(b)$ coupling determined by the higher order terms in the couplings of the RG transformation.

Concepts

The scale of observation does not modify the properties of the critical state

nor the disordered or order character of the phases

Kadanoff blocks modify the physical state by coarse-graining but,

the modern RG (Wilson) transforms the form of the model until reaching the one that describes the critical state

At criticality and close to it the microscopic details are not important

The critical points are repulsive fixed points of the RG

Above the critical temperature, evolution to the free fixed point, $T \to \infty$

Below the critical temperature, evolution to $T = 0$

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At criticality and close to it the microscopic details are not important

A renormalisation transformation is a scale transformation that leaves the partition function invariant. Since the thermodynamic properties of a system are governed by the partition function, the physics is preserved.

Concepts

The critical behaviour does not depend on the type of subcritical order but on the number n of components of the order parameter and the dimension of space d.

Renormalisation enables the classification in universality classes with the same critical properties, depending on (n, d)

Scaling laws apply to global, macroscopic properties of systems containing a large number of elementary microscopic units.

They are found in other domains, *e.g.* finance, biology, etc.

Exponents again

Comparison

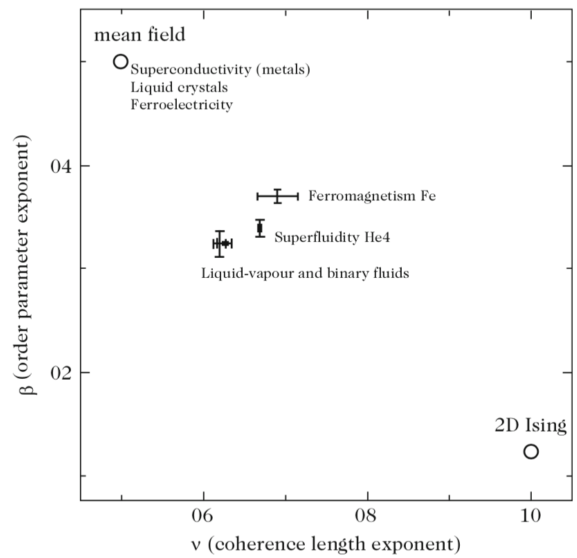

Fig. 1.22 Critical exponents β and ν: experimental results obtained for seven difference families of transitions, compared to values predicted by the mean field and 2D Ising models

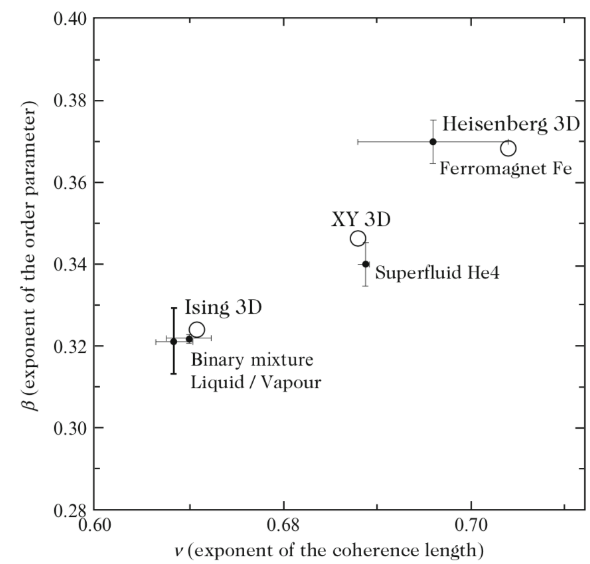

Fig. 1.23 Critical exponents measured for four families of transitions, compared to values predicted by the three corresponding models that take into account the scale invariance of the critical state

Images from

A. Lesne & M. Lagües, *Scale invariance from phase transitions to turbulence* (Springer-Verlag, 2012)