

2 - CLASSICAL PHASE TRANSITIONS

2.1 PHENOMENA

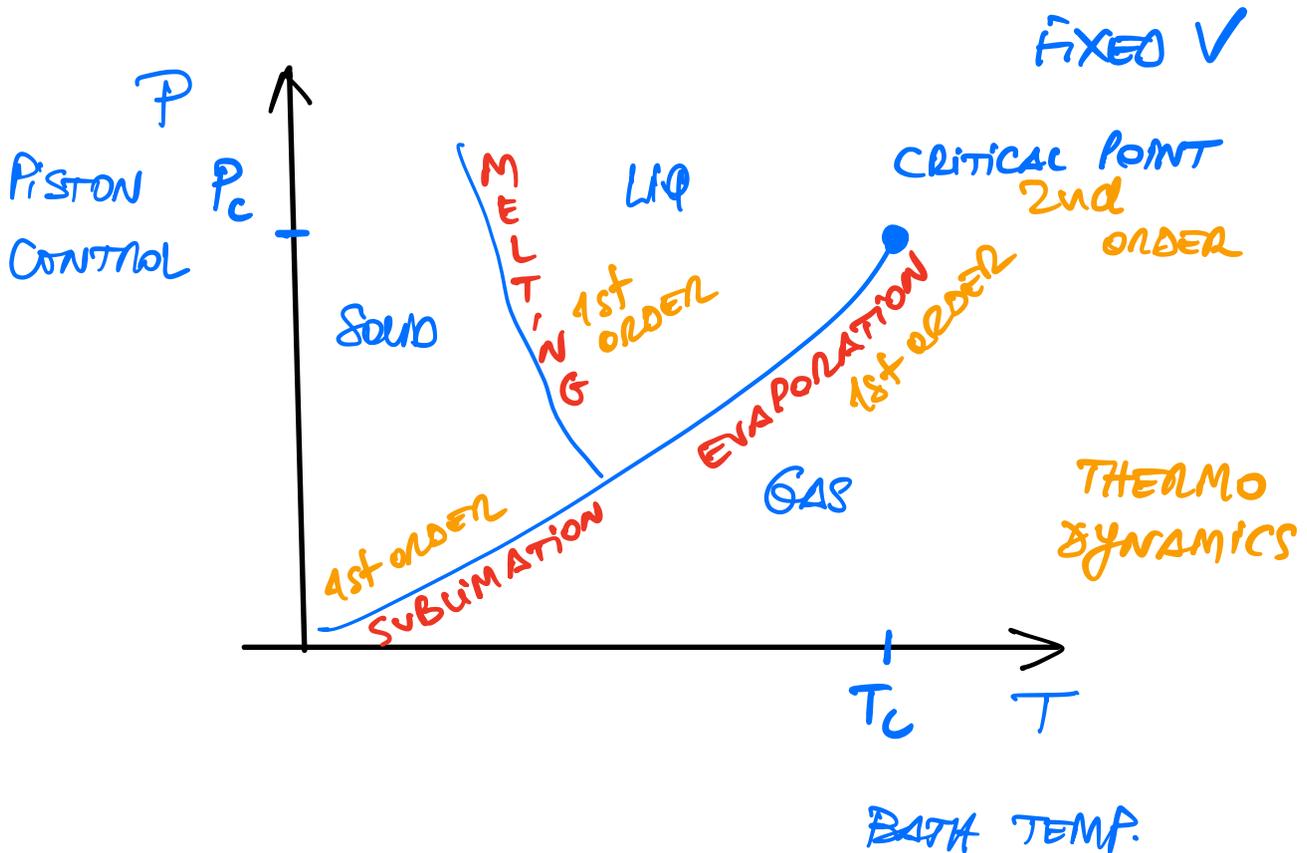
- USUALLY, SYSTEM IN CONTACT W/
ENVIRONMENT
(BUT ALSO CLOSED POSSIBLE)
- GLOBAL OBSERVABLES
eg. $\frac{\text{ENERGY}}{N}$, $\frac{\text{MAGN}}{N}$, ETC
DENSITIES
- CONTROL PARAMETERS
eg. TEMP., MAGN FIELDS, ETC

TUNE CONTROL PARAM \rightarrow MEASURE OBS

PHASE TRANS \equiv SHARP CHANGES IN
MACROSC BEHAV.

COLLECT PHENOMENA
IN $N \rightarrow \infty$

PHASE DIAGRAMS



IN ISOLATION, E FIXED (REPLACES T)
 VOLUME ALSO FIXED
 CHANGES IN PARAM IN HAMILTONIAN

PHASES IDENTIFIED BY PARAM DEP. OF
 ORDER PARAMETERS



GLOBAL OBS. JUDICIOUSLY CHOSEN

PHASE TRANSITIONS

SHARP CHANGES (INTUITIVE)

NON-ANALYTICITIES OF RELEVANT
THERMODYN POTENTIALS
WITH RESPECT TO CONTROL PARAM.

e.g. FREE-ENERGY

$$-\beta f(\beta J) = \frac{1}{N} \ln Z(\beta J)$$

$$Z(\beta J) = \sum_c e^{-\beta H_J(c)}$$

LOCATE w/ THEM THE CRITICAL CURVES

GOALS

- ORDER PARAM AS fcts of CONTROL PARAM
- CRITICAL CURVES IN PH. DIAG.
- CRITICAL BEHAVIOUR
- THERMODYN PROPS

METHODS

— MEAN FIELD

QUANTITATIVE UNDERSTANDING ✓

QUANTITATIVE APPROX ONLY ~

CRIT BEH TYP. WRONG X

— SCALING TH

PHENOM OF CRIT PHEN

— RANDOM. GROUP

CRIT BEH ✓

BEYOND STANDARD PH. TRANS

TOPOLOGICAL ONES

QUANTUM PHASE TRANS (LATER)

DYN PHASE TRANS. (NOT

DISCUSSED)

CONTEXT

PART: IN INTERACT

SPINS IN INTERACT

EASIER TO TREAT

2.2 ISING MODELS & EXTENSIONS

SIMPLIFIED REPRESENTATION OF A
MAGN. SYST

MAGN MOM. \nearrow \vec{S}_i LOCALIZED
ARROW w/
N-COMP.

$$H[\{\vec{S}_i\}] = -\frac{J}{2} \sum_{ij} \vec{S}_i \cdot \vec{S}_j - \sum_i \vec{h} \cdot \vec{S}_i$$

$J > 0$ FM INTERACTIONS //
 $J < 0$ AF ANTI- //

\sum_{ij} DECIDE WHETHER SPINS SIT ON A
LATTICE / A GRAPH

\vec{h} UNIFORM MAGN FIELD

d DIM. OF SPACE n DIM OF SPIN

FACTS

SIMPLEST MODEL

- 1 - ISING $J > 0$ SOLVED IN $d=1, 2$ ($R=0$), ∞
UNSOLVED ANALYTICALLY IN $d=3$
 - $d \geq d_p=2$ PH. TRANS AT $\beta_c J$ FINITE ($R=0$)
BTW FM & PM
 - d_u UPPER CRIT DIM: MFKK $d > d_u=4$
FAILS BELOW.
 - SCALING & RG AT WORK

3 - FRUSTRATION AF ON Δ LATT.

4 - QUENCHED DISORDER $J \rightarrow J_{ij}$ pdfs
 $h \rightarrow h_i$

RANDOM BOND FM $J_{ij} > 0$ SPIN-GLASS $J_{ij} \geq 0$

5 - $n=1$ ISING

$n=2$ XY

$n=3$ HEISENBERG

$n \rightarrow \infty$ $O(n)$

$$\sum_{a=1}^n (s_i^a)^2 = n$$

6 - STOCHASTIC DYNAMICS / KINETIC ISING
 n COMP MODELS / DOMAIN GROWTH

2.3 CONCEPTS

THERMODYN LIMIT

$N \rightarrow \infty$ NECESSARY TO GET NON-ANAL
 $f(\beta J, \beta h)$

OTHERWISE JUST \sum_e POSIT. TERM = ∞

AND THIS IS CONTINUOUS IN THE PARAM.

STRESS

ROLE OF ONSAGER'S SOL 2dIM

A PROOF IN A SOLVABLE CASE

ORDER PARAMETERS

A MEASURE OF THE DEGREE OF ORDER

STATISTICAL AVERAGE OF AN OBSERVABLE

$$\langle \theta \rangle$$

WHICH TAKES DIFF VALUES IN \neq PHASES

eg. $\tilde{\rho}(\vec{k}) = \int d^d r \rho(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$

FOR SOLID LIQUID

↑
PART
DENSITY

$$\vec{m} \equiv \left\langle \frac{1}{N} \sum_i \vec{s}_i \right\rangle \text{ FOR FM-PM}$$

SUBTLETY ABOUT SYMM TO BE DISCUSSED
LATER

TRANSITION CONTINUOUS VS. DISCONT.

↓
CONT DEPARTS FROM 0

↓
JUMP IN $\langle \theta \rangle$

SYMMETRIES

CONTINUOUS $O(n)$ SYMMETRY

$$\vec{h} = 0$$

UNIFORM ROTATION OF ALL SPINS

$$\begin{aligned} H[\{\vec{s}_i'\}] &= -\frac{J}{2} \sum_{ij} \vec{s}_i' \cdot \vec{s}_j' \\ &= -\frac{J}{2} \sum_{ij} R^{ab} s_i^b R^{ac} s_j^c \end{aligned}$$

SUM OVER SPIN COMP.

b AND c ASSUMED

$$R R^t = R^t R = \mathbb{1} \quad \Rightarrow$$

$$R^{ab} R^{cb} = R^{cb} R^{ab} = \delta^{ac}$$

$$= -\frac{J}{2} \sum_{ij} s_i^b s_j^b = H[\{\vec{s}_i\}]$$

DISCRETE \mathbb{Z}_2 UP-DOWN SYMM

FOR $n=1$, THE ISING CASE

$$\begin{aligned} H[\{s'_i\}] &= -\frac{J}{2} \sum_{ij} s'_i s'_j \\ &= -\frac{J}{2} \sum_{ij} (-s_i)(-s_j) \\ &= H[\{s_i\}] \end{aligned}$$

- NOTE THAT THESE SYMM HOLD FOR ALL LATTICES / GRAPHS
- BROKEN BY $\vec{h} \neq 0$ OR $h \neq 0$

FOR PARTICLE SYST AND SOLID-LIQUID INVARIANCE IS SPACE TRANSLATIONS.

$$\vec{r}' = \vec{r} + \vec{\Delta}$$

Typically $V(\vec{r}_i, \vec{r}_j) = V(|\vec{r}_i - \vec{r}_j|)$

PINNING FIELD

$$\langle s_i \rangle = \sum_{\{s_j = \pm 1\}} \frac{s_i e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

$$= \sum_{\{s_j' = \pm 1\}} \frac{s_i' e^{-\beta H(\{s_j'\})}}{Z(\beta)}$$

CHANGE
of

NAME

$\{s_i \rightarrow s_i'\}$

$$= \sum_{\{s_j = \pm 1\}} \frac{(-s_i) e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

$s_i' = -s_i$

AND

$H(\{s_j\}) =$

$H(\{-s_j\})$

$$= -\langle s_i \rangle$$

$$\Rightarrow \langle s_i \rangle = 0 \quad \forall i$$

So, how can m be $\neq 0$?

PINNING FIELD, WHEN GOING ACROSS
CRITICALITY

RESIDUAL OR INTENTIONAL

THE ACTUAL PROCEDURE IS

$$\langle s_i \rangle = \lim_{h \rightarrow 0^+} \lim_{N \rightarrow \infty} \langle s_i \rangle_h$$

(or $h \rightarrow 0^-$) AND THIS FORM YIELDS

$$\langle s_i \rangle \geq 0 \quad \text{AT } (T/J) < \text{CRIT}$$

$$\langle s_i \rangle = 0 \quad \text{AT } (T/J) > \text{CRIT}$$

BROKEN ERGODICITY

WE DEFINED ERGOD FROM

$$\overline{A} = \langle A \rangle$$

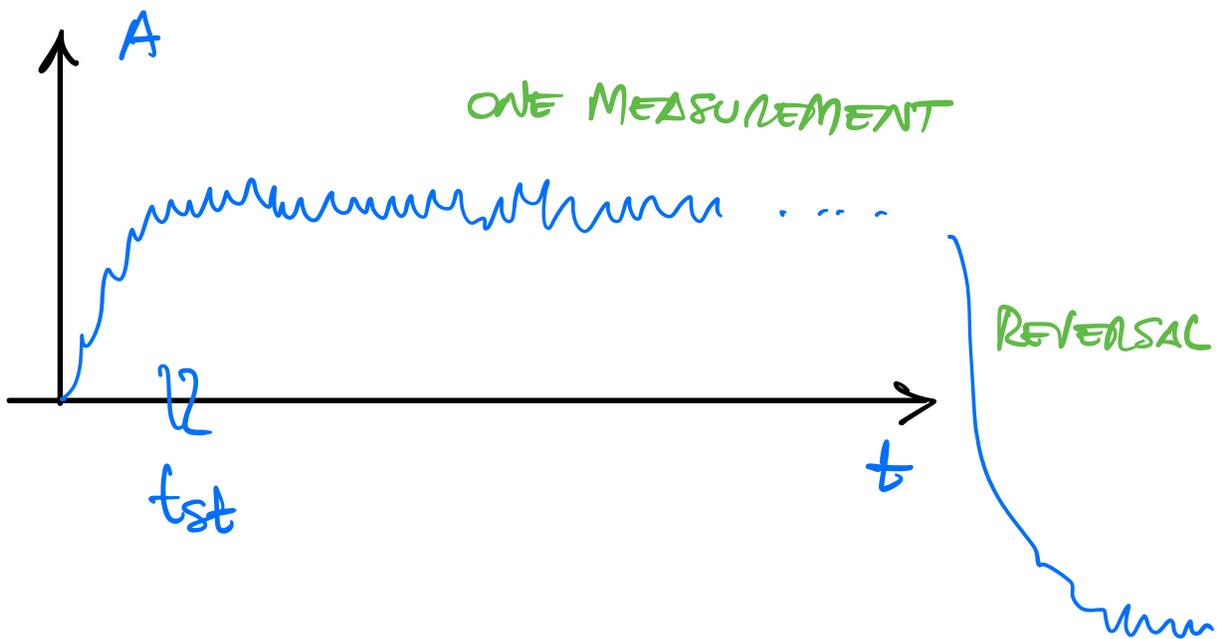
FOR GLOBAL OBSERV.

NOW, WHAT HAPPENS W/ THIS EQUAL.
IS A QUESTION OF TIME SCALES AND
DEFINITION OF $\langle \cdot \rangle$

EXPERIMENT

$$\bar{A} = \lim_{\tau \gg t_0} \lim_{t > t_{st}} \frac{1}{\tau} \int_{t_{st}}^{t_{st} + \tau} dt' A(t')$$

IN THE MAGNETIC (NICKEL) EXAMPLE $A =$ GLOBAL MAGN.



IF WE COMPARE :

- \bar{A} WITH $\tau < t_{\text{rev}}$ TO $\langle A \rangle$ WITH NO PINNING FIELD
 $\bar{A} \neq \langle A \rangle$
- \bar{A} WITH $\tau \gg t_{\text{rev}}$ TO $\langle A \rangle$ WITH NO PINNING FIELD
 $\bar{A} = \langle A \rangle$
- \bar{A} WITH $\tau < t_{\text{rev}}$ TO $\langle A \rangle$ W/ h
 $\bar{A} = \langle A \rangle$
- \bar{A} WITH $\tau \gg t_{\text{rev}}$ TO $\langle A \rangle$ W/ h
 $\bar{A} \neq \langle A \rangle$

ERGODICITY DEPENDS ON TIME SCALES AND HOW THE STAT AVERAGES ARE COMPUTED

ORDER OF LIMITS
 $\hbar \rightarrow 0$ AND $N \rightarrow \infty$

MATTERS

SPONTANEOUS SYMM BREAKING

By CHOOSING ONE AMONG TWO (OR MORE) EQUAL PHASES WHILE GOING ACROSS THE PH. TRANS

⇒ SPONTANEOUS SYMM BROKEN

$$\lim_{h \rightarrow 0^+} \lim_{N \rightarrow \infty} \langle S_i \rangle = - \lim_{h \rightarrow 0^-} \lim_{N \rightarrow \infty} \langle S_i \rangle$$

$\neq 0$

ENTROPY VS. ENERGY - PETERLS

FROM THERMODYN WE KNOW THAT IN EQUIL

- ENERGY TENDS TO BE MINIMIZED
- ENTROPY MAXIMIZED.

IN OPEN CANONICAL CONDITIONS THE RELEVANT THERMODYN. POTENTIAL TO BE MINIMIZED IS THE FREE-ENERGY

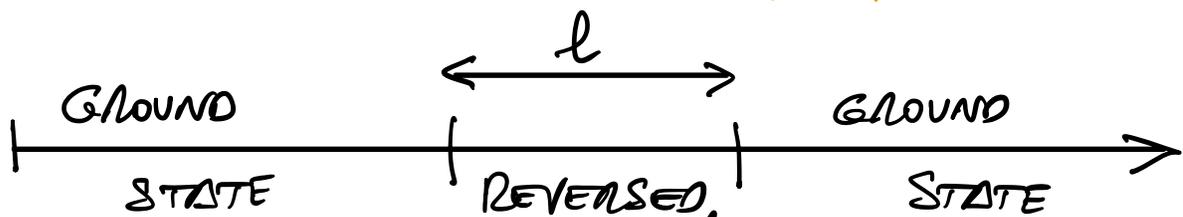
$$F = U - TS$$

$U = \langle H \rangle$ INTERNAL ENERGY

S ENTROPY

$$H = -J \sum_i S_i S_{i+1} \quad \text{EACH BOND ONCE}$$

EX: 1d ISING CHAIN W/ PBC



IS THE OTHER GROUND STATE

WE STUDY WHETHER $T=0$ GROUND STATE
IS STABLE
AGAINST TEMP FLUCT
{ $S_i = +1$ }

LOWEST ENERGY EXCITATION :
ONE DOMAIN, TWO DOMAIN WALLS

TO SIMPLIFY DISCUSSION TAKE ONE
BACKGROUND - SAY ALL UP - AND
REVERSE A DOMAIN TO DOWN WITHIN

$$F_0 = U_0 - T S_0 \quad \text{GROUND STATE}$$

$$F_2 = U_2 - T S_2 \quad \text{LOWEST EXC.}$$

(THE SUBSCRIPT 2 \equiv 2 DWs)

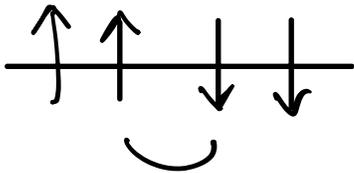
$$\Delta F = F_2 - F_0 \geq 0 \quad ?$$

IF $< 0 \Rightarrow$ ORDERED WOULD
BY FLUCT

$$\Delta F = \Delta U - T \Delta S$$

$$\Delta U = U_2 - U_0 = 2 \cdot 2J = 4J$$

THE TWO WALLS OF INTERVAL l



$$\begin{array}{l} \text{DW} \Rightarrow +J \\ \text{NO DW} \Rightarrow -J \end{array} \left. \vphantom{\begin{array}{l} \text{DW} \\ \text{NO DW} \end{array}} \right\} \text{DIFF: } 2J$$

ON ALL OTHER UNITS $U_2 - U_0 = 0$ locally

ΔS ONE CONF WITHOUT DW $S_0 = k_B \ln 1$
 N CONFS W/ ONE DOMAIN
 OF LENGTH l

$$S_2 = k_B \ln N$$

$$\Delta S = k_B \ln N - k_B \ln 1$$

$$\longrightarrow k_B \ln N$$

$T \neq 0$

$$\Delta F = 4J - k_B T \ln N \longrightarrow -k_B T \ln N$$

$N \gg 1$
 J FINITE

if $T \neq 0$ ΔF IS NEGATIVE AND LOGAR
LARGE IN N

\Rightarrow IT'S CONVENIENT TO HAVE
THE REVERSED DOMAIN

NEXT REPEAT THE ARGUMENT W/
ANOTHER REVERSED DOMAIN



1dIM DISORDERS AS SOON AS
 $T > 0$

BUT THERE CAN BE ORDER AT FINITE T :

1dIM W/ POWER LAW DECAYING
INTERACTIONS & FBC

RECALL $\alpha > d = 1$ ARGUED IN CH. 1
WITH ENERGY INTEGRAL
TO HAVE SHORT-RANGE INTER.

SEE CALC. IN THE NOTES

FOR SUFF. LONG-RANGE INT $\alpha < 1$

LONG-RANGE ORDER IS POSSIBLE

MANY MATH PAPERS w/ COND
FOR ABSENCE OF ORDER

2.4 OTHER METHODS

(WE WILL NOT TREAT THEM)

• LOW TEMP EXPANSIONS

- IDENTIFY GROUND STATES
(MINIMA OF THE ENERGY)
AND 1ST EXCITED STATES
∑ SUM OVER THESE CONFS ONLY
- MORE SYSTEMATIC

$$e^{-\beta H}$$

- SADDLE POINT APPROX $\beta \rightarrow \infty$ ($T \rightarrow 0$)
- EXPAND AROUND THE MINIMUM
∑ TREAT NON-QUAD. TERMS
AS PERTURB w/ WICK.

• HIGH TEMP EXPANSION

EXPLOIT SOME RE-WRITING

$$e^{\beta J S_i S_j} = a (1 + b S_i S_j)$$

AND ORGANIZE THE PARTITION SUM
⇒ CLUSTER EXP

- DUALITY MAPPING $g \rightarrow \frac{1}{g}$
STRONG \leftrightarrow WEAK COUPLING

- TRANSFER MATRIX
SPECIFIC TO 2d-

WE WILL SEE IT WHEN DISCUSSING
QUANTUM CHAINS AND THEIR MAPPING
TO CLASSICAL 2d MODELS.

2.5 MEAN-FIELD

WE USE SPIN MODELS SINCE THEY ARE SIMPLER TO TREAT THAN PARTICLE ONES

YOU NEED TO LEARN CONCEPTS AND METHODS, AND HOW TO USE THEM

BACK TO SPIN MODELS, THE ONES WITH FM INTERACTIONS ARE EASIER.

ONE CAN GUESS THEIR HIGH/LOW T PHASES

BUT SOLVING THEM, e.g. FINDING ANALYT. EXPRESSIONS FOR T_c , ORDER PARAM. CRITICAL BEHAVIOUR : DIFFICULT

DONE IN $d = 1, 2, \infty$

MF METHODS GET APPROX RESULTS FOR T_c , ORDER PARAMETERS, CRITICAL BEHAVIOUR

THESE RESULTS ARE GENERICALLY
CORRECT

QUANTITATIVELY

AND TO A CERTAIN EXTENT ALSO

QUANTITATIVELY

BUT, e.g. CRITICAL BEHAVIOUR WRONG
BELOW d_u
SOMETIMES PHASE TRANS. PREDICTED
WHICH \neq , etc.

GENERIC MODEL

p-SPIN WITH FM INTER.

$$H(\{s_i\}) = -J \sum_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} - h \sum_i s_i$$

FOCUS ON $p=2$, BUT $p \geq 2$ ALSO INTERESTING

POPULAR, EXTENSIONS $p > 2$
GLASSY PHYSICS
OPTIM. FBM
EVEN BLACK HOLE PHYSICS
IF QUANTUM

$p=2$ TWO BODY INTERACTION

AND KEEP J_{ij} & h_i LOCAL
IN DERIVATION BELOW

$$H = - \frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j$$

DOUBLE SUM NO SELF-COUPLING KEEP IT GENERAL

$$\frac{1}{2} \sum_{i \neq j} = \frac{1}{2} \sum_i \sum_{j (\neq i)} = \frac{1}{2} \sum_i \sum_{\partial_i}$$

EQUIVALENT NOTATIONS
 $\frac{1}{2}$ ENSURES THAT EACH PAIR \Rightarrow ONCE

FACTORS AND NOTATION

$$\sum_{ij} \mapsto \sum_{i \neq j} \quad \text{SINCE THERE'S NO SELF-INTERACTION}$$

$i \neq j$
only
ONCE

$$\frac{1}{2} \sum_{i \neq j} \dots = \frac{1}{2} \sum_i \sum_{\neq i}$$

TO AVOID SUMMING TWICE EACH PAIR

eg $d=1$

$$\sum_{i \neq j} s_i s_j = \sum_i s_i s_{i+1} = \dots s_1 s_2 + s_2 s_3 + \dots$$

$$\frac{1}{2} \sum_i \sum_{j \neq i} s_i s_j = \frac{1}{2} \left[\begin{array}{l} s_1 (s_0 + \underline{s_2}) + \\ + \underline{s_2} (s_1 + s_3) + \\ + s_3 (s_2 + s_4) \end{array} \right] +$$

TERMS WITH s_2

$s_1 s_2$ APPEARS TWICE IN THE SUM

STANDARD INTRODUCTION OF MF

$$s_i = m_i + \delta s_i \quad \delta s_i \ll m_i$$



FUCT

NON FUCT

CANDIDATE ORDER PARAMETER

eg. TWO BODY INTERACTION

$$s_i s_j = (m_i + \delta s_i) (m_j + \delta s_j)$$

$$\approx m_i m_j + m_i \delta s_j + m_j \delta s_i$$

2nd ORDER IN δs DROPPED

USING AGAIN $\delta s_i = s_i - m_i$

$$s_i s_j \approx m_i m_j + m_i (s_j - m_j) + i \leftrightarrow j$$

$$= m_i s_j + m_j s_i - m_i m_j$$

USE GENERIC TWO BODY COUP. J_{ij} & FIELDS ψ_i

APPROXIMATION ←

$$\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j \approx \frac{1}{2} \sum_{i \neq j} J_{ij} \left(\underbrace{m_i s_j + m_j s_i}_{\text{SPINS IN A FIELD}} - \underbrace{m_i m_j}_{\text{CONST WRT } \{s_i\}} \right)$$

IN BOLTZMANN FACTOR $\exp \left\{ -\beta \left[\left(-\frac{1}{2}\right) \sum_{i \neq j} J_{ij} s_i s_j - \sum_i h_i s_i \right] \right\}$

$$Z(\beta J_{ij}, \beta h_i) = e^{-\frac{\beta}{2} \sum_{i \neq j} J_{ij} m_i m_j} \sum_{\{s_i = \pm 1\}} e^{+\frac{\beta}{2} \sum_{i \neq j} J_{ij} m_i s_j} e^{+\beta \sum_i h_i s_i}$$

JUST INDEED SPINS UNDER LOCAL FIELDS

$$h_i^{\text{loc}} = \underbrace{\sum_{j \neq i} J_{ij} m_j}_{\text{GENERATED BY THE OTHER ONES}} + \underbrace{h_i}_{\text{THE EXT. FIELD PINNING SOURCE}}$$

IT IS AS IF THE HAMILTONIAN BECAME AN EFFECTIVE ONE

$$H_{\text{eff}} = - \sum_i h_i^{loc} s_i \quad \text{AND THE } \{s_i\} \text{-DEP} \\ \text{BOLTZMANN WEIGHT} \\ e^{\beta \sum_i h_i^{loc} s_i}$$

USING NOW $\sum_{s_i = \pm 1} e^{\beta h_i^{loc} s_i} = 2 \text{ch}(\beta h_i^{loc})$ FOR EACH i

$$\mathcal{Z}(\beta J_{ij}, \beta h_i) = e^{-\beta \sum_i \sum_{j \in \partial i} J_{ij} m_i m_j} \\ \prod_i 2 \text{ch} \beta \left(\sum_{j \in \partial i} J_{ij} m_j + h_i \right)$$

RECALL
NOTATION

$\sum_{\partial i}$: SUM
OVER
NEIGHS
OF SITE i

THE FREE-ENERGY DENSITY IS

$$-\beta f = \frac{1}{N} \ln \mathcal{Z}$$

SO TAKING THE \ln :

(1)

$$-\beta f(\{\beta J_{ij}, \beta h_i\}) = -\frac{1}{N} \beta \sum_i \sum_{j \in \partial_i} J_{ij} m_i m_j + \frac{1}{N} \sum_i \ln \left[2 \cosh \beta \left(\sum_{j \in \partial_i} J_{ij} m_j + h_i \right) \right]$$

THE SELF-CONSISTENCY IS

$$\langle s_i \rangle = m_i$$

WITHIN THIS APPROX

QUANTITY $\mathcal{O}(1)$, SINGLE SITE ONE

$$\langle s_i \rangle = \frac{\sum_{\{s_j = \pm 1\}} s_i e^{\beta \sum_j J_{ij} s_j}}{\sum_{\{s_j = \pm 1\}} e^{\beta \sum_j J_{ij} s_j}}$$

IN J_{ij} THERE'S J_j AS ONE TERM

\Rightarrow

$$\frac{1}{\beta} \frac{\partial \ln Z(\beta J_{ij}, \beta h_i)}{\partial h_i} = \langle S_i \rangle$$

THE \ln IS HERE TO GET THE NORM. & $1/\beta$ TO GET RID OF βS_i

ACTING AS A SOURCE

USING NOW

$$-\beta f = \frac{1}{N} \ln Z$$

\Rightarrow

$$\frac{-\partial(Nf)}{\partial h_i} = \langle S_i \rangle$$

WE THEREFORE HAVE

$$m_i = \langle S_i \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial h_i} = - \frac{\partial(Nf)}{\partial h_i}$$

WITHIN THE MF APPROX.

BACK TO THE EXPRESSION ABOVE FOR $(-\beta f)$ (1)

THE 1ST TERM IS INDEP. OF h_i

THE 2ND TERM DOES DEP. ON h_i

WE FIND:

$$m_i = \frac{1}{\beta} \frac{\partial}{\partial h_i} \sum_k \ln \left[z \operatorname{ch} \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right) \right]$$

$$= \frac{1}{\beta} \sum_k \frac{z \operatorname{sh} \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right)}{z \operatorname{ch} \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right)} \delta_{ki} \beta$$

$$m_i = \operatorname{th} \beta \left(\sum_{j \in \partial i} J_{ij} m_j + h_i \right)$$

IN THE CONTEXT OF DISORDERED SYSTEMS, WHERE J_{ij} DEPENDS ON ij AND TAKEN FROM A PDF :
NAIVE MF EQS (TAP)

A GENERALIZATION OF THE USUAL FORM, RECOVERED FOR $J_{ij} = J$ FOR $m_i = m$
 $h_i = h$ WHICH

$$m = \operatorname{th} \left(\beta J \sum_{\partial i} m + \beta h \right) \quad (2)$$

$\sum_{\partial i} \bullet = z$ # NEIGHBOURS OF SITE i

DEPENDS ON THE LATTICE eg $z=4$ FOR $d=2$ \square LATTICE

WE REPLACED THE TWO-BODY INTER BY A ONE
SITE PROBLEM DECOUPLED FROM OTHERS, SEEN
"ON AVERAGE"

THE NUMERICAL FACTORS. — A CHECK

- LET'S CHECK WHETHER WE GET $m = \pm 1$
FOR $T \rightarrow 0$ AND $R=0$ FROM THE EQ.

$$m = \lim_{\beta J \rightarrow \infty} (\beta J z m + \beta h)^{z-1}$$

$$\begin{array}{l} \text{— FOR } \beta J \rightarrow \infty \\ \quad m > 0 \end{array} \quad \lim_{\beta J \rightarrow \infty} \beta h \rightarrow 1$$
$$\implies m = 1 \quad \underline{\text{OK}}$$

$$\begin{array}{l} \text{— FOR } \beta J \rightarrow \infty \\ \quad m < 0 \end{array} \quad \lim_{\beta J \rightarrow \infty} \beta h \rightarrow -1$$
$$\implies m = -1 \quad \underline{\text{OK}}$$

- DO WE GET $m \rightarrow 0$ FOR $\beta \rightarrow 0$ AND ANY h ?

$$m = \tanh(\beta J z m + \beta h)$$

$$= \tanh 0 = 0 \Rightarrow m = 0 \quad \underline{OK}.$$

- m IS IN $[-1, 1]$ SINCE RHS IN $[-1, 1]$
- NOTE THAT THE "TRICKY" FACTOR

$$\beta J z$$

IS HARD TO CHECK. ONE MEMOTECHNIQUE RULE

$$z \rightarrow N-1 \quad J \rightarrow J/N$$

$$\beta J z \rightarrow \beta J \Rightarrow T_c = J$$

FULLY CONNECTED MODEL WITH EACH PAIR COUNTED ONCE.

FREE-ENERGY LANDSCAPE

ONE CAN DEF. AN ORDER PARAM.
AND CONTROL PARAM. DEP. FUNCTION

$$f(m; \beta J, \beta h)$$

FROM WHICH THE EQ. (2) COMES TO

$$\frac{\partial f(m; \beta J, \beta h)}{\partial m} = 0$$

$$f(m) = \frac{Jzmm^2}{2} - k_B T \ln 2 \cosh(\beta Jzmm + \beta h)$$

I WRITE ONLY THE
ORDER PARAM. DEP.

LET'S CHECK IT:

$$\frac{\partial f(m)}{\partial m} = 0 =$$

$$\cancel{\frac{2Jz_m}{2}} - \cancel{h_{BT}} = \frac{\cancel{2} \cancel{8} h (\beta J m z + \beta h)}{\cancel{2} \cancel{c} h (\beta J z m + \beta h)} \cancel{\beta J z}$$

$$\Rightarrow m = \tanh(\beta J m z + \beta h)$$

AND IT IS OK

IT CAN BE GENERALIZED TO

$$f(z_{m,j}; \beta J_j, \beta h_j)$$

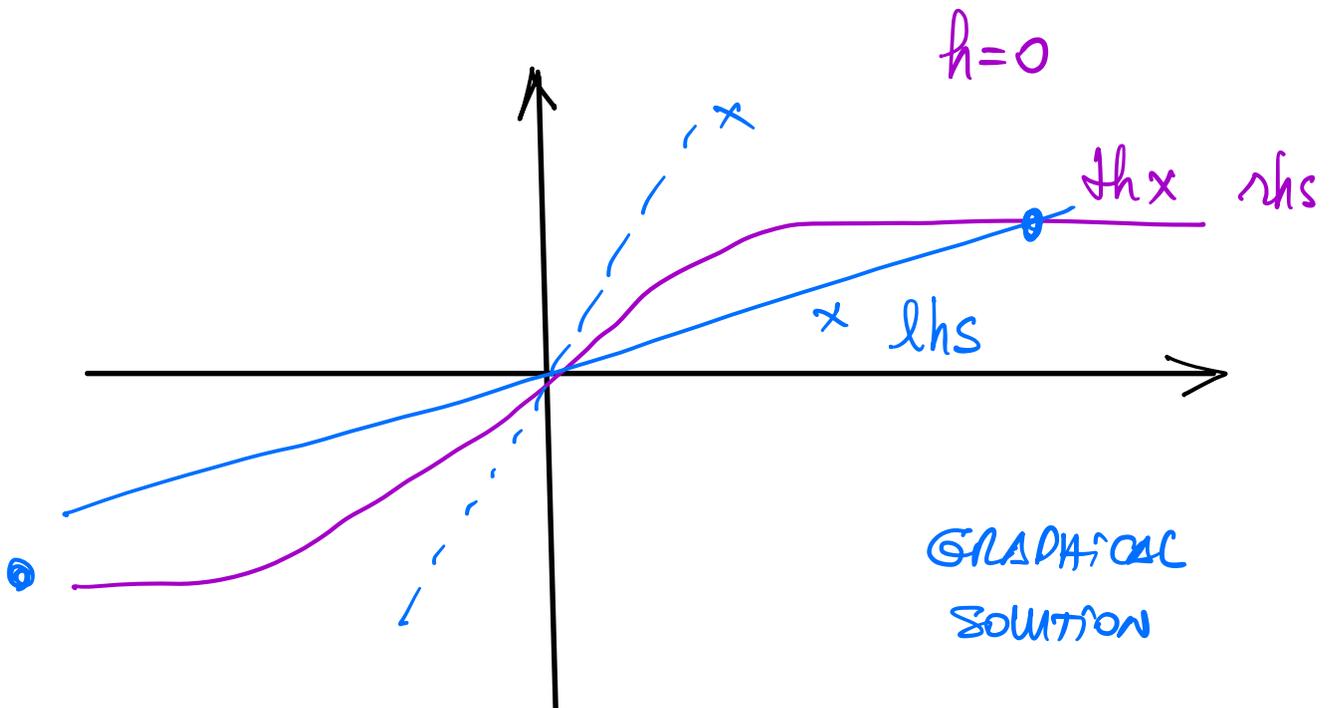
MEAN-FIELD EQ

ANALYSIS



MEAN FIELD EQUATION ANALYSIS

$$m = \tanh(\beta z J m + \beta h)$$



CHANGE in # SOLS WHEN SLOPE AT $m \approx 0$ OF lhs = SLOPE rhs

- ONE SOLUTION $m=0$ if $zJ\beta < 1$
- THREE SOLUTIONS $m=0, m_+ = -m_-$ OTHERWISE

$$zJ\beta z = 1$$

NB NUM.
FACTORS DEP
ON LATTICE (z)
AND CONNECTION

PROPERTIES

- HIGH T only $m=0$ NO ORDER
- LOW T symm $m \leftrightarrow -m$

WHICH ONE WINS? $|m| \neq 0$ OR $m=0$
WHO IS STABLE?

ITERATIVE SOLUTION & CONVERGENCE

$$m = \tanh(\beta J z m) \quad h=0$$

ITERATIVE SOLUTION

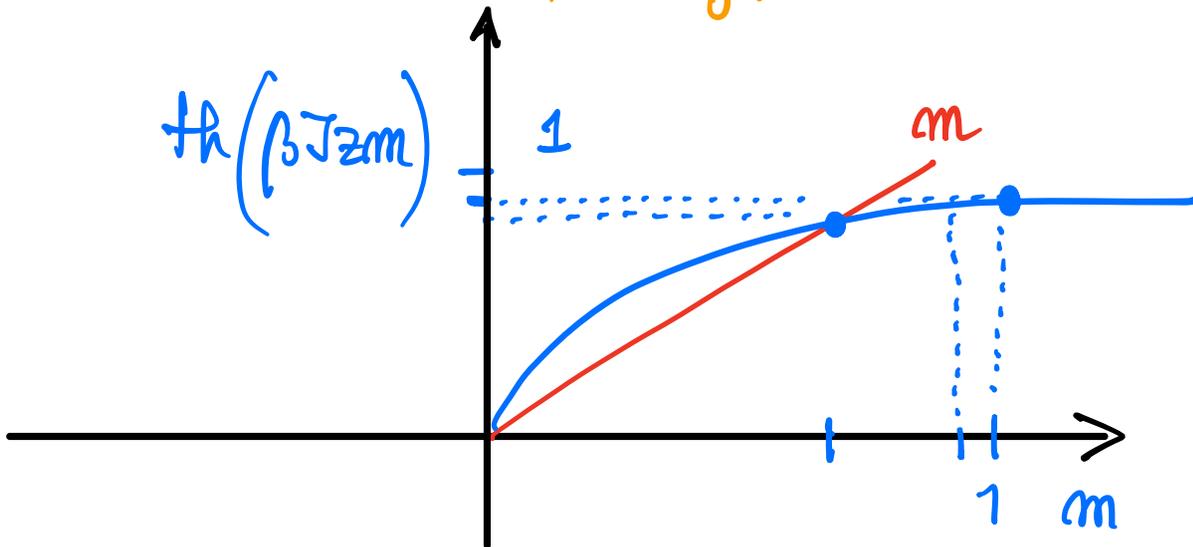
SAY YOU START FROM $m_0 = 1$

$$m_1 = \tanh(\beta J z m_0) < 1 \quad \text{IF } \beta J z > 1$$

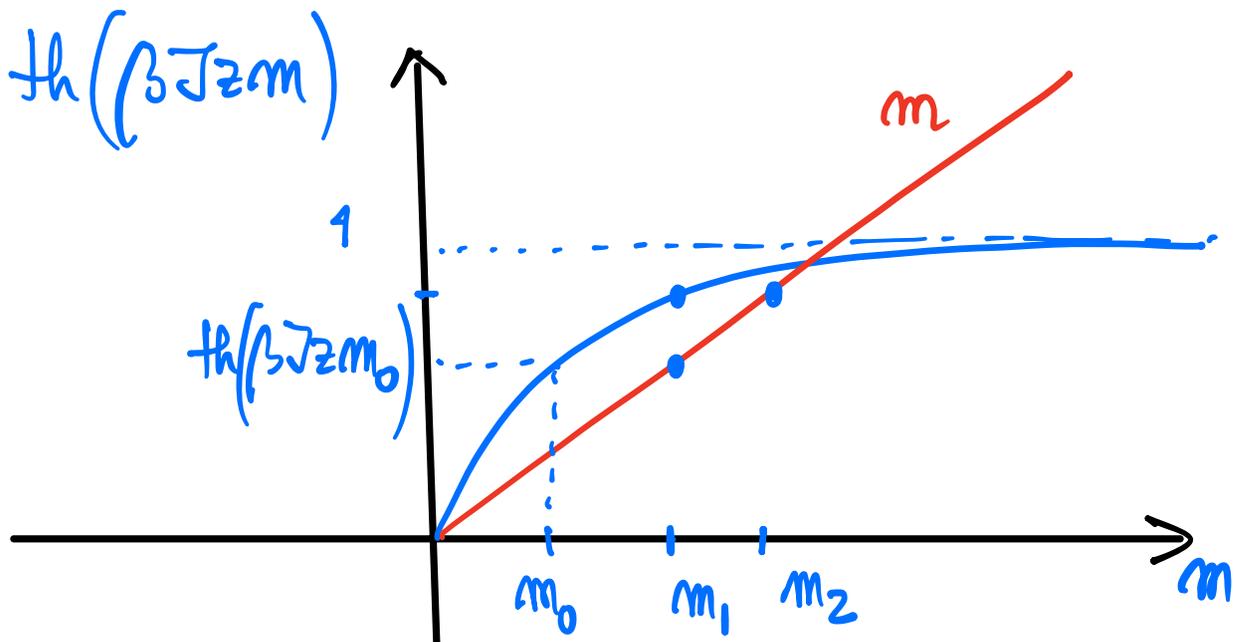
$$m_2 = th(\beta J z m_1) < m_1 \quad \text{EX UNTIL}$$

FIXED POINT

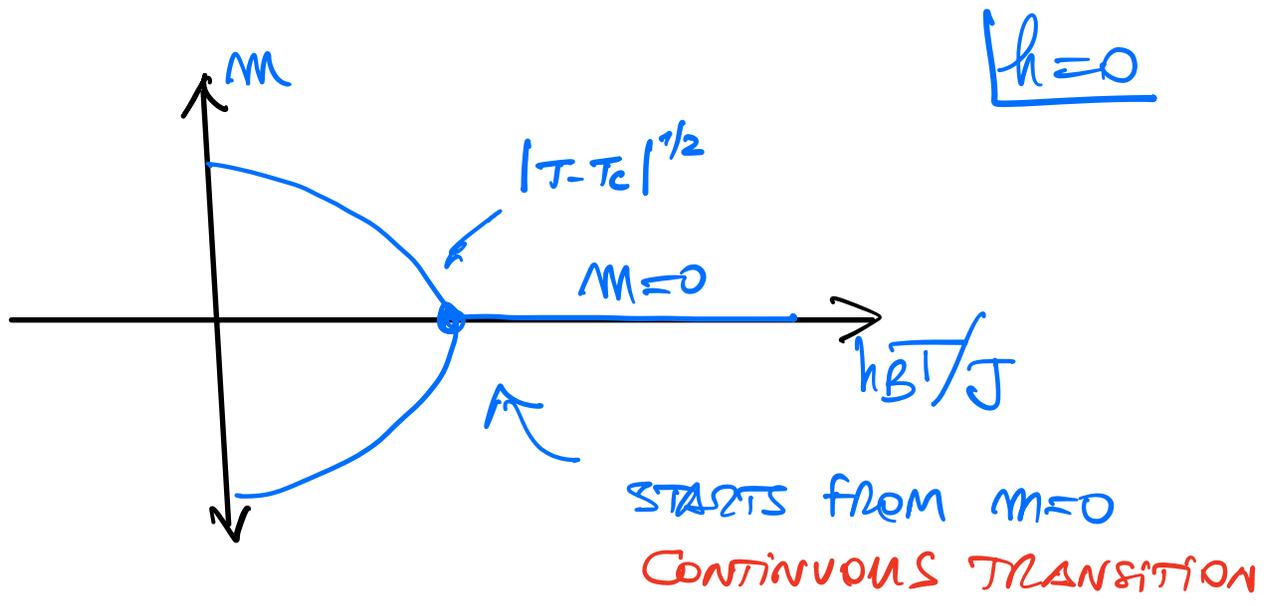
EX STARTING FROM $m_0 > m^*$



Now, FROM $m_0 < m^*$



THE CONTROL PARAM. DEP. OF THE ORDER
PARAM. m



IF WE INTERPRET CHANGE IN # SOL AS
THE PHASE TRANSITION
BUT IF SUM OVER TWO BRANCHES
NEEDED AT LOW $T \Rightarrow 0$ AGAIN!
SEE PINNING FIELD BELOW

EXPERIMENTS WERE SHOWING MEASUREMENT
WITH SMALLER VALUES OF β IN $d=3$
SAMPLES

CASTING DOUBTS ABOUT MF THEORY
CONNECTNESS CLOSE TO T_c

THE EXTERNAL PINNING FIELD

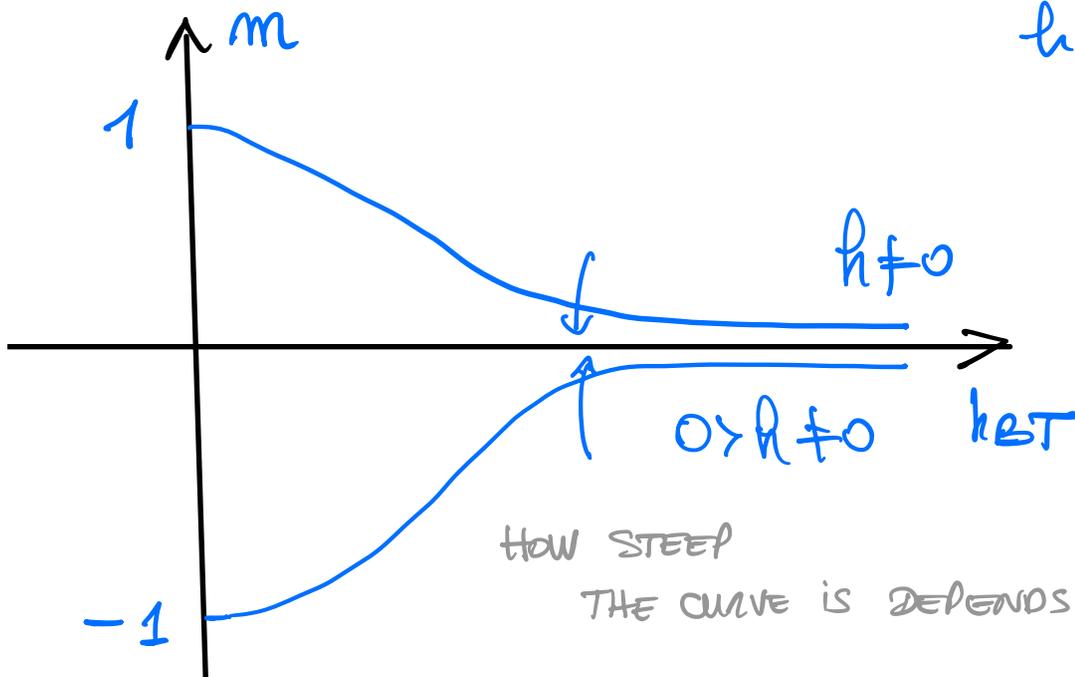
$$m = \tanh \beta (Jz m + h)$$

IS $m=0$ A SOLUTION IF $h \neq 0$? NO!

$$0 = \tanh \beta h \neq 0$$

\Rightarrow $m \neq 0$ $\forall k_B T < \infty, h \neq 0$

only for
 $k_B T \rightarrow \infty$
h finite
 $m \rightarrow 0$

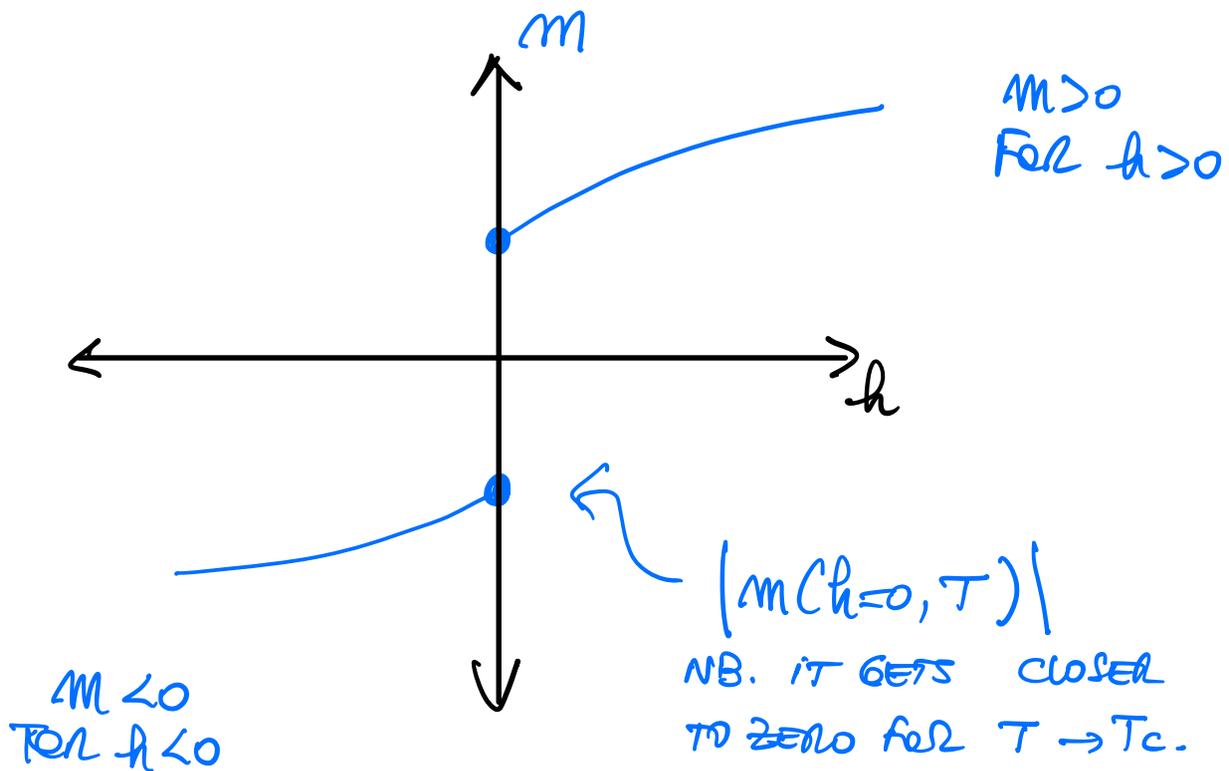


$$\lim_{h \rightarrow 0^+} m(h) = m > 0$$

$$T < T_c (h=0)$$

SIMIL. FOR

$$h \rightarrow 0^- \quad m < 0$$



NOTE DISCONTINUITY OF m AT $h=0$

HOW DOES $m \rightarrow 0$ WITH $h \rightarrow 0$
AT T_c ? LATER

MEANING OF THE EQUILIBRIUM STATE

WITH A POSITIVE PINNING FIELD

- AT $T=0$ THERE IS A SINGLE CONF
 $\forall S_i = +1$ AND $m = 1$
WHICH IS THE GROUND STATE
- AT $T = \epsilon > 0$ FLUCTUATIONS ARE
SWITCHED ON & $m \lesssim 1$

ALREADY N CONFS CONTRIBUTE TO
 $m = 1 - 2/N$

THE NON-VANISHING T EQUIL. STATES
ARE LARGE ENSEMBLES OF
MICRO CONFS.

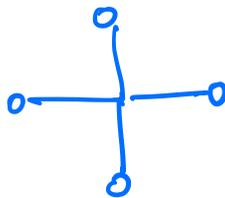
MACRO STATES VS. MICROSTATES

HOW DOES T_c DEPEND ON LATTICE?

$$k_B T_c = zJ \quad \uparrow \text{ if } J \uparrow \text{ ok}$$

IF MODEL ON A SQUARE LATTICE

$$z = 4$$



$T_c \uparrow$ if $z \uparrow$ WHICH IS
EXPECTED
 \Rightarrow ORDERED
FAVOURABLE BY
MORE INTER.

BUT $z = 2$ in $d = 1 \Rightarrow k_B T_c = 2J$
 \Rightarrow PROBLEM !

MF FAILS AT LOW d !

THE ORDER PARAM (h=0) - β EXPONENT

$$m = \frac{1}{k_B T} \beta (J_z m + h)$$

$$m \approx \beta J_z m + (\beta J_z m)^3$$

$\xrightarrow{0}$
 $\xrightarrow{m^2}$

$$m^2 \approx \frac{1}{(\beta J_z)^3} [1 - \beta J_z]$$



if $\beta = \beta_c \Rightarrow 1$

$(k_B T - k_B T_c)$

$$m \sim |k_B T - k_B T_c|^{1/2}$$

CRITICAL EXPONENT
CONTROLS HOW

$$\beta = 1/2$$

THE ORDER PARAM VANISHES

THE MAGN. SUSCEPT - γ EXPONENT

VARIATION OF THE ORDER PARAM
WRT CONJUGATE FIELD

$$m = Jh \beta (Jz m + h)$$

THE SOLUTION IS $m(h)$

COMPUTE $\frac{\partial m(h)}{\partial h}$ AND TAKE
 $h \rightarrow 0$

CALLING $m(h) = m_h$ & TAKING $\frac{\partial m_h}{\partial h}$

$$\frac{\partial m_h}{\partial h} = \frac{\beta (Jz \frac{\partial m_h}{\partial h} + 1)}{ch^2 \beta (Jz m_h + h)} \Rightarrow$$

$$\frac{\partial m_h}{\partial h} = \frac{\beta}{[ch^2 \beta (Jz m_h + h) - \beta Jz]}$$

RECALL $k_B T_c = Jz = 1/\beta_c$

• $T > T_c$ χ_+ $m_h = 0, h = 0$

$$\chi_+ = \left. \frac{\partial m_h}{\partial h} \right|_{h=0} = (k_B T - k_B T_c)^{-1}$$

$$\boxed{\chi_+ = 1}$$

• $T < T_c$ χ_- $m_h \propto (T_c - T)^{-\beta}$

$$\chi_- = \left. \frac{\partial m_h}{\partial h} \right|_{h=0} \approx \text{EXPAND } m_h \approx 0$$

$$\text{ch } x \approx \frac{e^x + e^{-x}}{2} =$$

$$= \frac{1}{2} \left[(1 + x + x^2/2) + (1 - x + x^2/2) \right] + \dots$$

$$= 1 + \frac{x^2}{2} + \dots$$

$$ch^2x \approx \left(1 + \frac{x^2}{2} + \dots\right)^2 \approx 1 + x^2 + \dots$$

$$X_- \approx \frac{\beta_c}{\cancel{1 + \beta^2} (J_z m_h + h)^2 - \cancel{1}}$$

\nearrow
 AT $\beta = \beta_c$

SET $h=0$ AND $m_{h \rightarrow 0} \approx |T - T_c|^\beta$
 $\beta = \beta_c$

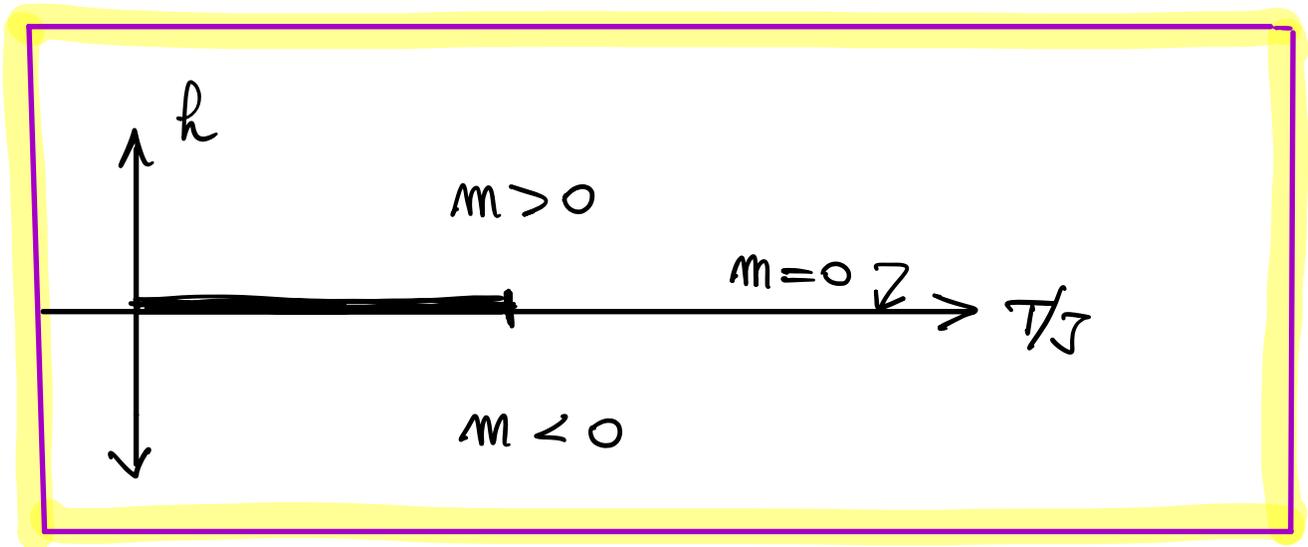
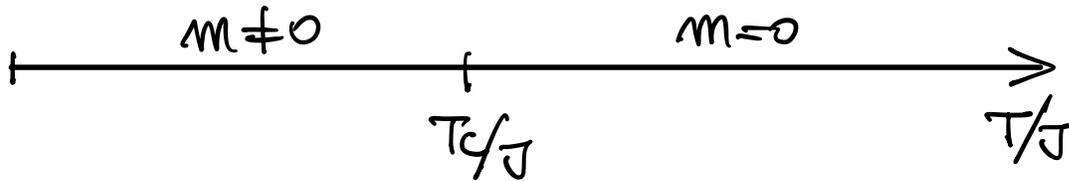
$$X_- = \frac{\cancel{\beta_c}}{\underbrace{\beta_c^2}_{1} J_z (T_c - T)^{2\beta}}$$

$$= (T_c - T)^{2\beta - 1}$$

$2\beta - 1 = 1$

DEPENDENCE ON FIELD - δ EXPONENT

$h=0$



WHAT HAPPENS GOING ACROSS $h=0$
AT FIXED $T < T_c (h=0)$?

A JUMP IN $m > 0$ TO $m < 0$
OR $m < 0$ TO $m > 0$



DISCONTINUOUS
TRANSITION

$$m \sim \beta (Jz_{m+h}) - \frac{\beta}{3} (Jz_{m+h})^3 \quad \beta = \beta_c$$

LEFT: $0 \sim h - \frac{\beta_c^2}{3} (Jz_{m+h})^3$

if $m \sim h^{1/\delta}$ AND $1/\delta < 1 \Rightarrow$ DROP h WRT m :

$$h \sim \frac{\beta_c^2}{3} (Jz)^3 m^3 = 0$$

$$m \sim h^{1/3}$$

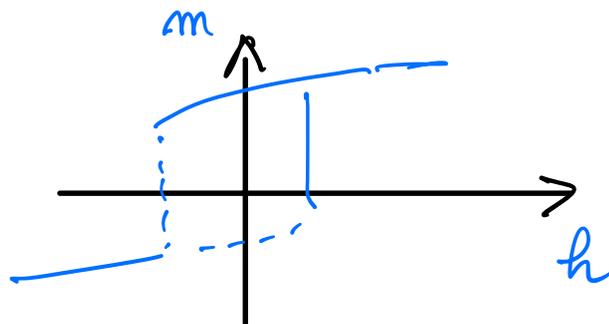
CONSISTENTLY

$$1/\delta < 1$$

$$\delta = 3$$

Hysteresis

IN AN ACTUAL EXP., SINCE THE "INITIAL" STATE IS STILL STABLE AFTER GOING ACROSS THE TRANS. \Rightarrow ABRUPT CHANGE CAN OCCUR LATER



SUMMARY OF MEAN-FIELD (SO FAR)

- MEAN-FIELD FOR UNIFORM m

(IN NOTES, DONE FOR POSSIBLY LOCAL m ; IF J_{ij} NOT ALL THE SAME, IE HETEROGENEOUS)

$$m + \delta s_i = s_i \quad \langle \delta s_i \rangle = 0$$

DROP $\delta s_i \delta s_j$ TERM IN BOLTZMANN WEIGHT

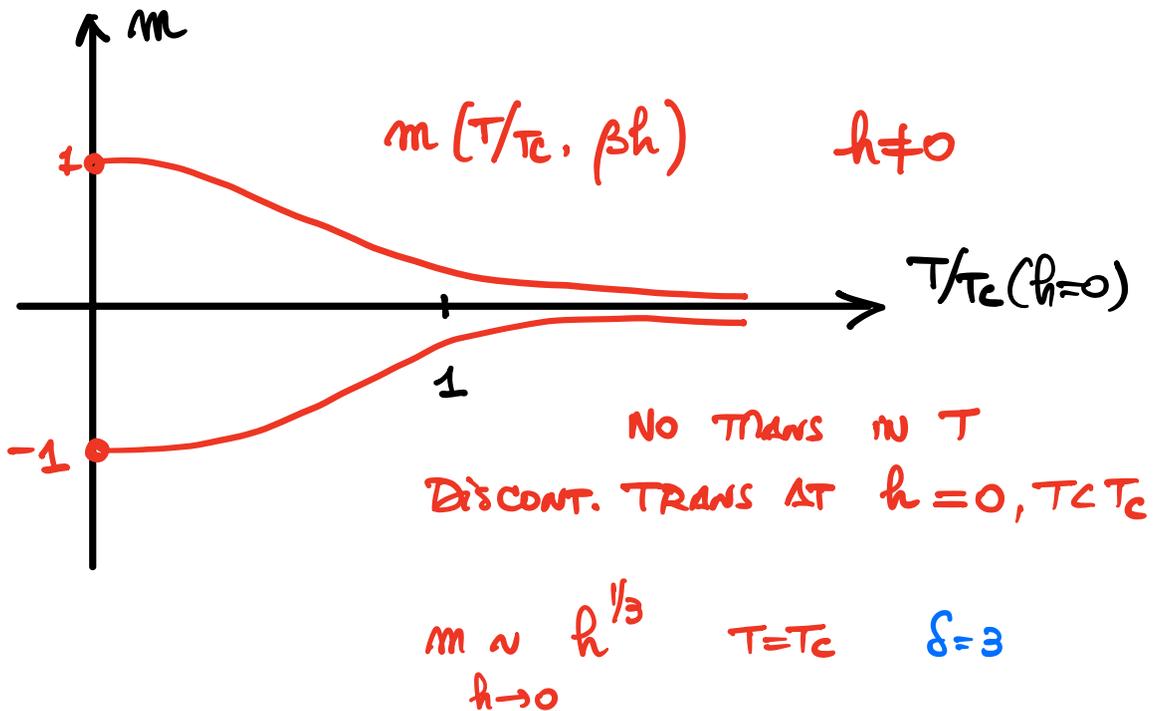
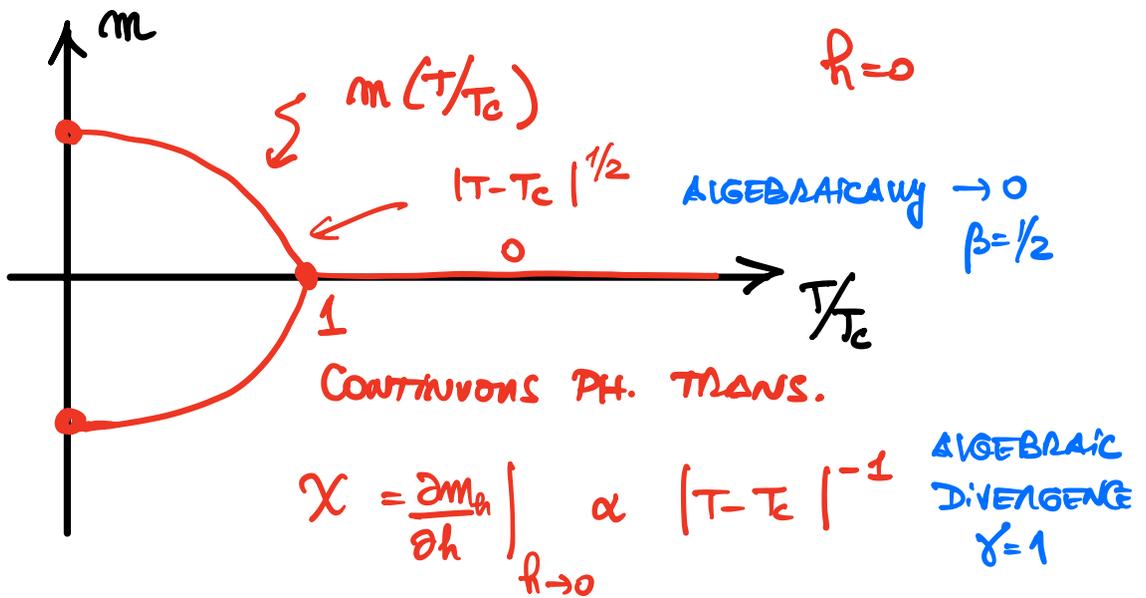
- $f(m) = \underbrace{\frac{Jzm^2}{2} - hm}_{\text{FROM ENERGY}} - \underbrace{k_B T \ln [2 \cosh \beta (Jzm+h)]}_{\text{FROM T. ENTROPY}}$

- MF EQ FROM $\frac{\partial f(m)}{\partial m} = 0$

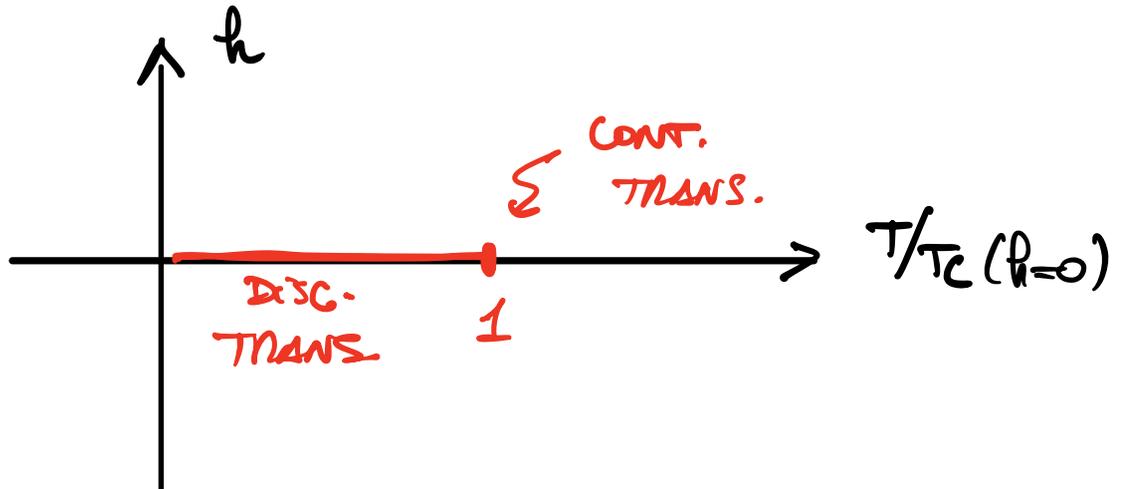
- MF EQ $m = fh (\beta J z m + \beta h)$

2.5.3 ORDER PARAMETER

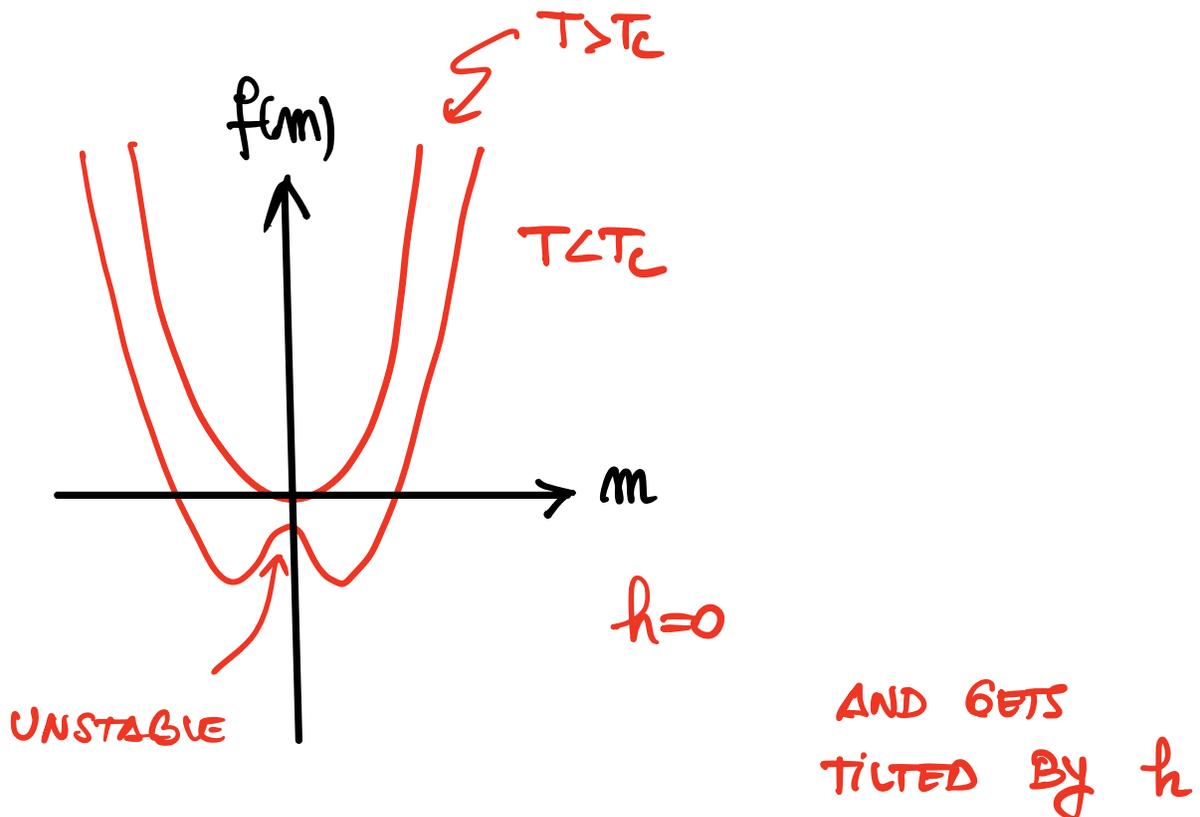
$$h_0 T_c(h=0) = J z$$



PHASE DIAGRAM

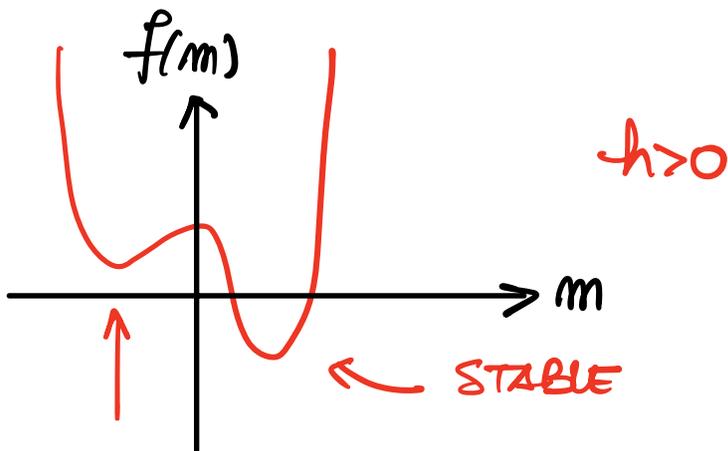


FREE-ENERGY DENSITY AS FCT ORDER PARAM



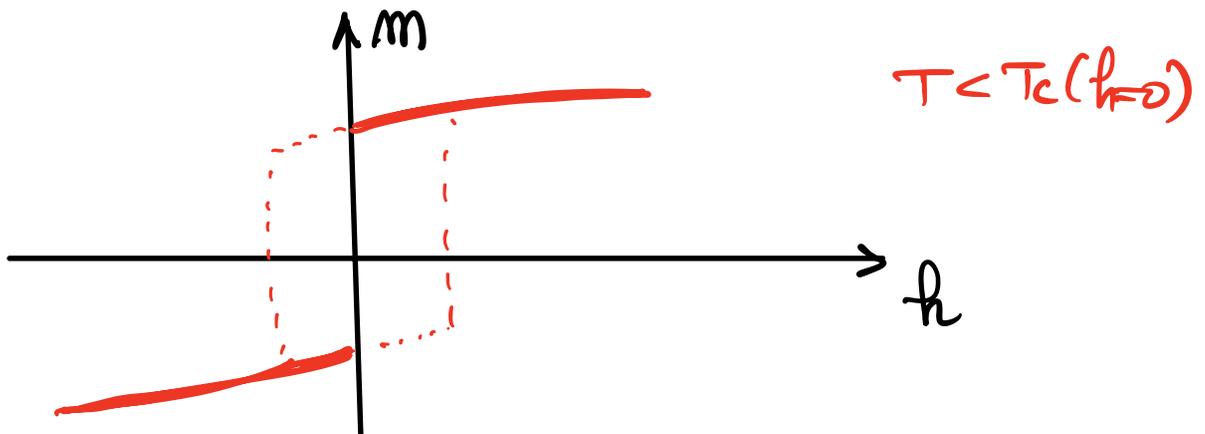
EFFECT OF A FIELD

- DISCONTINUOUS TRANS
- HYSTERESIS



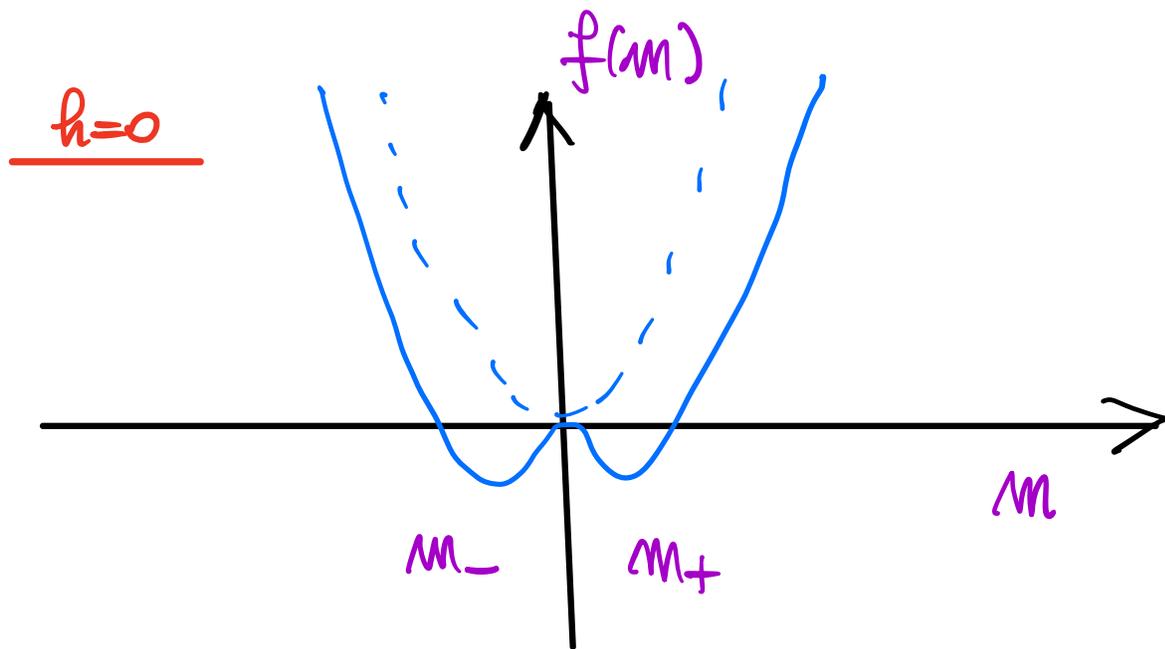
METASTABLE

THE SYST CAN BE NEGATIV.
MAGN. UNDER $h > 0$ IF
PREPARED AT $m < 0$



15 min

THE FULL FREE-ENERGY LANDSCAPE



HERE $f(m)$ IS A FUNCT OF ALL POSSIB
 m VALUES, i.e. $m \in [-1, 1]$

BELOW T_c $|m| \neq 0$ ARE BELOW $m=0$
 \Rightarrow NON TRIVIAL EQ. SOLUTIONS

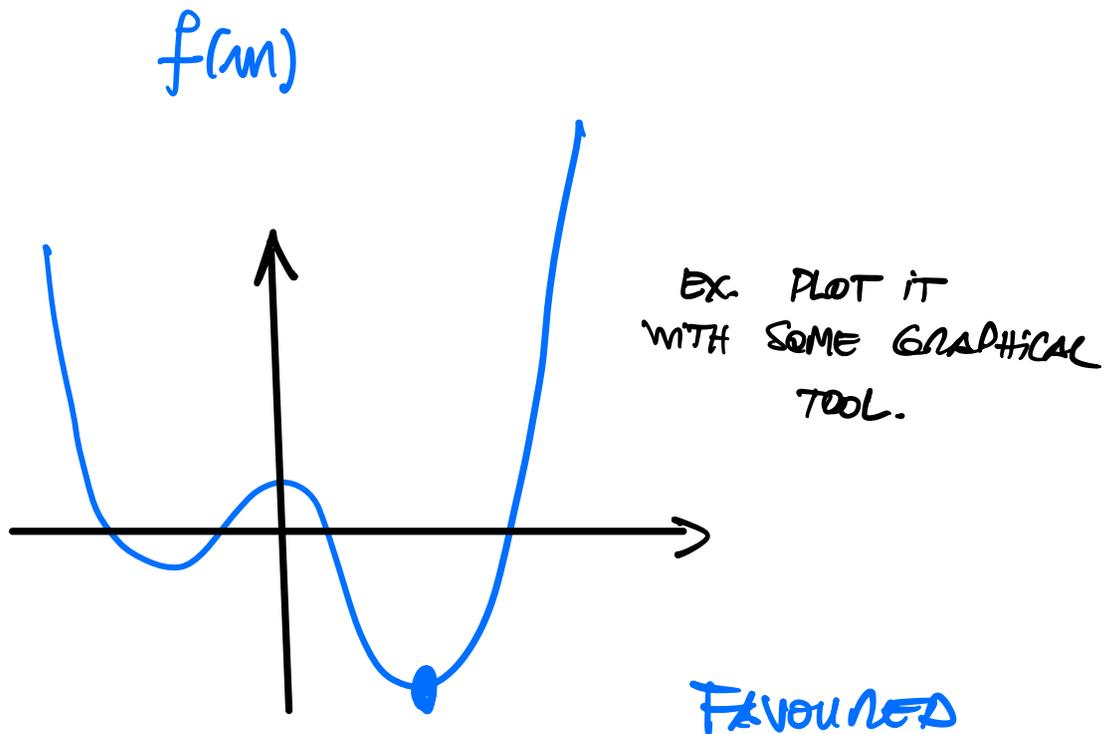
$$f(m) = \underbrace{\frac{J}{2} 2m^2}_{\text{ENERGY}} - \underbrace{k_B T \ln 2 \cosh \beta (Jz m)}_{T \text{ ENTROPY}}$$

THE PINNING FIELD ROLE

EFFECT OF FIELD $-hm \Rightarrow$ TILT
 $f(m)$ TO MAKE $m > 0$ LOWER IF $h > 0$
 $m < 0$ LOWER IF $h < 0$

$h \rightarrow 0$ SPONTANEOUS SYMM BREAKING

IT TILTS THE FREE-ENERGY



5min

THE $\lambda\phi^4$ POTENTIAL

$$f(m) = \frac{zJ}{2} m^2 - k_B T \ln 2 \operatorname{ch} \beta (Jz m)$$

IF WE ASSUME $m \approx 0$ CLOSE TO T_c (ie $T \approx T_c$)

TAYLOR EXPAND 2nd TERM

$$\ln \left[\operatorname{ch} y \approx 1 + \frac{y^2}{2} \right] \approx \frac{y^2}{2} - \frac{y^4}{12}$$

$$f(m) \approx \frac{zJ}{2} m^2 - \frac{k_B T}{2} (\beta J z m)^2 + \frac{k_B T}{12} (\beta J z m)^4 - k_B T \ln 2$$

$$= \frac{zJ}{2} m^2 \left[1 - \beta J z \right] + \frac{\beta^3 J^4 z^4}{12} m^4 - k_B T \ln 2$$

$$f(m) \approx \frac{z\beta J}{2} \left(k_B T - k_B T_c \right) m^2 + \frac{\lambda m^4}{4!}$$

THE CURVATURE CLOSE TO $m=0$
CHANGES AT T_c

$\lambda=0$

+ ADDIT
CONST $-k_B T \ln 2$

5min

THE SPECIFIC HEAT

DISCONTINUOUS AT T_c

$$C_v = \frac{\partial E}{\partial T} = -\frac{1}{k_B T^2} \frac{\partial^2 (\beta F)}{\partial \beta^2} = -2k_B \frac{\partial \beta F}{\partial T} - k_B T^2 \frac{\partial^2 (\beta F)}{\partial T^2}$$

$$\ln Z = -\beta F = -\beta N f \quad E = \langle H \rangle$$

$$f(m) \sim -k_B T \ln 2 + \frac{1}{2} (k_B T - k_B T_c) m^2$$

WHAT IS f AS A FUNCT. OF T ? DROPPES $O(m^4)$ TERMS

- $T > T_c$ $m=0$

$$\beta f(m(T)) = -\ln 2$$

$$\Rightarrow \frac{\partial^2 (\beta f(m(T)))}{\partial T^2} = 0$$

- $T \lesssim T_c$ $m \sim (T_c - T)^\beta$

$$\beta f(m(T)) \sim -\ln 2 - \frac{1}{2} \beta (T_c - T)^{1+2\beta}$$

$$1+2\beta=2 \iff \text{EXP. } \beta=1/2$$

$$\frac{\partial(\beta f)}{\partial T} = +\frac{1}{2} \frac{(T_c - T)^2}{(k_B T)^2} - \frac{1}{2} \beta (T_c - T) = 0 \quad T=T_c$$

$$\frac{\partial^2(\beta f)}{\partial T^2} = \frac{1}{2} \frac{(-2)}{(k_B T)^3} + \frac{1}{2} \frac{2(T_c - T)(-1)}{(k_B T)^2}$$

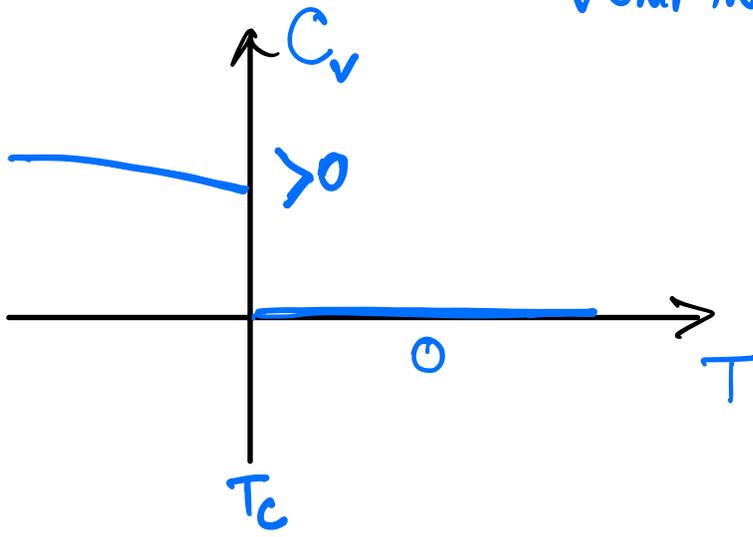
$$+ \frac{1}{(k_B T)^2} (T_c - T) + \frac{1}{k_B T} (-1)$$

$$\Delta T \ll T_c : \quad = -\beta_c$$

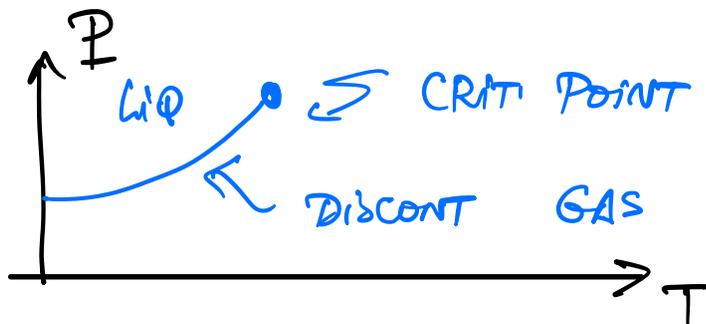
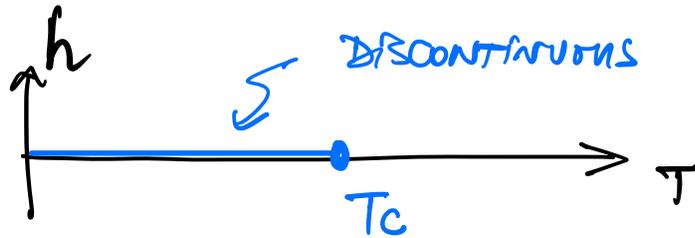
$$C_V = \beta_c^2 \frac{\partial^2 \ln Z}{\partial T^2} = \beta_c^2 \frac{\partial^2 (-\beta N f)}{\partial T^2}$$

$$\downarrow \Delta T \ll T_c = \begin{cases} 0 & T > T_c \\ \text{CONST} & T \ll T_c \end{cases}$$

JUMP IN C_V AT T_C



2.5.7 HINTS ON UNIVERSALITY



VERY SIMILAR

MONODOMER

SPIN MODEL

LATTICE GAS

$$S_i^0$$

$$n_i^0 = \frac{S_i^0 + 1}{2}$$

$$\pm 1$$

$$1, 0$$

VERY \neq PROBLEMS w/ SIMILAR PHYSICS

\Rightarrow UNIVERSALITY? SAME EXPONENTS $\alpha, \beta, \gamma, \delta$
MORE LATER

WHAT DO WE MISS WITH THE MF APPROX?

2.5.1 ABSENCE OF CORR

CONNECTED CORRELATIONS: $C_{ij}^{\text{CONN}} \equiv$

$$\langle (s_i - \langle s_i \rangle) (s_j - \langle s_j \rangle) \rangle = \langle \delta s_i \delta s_j \rangle$$

UNDER
THE ASSUMPTIONS MADE

RECALL WE DROPPED THIS QUAD. TERM

DO WE GET ZERO FOR THIS CORR?

SEVERAL WAYS TO DO IT. BY DEF: ($h_i = 0$):

$$C_{kl}^{\text{CONN}} = \frac{\sum_{\{s_i\}} s_k s_l e^{-\beta H(m_i, s_i)}}{\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}}$$

$$= \frac{\sum_{\{s_i\}} s_k e^{-\beta H(m_i, s_i)}}{\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}} \frac{\sum_{\{s_j\}} s_l e^{-\beta H(m_j, s_j)}}{\sum_{\{s_j\}} e^{-\beta H(m_j, s_j)}}$$

WITH $H(m_i, s_i) = -\frac{J}{2} \sum_{i \neq j} (m_i m_j - 2 s_i m_j)$

NOW, MANY CANCELLATIONS BETWEEN NUM
& DEN. FROM THE SUMS $\sum_{\substack{3s_i \\ y}}$... ONLY
TERMS WHERE

$s_k \neq s_l$
APPEAR WILL REMAIN.

MOREOVER, IF k AND l ARE \neq
AND DON'T SHARE NEIGHBORS

\Rightarrow FACTORIZE 1st SUM

\Rightarrow TWO TERMS IN SUM ARE THE
SAME AND RESULT
VANISHES IDENTICALLY

$$C_{kl}^{\text{CONN}} = 0$$

NO CORRELATION BEYOND
LATTICE DISTANCE

EVEN MORE CLEARLY SEEN IN THE FOLLOWING
WAY OF OBTAINING THE SAME DERIVATION:

2.5.2

ANOTHER WAY TO OBTAIN THE SAME EQS.

$$P(\{s_i\}) = \prod_i P_i(s_i)$$

$$P_i(s_i) = \frac{(1+m_i)}{2} \delta_{s_i, 1} + \frac{(1-m_i)}{2} \delta_{s_i, -1}$$

NOTATION: WE KEEP A POSSIBLE i -DEP. IN m_i
IN CASE WE WANT TO CONSIDER J_{ij} HETEROG.

CHECK $\sum_{\{s_i = \pm 1\}} P_i(s_i) = 1$ NORM.

$\sum_{\{s_i = \pm 1\}} s_i P_i(s_i) = m_i$ DEF OF $\{m_i\}$

FOR USUAL ISING
 $J_{ij} = J \delta_{m_i = m_j}$

$F = U - TS$ FREE-ENERGY

WHAT IS U ? $\langle H \rangle_P$

WHAT IS S ? $-k_B \langle \ln P \rangle_P$

FIND $F(\{m_i\})$ sgu $P < 1, \ln P < 0, S > 0$

ASK IT TO BE A MINIMUM WRT $\{m_i\}$

\Rightarrow FIND EQS ABOVE FOR $\{m_i\}$

$$\langle H \rangle_P = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \langle s_i s_j \rangle_P = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \langle s_i \rangle_P \langle s_j \rangle_P$$

$$= -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} m_i m_j$$

IF ALSO LOCAL FIELDS $\langle \sum_i h_i s_i \rangle_P = \sum_i h_i \langle s_i \rangle_P = \sum_i h_i m_i$

$$\langle \ln P \rangle_P = \left\langle \sum_i \ln \left(\frac{1+m_i}{2} \delta_{s_i, 1} + \frac{1-m_i}{2} \delta_{s_i, -1} \right) \right\rangle_P =$$

$$= \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \quad (< 0)$$

$$\beta f(m_i) = -\frac{\beta}{2N} \sum_{\langle ij \rangle} J_{ij} m_i m_j - \frac{\beta}{N} \sum_i h_i m_i$$

$$+ \frac{1}{N} \sum_i \left(\frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \right)$$

LOOKS A BIT DIFF. BUT IT'S EQUIVALENT

SHOW

$$P_i(s_i) = \frac{e^{\beta h_i^{loc} s_i}}{2 \cosh \beta h_i^{loc}}$$

EXERCISE

$$h_i^{loc} = h_i^{eff} + h$$

ARE THE TWO EXPRESSIONS OF $F(m)$ THE SAME?

$$-\frac{1}{2} J_2 m^2 - h m + k_B T \left[\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

?

=

$$\frac{J_2}{2} m^2 - h m - k_B T \ln 2 \cosh(\beta J_2 m + \beta h)$$

- CHECK $\frac{\partial f(m)}{\partial m} = 0$ FROM 1ST EXPRESSION
SET $h=0$ TO MAKE EXPRESSIONS SHORTER

$$0 = -J_2 m + k_B T \left[\frac{1}{2} \ln \frac{1+m}{2} - \frac{1}{2} \ln \frac{1-m}{2} \right]$$

$$+ k_B T \left[\frac{\cancel{1+m}}{2} \frac{1/2}{\cancel{1+m}/2} + \frac{\cancel{1-m}}{2} \frac{(-1/2)}{\cancel{1-m}/2} \right]$$

= 0

$$0 = -Jz m + \frac{k_B T}{z} \ln \frac{\frac{1+m}{z}}{\frac{1-m}{z}}$$

$$\frac{1+m}{1-m} = \exp\left(\frac{-2Jz m}{k_B T}\right)$$

$$1+m = (1-m) \exp(-2\beta Jz m)$$

$$1 - e^{-2\beta Jz m} = m (e^{-2\beta Jz m} + 1)$$

$$m = \frac{1 - e^{-2\beta Jz m}}{1 + e^{-2\beta Jz m}}$$

$$m = \tanh(\beta Jz m)$$

- Check now $\frac{\partial f(m)}{\partial m} = 0$ from 2nd exp.

$$\frac{J_z m^2}{2} - k_B T \ln 2 \cosh(\beta J_z m)$$

$$\Rightarrow \frac{\cancel{J_z m} - \cancel{k_B T} \frac{2 \cancel{\cosh}(\beta J_z m)}{2 \cancel{\cosh}(\beta J_z m)}}{\cancel{2 \cosh}(\beta J_z m)} \cdot \cancel{\beta J_z} = 0$$

$$m = \tanh(\beta J_z m) \quad \frac{\text{SAME EQ.}}{\underline{\text{OK}}}$$

CAN ALSO TRANSFORM ONE $f(m)$ INTO ANOTHER.

SAME $m \neq 0$ EXPANSION

2.5.5 THIRD WAY OF GETTING THE MEAN-FIELD EDS

- IN NOTES GENERALIZED TO p -SPIN MODELS
- IN TD APPLIED TO BUWME-CAPPEL SPIN 1 MODEL

IN ISING SPIN MODEL

$$Z = \sum_{\{s_i = \pm 1\}} e^{\frac{\beta}{2} \sum_{ij} J_{ij} s_i s_j + \beta \sum_i h_i s_i}$$

$J_{ij} = J$ $\sum_{i \neq j}$ FULLY CONNECTED GRAPH
SUM OVER ALL PAIRS ON COMPLETE GRAPH

$$J \rightarrow J/N \text{ OTHERWISE } H = \Theta(N^2)$$

$$\begin{aligned} \sum_{i \neq j} s_i s_j &= \sum_{ij} s_i s_j - \sum_i s_i^2 \\ &= \left(\sum_i s_i \right) \left(\sum_j s_j \right) - \sum_i 1 \\ &= N \left(\frac{1}{N} \sum_i s_i \right) \left(\frac{1}{N} \sum_j s_j \right) - N \end{aligned}$$

TRANSFORM SUM OVER $\{s_i = \pm 1\}$ IN SUM OVER

$$x = \frac{1}{N} \sum_i s_i \in [-1, 1], \quad \delta x = \frac{2}{N} \ll 1$$

$$x = -1, -1 + \frac{2}{N}, -1 + \frac{4}{N}, \dots, 1$$

$$\mathcal{Z} = \sum_x \mathcal{N}(\{s_i\} / \frac{1}{N} \sum_i s_i = x) e^{\frac{\beta N}{2} x^2}$$

WHAT IS THE DEGENERATION \mathcal{N} ?

$$\mathcal{N} = \binom{N}{N_+}$$

N_+ : # + SPINS

$$= \frac{N!}{N_+! N_-!}$$

STIRLING \Rightarrow

$$\mathcal{N} = e^{N J(x)}$$

WITH

$$-J(x) = \frac{1+x}{2} \ln \frac{1+x}{2} + \frac{1-x}{2} \ln \frac{1-x}{2}$$

ENTROPY
OF ISING
SPINS

$$\mathcal{Z} \longrightarrow \int dx e^{-\beta N f(x)}$$

$$f(x) = -\frac{x^2}{2} - h x - k_B T J(x)$$

THE SADDLE-POINT VALUES EQUAL THE AVERAGE

$$\bar{z} \sim \sum_{\alpha} e^{-\beta N f(x_{sp}^{\alpha})}$$

LABELLING
THE SADDLE POINTS (if SEVERAL)

$$x_{sp} = - \frac{\partial f(x)}{\partial h} \quad \Bigg| \quad = \langle x \rangle$$

↓
JUST FROM THE
LINEAR COUPLING
BTW h & x

x_{sp} ↓
USING h AS
A SOURCE WITHIN
PARTITION SUM

⇒

$$x_{sp} = \langle x \rangle$$

PROOF BELOW

if MORE THAN ONE SADDLE POINT ⇒
AVERAGE OVER THE CORRESP "ERGODIC
COMPONENT" OR "STATE"

CAN SELECT THEM w/ PINNING FIELD

HOW DOES THE "SOURCE" METHOD WORK?

$$\bar{z} = \frac{1}{\alpha} e^{-\beta N f(x_{sp}^\alpha)} \Rightarrow e^{-\beta N f(x_{sp}^\alpha)}$$

JUST ONE
TO SIMPLIFY

$$\frac{\partial \bar{z}}{\partial h} = -\beta N \frac{\partial f(x_{sp}^\alpha)}{\partial h} e^{-\beta N f(x_{sp}^\alpha)}$$

$$\text{if } f(x_{sp}^\alpha) = \underbrace{f^{(0)}(x_{sp}^\alpha)}_{\text{NOT DEP ON } h} - \underbrace{h x_{sp}}_{\text{THE FIELD DEP. TERM}}$$

$$\frac{\partial f(x_{sp}^\alpha)}{\partial h} = -x_{sp}$$

NB NO "CHAIN RULE" CONTRIB SINCE

$$\underbrace{\frac{\partial f(x_{sp}^\alpha)}{\partial x_{sp}^\alpha}}_{\text{"0 by def of } x_{sp}^\alpha} \frac{\partial x_{sp}^\alpha}{\partial h} = 0$$

"0 by def of x_{sp}^α .

$$\text{THEN } \frac{1}{\bar{z}} \frac{\partial \bar{z}}{\partial h} = \frac{\beta N x_{sp} e^{-\beta N f(x_{sp}^\alpha)}}{e^{-\beta N f(x_{sp}^\alpha)}} = \beta N x_{sp}$$

THE LEFT-HAND-SIDE IS $\frac{\partial \ln Z}{\partial h}$

$$\text{BUT } Z = \int dx e^{-\beta N f(x) + \beta N h x}$$

BEFORE SADDLE-POINT

$$\Rightarrow \frac{\partial \ln Z}{\partial h} = \frac{1}{Z} \int dx e^{-\beta N f(x)} \beta N x$$

$$= \beta N \langle x \rangle$$

FINALLY, COMPARING

$$\frac{\partial \ln Z}{\partial h} = \beta N x_{sp} = \beta N \langle x \rangle$$

$$\Rightarrow x_{sp} = \langle x \rangle$$

PROBLEM THE EXPONENTS MEASURED ARE NOT THE MEAN-FIELD ONES

SHOW
IN $d=3$
SAMPLES

$$m \sim (T_c - T)^\beta \quad \beta \approx 0.3$$
$$C \sim (T_c - T)^\alpha \quad \alpha \approx 0.11$$

ONSAGER VALUES IN $d=2$

$$\beta = 1/8 \quad \alpha = 0$$

VS $\beta_{MF} = 1/2 \quad \alpha_{MF} = 0 \quad \forall d$

SO?

SOLUTION LATER

NOT COMPLETELY WRONG

ORDER OK

PHASE DIAGRAM OK

$d \geq 4 = d_u$ EVERYTHING IS OK

$d=2,3$

WHAT HAVE WE DONE ?

- MEAN-FIELD WITH NO SPATIAL STRUCT
FOUND PHASE TRANS



CRITICAL EXP
INDEX OF d
SIMPLE FRACTIONS

SUMMARY 4th LECTURE

- FINISHED ANALYSIS OF MF EQS & OBTAINED ALL EXPONENTS

$$\alpha, \beta, \gamma, \delta$$

$$S(m) = -k_B \ln P$$

NO CORRECTIONS TO T_C
 $T_C = J$

- COMPARED TO EXP. DATA AND SAW THAT THE INTEGER VALUES - INDEP. OF d - WERE NOT MEASURED
- SHOWED $\langle \delta s_i \delta s_j \rangle = 0$
 $i \neq j$
- STARTED ADDING SPACE DEP.

5ft LECTURE

SHOW TRANSPARENCIES W/ EXP. DATA.

THIS SHOWS THAT THESE CALC. ARE NOT EXACT IN $d=3$ (WHERE EXPS. SHOWN ARE DONE) BUT NOT TOO BAD.

2.3.8

PERIALLS HERE

p. 16

THESE NOTES

- LANDAU'S PRESCRIPTIONS
- HOMOGENEOUS S.P.
DOMAIN WALL / KINKS
- CORR. OF FUCT $\langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$
CORR. LENGTH $\xi \sim |T - T_c|^{-\nu}$
- GINZBURG CRITERION
 $\langle \delta\phi_{oh}^2 \rangle$ vs. $\langle \phi \rangle^2$
- 1st ORDER

- WE NEED SOME SPACE DEPENDENCE



2.6 GINZBURG-LANDAU THEORY

1- LANDAU TH. $F(\phi)$

2- CORR FCT. $C_{\text{CORR}}(r)$

3- GINZBURG CRIT

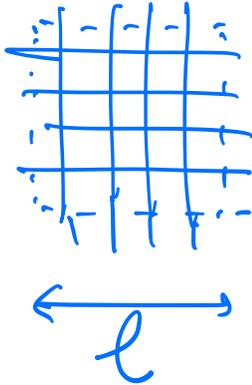
ESTIMATE "STRENGTH" OF
FLUCTUATIONS

$$d_u \leq d \text{ MF OK}$$

2.6 LANDAU THEORY

- WE HAD NO SPATIAL INFO SO FAR
- LANDAU \rightarrow GO A BIT BEYOND, SIMILAR APPROACH TO WHAT DONE FOR FULLY-CONNECTED

- COARSE-GRAIN



$$\phi(\vec{x}) = \frac{1}{\sqrt{V_{\vec{x}}}} \sum_{i \in V_{\vec{x}}} s_i$$

SUM OVER SPINS IN BOX

$$a \ll l \ll L$$

$\phi(\vec{x})$ TAKES DISCRETE VALUES BUT IN THIS LIMIT \Rightarrow CONTINUOUS FCT OF SPACE

OVERLAPPING	BOXES	$\delta x = a$
NON-OVERL.	BOXES	$\delta x = l$

$$Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$$

$$= \sum_{\alpha} \sum_{\{s_i/\alpha\}} e^{-\beta H_{\alpha}(\{s_i\})}$$

SAME AS WHAT WE
DID FOR FC MODEL,
GROUP SPINS ACCORD.
TO THE VALUES OF
SOME AUX. VAR. &
SUM OVER THEM.
GENERIC α INDEX

APPLY TO α BEING ϕ

$$= \int \mathcal{D}\phi \sum_{\{s_i/\phi\}} e^{-\beta H_{\phi}(\{s_i\})}$$

ALL CONF OF SPINS
YIELDING THE
SAME ϕ

$$Z = \int \mathcal{D}\phi e^{-\beta F(\phi)}$$

WRITTEN AS A
FIELD DEP. (β FREE ENERGY)
GINZBURG-LANDAU
FREE ENERGY
(NB ITS MACROSCOPIC)

$$\int \mathcal{D}\phi ?$$

MEANS THAT ONE HAS TO SUM OVER ALL POSSIBLE VALUES OF ϕ AT EACH POINT IN SPACE \vec{x} LABELS THE BOXES

$$\int \mathcal{D}\phi = \prod_{\vec{x}} \int_{-\infty}^{\infty} d\phi(\vec{x})$$

WE LIFT THE CONSTRAINT $-1 \leq \phi(\vec{x}) \leq 1$

THE BEST WAY TO UNDERSTAND IT IS TO THINK ABOUT \vec{x} AS A LABEL

$F(\phi)$ IS A MACROSC. FREE ENERGY FUNCTIONAL
(A FUNCTION OF A FUNCTION)
F ϕ

THE INTEGRAL IS A FUNCTIONAL INTEGRAL

RECALL $x = \frac{1}{N} \sum_i s_i$ CURIE-WEISS

HERE $\phi(\vec{x}) = \frac{1}{N} \sum_{i \in V_{\vec{x}}} s_i$ SPACE-DEP.

GUESSES FOR $F(\phi)$

1- LOCAL $F(\phi) = \int d^d x f(\phi(\vec{x}))$
 \supset MACROSCOPIC

2- SPATIAL SYMMETRIES PRESERVED
 TRANSL \supset ROTAT OF LATTICE
 \Rightarrow CONT TH. TOO

3- SPIN REVERSAL $\Rightarrow \phi \rightarrow -\phi$
 SYMM.
 $F(\phi) = F(-\phi)$

4- ANALYTIC FUNCTIONAL
 $F(\phi)$ TAYLOR EXPANDABLE IN
 POWERS OF $(\phi, \vec{\nabla}\phi)$

5- SMOOTH VARIATIONS IN SPACE
 KEEP ONLY $(\vec{\nabla}\phi)^2$

$$F(\phi) = \int d^d x \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + \frac{\bar{a}}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

a controls small ϕ BEHAVIOUR

$$\bar{a} > 0 \quad T > T_c$$

$$\bar{a} < 0 \quad T < T_c$$

$$\bar{a} = a \cdot \frac{T - T_c}{T_c}$$

cf FULLY CONNECTED ISING MODEL
FOR SMALL ORDER PARAMETER (x)
THE SAME + $(\nabla \phi)^2$ TERM

DIMENSIONS

$$[c] \frac{[\phi^2]}{L^2} = [a] [\phi^2] = [\lambda] [\phi^2]$$

$$[c] = [a] L^2$$

$$[c] = [a] [L^2]$$

$$[\phi^2] [\lambda] = [a]$$

$$[c/a]^{1/2} = [L]$$

NB THE RÔLE PLAYED BY N IS NOW PLAYED BY
THE VOLUME OF SPACE $\int d^d x \rightarrow$ Volume

SADDLE POINT

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = 0$$

HOW DO YOU CALCULATE FUNCTIONAL DERIV. ?
HOW MANY OF YOU KNOW HOW TO DO IT ?

$$F(\phi) = \int d^d x f(\phi(\vec{x}))$$

MATH SUPPORT

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = \int d^d x \frac{\delta f(\phi(\vec{x}))}{\delta \phi(\vec{y})}$$

if $f(\phi(\vec{x}))$
JUST A fct
OF $\phi(\vec{x})$

$$= \int d^d x \quad f'(\phi(\vec{x})) \underbrace{\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})}}_{\delta^d(\vec{x}-\vec{y})}$$

$$= f'(\phi(\vec{y}))$$

NB: IT SETS \vec{x} TO \vec{y}
AND TAKES A DERIVATIVE
(NORMAL DERIV.)

WHAT HAPPENS WITH TERMS WITH $\vec{\nabla}\phi(\vec{x})$?

$$\text{eg. } F_{\text{grad}}(\phi) = \int d^d x \quad \frac{1}{2} (\vec{\nabla}\phi(\vec{x}))^2$$

$$= \int d^d x \quad \frac{1}{2} \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}\phi(\vec{x})$$

$$\frac{\delta F_{\text{grad}}(\phi)}{\delta\phi(\vec{y})} = \int d^d x \left[\frac{1}{2} \vec{\nabla} \frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})} \cdot \vec{\nabla}\phi(\vec{x}) + \frac{1}{2} \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla} \frac{\delta\phi(\vec{x})}{\delta\phi(\vec{y})} \right]$$

TWICE THE SAME TERM

$$\begin{aligned}
&= \int d^d x \vec{\nabla} \mathcal{E}^d(\vec{x}-\vec{y}) \cdot \vec{\nabla} \phi(\vec{x}) \\
&= - \int d^d x \mathcal{E}^d(\vec{x}-\vec{y}) \nabla^2 \phi(\vec{x}) + \text{BORDER TERMS} \\
&= - \nabla^2 \phi(\vec{y}) \quad \begin{array}{l} \propto \vec{\nabla} \phi(\vec{y}) \\ \parallel \\ 0 \end{array}
\end{aligned}$$

ALL TOGETHER

$$0 = \frac{\delta F(\phi)}{\delta \phi(\vec{y})} = -c \nabla^2 \phi(\vec{y}) + \frac{\lambda}{3!} \phi^3(\vec{y}) + a \left(\frac{T-T_c}{T_c} \right) \phi(\vec{y})$$

$$\frac{\lambda}{3!} \phi_{sp}^3 + a \frac{T-T_c}{T_c} \phi_{sp} = 0$$

$$\phi(\vec{x}) = \phi_{sp} \text{ CONST}$$

$$\phi_{sp} = 0 \quad \text{or}$$

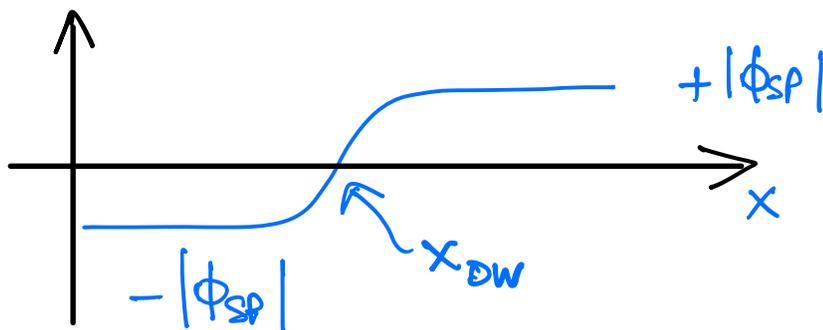
$$\phi_{sp} = \pm \sqrt{\frac{3! a (T_c - T)}{\lambda T_c}}$$

UNITS OK

UNTIL HERE \rightarrow REALLY SIMILAR! SAME EXPONENTS

DOMAIN WALLS

SPATIAL VARIATION / INFO



ADDING $\nabla^2\phi$ TO THE SP. EQ
WITH $\lambda\phi^4$ "POTENTIAL"

\Rightarrow SOLUTION VARIATION ALONG DIRECT. x

$$\phi_{DW}(x) = |\phi_{sp}| \tanh\left(\frac{x - x_{DW}}{w}\right)$$

XSP FREE, ϕ_{SP} THE SOL WITHOUT $\nabla^2 \phi$

$$W = \left(\frac{c T_c}{\alpha(T_c - T)} \right)^{1/2} \text{ WIDTH}$$

$$\text{UNITS } \left(\frac{[c]}{[\alpha]} \right)^{1/2} = \left(\frac{[\cancel{\alpha}][L^2]}{[\cancel{\alpha}]} \right)^{1/2} = [L^2]^{1/2} = [L]$$

OR, IT'S
A LENGTH.

W DIVERGES AT $T \rightarrow T_c^-$

DEPENDS ON C ELAST CONST.

INTUITIVELY: PROLIFERATION OF DW AT $T \rightarrow T_c^-$
DIVERGENCE OF W

$W \sim \xi$ CORRELATION LENGTH

WE'LL SEE IT LATER
FROM ITS DEF. FROM THE
CORR. FUNCTION.

CORRELATION FUNCTIONS

THE CONNECTED CORRELATION

$$C_{\text{CONN}}(\vec{x}, \vec{y}) = \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

$$\delta\phi(\vec{x}) = \phi(\vec{x}) - \langle \phi(\vec{x}) \rangle$$

OF THE FLUCTUATIONS CAN BE CALCULATED
ADDING A SPACE DEP. FIELD AS A SOURCE

$$\mathcal{Z}_h = \int \mathcal{D}\phi \ e^{-\beta F(\phi) + \beta \int d^d x \ h(\vec{x}) \phi(\vec{x})}$$

RECALL PROPS OF CUMULANTS (MATH SUPPORT)

$$\left. \frac{\delta^2 \ln \mathcal{Z}_h}{\delta h(\vec{x}) \delta h(\vec{y})} \right|_{h=0} = \beta^2 \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

HOW DO WE ISOLATE THE h -DEP IN \mathcal{Z}_h ?

- WE DROP THE $\phi^4(\vec{x})$ TERM IN F_h
ASSUMING WE WORK CLOSE TO T_c

- WE FOURIER TRANSFORM THE FIELD TO "DECOUPLE" THE GRADIENT TERM

$$\phi(\vec{k}) = \int d^d x e^{i\vec{k} \cdot \vec{x}} \phi(\vec{x})$$

$$\phi(\vec{x}) \in \mathbb{R} \Rightarrow \phi(\vec{k}) \in \mathbb{C} \quad \text{BUT}$$

$$\phi^*(\vec{k}) = \phi(-\vec{k}) \quad \begin{array}{l} \text{NO} \\ \text{DOUBLING} \\ \text{OF d.o.f.} \end{array}$$

IN TERMS OF $\phi(\vec{k})$ THE FREE-ENERGY IS
($\phi(\vec{k}) \approx 0$)

$$F_{\hbar}(\phi) = \int \frac{d^d k}{(2\pi)^d} \left[\frac{ck^2}{2} |\phi(\vec{k})|^2 + \frac{a(T-T_c)}{2 T_c} |\phi(\vec{k})|^2 \right]$$

$$- \int \frac{d^d k}{(2\pi)^d} \hbar(\vec{k}) \phi^*(\vec{k})$$

FOR THE MOMENT SHORT-HAND NOTATION:

$$\tilde{J}^{-2} \equiv a \cdot \frac{T-T_c}{T_c}$$

WE IDENTIFY AS A FACTOR OF THE

QUADRATIC TERM WE'LL SEE LATER IT'S

PROP TO COIL LENGTH (\neq DIM.)

$$\nu = 1/2$$

$$F_h(\phi) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} (ck^2 + \xi^{-2}) |\phi(\vec{k})|^2 - \int \frac{d^d k}{(2\pi)^d} h(\vec{k}) \phi^*(\vec{k})$$

NB EACH k ON ITS OWN \Rightarrow DECOUPLED

NOW, WE NEED TO EXTRACT THE $h(\vec{k})$ DEPENDENCE FROM THE FUNCT. INT. OVER $\phi(\vec{k})$

WE USE A TRICK:

TRANSLATE THE FIELD

$$\varphi(\vec{k}) = \phi(\vec{k}) - \frac{h(\vec{k})}{ck^2 + \xi^{-2}} \quad \begin{array}{l} A \downarrow \\ \text{NUM } \mathbb{R} \\ \text{CONST TO FIX} \end{array}$$

AND REWRITE F IN TERMS OF $\varphi(\vec{k})$:

$$F_h[\phi] = F[\varphi] + \text{TERM IN } h$$

THE
SQUARE
TERM

$$\begin{aligned} |\phi(t)|^2 &= \left| \varphi(t) + \frac{h(t)A}{ch^2 + \xi^{-2}} \right|^2 \\ &= |\varphi(t)|^2 + \left(\frac{\varphi(t)h^*(t)}{ch^2 + \xi^{-2}} + \text{c.c.} \right) A \\ &\quad + \frac{|h(t)|^2 A^2}{(ch^2 + \xi^{-2})^2} \end{aligned}$$

THE FIELD-DEP. TERM

$$\begin{aligned} h(t)\phi^*(t) &= h(t) \left[\varphi(t) + \frac{h(t)A}{ch^2 + \xi^{-2}} \right]^* \\ &= h(t)\varphi^*(t) + \frac{|h(t)|^2 A}{ch^2 + \xi^{-2}} \end{aligned}$$

• PUTTING ALL TOGETHER:

$$\begin{aligned} (ch^2 + \xi^{-2}) |\phi(t)|^2 - h(t)\phi^*(t) &= \\ = (ch^2 + \xi^{-2}) \left[|\varphi(t)|^2 + \left(\frac{\varphi(t)h^*(t) + \text{c.c.}}{ch^2 + \xi^{-2}} \right) A \right. \\ &\quad \left. + \frac{|h(t)|^2 A^2}{(ch^2 + \xi^{-2})^2} \right] \end{aligned}$$

$$- r(k) \varphi^*(k) - \frac{|r(k)|^2 A}{ck^2 + \xi^{-2}}$$

$$= (ck^2 + \xi^{-2}) |\varphi(k)|^2 \left(- r(k) \varphi^*(k) \right)$$

SAME EXP AS ORIGINAL ONE
BUT IN $\varphi(k)$

$$+ \left(\varphi(k) r^*(k) + \varphi^*(k) r(k) \right) A$$

$$\begin{aligned} & \Rightarrow \varphi(-k) r(k) \\ & \Rightarrow \varphi(k) r(-k) \rightarrow \text{UNDER} \\ & = \varphi(k) r^*(k) \quad \text{INT. OVER} \\ & \int d^d k \end{aligned}$$

$$- \frac{|r(k)|^2 A}{ck^2 + \xi^{-2}}$$

IT DOESN'T DEP. ON $\varphi(k)$
 \Rightarrow GOES OUTSIDE INTEGRAL

WE CAN CANCEL THE TERMS IN THE BUBBLES
TAKING $A = 1/2$

THEN, WE HAVE

$$F_h(\varphi) = \underbrace{\frac{1}{2} \int d^d k \left[c k^2 |\varphi(\vec{k})|^2 + \tilde{\xi}^{-2} |\varphi(\vec{k})|^2 \right]}_{F[\varphi]} - \frac{1}{2} \int d^d k \frac{|h(\vec{k})|^2}{c k^2 + \tilde{\xi}^{-2}}$$

AND THE VARIATIONS WRT $h(\vec{x})$ ONLY TOUCH THE LAST TERM IN $F_h(\varphi)$.

$$Z_h = \underbrace{\int \mathcal{D}\varphi e^{-\beta F(\varphi)}}_B e^{\frac{\beta}{2} \int d^d k \frac{|h(\vec{k})|^2}{c k^2 + \tilde{\xi}^{-2}}}$$

• NOW, FOURIER TRANSFORM BACK TO \vec{x} SPACE

$$Z_h = B e^{\frac{\beta}{2} \int d^d x \int d^d y h(\vec{x}) G(\vec{x}-\vec{y}) h(\vec{y})}$$

$$G(\vec{x}-\vec{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}(\vec{x}-\vec{y})}}{c k^2 + \tilde{\xi}^{-2}}$$

- Firstly,

$$\ln Z_h = \ln B + \frac{\beta}{2} \int d^d \vec{x} d^d \vec{y} h(\vec{x}) G(\vec{x} - \vec{y}) h(\vec{y})$$

$$\frac{\delta^2 \ln Z_h}{\delta h(\vec{x}) \delta h(\vec{y})} = \beta G(\vec{x} - \vec{y}) = \beta^2 \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

↑
WE ALREADY KNEW IT

AND THEN

$$\langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle = k_B T G(\vec{x} - \vec{y})$$

NOW WE HAVE TO GO BACK TO THE DEF OF

$$G(\vec{x} - \vec{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\vec{k}(\vec{x} - \vec{y})}}{ck^2 + \xi^{-2}}$$

AND TRY TO CALCULATE THE INTEGRAL

* HERE, FOLLOW THE CALC. IN THE LECTURE NOTES, VERY DETAILED !

• $r \gg \xi$ $G(r) \sim \exp(-r/\xi)$

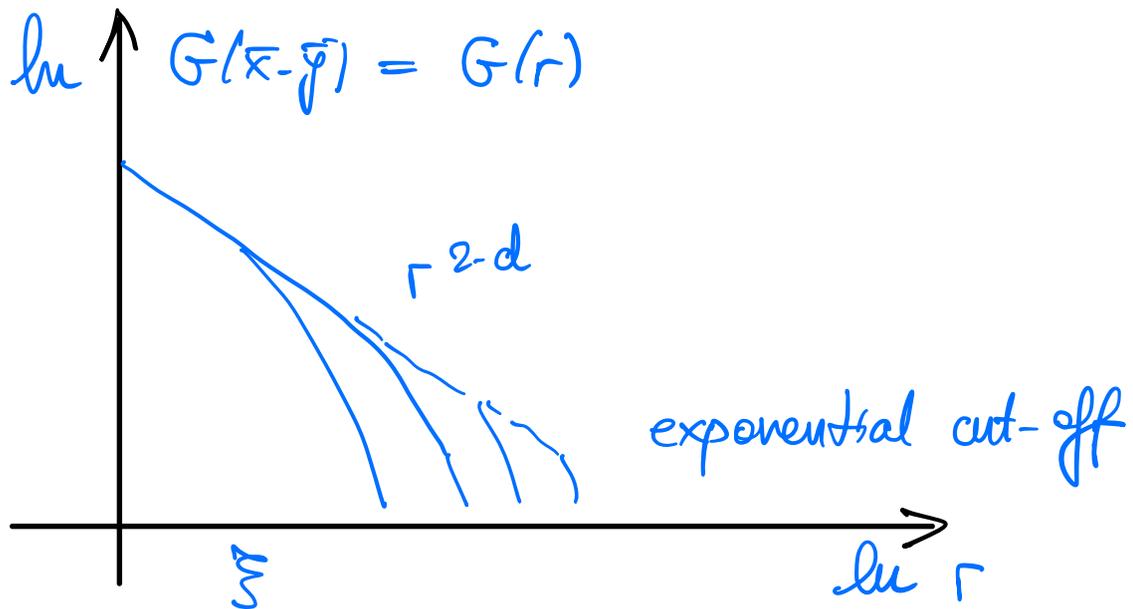
• $r \ll \xi$

$G(\vec{x}-\vec{y}) \propto r^{2-d}$ $r \ll \xi$

AND NOW FOR BOTH
 $r \ll \xi$ AND
 $r \gg \xi$

$G(\vec{x}-\vec{y}) \sim r^{2-d} e^{-r/\xi}$

DISCUSS THE FORM OF $G(\bar{x}-\bar{y})$



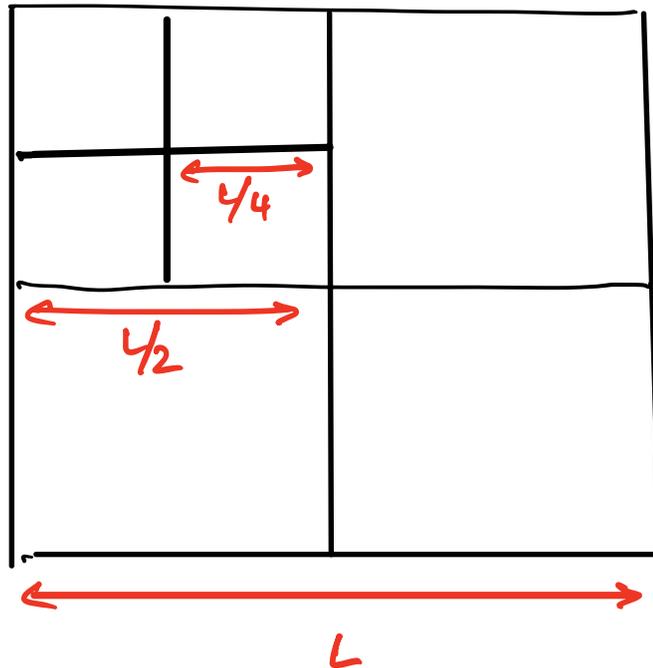
ξ DEPENDS ON $T-T_c$ $\xi \sim (T-T_c)^{-1/2}$

- FROM LEFT TO RIGHT T GETS CLOSER TO T_c
- WE SEE THAT $d=2$ PLAYS A SPECIAL ROLE IN THIS THEORY.

THE CORRELATION LENGTH

DISTANCE OVER WHICH THE
FLUCT. OF THE ORDER PARAMETER
ARE CORRELATED

80



MEASURE OBSERV IN $L^d, (L/2)^d, \dots$
SAME RESULT UNTIL

$$L \sim \xi$$

WE SEE THERMAL FLUCTUATIONS IN SPIN
CONF - SNAPSHOTS

THE AVERAGED SIZE OF THESE
REVERSED DROPLETS GIVE AN IDEA OF

ξ

ALTHOUGH IT'S NOT EXACTLY EQUAL TO IT

CONSEQUENTLY

SYST WITH $\xi \rightarrow \infty$ HAVE DROPLETS
OF ALL SIZES.

TRANSLATIONAL INVARIANCE \rightarrow
SCALE INVARIANCE

MORE LATER

GINZBURG CRITERION

GAUSSIAN FLUCT AROUND THE HOMOG. S.P.

$$\langle \delta\phi_{\text{coh}}^2 \rangle \sim \langle \phi^2 \rangle = \phi_{\text{SP}}^2$$

$$\delta\phi_{\text{coh}} \equiv \frac{1}{V_{\text{coh}}} \int d^d x (\phi(\vec{x}) - \phi_{\text{SP}})$$

$$V_{\text{coh}} = \xi^d \quad \text{WITH } \xi \text{ CORREL. LENGTH}$$

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{1}{V_{\text{coh}}^2} \int d^d x \int d^d y \underbrace{(\phi(\vec{x}) - \phi_{\text{SP}})(\phi(\vec{y}) - \phi_{\text{SP}})}_{\text{DEFINITION OF } C^{\text{CONN}}(\vec{x}, \vec{y})}$$

FOR A HOMOGENEOUS PROBLEM

TRANSL. INV.

ISOTROPY

EXPECTED \Rightarrow

$$C^{\text{CONN}}(\vec{x}, \vec{y}) = C^{\text{CONN}}(\|\vec{x} - \vec{y}\|)$$

Call $r = \|\vec{x} - \vec{y}\|$

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{1}{V_{\text{coh}}} \int d^d r C^{\text{CONN}}(r)$$

$$= \frac{1}{V_{\text{coh}}} \Omega_d \int dr r^{d-1} C^{\text{CONN}}(r)$$

WE SHOWED IT ABOVE.

if

$$C^{\text{CONN}}(r) \sim r^{2-d-\eta} e^{-r/\xi}$$

CLOSE TO T_c

WITH $\eta=0$

WITHIN LANDAU

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{\Omega_d}{\xi^d} \int dr r^{d-1} r^{2-d} e^{-r/\xi}$$

$$= \frac{\Omega_d}{\xi^d} \int \frac{dr}{\xi} \frac{r}{\xi} e^{-r/\xi} \xi^2$$

$$= \xi^{2-d} \underbrace{\int_0^L dy y e^{-y} \Omega d}_A$$

A NUMBER

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \xi^{2-d} A$$

CLOSE TO THE CRITICAL POINT

$$\langle \phi \rangle = \phi_{\text{SP}} \simeq |T_c - T|^\beta \quad (\text{LANSDAU})$$

$$\text{if } \xi \sim |T - T_c|^{-\nu} \implies |T - T_c| \sim \xi^{-1/\nu}$$

$$\implies \phi_{\text{SP}} \sim \xi^{-\beta/\nu}$$

WITHIN THE LANSDAU APPROX

$$\beta = 1/2$$

$$\nu = 1/2$$

$$\langle \phi \rangle = \phi_{\text{SP}} \sim \xi^{-1}$$

BACK TO THE GINZBURG CRITERION

$$\langle \delta\phi_{\text{coh}}^2 \rangle \sim \langle \phi \rangle^2 \Rightarrow$$

$$\xi^{2-d} \sim \xi^{-2} \Rightarrow \xi^{4-d} \sim 1$$

$$d \geq 4 \quad \Delta \quad d_u = 4$$

- if $d \geq 4$ THEN $\langle \delta\phi_{\text{coh}}^2 \rangle \approx \langle \phi \rangle^2$ & MEAN FIELD OK
- if $d < 4 \Rightarrow$ NO!

FIRST ORDER PHASE TRANS. WITHIN LANDAU'S FIELD THEORY

- FIELD DRIVEN

$$- \int d^d x \ h(\vec{x}) \phi(\vec{x})$$

- TEMP DRIVEN

$$\int d^d x \ \phi^3(\vec{x})$$

LIKE IN $p=3$ FM MODEL

APPLICATIONS

- COND-MATTER
- HIGH ENERGY ϕ HIGGS FIELD
 A_μ FIELD
- etc.

EXTENSIONS TO VECTOR VARIABLES

- VECTOR FIELDS

$$\vec{\phi}(\vec{x})$$

LATER : 2dXY MODEL \Leftrightarrow
 $O(2)$ FIELD TH. IN
 $d=2$

SUMMARY

TO BE SHOWN WITH SLIDES

LANDAU $\phi(\vec{x})$

$F(\phi)$ BASED ON ASSUMPTIONS

2nd \swarrow \searrow 1st

GINZBURG

LIMITS OF VALIDITY $d > d_m$ EXACT

$d \leq d_m$ FAR FROM T_c ok.

$$\xi \sim |T - T_c|^{-\nu}$$

$$G_{\text{CONN}}(\vec{r}) \sim r^{2-d-2} f(r/\xi)$$

$f(r/\xi)$ CUT-OFF $r \gg \xi$

$$\chi = \int d^d x \int d^d y \chi(\vec{x}, \vec{y}), \text{ FDT,}$$

$$\propto \beta \xi^{2-\eta} \quad \text{DIVERGES WITH } \xi$$

PROOF BELOW

CORR FUNCT & GLOBAL SUSCEPTIBILITY

BACK TO GENERICS. $\Theta \Rightarrow$ ORDER PAR. $\langle \Theta \rangle$

$$C(\vec{x}, \vec{y}) = \langle \delta\Theta(\vec{x}) \delta\Theta(\vec{y}) \rangle \quad \text{CORR} \\ \text{FUNCT}$$

(I WILL NOT WRITE "CONN" ANYMORE)

$$\sim r^{2-d-\eta} \underbrace{f(r/\xi)}$$

CUT-OFF FUNCT

η ANOMALOUS EXP.

{ TRANS INV.
ISOTROPY

$$\xi \sim |T - T_c|^{-\nu}$$

CORR LENGTH

$$\chi(\vec{x}, \vec{y}) = \frac{\delta \langle \Theta(\vec{x}) \rangle}{\delta h(\vec{y})} \Big|_{h \rightarrow 0} \quad \text{LINEAR} \\ \text{SUSCEP.}$$

$$H \rightarrow H - \int d^d x \ h(\vec{x}) \Theta(\vec{x})$$

$$\chi(\vec{x}, \vec{y}) = \beta C(\vec{x}, \vec{y})$$

LOCAL
FDT

DIVERGENCE OF $\xi \Rightarrow$ DIVERGENCE OF χ
INTEGRATED OVER FULL VOLUME

Proof

GLOBAL OF
TOTAL SUBE

$$\int d^d x \int d^d y \underbrace{\chi(\vec{x}, \vec{y})}_{\chi(r)} = \chi$$

INTEGRATING OVER VOLUME TWICE

$$= V \Omega_d \int dr r^{d-1} \chi(r)$$

$$= V \Omega_d \beta \int dr r^{d-1} C(r)$$

ONE VOLUME FACTOR ONE VOLUME INT

HOM / ISOT

FDT

if $C(r) \sim r^{2-d-\eta} f(r/\xi) \Rightarrow$

$$\chi = V \Omega_d \beta \int dr r^{d-1} r^{2-d-\eta} f(r/\xi)$$

$$= V \Omega_d \beta \int dr r^{1-\eta} f(r/\xi)$$

$$= V \Omega d \beta \xi^{2-\eta} \int \frac{dr}{\xi} \left(\frac{r}{\xi}\right)^{4-\eta} f(r/\xi)$$

THIS ONE IS INTEGRATED TWICE OVER VOLUME

CONST A

$$X = V \Omega d A \beta \xi^{2-\eta}$$

IF $\eta < 2 \Rightarrow$
 ξ DIVERG. \Rightarrow
 X DIVERG.

POWER LAW DECAY OF CORR FOR

$\Gamma < \xi \Rightarrow$ DIVERGENCE OF X
 WITH $\xi^{2-\eta}$ (TWICE INTEGRATED)

CHECK $X \sim V \xi^{2-\eta}$
 \downarrow ONE VOLUME L^d $\left\{ \begin{array}{l} \frac{X}{L^d} \sim \xi^{2-\eta} \\ \sim |T-T_c|^{-\nu(2-\eta)} \end{array} \right.$
 DIVERGENCE CHANGED WRT
 $X \sim |T-T_c|^{-\gamma}$

EXPONENTS \Rightarrow UNIVERSALITY

$$\alpha = \alpha' = 0 \quad \beta = 1/2 \quad \gamma = \gamma' = 4 \quad \delta = 3$$

$$\nu = 1/2 \quad \eta = 0 \quad \underline{MF}$$

$$\alpha = \alpha' = 0 \quad \beta = 1/8 \quad \gamma = \gamma' = 7/4 \quad \delta = 15$$

$$\nu = 1 \quad \eta = 1/4 \quad \underline{d=2 \text{ ONSAGER}}$$

$$\alpha = \alpha' = 0.11 \quad \beta = 0.325 \quad \gamma = 1.24 \quad \delta = 4.82$$

$$\nu = 0.63 \quad \eta = 0.032 \quad \underline{d=3 \text{ APPROX}}$$

THE SAME VALUES IN MANY MANY \neq SYST
MICROSC DOESN'T MATTER
APART FROM FEATURES WE WILL SEE DO
MATTER (DIM, DIM ORDER, SYMM.)
PARAM

d n SYMM

SCALING AND UNIVERSALITY

SHOW THAT THE SINGULAR PART OF f

DEPENDS ON

THE CONTROL PARAMETERS AS.

$$f_{\text{SING}} = |t|^{2-\alpha} \quad g_f \left(\frac{h}{|t|^\nu} \right)$$

$$t = \frac{T - T_c}{T_c}$$

⇒ FIND RELATIONS BETWEEN EXPONENTS

$\alpha, \beta, \gamma, \delta, \nu, \eta$ NOT ALL INDEP.

IIA CURIE-WEISS MODEL

$$f(t, h) = \min_m \left(\frac{t}{2} m^2 + u m^4 - h m \right)$$

I MAY BE CHANGING NUMERICAL FACTORS:

IT IS NOT IMPORTANT

$$t \equiv \frac{T - T_c}{T_c}$$

$$u = \frac{\lambda}{4}$$

(IN PREVIOUS
NOTATION λ
NO FACTOR $1/2$)

USE THE EQ. FOR THE ORDER PARAMETER

$$t m + 4 u m^3 - h = 0$$

AND WRITE $f(m; t, h) = f(m_{\text{sol}}; t, h)$

$$= f(t, h) \quad \text{FOR SING. PART}$$

① For $\begin{cases} t < 0 \\ h = 0 \end{cases}$ $4u m^3 + t m = 0 \Rightarrow$

$$m^2 = \frac{-t}{4u} \Rightarrow$$

$$f(t, h) = \frac{t}{2} \frac{(-t)}{4u} + u \frac{t^2}{16 u^2}$$

$$f(t, h) = \frac{-t^2}{16 u}$$

PARAMETER DEPENDENCE
OF FREE-ENERGY DENSITY

② For $\begin{cases} h \neq 0 \\ t = 0 \end{cases}$ $4u m^3 - h = 0 \Rightarrow$

$$m = \left(\frac{h}{4u} \right)^{1/3}$$

$$f(t, h) = u \left(\frac{h}{4u} \right)^{4/3} - h \left(\frac{h}{4u} \right)^{1/3}$$

$$f(t, h) = \frac{3}{4^{4/3}} \frac{h^{4/3}}{u^{1/3}}$$

IN CURIE WEISS

$$f_{\text{SING}}(t, h) = |t|^2 g_f\left(\frac{h}{|t|^\Delta}\right)$$

THIS IS AN EXAMPLE OF AN HOMOGENEOUS FUNCTION OF ITS ARGUMENTS

WITH $g_f(y=0) = -\text{CONST}$ ✓

$g_f(y \rightarrow \infty) = y^{4/3} \Rightarrow$

$$|t|^2 \frac{h^{4/3}}{|t|^{\frac{4\Delta}{3}}} \sim h^{4/3} \Rightarrow 2 - \frac{4\Delta}{3} = 0 \Rightarrow$$

$$\Delta = \frac{3}{2}$$

SINGULAR PART OF THE FREE-ENERGY DENSITY
AS A FUNCT OF CONTROL PARAM.

IN CASES WITH $\alpha \neq 0$ & EQUAL ON BOTH SIDES OF TRANS

$$f_{\text{SING}}(t, h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^\Delta}\right)$$

RECALL $\langle H \rangle = - \frac{\partial \ln Z}{\partial \beta}$

$$C_V = \frac{\partial \langle H \rangle}{\partial T} = \frac{\partial}{\partial T} \left(- \frac{\partial \ln Z}{\partial \beta} \right) = \frac{\partial}{\partial T} \frac{\partial}{\partial \beta} (\beta F)$$

$$= \frac{\partial}{\partial T} \left[F + \beta \frac{\partial F}{\partial \beta} \right]$$

$$= \frac{\partial F}{\partial T} + \frac{\partial \beta}{\partial T} \frac{\partial F}{\partial \beta} + \underbrace{\beta \frac{\partial}{\partial T} \frac{\partial F}{\partial \beta}}_{\text{MOST SING. PART}}$$

$$C_V \sim \beta \frac{\partial}{\partial T} \frac{\partial F}{\partial \beta} \Rightarrow \text{THE MOST SING PART COMES FROM } \frac{\partial^2}{\partial T^2}$$

$$C_V \sim \frac{\partial^2 F}{\partial T^2} \sim N \frac{\partial^2 f_{\text{SING}}}{\partial T^2}$$

$$C_V \sim |t|^{-\alpha} \Leftrightarrow f_{\text{SING}} \sim |t|^{2-\alpha} \underbrace{g\left(\frac{h}{|t|^\lambda}\right)}$$

$$\frac{\partial^2 f_{\text{SING}}}{\partial T^2} \sim |t|^{-\alpha} \quad \text{OK.}$$

THIS PART DOESN'T CONT

$h \rightarrow 0$: CONST

DONE WE JUSTIFIED f_{SING} BEYOND CURIE-WEISS

CONSEQUENCES:

THE MAGN DENSITY

$$m = \frac{\partial f}{\partial h} = |t|^{2-\alpha} g'_f \left(\frac{h}{|t|^\Delta} \right) \frac{1}{|t|^\Delta}$$
$$= |t|^{2-\alpha-\Delta} g_m \left(\frac{h}{|t|^\Delta} \right)$$

WE KNOW $m \sim |t|^\beta$ FOR $\frac{h}{|t|^\Delta} \rightarrow 0$

$$\Rightarrow \boxed{2-\alpha-\Delta = \beta} \quad \& \quad g_m(y) \sim 1 \quad y \rightarrow 0$$

WE ALSO KNOW $m \sim h^{1/\delta}$ FOR $\frac{h}{|t|^\Delta} \rightarrow \infty$

$$\Rightarrow g_m(y) \sim y^{1/\delta} \quad y \rightarrow \infty$$

$$\boxed{2-\alpha-\Delta - \Delta/\delta = 0}$$

FROM NO DEP.
ON $|t|$ LEFT

PUTTING THE TWO TOGETHER

$$\Delta = \beta \delta \quad (1)$$

$$2 - \alpha - \beta \delta = \beta \Rightarrow$$

$$\beta = \frac{2 - \alpha}{1 + \delta} \quad (2)$$

IN THE MF CASE

$$\beta = 1/2 \quad \alpha = 0 \quad \delta = 3 : \quad \frac{1}{2} = \frac{2}{4} \quad \checkmark$$

SUSCEPTIBILITY

$$\chi = \frac{\partial^2 f}{\partial h^2} \Big|_{h \rightarrow 0} = \frac{\partial m}{\partial h} \Big|_{h \rightarrow 0}$$

BOTH m AND h
ARE "INTENSIVE"
NOT SCALING WITH
 L^d

$$= |t|^{2-\alpha} \left[g_m'' \left(\frac{h}{|t|^\Delta} \right) \frac{1}{|t|^\Delta} \frac{1}{|t|^\Delta} \right]$$
$$= |t|^{2-\alpha-2\Delta} g_m''(0) \quad h \rightarrow 0$$

SHOULD BE EQUAL TO $|t|^{-\gamma}$

$$\gamma = \alpha + 2\Delta - 2$$

$$\gamma = \alpha + 2\beta\delta - 2 \Rightarrow (3)$$

PROOF OF WIDOM'S

FROM (2) $\beta = \frac{2-\alpha}{1+\delta} \Rightarrow \alpha = 2 - \beta(1+\delta)$

IN (3) $\gamma = \cancel{2} - \beta - \beta\delta + 2\beta\delta - \cancel{2} = \beta(\delta - 1)$

WIDOM'S
IDENTITY

$$\delta - 1 = \gamma/\beta$$

(4)

FINALLY, ANOTHER ONE WITH A NAME:

REPLACE WARDOM'S δ IN (3)

$$\gamma = \alpha + 2\beta \left(1 + \frac{\delta}{\beta}\right) - 2$$

$$\gamma = \alpha + 2\beta + \cancel{\delta} - 2$$

$$2 = \alpha + 2\beta + \gamma$$

RUSHBROOKE'S
IDENTITY

$$\alpha + 2\beta + \gamma = 2$$

IN MF CURSE-WEISS $\alpha = 0$ $\beta = 1/2$ $\gamma = 1$

0 + 1 + 1 = 2 ok. ✓

RUSHBROOKE'S

WIDOM'S

$$\alpha + 2\beta\delta + \gamma = 2$$

$$\delta - 1 = \gamma/\beta$$

THESE ARE, FOR THE MOMENT, INDEP OF d

FROM 6 EXPONENTS NOW WE ARE LEFT WITH 4
INDEP. ONES ONLY.

SPATIAL INFO IN C AND ξ START USING ξ

ASSUME SCALING OF CORRELATION LENGTH

$$\xi(t, r) \sim |t|^{-\nu} g_{\xi} \left(\frac{r}{|t|^{\Delta}} \right)$$

$$g_{\xi}(y \rightarrow 0) = \text{ct.} \quad g_{\xi}(y \gg 1) \sim y^{-\nu/\Delta}$$

$$\hookrightarrow \xi \sim h^{-\nu/\Delta}$$

$$-\beta f_{\text{sing}}(t, h) \sim L^{-d} \underbrace{\ln Z}_{\text{ADIM}}$$

$$\sim L^{-d} \left(\frac{L}{\xi}\right)^d \quad \begin{array}{l} \text{UNION OF} \\ (L/\xi)^d \text{ INDEP} \\ \text{PIECES} \end{array}$$

$$\sim \xi^{-d}$$

$$\sim |t|^{+d\nu} \left[g_{\xi} \left(\frac{h}{|t|^{\Delta}} \right) \right]^{-d}$$

THIS HAS TO BE EQUAL
TO THE PREVIOUS FORM WE USED

$$f_{\text{sing}}(t, h) \sim |t|^{2-d} g_f \left(\frac{h}{|t|^{\Delta}} \right)$$

⇒

$$2 - \alpha = d\nu$$

JOSEPHSON'S
IDENTITY

A RELATION WITH d IN IT. CALLED

HYPERSCALING

STILL HAVE TO RELATE η TO THE OTHER EXPONENTS
WE KNOW FROM FDT AND DIVERG OF GLOBAL χ :

$$C(r) \sim r^{2-d-\eta} \Rightarrow \frac{\chi}{L^d} \sim \xi^{2-d}$$

$$\frac{\chi}{L^d} \sim |t|^{-\gamma} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} \Rightarrow$$

$$\gamma = \nu(2-\eta)$$

WE PROVED FOUR RELATIONS BETWEEN
EXPONENTS

IN CONCLUSION, THERE ARE ONLY TWO
INDEPENDENT EXPONENTS.

ALL OTHER ONES CAN BE WRITTEN IN
TERMS OF THEM.

eg. ν AND η

IT WAS BELIEVED THAT $\alpha = \alpha'$ $\gamma = \gamma'$ ON TWO
SIDES OF TRANS. BUT THIS IS NOT NECESSARILY
TRUE \Rightarrow APPLY ABOVE ON EACH SIDE.

ALL THIS IN 45 MIN CIRCA

HOMOGENEOUS FUNCTIONS

(x_i PARAMETERS)

THEY WILL BE THE CENTRAL PARAM OF THE TRANSITION)

$$f(x_1, x_2, \dots, x_n) b^{\frac{P_f}{f}} =$$

$$f(b^{P_1} x_1, b^{P_2} x_2, \dots, b^{P_n} x_n)$$

$\forall b \in \mathbb{R}$

TAKE $x_n = b^{-P_n} \Rightarrow$ WITH ARGUMENT $= 1$

$$b = x_n^{-1/P_n}$$

$$f(x_1, x_2, \dots, x_n) =$$

$$x_n^{P_f/P_n} f\left(x_n^{-P_1/P_n} x_1, x_n^{-P_2/P_n} x_2, \dots, 1\right)$$

ONE LESS INDEP VARIABLE

& ALL APPEAR AS

$$\frac{x_i}{x_n^{P_i/P_n}}$$

CALL

$$\frac{P_i}{P_n} = \Delta \Rightarrow$$

$$\frac{x_i}{x_n^\Delta}$$

$n-1$ SCALING VARIABLES
 g SCALING FUNCTION.

EXAMPLE CURIE WEISS g is $f_{\text{sing}}(h, |t|)$

$$x_1 = h \quad x_2 = |t|$$

$$\frac{x_1}{x_n^\Delta} = \frac{h}{|t|^\Delta} \quad \frac{p_1}{p_2} = \Delta$$

$$f_{\text{sing}} = |t|^2 g\left(\frac{h}{|t|^\Delta}\right) \quad \Delta = \frac{3}{2}$$

WONDERS OF SCALING : $\left\{ \begin{array}{l} \text{SAME EXPONENTS } \Delta \\ \text{SAME SCALING FUNCTIONS} \end{array} \right.$

SCALE INVARIANCE

$L \rightarrow \infty$, JUST ξ AS ONLY SCALING LENGTH
 AT CRIT POINT $\xi \rightarrow \infty \Rightarrow$ NONE LEFT !

$$\lambda^p \cdot C(\vec{x}, \vec{y}) = C(\lambda \vec{x}, \lambda \vec{y})$$

WE ZOOM IN & OUT & STATISTICALLY WE SEE THE SAME PICTURES

FINITE SIZE SCALING

$a \ll \xi \ll L$ IN REAL LIFE

SCALING $\frac{\xi}{a}, \frac{L}{\xi}$ ONLY RELEVANT

SITUATION

$\frac{\xi}{a} \rightarrow \infty$ $\frac{L}{\xi}$ FINITE

QUESTION : WHAT HAPPENS WITH c_f
 $\chi(t)$, A QUANTITY THAT
 WOULD DIVERGE AT T_c
 IF $L \rightarrow \infty$

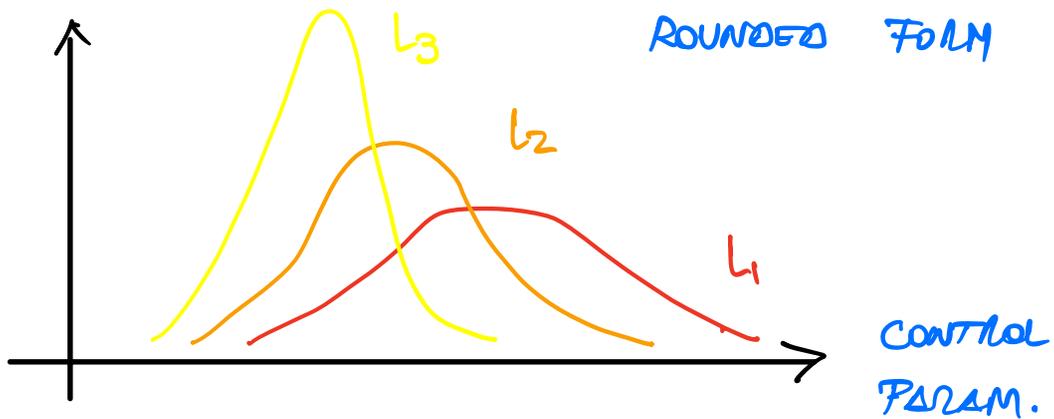
?

OBSERVATIONS FROM EXPS & NUM SIMUL.

THIS IS THE LOCAL ONE, OR AN INTENSIVE ONE THAT SHOULD

χ

DIVERGE AS $|T - T_c|^{-\gamma}$ FOR $L \rightarrow \infty$



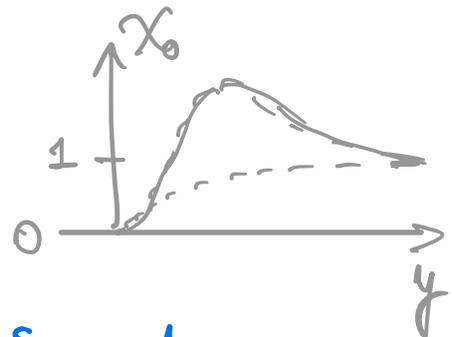
- MAXIMUM MOVES TO THE "LEFT" FOR INCREASING L
- WIDTH OF CURVE DIMINISHES WITH $L \uparrow$
- HEIGHT OF MAX \uparrow WITH $L \uparrow$

ALL \sim POWER LAWS

HYPOTHESIS

$$\chi \sim \xi^{\delta/2} \chi_0 \left(\frac{L}{\xi} \right)$$

$$\chi_0(y) \sim \begin{cases} \text{cst} & y \gg 1 \\ y^{\delta/2} & y \rightarrow 0 \end{cases}$$



ENSURING $\left\{ \begin{array}{l} \xi^{\delta/2} \text{ FOR } L \gg \xi ; L \rightarrow \infty \\ L^{\delta/2} \text{ FOR } \xi \gg L \end{array} \right.$

IN TERMS OF $|t|$ AGAIN, USE $\xi \sim |t|^{-\nu}$
TO REWRITE χ AS A FCT OF $|t|$ AND L

ONE CAN GUESS IT

$$\chi(|t|, L) \rightarrow L^{\gamma/b} \tilde{\chi}(L^{\nu/b} |t|)$$

$$L^{\gamma/b} \tilde{\chi}(L^{\nu/b} |t|)$$

WHERE WAS ξ NOW THERE L (IN FRONT) AND ALIGNMENT
CHANGE IS QUITE OBVIOUS

HOW TO MAKE A CHECK OF THIS?

PLOT $L^{-\gamma/b} \chi_L$ vs. $L^{\nu/b} |t|$ FOR DIFF L .

DO THE DATA POINTS FALL ON TOP OF EACH OTHER?

CAREFUL WITH NOTATION $\chi = \frac{\partial M}{\partial h}$ CAN BE A LOCAL QUANTITY
OR A GLOBAL ONE IF INTEGRATED OVER SPACE
THERE'S AN L^d DIFF BTW THE TWO

HOMEWORK: ON FINITE SIZE SCALING

SHOW SLIDES

THE RENORMALIZATION GROUP

- LARGE CORRELATED SCALES (ξ)
- MICROSC. DETAILS SHOULD NOT MATTER

\Rightarrow COLLECTIVE PHENOM.
SCALE INV.
UNIVERSALITY

VIA BLOCK SPINS

FLOW OF COUPLING CONSTS.

THE 1D ISING CHAIN

$$K \equiv \beta J$$

$$\mathcal{Z} = \sum_{\{S_i\}} e^{-H_K(\{S_i\})} = \sum_{\{S'_i\}} e^{-H_{K'}(\{S'_i\})}$$

SAME FCT FORM BUT \neq COUPLING STRENGTH

BOTH SUMS ON ISING VARIABLES

$$i = 1, \dots, N$$

$$I = 1, \dots, N'$$

$H_k \neq H_{k'}$ SAME FUNCT FORM BUT
DIFF COUPLING CONST.

N ODD eg $N=7$
 $N-1$ EVEN eg $N-1=6 \Rightarrow \frac{N-1}{2} = 3$

$$\mathcal{Z} = \sum_{\{s_i\}} e^{k [s_1 s_2 + s_2 s_3] + k [s_3 s_4 + s_4 s_5] + \dots}$$

THE EVEN SPINS APPEAR WITHIN THE [...] AND ONLY ONCE IN EACH

\Rightarrow SUM OVER THEIR CONFS.

$$\mathcal{Z} = \sum_{\{s_i\}} 2 \operatorname{ch} k (s_1 + s_3) 2 \operatorname{ch} k (s_3 + s_4) \dots$$

REWRITE EACH FACTOR IN THE FORM OF THE 1DIM CANONICAL WEIGHT, e.g.

$$2 \operatorname{ch} k (s_1 + s_3) = f(k) e^{k' s_1 s_3}$$

$S_1 S_3 = 1$ TWO POSSIBILITIES OUT OF
 $S_1 S_3 = -1$ $S_1 = \pm 1$ $S_3 = \pm 1$ \Rightarrow

$$2 \cosh 2k = f(k) \exp k'$$

$$2 = f(k) \exp(-k')$$

TAKE THE RATIO BTW THE TWO EXPRESSIONS

$$\cosh 2k = \exp(2k') \quad \Rightarrow$$

$$k' = \frac{1}{2} \ln \cosh 2k$$

$$f(k) = 2 \exp k' = 2 \exp \frac{1}{2} \ln \cosh 2k$$

$$f(k) = 2 (\cosh 2k)^{1/2}$$

SO WE HAVE FOUND A REWRITING SUCH
THAT

$$Z(k, N) = \left(f(k) \right)^{\frac{N-1}{2}} \quad \leftarrow \quad \# \text{ DECIM. SPINS}$$

$\uparrow \quad \uparrow$
 $\# \text{ SPINS}$

ORIGINAL COUPLING STRENGTH

$$\times \sum_{\{S_I = \pm 1\}} e^{k' \sum_I S_I S_{I+1}}$$

$$Z(k, N) = \left(f(k) \right)^{\frac{N-1}{2}} Z(k', \frac{N+1}{2})$$

RELATION BETWEEN PARTITION
FUNCTIONS

\uparrow
 $\# \text{ REMAINING SPINS}$

AFTER ONE STEP OF DECIMATION

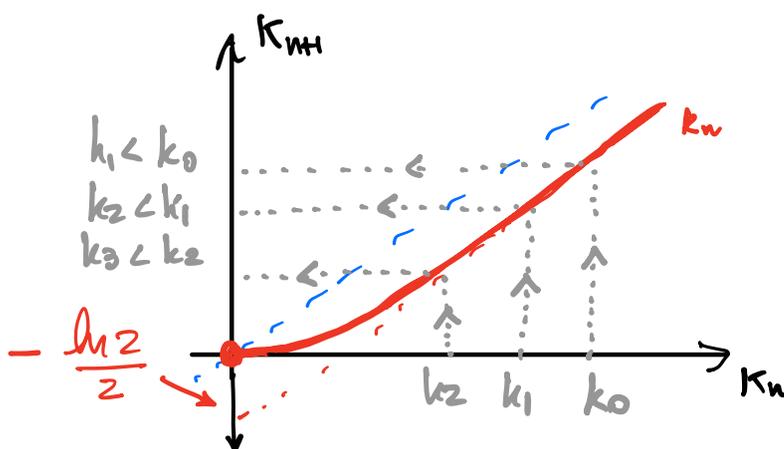
HOW IS THE EVOLUTION OF THE COUPLING
CONSTANTS GOING ?

$$k' = \frac{1}{2} \ln ch 2k$$

IT CAN BE THOUGHT OF AS A RECURRENCE

$$k_{n+1} = \frac{1}{2} \ln ch 2k_n$$

WITH n LABELING THE STEP OF THE
DECIMATION.



FUNCTIONAL
FORM OF
ITERATION

$$k_n = 0 \Rightarrow k_{n+1} = 0$$

$$k_n \rightarrow \infty \Rightarrow$$

$$\begin{aligned} k_{n+1} &\rightarrow \frac{1}{2} \ln \left(\frac{e^{2k_n}}{2} \right) = \frac{1}{2} (2k_n - \ln 2) \\ &= k_n - \frac{\ln 2}{2} \end{aligned}$$

$$\frac{d}{dk_n} \left(\frac{1}{2} \ln \left(\frac{e^{2k_n}}{2} \right) \right) = \frac{1}{2} \frac{1}{e^{2k_n}} \cdot 2e^{2k_n} = 1$$

$$\xrightarrow{k_n \rightarrow 0} 0 \quad ; \quad \xrightarrow{k_n \rightarrow \infty} 1$$

TWO FIXED POINTS $k^* = 0$ ATTRACTIVE
 $k^* \rightarrow \infty$ REPULSIVE

BACK TO THE PARTITION FUNCTION, TAKE \ln

$\ln Z = \sum N$ SHOULD BE $\propto N$
THAT IS $\sum = \Theta(1)$
(IT'S PROP OF FREE-ENERGY)

$$\ln Z(N, k) = \frac{N-1}{2} \ln f(k) + \ln Z(k', \frac{N+1}{2})$$

\uparrow # DECIMATED SPINS
 \uparrow # REMAINING SPINS

$\Rightarrow N \gg 1$

$$S(k) = \frac{1}{2} \ln f(k) + \frac{1}{2} S(k')$$

WITH $f(k)$ WRITTEN ABOVE. FROM HERE, $S(k')$ OBTAINED FROM $S(k)$ AND k .

ITERATE IN OPPOSITE DIRECTION. START FROM k' VERY SMALL \rightarrow GET $k > k'$
 (HIGH) (LOW)

APPROACH $T \rightarrow 0$ LIKE THIS & CONSTRUCT $S(k)$. EXCELLENT AGREEMENT W/ EXACT VALUES. SEE TABLE.

NOTE: WE HAVEN'T SEEN THE EFFECT OF $a \rightarrow 2a$ IN THIS CALC. BUT, IN $C(\bar{x}, \bar{y})$ WE DO.

RECAP

WE SAW SCAUNG AND
A VERY QUICK ANALYSIS
OF THE 1DIM \Rightarrow REPEAT IT

SCAUNG RELATIONS

$$f_{\text{SING}}(t, h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^\Delta}\right)$$

$$\zeta(t, h) \sim |t|^{-\nu} g_\zeta\left(\frac{h}{|t|^\Delta}\right)$$

\Rightarrow $\left\{ \begin{array}{l} \text{RELATIONS BETWEEN EXPONENTS} \\ \text{ONLY TWO INDEPENDENT EXPONENTS} \end{array} \right.$

FIRST ENCOUNTERED WITH RG IDEAS

1d IM - $h=0$

DECIMATION \Rightarrow INTEGRATE AWAY ONE
OUT OF TWO SPINS

$$K_{n+1} = \frac{1}{2} \ln ch 2K_n$$

FLOW OF COUPLING CONSTANT w/ TWO
FIXED POINTS

$$K^* = 0$$

$$T \rightarrow 0$$

STABLE

$$K^* \rightarrow \infty$$

$$T = 0$$

UNSTABLE



CRITICAL POINT

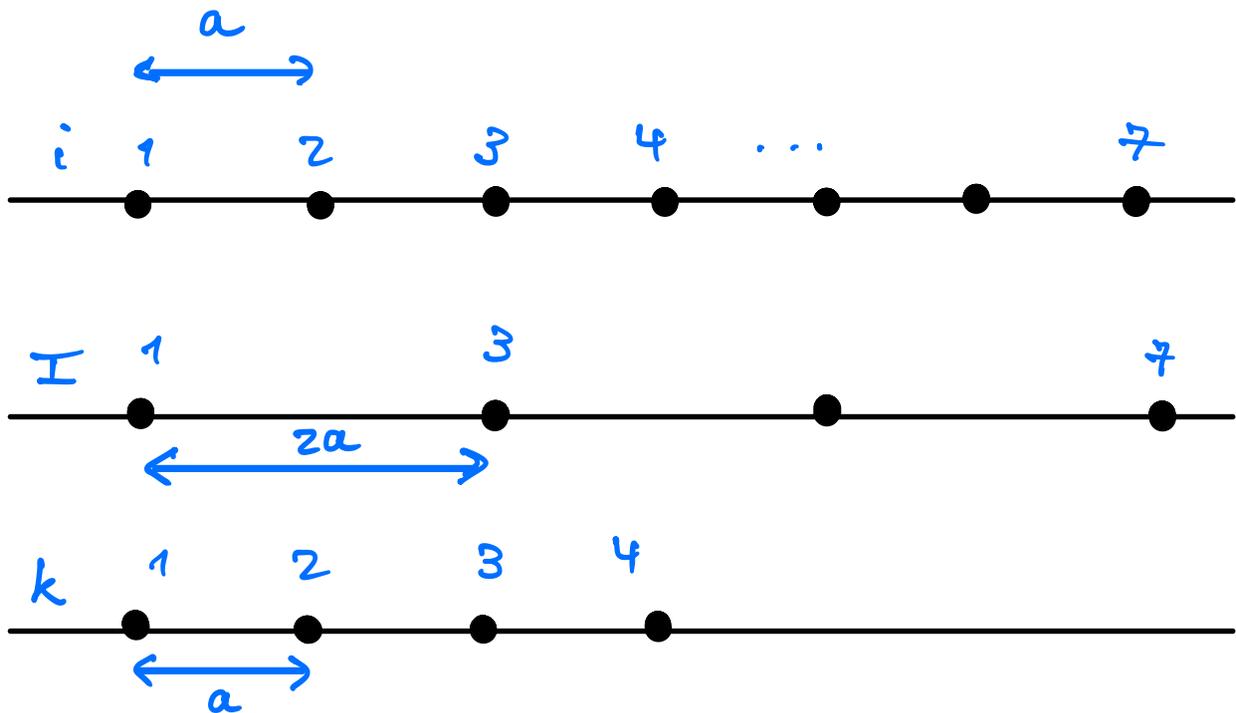
$$\mathcal{S}(k) = \frac{1}{2} \ln f(k) + \mathcal{S}(k')$$

WHERE

$$\mathcal{J}(k) = N \ln Z(k)$$

(NOTE $\mathcal{J}(k) = -k_B T F(k)$)

THE SKETCH OF TRANSFORMATIONS



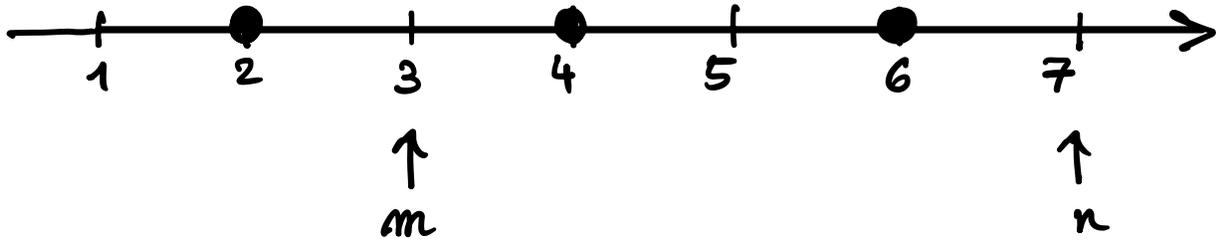
$i = 1, \dots, N$ SPINS DECIMATE $\frac{1-N}{2}$ SPINS:

$I = 1, \dots, \frac{N+1}{2}$ SPINS RELABEL REMAINING:

$k = \frac{I+1}{2} = 1, \dots, \frac{N+1}{2}$ SPINS SPACE RESCALING

$$2a \rightarrow a$$

NOTE THAT IN THE LAST TRANSFORMATION
THE DISTANCES HAVE BEEN
DIVIDED BY 2



● DECIMATED

m, n LABELS OF SPINS TO BE CORRELATED

$$\begin{aligned}
 C_{mm} &= \langle S_m S_n \rangle \\
 &= \frac{1}{Z} \sum_{\{s_i = \pm 1\}} S_m S_n e^{-\beta H} \\
 &= C(\vec{x}, \vec{y})
 \end{aligned}$$

$$\vec{x} = a m \qquad \vec{y} = a n$$

THE CORRELATION LENGTH

FROM $C(\vec{x}, \vec{y}) \sim e^{-|\vec{x}-\vec{y}|/\xi(k)}$

$[\xi(k)] = L$ LENGTH UNITS

DEPENDS ON DIMENS. PARAM k

$$C(\vec{x}, \vec{y}) = \frac{1}{Z(N, k)} \sum_{\{S_i = \pm 1\}} S_m S_n e^{+k \sum_{i=1}^N S_i S_{i+1}}$$

NOT SUMMED OVER

- WE USE THE RELATION BETWEEN PARTITION SUMS BEFORE & AFTER DECIMATION
- WE DECIMATE THE EVEN LABELED SPINS.

$$= \frac{1}{\cancel{\left(\frac{1}{Z(k)}\right)^{\frac{N-1}{2}} Z\left(\frac{N+1}{2}, k'\right)} \sum_{\substack{\{S_I = \pm 1\} \\ I \text{ ODD}}} S_m S_n \cancel{\left(\frac{1}{Z(k)}\right)^{\frac{N-1}{2}}} e^{k' \sum_{I \text{ ODD}} S_I S_{I+1}}$$

- WE RENAME $k = \frac{I+1}{2} = 1, \dots, \frac{N+1}{2}$

$$= \frac{1}{Z(N+1, k')} \sum_{\left. \begin{array}{l} S_u = \pm 1 \\ k=1, \dots, \frac{N+1}{2} \end{array} \right\}} S_{\frac{m+1}{2}} S_{\frac{N+1}{2}} e^{k' \sum_{k=1}^{\frac{N+1}{2}} S_u S_{u+1}}$$

THE CORREL FCT CAN BE COMPUTED IN THE TWO WAYS & GET THE SAME RESULT

- IN THE SYST WITH (k, N, a)
- IN THE SYST WITH $(k', \frac{N+1}{2}, 2a)$

$$C(\vec{x}, \vec{y}) \Big|_{N, k, a} = C(\vec{x}, \vec{y}) \Big|_{\frac{N+1}{2}, k', 2a}$$

SAME FUNCTIONAL FORM

CLOSE TO THE CRIT POINT EXP. DECAY
WITH A CORRELATION LENGTH ξ

$$[\xi] = L \quad \text{LENGTH DIMENSION}$$

$$\overline{\xi}(k) a = \overline{\xi}(k') 2a$$

$$\text{say it's } \xi = 10\text{cm} = \overline{\xi}(k) 1\text{cm} = \overline{\xi}(k') 2\text{cm}$$

$$\Rightarrow \overline{\xi}(k') = \overline{\xi}(k) / 2$$

$$\overline{\xi}(k') = \frac{\overline{\xi}(k)}{2}$$

HOW DOES ONE GET $\overline{\xi}(k)$ FOR $k \rightarrow \infty$?

$$\text{NOTICE THAT } k_{n+1} = \frac{1}{2} \ln \text{ch } 2k_n$$

$$\rightarrow \frac{1}{2} \ln e^{2k_n} / 2$$

$$= \frac{1}{2} (2k_n - \ln 2)$$

$$k_{n+1} - k_n = \frac{-\ln 2}{2} \quad \text{DECREASE PER ITERATION STEP}$$

SAY ONE STARTS FROM $k_0 = k$. HOW LONG IN \bar{n} DOES IT TAKE TO GET TO $k_f \sim \Theta(1)$?

$$k_f = \Theta(1) = k - \frac{\ln 2}{2} \cdot \bar{n} \Rightarrow$$

$$\bar{n} \sim \frac{2k}{\ln 2}$$

THEN $\bar{\Sigma}(k')$ AFTER \bar{n} STEPS

$$\bar{\Sigma}(k') = \frac{\bar{\Sigma}(k)}{2}$$

\Rightarrow

$$\bar{\Sigma}(k_f) = \frac{\bar{\Sigma}(k)}{2^{\bar{n}}} = \Theta(1)$$

$$\bar{\Sigma}(k) \sim 2^{\bar{n}} = 2^{\frac{2k}{\ln 2}} \sim e^{2k}$$

EXPON. DIVERGENCE OF $\bar{\Sigma}(k)$

THE LAST IDENTITY IS A CONSEQ. OF

$$2 = e^{\ln 2} \Rightarrow$$

$$2^{\frac{2k}{\ln 2}} = \left(e^{\ln 2} \right)^{\frac{2k}{\ln 2}}$$

$$= e^{\ln 2 \cdot \frac{2k}{\ln 2}} = e^{2k}$$

$$\bar{S}(k) \sim e^{2k}$$

NOTE THAT THIS IMPLIES

$$\bar{S}(k) \sim a e^{2k}$$

THE LENGTH UNITS ARE RECOVERED. WHAT'S
IMPORTANT IS THE EXP DIVERGENCE
AS $T \rightarrow 0$

th

KADANOFF BLOCK SPINS & RENORMALIZATION

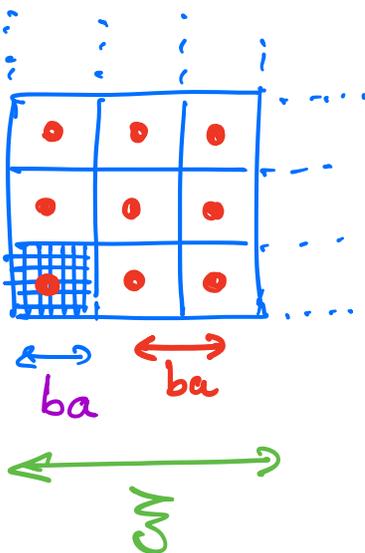
AS LANDAU, BUILD BLOCK SPINS

$$S_I' = b^{-x} \sum_{i \in B_I} s_i$$

NORMALIZATION

SINCE SOME SPINS IN B_I ARE REVERSED
THE SUM IS NOT EQUAL TO B_I

$$\Rightarrow b^{-x} \text{ WITH } x < d$$



- S_I' ISING VARIABLES
- NEW LATTICE CONST. MAGNIFIED
- $a' = ba$ (NB $b > 1$ A NUMBER)

$$\bar{\xi} = \bar{\xi} \cdot a = \bar{\xi}' a' = \bar{\xi}' ba$$

CONTR. LENGTH IN ABSOLUTE
UNITS (eg. cm)

$$\bar{\xi} = \bar{\xi}' b \Rightarrow \bar{\xi}' = \bar{\xi} / b$$

$\bar{\xi}'$ REDUCED ! IT'S AS SYST FURTHER
AWAY FROM CRITICALITY

SCALE TRANSF. TO GET BACK TO ORIGINAL

$$x'_\alpha = \frac{x_\alpha}{b} \quad \alpha = 1, \dots, d$$

BLOCK SPINS : ISING VARIABLES OCCUPYING
THE SITES OF A LATTICE W/ SAME LATTICE
SPACING a AFTER DISTANCE RE-SCALING
BUT LESS SPINS : $N' = N/b^d$

FOR THE 1D ISING CHAIN WE KNEW (CALCULATED
BY DECIMATION THE NEW HAMILTONIAN

$$N \rightarrow \frac{N+1}{2}, \quad k \rightarrow k', \quad ba = 2a \rightarrow a$$

IN GENERAL, WE DON'T KNOW HOW TO DO IT.

PROPOSE

$$H'(3S_i) = R_b H(3S_i)$$

FOR A TRANSFORMATION WITH SCALE FACTOR b

PROPERTIES

COMPOSITION

- APPLY TRANSF b & APPLY TRANSF b SHOULD BE EQUIV TO APPLYING TRANSF b^2 . IT'S MULTIPLICATIVE.

$$R_b^2 = R_b R_b \implies R_b^n = (R_b)^n$$

THIS TRANSF SHOULD HAVE Δ FIXED POINT AT CRITICALITY

$$H'(z_{SI}^*) = H(z_{SI}^*)$$

FOR $\xi \gg r \gg ba \gg a$

SAME FUNCT FORM ^w/SAME PARAMETERS
DESCRIBING LONG DIST PROP.

$$R_b H^* = H^*$$

COUPLING CONST FLOW

$$H(\{s_i\}) = - \sum_A J_A S_A$$

ALL POSSIBLE GROUPS OF SPINS

(NOTE THAT IN A GENERIC PROBLEM ONE
COULD START WITH TWO-BODY INTERACTIONS
AND GENERATE MANY-BODY ONES AFTER
DECIMATION)

$$\beta J_A = k_A \quad \Rightarrow \quad [k] = (k_1, k_2, \dots)$$

$$[k'] = R_b [k]$$

TRANSFORMATION OVER FINITE LENGTH $b \ll \infty$

ANALYTIC FUNCTION OF COUPLING CONSTANTS RG TRANSFORM

- EXPAND IN TAYLOR

$$k'_\alpha = \sum_{\beta} R_{\alpha\beta}(b) k_{\beta} + \dots$$

MEASURING k_{α} FROM CRIT POINT

MEANING $k_{\alpha} = \tilde{k}_{\alpha} - \tilde{k}_{\alpha}^c$

eg. $\frac{T-T_c}{T_c}$, βh IN FM-PM PROBLEM

$k_{\alpha}^c = 0 \quad \forall \alpha. \implies$ NO ZEROTH ORDER TERMS

$$k' = A(b) k + B(b) H + \dots$$

$$H' = C(b) k + D(b) H + \dots$$

FOR $b=1$, NO TRANSF \Rightarrow

$$\begin{aligned} A(1) &= 1 & B(1) &= 0 \\ C(1) &= 0 & D(1) &= 1 \end{aligned}$$

USE SYMM. TO KILL SOME TERMS
eg. $\int \sin x \} s_i \rightarrow -s_i \} \int H \rightarrow -H \}$
 $\} k \rightarrow k \}$

$$\Rightarrow \begin{aligned} B(b) &= 0 \\ C(b) &= 0 \end{aligned}$$

$$K' = A(b)K + \dots$$

... h.o.t.

$$H' = D(b)H + \dots$$

FINALLY, IMAGINE YOU APPLY TWO TRANSF
IN A ROW W/ SCALE b . SHOULD BE THE
SAME AS APPLYING A TRANSF WITH $b \cdot b$

$$A(b)A(b) = A(b^2)$$

$$D(b)D(b) = D(b^2)$$

\Rightarrow

$$A(b) = b^{\gamma_k}$$

$$D(b) = b^{\gamma_H}$$

$$\gamma_k > 0$$

$$\gamma_H > 0$$

TO MOVE AWAY FROM CRIT
w/ TRANSF. PERFORMED

IN SUMMARY

$$\begin{array}{l} k' = b^{\gamma_k} k \\ H' = b^{\gamma_H} H \end{array}$$

DROPPING h.o.t.

NOW, IN FREE ENERGY

$$F(k', H') = F(k, H) \quad \text{AT CRITICALITY}$$

$$N' f(k', H') = N f(k, H)$$

$$f(k, H) = \left(\frac{N'}{N} \right) f(k', H')$$

$$f(k, H) = b^{-d} f(b^{j_k} k, b^{j_H} H)$$

HOMOGENEITY PROP. PROVEN!

CHOOSE NOW $b = k^{-1/j_k}$

$$f(k, H) = k^{d/j_k} f\left(1, \frac{H}{k^{j_H/j_k}}\right)$$

$$\Delta \equiv \frac{j_H}{j_k} \quad \frac{d}{j_k} = 2 - \alpha$$

$$f(k, H) = k^{2-\alpha} g_f\left(\frac{H}{|k|^4}\right)$$

WE FOUND THE SCALING FORM.

RELATION BETWEEN PHYSICAL EXPONENTS
AND THE POWERS IN PARAM TRANSF

$$\frac{\xi(R_b(k))}{\xi(k)} = \frac{1}{b}$$

RECALL 1DIM
CALCULATION
& ARGUMENT

ASSUMING $\xi \sim (k - k_c)^{-\nu}$ POWER LAW DIVERGENCE

$$\begin{aligned} \frac{1}{b} &= \left(\frac{R_b(k) - R_b(k^*)}{k - k^*} \right)^{-\nu} = \left(\frac{R_b(k) - k^*}{k - k^*} \right)^{-\nu} \\ &= \left(\left. \frac{dR_b(k)}{dk} \right|_{k=k^*} \right)^{-\nu} \end{aligned}$$

FROM HERE,

$$-\ln b = -\nu \ln \left. \frac{dR_b}{dk} \right|_{k=k^*}$$

$$\nu = \frac{\ln b}{\ln \left. \frac{dR_b}{dk} \right|_{k=k^*}}$$

FOR THE POWER LAW $R_b \sim b^{\gamma_k} k$

$$\frac{dR_b(k)}{dk} = \frac{d}{dk} (b^{\gamma_k} k) = b^{\gamma_k}$$

$$\ln \frac{dR_b}{dk} = \gamma_k \ln b$$

\Rightarrow

$$v = \frac{1}{\gamma_k}$$

THE TRANSF OF $\xi \Rightarrow$

POWER LAW DIVERGENCE OF ξ

$$\bar{\xi}(k_n) = \bar{\xi}(b^n \gamma_k k)$$

REPLACING k_n

$$= b^{-n} \bar{\xi}(k)$$

USING FLOW AWAY
FROM CRITICALITY

MULTIPLYING BY b^n

$$b^n \bar{\xi}(b^n \gamma_k k) = \bar{\xi}(k)$$

CHOOSING $b^n = \left(\frac{b}{k}\right)^{\frac{1}{\gamma_k}} = (b k^{-1})^{\frac{1}{\gamma_k}}$

ONE HAS $b^{1/y_k} k = (b k^{-1})^{1/y_k} k = b$

∑ REPLACING ABOVE

$$(b k^{-1})^{1/y_k} \bar{\xi}(b) = \bar{\xi}(k)$$

ISOLATING THE k DEPENDENCE (THE REST IS
A NUMBER)

$$\bar{\xi}(k) \sim k^{-1/y_k}$$

∑ SHOWED POWER LAW DIVERGENCE

WITH

$$\nu = 1/y_k$$

GENERAL

$$[k'] = R_b [k]$$

TRANSF PARAM.

$$\sum(k') = \frac{\sum(k)}{b}$$

TRANSF CORR LENGTH

FIXED POINT FROM CORR LENGTH

$$\sum^* \rightarrow 0$$

or

$$\sum^* \rightarrow \infty$$

TRIVIAL

INTERESTING

STABLE

UNSTABLE UNBOD
FLOW

ATTRACTIVE

REPULSIVE

CLOSE TO THE FIXED POINT

$$[k'] = R_b [k] \Rightarrow$$

$$K'_\alpha(b) - k_\alpha^* = R_{\alpha\beta}(b) (k_\beta - k_\beta^*)$$

LINEAR FORM CLOSE TO $[k^*]$

$$\delta K'_\alpha(b) = R_{\alpha\beta}(b) \delta k_\beta$$

$$R_{\alpha\beta}(b) = \left. \frac{\partial K'_\alpha(b)}{\partial k_\beta} \right|_{[k^*]} \text{ AT}$$

FIXED
POINT

MATRICIAL FORM

$$\delta K' = R \delta k$$

DIAGONALIZE WITH $U U^T = U^T U = \mathbb{1}$

$$U \delta k' = \underbrace{U R U^T}_D U \delta k$$

$$\delta \kappa'(b) = D(b) \delta \kappa \quad \begin{array}{l} \text{KAPPA} \\ \text{ROTATED} \\ \text{BY } U \end{array}$$



DIAGONAL MATRIX WITH
EIGENVALUES IN DIAGONAL

FROM TRANSF COMPOSITION RULE
(MULTIPLICATIVE)

$$\lambda_i(b) = b^{y_i} \quad \Rightarrow \quad y_i = \frac{d \ln \lambda_i}{d b}$$

DERIVATIVE JUSTIFIED SINCE b CONT.
& CLOSE TO 1 ASSUMED.

THE TRANSF OF PARAM. YIELDS $\lambda_i(b)$
 $\Rightarrow y_i$

\Rightarrow DIVERGENCE OF
OBSERV. eg

$$\xi(k) \sim k^{-y_k}$$

CLASSIFICATION OF PARAMETERS

- $y_i > 0 \Rightarrow$ RELEVANT (INCREASES)
- $y_i < 0 \Rightarrow$ IRRELEVANT (VANISHES)
- $y_i = 0 \Rightarrow$ MARGINAL (h.o.t.)

COMMENTS

- RG TRANSFORM MOVES WITHIN A PHASE
- REPULSIVE FIXED POINTS \Rightarrow CRIT. POINTS (ξ DECREASES UNDER RG)
 - IN DISORDERED PHASE RG TO $T^* \rightarrow \infty$
 - IN ORDERED PHASE RG TO $T \rightarrow 0$

- Check β factor in H
- POTS FACTEUR 2
- BIBLIOG FINI TROP VITE