Advanced Statistical Physics

Exam

9th of January, 2023

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English or French, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but also the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The answers must be written neatly within the boxes.

The problems roughly follow the order of the chapters in the Lecture Notes but are not necessarily of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

Ergodicity

Take a time-dependent process given by $x(t) = \cos(\omega t + \theta)$ with the angle θ uniformly distributed $P(\theta) = 1/(2\pi)$ in $[0, 2\pi]$.

1. Focus on the average of this process. Is it compatible with the process being ergodic?
2. But, is the full process ergodic? Hint: think about other functions of the process x.

The ferromagnetic Ising p-spin model

Take the so-called ferromagnetic p Ising spin model

$$H_J[\{s_i\}] = -\frac{J_0}{p} \sum_{i_1 \neq \dots \neq i_p} s_{i_1} \dots s_{i_p} \tag{1}$$

with $i = 1, \dots, N$ classical Ising spins, $s_i = \pm 1$, and $J_0 > 0$. The sum runs over all possible groups of integer p spins. For example, if $p = 3$, the terms in the sum are of the form $s_1 s_2 s_3$, $s_1 s_4 s_5$, etc. We will consider the statistical properties of this model coupled to an equilibrium bath at temperature T.

1. How do you recover the standard fully connected Ising model with pair interactions?

2. How do you render the energy (1) extensive for any finite integer p?

3. Which are the adimensional parameters that control the equilibrium properties of this model?

4. Which condition should the ground state/s satisfy?

5. Which is/are the ground state/s?

6. Which configurations should be the most favourable at $T \to \infty$?

7. Give a necessary condition to allow for a phase transition.

8. For fixed and finite p, do you expect to find a phase transition? As a function of which parameter?

9. Which order parameter would you propose?

10. Write the definition of the partition function.

11. Write the definition of the free-energy density.

12. In the large N limit, find the expression of the free-energy density as a function of the control parameters and a variable which plays the role of your chosen order parameter. Use the method that you prefer to derive this expression but explain clearly the steps that you follow and how you derive it.

13. Draw, side-by-side, this free-energy density for $p = 2$ and several representative values of the control parameter, at and on both sides of the phase transition, if you think there will be one.

14. Draw this free-energy density for $p = 3$ and several representative values of the control parameter.

15. What do you observe? Discuss similarities and differences between the cases $p = 2$ and $p = 3$.

16. Explain to which quantities are the exponents β and γ associated to.

17. Evaluate the critical exponents β and γ when applicable in the p spin model.

Scaling

Figure 1 (a) displays the inverse shear viscosity η^{-1} as a function of the shear stress σ of a system of particles confined in a box, with different global densities ρ given in the key, close to its jamming transition.

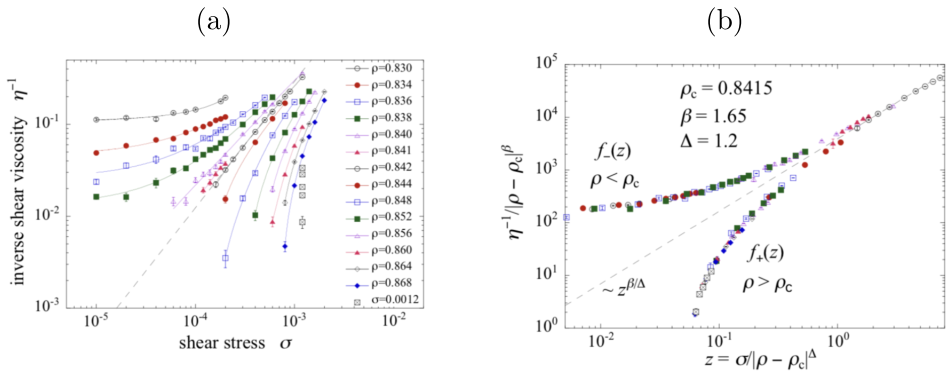


Figure 1: (a) Inverse shear viscosity in a jamming system. (b) Analysis of the data in (a).

1. In your opinion, which is the order of this phase transition?

2. Explain what has been done in panel (b) in the same figure.

3. Give the meaning of ρ_c, β and Δ.

The $2d$ Ising model

Figure 2 shows three representative snapshots of the $2d$ Ising model

$$H = -\frac{J}{2}\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_i \tag{2}$$

in equilibrium. The spins $s_i = 1, -1$ are represented with dark and white dots in the images. The first term represents two-body nearest neighbour interactions on the lattice (the factor $1/2$ is there to ensure that each pair interaction is summed only once). The coupling strength is positive, $J > 0$. The second term represents the effect of a uniform applied field which for the snapshots equals zero. The system is coupled to an equilibrium bath at three representative temperatures T.

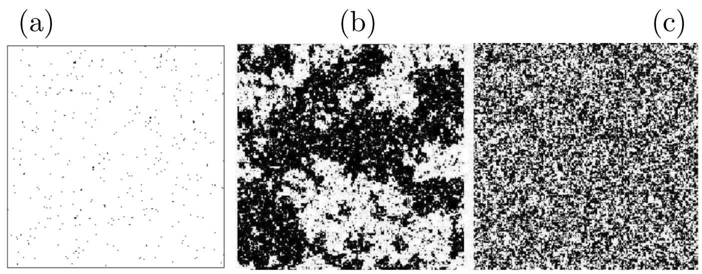


Figure 2: Snapshots of the $2d$ Ising model in equilibrium at three representative temperatures. No magnetic field is applied, $h = 0$. The $+1$ and -1 spins are shown in black and white.

1. According to you, how does the temperature compare to the critical one in the three snapshots (a), (b) and (c)?

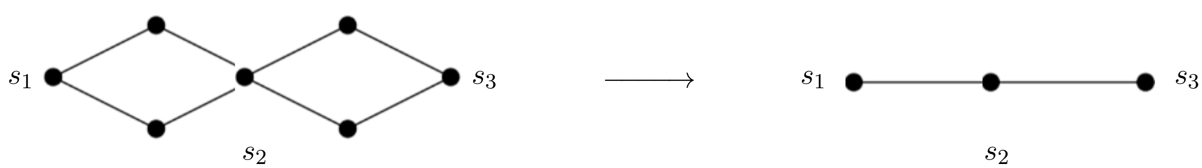
2. Draw, on the figures above, your estimate for the correlation length. Give the definition of the correlation length and justify your answer.

3. Explain what a Kadanoff block transformation would do to these configurations, and which would be the configurations after several such transformations in each of the three cases.

4. When the model is defined on a hierarchical lattice, the flow equations for the adimensional couplings $K = \beta J$ and $\Delta = \beta h$ read

$$
\begin{aligned}
K' &= \frac{1}{2}\ln\left[\frac{\cosh(2K+\Delta)\cosh(2K-\Delta)}{\cosh^2\Delta}\right], &(3)\\
\Delta' &= \Delta + \frac{1}{2}\ln\left[\frac{\cosh(2K+\Delta)}{\cosh(2K-\Delta)}\right]. &(4)
\end{aligned}
$$

Explain how would you proceed to obtain these equations (without doing the full calculation) by decimation of the following structure:



Call the central spin s_2 in both cases, and the unnamed in the figure spins, s_4, s_5, s_6, s_7.

5. Consider the case $\Delta = 0$ and study (graphically) the fixed points of the renormalisation group transformations (3)-(4) for the coupling K together with their stability.

6. Is there a finite temperature phase transition in the Ising model on the hierarchical lattice?

7. What is the scale numerical factor b of the transformation in the figure above that was used to find the transformation rules (3)-(4)? Hint: think in terms of number of degrees of freedom for each rhombus.

8. Still at $\Delta = 0$, give the expression for the critical exponent ν in this formalism and, next calculate it for the Ising model on the hierarchical lattice.

9. Linearize the flow equations for $0 < h \ll 1$ and sketch the RG flow in the full (K, Δ) plane. (This question may take you some time, you can leave it for the end.)

10. Write the Landau free energy density for the phase transition in this universality class. Explain the meaning of each term and give all necessary definitions.

11. Which is the criterium used to estimate the limit of validity of the saddle-point approximation in the Landau approach?

12. Describe which quantities should be calculated (without calculating them explicitly) to determine this limit of validity.

13. Give an idea of the constraints that this criterium imposes.

Quantum statistical physics

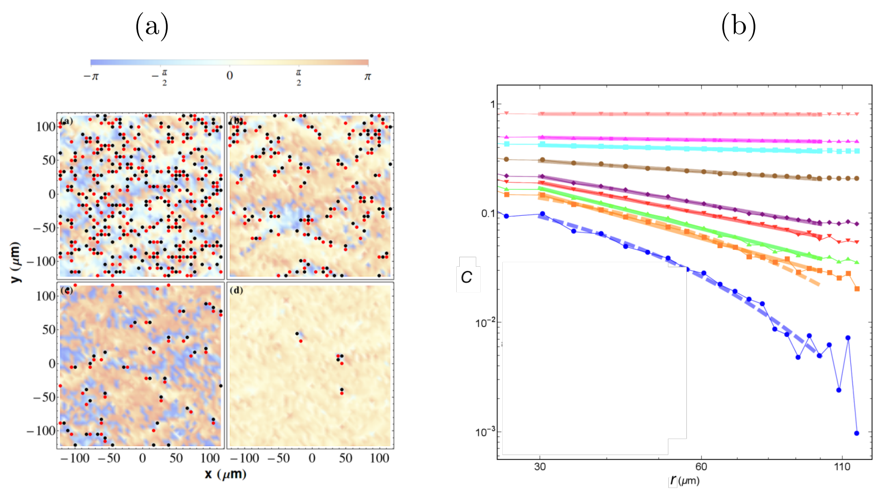


Figure 3: (a) Map of the phase of the complex field $\psi(\mathbf{r})$ according to the scale in the upper bar. The black and red dots are located at the singularities of ψ. (b) decay with distance of the field-field correlation function for increasing values of f_p from bottom to top.

Figure 3(a) shows the phase of a complex field $\psi(\mathbf{r})$ in the steady state of a two dimensional strongly driven and highly dissipative interacting many-body quantum light-matter system.[1] These are numerical results, obtained with periodic boundary conditions. Physical examples are polaritons in semi-conductor microcavities or cold atoms in optical cavities. The color map corresponds to the scale in the upper bar. Black and red dots are located at singularities of the field. The four panels correspond to increasing values of the drive or pump intensity, which we call f_p.

Figure 3(b) presents the decay with distance r of a field-field correlation function which we call C. The curves correspond to different values of f_p increasing from bottom to top.

Explain, in detail, what you conclude about the behaviour of this system as f_p is varied.

[1] G. Dagvadorj, J. M. Fellows, S. Matyjaśkiewicz, F. M. Marchetti, I. Carusotto, and Szymańska, Phys. Rev. X 5, 041028 (2015)

Quantum - classical mapping

Take the quantum Ising chain

$$\hat{\mathcal{H}} = -J\sum_{i=1}^{L}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z - \Delta\sum_{i=1}^{L}\hat{\sigma}_i^x \tag{5}$$

1. What is the ground state at $\Delta = 0$?

2. What is the ground state at $J = 0$?

3. Do the first and second terms in Eq. (5) commute?

4. The Suzuki-Trotter formula is

$$e^{-\beta(\hat{A}+\hat{B})} = \lim_{L\to\infty} \left(e^{-\frac{\beta}{L}\hat{A}} \, e^{-\frac{\beta}{L}\hat{B}} \right)^L \tag{6}$$

Use it to transform the quantum partition function of the model defined by the Hamiltonian (5) at an inverse temperature β_q to a classical one. Discuss what you find.

5. Do you know another method to obtain the energy spectrum, and in particular the ground state energy, of this model analytically?