Advanced Statistical Physics

TD7: Quantum Ising model

December 2022

We consider the quantum Ising model in d dimensions made by N spins 1/2 interacting via ferromagnetic interactions in presence of a transverse field:

$$\hat{\mathcal{H}} = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - \Gamma \sum_i \hat{\sigma}_i^x \, ,$$

where $\langle i,j \rangle$ denotes all pairs of nearest neighbor on a d-dimensional lattice and $\hat{\sigma}_i^{x,y,z}$ are the Pauli matrices.

I. **Ground state properties**

(a) What is the energy of the ground state when $J = 0$? Is the ground state degenerate? Determine the expectation values of $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_z \rangle$ in this ground state.

(b) What is the energy of the ground state when $\Gamma = 0$? Is the ground state degenerate? Determine the expectation values $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_z \rangle$ in this ground state.

(c) Let us now restore both couplings J and Γ. Plot the qualitative behavior of $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_z \rangle$ as a function of Γ for $J = 1$.

II. **Mean-field approach à la "Curie-Weiss"**

(a) In the mean-field approach one replaces the instantaneous fluctuating fields by their average values. Consider a site i of the lattice and its neighbors $j = 1, \ldots, 2d$. We introduce the average magnetization per spin:

$$m = \frac{1}{N} \sum_i \langle \hat{\sigma}_i^z \rangle \, .$$

Replace the operators $\hat{\sigma}_j^z$ by their expectation values, and write the mean field Hamiltonian as:

$$\mathcal{H}_{\rm mf} = - \sum_i \vec{h}_{\rm mf} \cdot \vec{\sigma}_i \, .$$

Give the expression of the effective field $\vec{h}_{\rm mf}$ in terms of d, J, Γ, and m.

(b) What is the energy of the ground state of $\mathcal{H}_{\rm mf}$? Determine the expectation values $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_z \rangle$ in the ground state and find the self-consistent equation for m.

(c) Find the solution of the self-consistent equation for m and show that there exist a critical value of the transverse field Γ_c above which the only solution of this equation is $m = 0$.

(d) We now investigate the effect of thermal fluctuations. Find the self-consistent equation for m at finite T and express it in terms of m, Γ, Γ_c, and β only.

(e) Find the equation that gives the critical value of the field $\Gamma_c(\beta)$ above which the only solution of the self-consistent equation for m is $m = 0$ as a function of Γ_c and β and draw the mean-field phase diagram of the model.

III. **Fully-connected mean-field model**

We consider the mean-field fully-connected p-spin ferromagnet in a transverse field defined by the following Hamiltonian:

$$\hat{\mathcal{H}} = -\frac{J}{N^{p-1}} \sum_{i_i \neq i_2 \neq \ldots \neq i_p} \hat{\sigma}^z_{i_1} \hat{\sigma}^z_{i_2} \cdots \hat{\sigma}^z_{i_p} - \Gamma \sum_i \hat{\sigma}^x_i \, .$$

The N dependency of the coupling constant is chosen to ensure the extensivity of the model. We introduce the operators

$$\hat{m}_{x,y,z} = \frac{1}{N} \sum_i \hat{\sigma}^{x,y,z}_i \, . \tag{1}$$

(a) Find the commutation relation between the operators $\hat{m}_\alpha$.

(b) Write the Hamiltonian in terms of $\hat{m}_z$ and $\hat{m}_x$ only.

(c) Write the partition function of the model and use the Suzuki-Trotter formula to disentangle the two non-commuting terms in the Hamiltonian.

(d) Thanks to the mean-field character of the model we can reduce the problem to a single-spin one by defining $m(\tau) = \frac{1}{N}\sum_{i=1}^N \sigma_i(\tau)$, where $\sigma_i(\tau) = \pm 1$ are the eigenvalues of $\hat{\sigma}^z_i$ at the imaginary time τ, and imposing this identity by adding the exponential representation of the Dirac distribution at each time step. Write the partition function obtained after these manipulations.

(e) We make the natural assumption that the dominant contribution comes from the trajectories that are constant in imaginary time. Evaluate the partition function using the saddle-point method. Find the stationary conditions and the free-energy per spin.

(f) In the following we consider the case $p = 2$. Find the solution of the self-consistent equation for $\langle \hat{m}_z \rangle$ at $T = 0$ and show that there exists a critical value of the transverse field Γ_c above which the only solution of this equation is $\langle \hat{m}_z \rangle = 0$.

(g) Find the self-consistent equation for m at finite T, and determine the critical value of the field $\Gamma_c(\beta)$.