Statistical Physics – M2 – ICFP condensed matter

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TD 7 : Quenched randomness

The objective of this session is to better understand the different mechanisms that can trigger disorder in a given system, such as introduction of topological defects, dilution, random magnetic field, frustration and also detachment/emergence of domain walls and surface fluctuations.
For more details, see also References below [1].

Exercise 1 : diluted ferromagnet

Let us consider a spin lattice and assign to each site a certain probability p_i to be occupied or empty. If $p_i = 1\ \forall i$, we recover the Ising model back, otherwise if $p_i < 1$ we end up with a diluted model. We define a diluted system on a cubic lattice by the Hamiltonian :

$$H = -\sum J_{ij}\sigma_i\sigma_j + h\sum \sigma_i \tag{1}$$

where $\sigma_i = \pm 1$ and $J_{ij} = 1$ with probability p and 0 with probability $(1-p)$ respectively. For $h \neq 0$ there is no phase transition, whereas for $h = 0$ we can define the evolution in temperature $T_c(p)$ as a function of p.

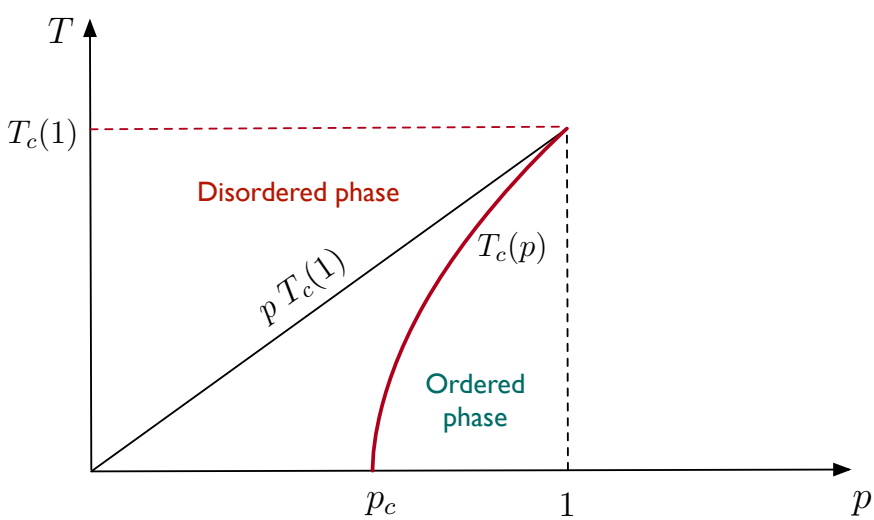


FIGURE 1 – Phase diagram of a diluted ferromagnet.

Clearly we recover : $T_c(1) = T_c^{\text{Ising}} = T_c^{\text{pure}}$. Furthermore,

1. R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969) ;
A. J. Bray, Phys. Rev. Lett. **60**, 720 (1988) ;
M. Randeria, James P. Sethna *et al.*, Phys. Rev. Lett. **54**, 1321 (1985).

— If $p < p_c$ (probability of link percolation), $T_c(p) = 0$;
— $T_c(p) < pT_c(1)$;
— In the disordered phase with $T > T_c(p)$ and $T < T_c(1)$, Griffiths singularities can occur leading to a divergent high-temperature expansion (even without undergoing a true phase transition).

These features for the different phases are summarized in Fig. 1 below.

1) What should you expect for the probability of site occupancy p ? What are the observables that might depend on p ?

In a randomly diluted ferromagnet (with dilution p) only a fraction of the sites are occupied with Ising spins. Interactions exist only between neighboring pairs of occupied sites. The probability p of occupation of a given site is independent of h and T as well as of the occupation of other sites. Conversely, it is fairly well established that $T_c(p)$ for a dilute ferromagnet varies with p.

2) Considering the Hamiltonian in Eq. (1), how can you define site dilution and link dilution respectively ?

$$H = -\sum_{i,j} \sigma_i J_{ij} \sigma_j + h \sum_i \sigma_i \tau_i \tag{2}$$

Site dilution defined by :

$$J_{ij} = \begin{cases} J\tau_i\tau_j & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

And

$$\tau_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases} \tag{4}$$

Link dilution defined by :

$$H = -\sum_{i,j} \sigma_i J_{ij} \sigma_j + h \sum_i \sigma_i \tag{5}$$

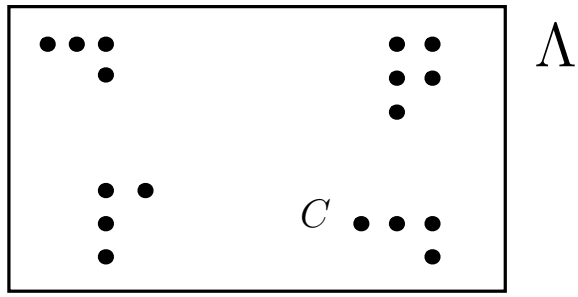
where

$$J_{ij} = \begin{cases} 0 & \text{if} \quad |i-j| > 1 \\ 1 & \text{with prob. } p, |i-j| = 1 \\ 0 & \text{with prob. } 1-p, |i-j| = 1 \end{cases} \tag{6}$$

3) We aim to show that upon increasing the system size increases with $T < T_c(1)$ – but not too small – the imaginary axis in the complex plane of h contains a singularity at $h = 0$. However, the spontaneous magnetization does not display any jump (indeed the magnetization could be C^∞ in h at $h = 0$).

Let us consider site dilution, *i.e.* j is occupied only if $\tau_j = 1$, otherwise the site is empty. Then we call :

— $C \equiv$ arbitrary configuration of occupied sites in the lattice region Λ ;
— $|C| \equiv$ number of sites belonging to C ;
— $P_{C,\Lambda} \equiv$ probability of occurrence of C, namely probability of obtaining C by site percolation ;
— $M_\Lambda \equiv$ average magnetization per site in Λ ;
— $M_C \equiv$ average magnetization per site in C.



Given the definitions above, what is the average magnetization per site in Λ (see Fig. below) ?

$$M_\Lambda = \frac{1}{|\Lambda|} \sum_{C \in \Lambda} M_C \, |C| \, P_{C,\Lambda} \tag{7}$$

We would like now to make the computation more explicit. Let us call $[C]$ all possible configurations and $\{C_p\}$ the subset of all possible configurations obtained by filling sites with probability p :

$$\sum_{C \in \Lambda} = \sum_{[C]_\Lambda} = \sum_{U_{p \in [0,1]} \{C_p\}_\Lambda} \tag{8}$$

where

$$P_{C,\Lambda} = \frac{1}{N_{\{C_p\}}} \sum_{\{C_p\}} \delta[C - C_p] \tag{9}$$

Using Eqs. (8)-(9), we can write

$$M_\Lambda = \frac{1}{|\Lambda|} \sum_{[C]_\Lambda} M_C |C| \frac{1}{N_{\{C_p\}}} \delta[C - C_p] = \frac{1}{N\{C_p\}} \frac{1}{|\Lambda|} \sum_{\{C_p\}} M_{C_p} |C_p| \; . \tag{10}$$

We also have for the probability :

$$\frac{1}{|\Lambda|} \sum_{[C]_\Lambda} |C| P_{C,\Lambda} = \frac{1}{|\Lambda|} \sum_{[C]} |C| \frac{1}{N_{\{C_p\}}} \sum_{\{C_p\}} \delta[C - C_p] = \frac{1}{|\Lambda| N_{\{C_p\}}} \sum_{\{C_p\}} |C_p| = \frac{|C_p|}{|\Lambda|} = p \; . \tag{11}$$

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4) If we introduce the weight $z = e^{-2\beta h}$, how would you express the average magnetization restricted to C in terms of the free energy f_C ? (Suggestion : use the thermodynamic relation between the order parameter and the free energy.)

$$f_C = -\frac{1}{\beta |C|} \log Z_C \tag{12}$$

Therefore, we can write the average magnetization in C as

$$M_C = \frac{\partial f_C}{\partial h} = \frac{\partial f_C}{\partial z} \frac{\partial z}{\partial h} = -2\beta z \frac{\partial f_C}{\partial z} \tag{13}$$

By using the following identity $e^{-\beta h \sigma_i} = e^{\beta h} e^{-\beta h(\sigma_i+1)}$ and summing over all values of σ_i in C, we can immediately recognize that $e^{-\beta h|C|} Z_C$ is a polynomial function in z of degree $|C|$.

In this way, $Z_C \propto z^{-\frac{|C|}{2}} \prod_{\alpha=1}^{|C|} (z - \zeta_\alpha(C))$ where ζ_α stands for the $\alpha-$th zero of the partition function Z_C.

Thanks to the Lee-Yang theorem we also have $|\zeta_\alpha(C)| = 1$, $\forall \alpha$.

From Eq. (13) then :

$$\begin{aligned} M_C =& -2\beta z \frac{\partial f_C}{\partial z} = -2\beta z \frac{\partial}{\partial z}\left(-\frac{1}{\beta|C|}\right) \log\left[\text{const.} z^{-\frac{|C|}{2}} \prod_{\alpha=1}^{|C|}(z-\zeta_\alpha(C))\right] = \\ =& \frac{2z}{|C|}\left[-\frac{|C|}{2z} + \sum_{\alpha=1}^{|C|} \frac{1}{z-\zeta_\alpha(C)}\right] = -1 + \frac{2z}{|C|}\sum_{\alpha=1}^{|C|}\frac{1}{z-\zeta_\alpha(C)} \end{aligned} \tag{14}$$

Plugging this expression in Eq. 10, we eventually get

$$\begin{aligned} M_\Lambda =& -p + \frac{1}{|\Lambda|}\sum_{C\subset\Lambda} 2z\, P_{C,\Lambda} \sum_{\alpha=1}^{|C|} \frac{1}{z-\zeta_\alpha(C)} = \\ =& -p + 2z\frac{1}{|\Lambda|}\sum_{a=1}^{N_\Lambda} \sum_{C:\zeta_\alpha(C)=\zeta_a} P_{C,\Lambda}\, m_a(C) \frac{1}{z-\zeta_a} \end{aligned} \tag{15}$$

We are basically summing first over all configurations with a given zero a times its multiplicity, and then over all the other different zeros.

$$M_\Lambda = -p + 2z \sum_{a=1}^{N_\Lambda} \eta_a \frac{1}{z-\zeta_a} \tag{16}$$

where we have used

$$\eta_a = \frac{1}{|\Lambda|} \sum_{C:\zeta_\alpha(C)=\zeta_a} P_{C,\Lambda}\, m_a(C) > 0 \tag{17}$$

$m_a(C)$ being for the multiplicity of ζ_a.

5) Optional exercise.

What can you claim about the average magnetization in the domain ? Is it bounded ? Using the expressions for M_Λ and $\eta_a(\Lambda) \equiv \frac{1}{|\Lambda|}\sum_{C:\zeta_\alpha(C)=\zeta_a} P_{C,\Lambda}\, m_a(C)$ used before, analyze M_Λ as a function of z.

$$\sum_{a=1}^{N_\Lambda} \eta_a(\Lambda) = \frac{1}{|\Lambda|}\sum_{C\subset\Lambda}\sum_{\alpha=1}^{|C|} P_{C,\Lambda} = \frac{1}{|\Lambda|}\sum_{C\subset\Lambda} P_{C,\Lambda}|C| = p \tag{18}$$

To prove that $M_\Lambda(z)$ is uniformly bounded in Λ, we consider first $|z| \neq 1$ and a circular domain of radius $r < 1$.

$$|M_\Lambda| = \left|-p + 2z\sum_{a=1}^{N_\Lambda}\eta_a \frac{1}{z-\zeta_a}\right| \leq p + 2|z|\sum_{a=1}^{N_\Lambda}\eta_a\frac{1}{|z-\zeta_a|} \leq p + \frac{2pr}{1-r} \tag{19}$$

On the other hand, if $|z| > 1$ we should consider a circle of radius $1/r > 1$. Therefore :

$$\begin{aligned} |M_\Lambda| =& p + 2|z| \sum_{a=1}^{N_\Lambda} \eta_a \frac{1}{|z - \zeta_a|} \leq p + 2|z| \sum_{a=1}^{N_\Lambda} \eta_a \frac{1}{||z| - |\zeta_a||} = \\ =& p + \frac{2|z|p}{|z| - 1} = p + \frac{2p}{1 - \frac{1}{|z|}} \leq p + \frac{2p}{1 - r} \ . \end{aligned} \tag{20}$$

We can conclude that

$$|M_\Lambda| \leq p + \frac{2p}{1 - r} \quad \forall \, |z| \neq 1 \ , r < 1 \ . \tag{21}$$

Then, for any positive z and $z \neq 1$, the average magnetization M_Λ converges to a finite value as Λ increases : this is due to the fact that the thermodynamic limit of the quenched free energy exists and is well-defined for positive z with $M_\Lambda(z) = -2\beta z \frac{\partial f_\Lambda}{\partial z}$.

Exercise 2 : Ising model in a random magnetic field

In the following we shall consider a simple Ising model to show the equivalence – close to the critical point – between a spin system in d-dimensions in a random magnetic field and a spin system in $(d-2)$-dimensions without field.

1) How would you define the free energy of the model with Lagrangian $\mathcal{L}(x) = -\frac{1}{2}\phi(x)\Delta\phi(x) + V(\phi(x))$ and averaged over the random field ? For the computation, assume the field to be Gaussian distributed.

$$F[h] = \log \int \delta\phi \exp\left\{ - \int d^d x \left[\mathcal{L}(x) + h(x)\phi(x)\right] \right\} \tag{22}$$

Integrating over the field then

$$F = \int dh\ F[h] \exp\left\{ -\frac{1}{2} \int d^d x\ h^2(x) \right\} \tag{23}$$

where $\mathcal{L}(x)$ is the Lagragian operator defined in the text.

2) If we take advantage of a tree-like diagram approximation, what would the expression for $F[h]$ look like ?

$$F[h] \sim \log \left[e^{-\mathcal{S}(\phi_0, h)} \right] \tag{24}$$

where the action $\mathcal{S}$ is such that $\frac{\delta \mathcal{S}}{\delta \phi} = 0$, *i.e.* $-\Delta\phi_0 + h + V^{'}(\phi_0) = 0$.
In the following we will assume $\phi_0 = 0$ as $h \to 0$, so that the solution will be unique.

3) If you consider again the tree-like approximation, how should you write the corresponding expression for the correlation $\langle \phi(x)\phi(0)\rangle_h$ and for $\langle \phi(x)\phi(0)\rangle$?

$$\langle \phi(x)\phi(0)\rangle_h = \frac{\int \delta\phi\ \phi(x)\phi(0) e^{-\mathcal{S}(\phi,h)}}{\int \delta\phi\ e^{-\mathcal{S}(\phi,h)}} \sim \frac{\phi_0(x)\phi_0(0) e^{-\mathcal{S}_0(\phi_0,h)}}{e^{-\mathcal{S}_0(\phi_0,h)}} = \phi_0(x)\phi_0(0)\ . \tag{25}$$

To get rid of h, we eventually integrate over the Gaussian field

$$\langle \phi(x)\phi(0)\rangle = \int \delta h\ \phi_0(x)\phi_0(0) \exp\left[-\frac{1}{2} \int d^d x\ h^2(x) \right]\ . \tag{26}$$

We introduce now a δ-function via

$$1 = \int \delta\left[-\Delta\phi + V^{'}(\phi) + h \right] \left| \det\left(-\Delta + V^{''}(\phi) \right) \right|\ \delta\phi \tag{27}$$

which allows us to safely replace ϕ_0 with ϕ. We assume that we can neglect the sign of the determinant (always non-zero), so that the correlation function becomes

$$\langle \phi(x)\phi(0)\rangle \sim \int \delta\phi\delta h\ \phi(x)\phi(0)\ \delta\left[-\Delta\phi + V^{'}(\phi) + h \right] \det\left(-\Delta + V^{''}(\phi) \right) \exp\left[-\frac{1}{2} \int d^d x\ h^2(x) \right]\ . \tag{28}$$

• To rewrite in a more convenient form the correlation, we use

$$\det\left[-\Delta + V^{''}(\phi) \right] = \int \delta c\ \delta\bar{c} \exp\left\{ - \int d^d x\ \ \bar{c}\left[-\Delta + V^{''}(\phi) \right] c \right\} \tag{29}$$

• together with the Fourier functional transform of the δ

$$\delta\left(-\Delta\phi + V^{'}(\phi) + h\right) = \int \delta\lambda \ \exp\left[i\int d^dx \ \lambda(x)\left(-\Delta\phi + V^{'}(\phi) + h\right)\right] \tag{30}$$

Gathering the two expressions together, we have

$$\langle\phi(x)\phi(0)\rangle \sim \int \delta\phi\delta h\delta\lambda \ \phi(x)\phi(0)\exp\left[-\frac{1}{2}\int d^dx \ h^2(x) + i\int d^dy\lambda(y)\left(-\Delta\phi + V'(\phi) + h\right)\right]\det\left[-\Delta + V^{''}(\phi)\right] \tag{31}$$

4) Comment the result about the last integral form. How would you solve the integral for the correlation function ?
Suggestion : use the variable change $h(x) = h^{'}(x) + \tilde{h}(x)$.

The integral is simply Gaussian in $h(x)$. It can be solved looking for the stationary points and using a change of variable. Therefore :

$$h(x) - i\lambda(x) = 0 \qquad \text{with} \qquad h(x) = h^{'}(x) + \tilde{h}(x) \tag{32}$$

We thus obtain

$$\int \delta h \ \exp\left[-\frac{1}{2}\int d^dx \left(h^2(x) - 2i\lambda(x)h(x)\right)\right] = \int \delta h^{'} \ \exp\left[-\frac{1}{2}\int d^dx \left((h^{'})^2 + \tilde{h} + 2h^{'}\tilde{h} - 2i\lambda h^{'} - 2i\lambda\tilde{h}\right)\right] =$$
$$= \exp\left(\frac{1}{2}i\lambda\tilde{h} \ d^dx\right) = \exp\left(-\frac{1}{2}\lambda^2(x) \ d^dx\right) \tag{33}$$

$\Rightarrow$

$$\langle\phi(x)\phi(0)\rangle \sim \int \delta\phi\delta\lambda\delta\bar{c} \ \exp\left(-\int d^dx\tilde{L}\right)\phi(0)\phi(x) \ , \tag{34}$$

where $\tilde{L} = \frac{1}{2}\lambda^2(x) - i\lambda(x)\left(-\Delta\phi + V^{'}(\phi)\right) + \bar{c}\left[-\Delta + V^{''}(\phi)\right]c$, which turns out to be invariant under super-symmetric transformations.

5) Optional point. Now we introduce the superspace defined by d dimensions x_i, which commute, and 2 dimensions θ, $\bar{\theta}$, which are Grassmann variables. We introduce then a **super-field** $\Phi(x,\theta,\bar{\theta}) = \phi(x) + \bar{\theta}c(x) + \bar{c}(x)\theta + \bar{\theta}\theta\lambda(x)$.
• Verify that the Lagragian operator satisfies the following relation

$$\int \tilde{L}(\phi, c, \bar{c}, \lambda) \ d^dx = \int \mathcal{S}_{\text{SUSY}}(\Phi) \ d^dxd\theta d\bar{\theta} \ , \tag{35}$$

with the action $\mathcal{S}_{\text{SUSY}}(\Phi) = -\frac{1}{2}\Phi\Delta_{\text{SS}}\Phi + V(\Phi)$ and the corresponding Laplacian $\Delta_{\text{SS}} = \Delta + \frac{\partial^2}{\partial\theta\partial\bar{\theta}}$.
• Check that the super-symmetric transformations are simply rotations in the aforementioned super-space leaving the metrics $x^2 + \theta\bar{\theta}$ invariant.

The introduction of the two Grassmann coordinates is equivalent to subtracting (rather than adding) 2 dimensions to the d-dimensional space in rotationally invariant integrals. To prove that we can consider an integral in the superspace $(x,\theta,\bar{\theta})$ and a test function which is rapidly decreasing at $\pm\infty$.

$$-\frac{1}{\pi}\int d^dxd\theta d\bar{\theta} \ f(x^2 + \theta\bar{\theta}) = -\frac{1}{\pi}\int d^dx \ f^{'}(x^2) \ . \tag{36}$$

Using the polar coordinates $x^2 = r$ and $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, we can write

$$-\frac{1}{\pi}\int d^d x\ f'(r^2) = -\frac{S_d}{\pi}\int r^{d-1}dr\ f'(r^2) = -\frac{S_d}{2\pi}\int r^{d-2}dr^2\ f'(r^2) \tag{37}$$

With a few manipulations, we end up with (indicating as $k = d-2$)

$$\frac{k}{2\pi}\frac{2\pi}{k}S_k\int r^{k-1}dr\ f(r^2) = \int d^k r\ f(r^2) = \int d^{d-2}x\ f(x^2) \tag{38}$$

and therefore the final expression

$$-\frac{1}{\pi}\int d^d x d\theta d\bar{\theta}\ f(x^2+\theta\bar{\theta}) = \int d^{d-2}x f(x^2)\ . \tag{39}$$

This expression allows us to conclude that the Green functions computed in the $(d-2)$-dimensional space are the same as those computed in the d-dimensional superspace. Therefore, the d-dimensional super-symmetric model – that is the model with random field – is equivalent to a purely bosonic model (*i.e.* a pure ferromagnetic model without random field) in $(d-2)$ dimensions.

6) Optional point – Imry and Ma argument.
Let us consider an Ising ferromagnet in a random magnetic field in d dimensions. It is defined by the following Hamiltonian

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i \tag{40}$$

with $P(h_i) \sim e^{-\frac{h_i^2}{2\epsilon}}$ and $h_i\bar{h}_j = h^2\delta_{ij}$.
We assume that the system without external field develops a spontaneous magnetization. • Explain on what basis the long-range order should be stable with respect to the formation of domain walls.

Because of the random field varying from point to point, there will be regions in which the magnetization will prefer to reorient itself so as to lower the energy due to the interaction with the local field. A domain will emerge then provided that the energy cost associated to the wall is not too high.
The field has Gaussian fluctuations meaning that its average value on a single domain is $O(L^{d/2}$. In the re-orientation, the domain has an energy gain proportional to $L^{d/2}$.
On the other hand, due to the wall, its energy loss is proportional to L^{d-1}. The energy cost of a domain is :

$$\mathcal{E} = c_1 L^{d-1} - c_2 L^{d/2} = c_1 L^{d-1}\left(1 - \frac{c_2}{c_1}L^{-\frac{d-2}{2}}\right) \tag{41}$$

The lower critical dimension is then $d_L^c = 2$. For $d = 2$ and sufficiently large size L, the negative term gives still a dominant contribution to the energy : it accounts for a volumetric contribution to be compared with a surface energy.
For $d > 2$ and $L \gg 1$, the energy contribution will be positive preventing the formation of a new domain (long-range order is thus stable).
For $d < 2$ the opposite condition takes place, with $\mathcal{E} < 0$: the domain formation is very likely (no long-range order).