Effective temperatures

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Plan

1. Equilibrium *vs.* out of equilibrium classical systems.

   How can a classical system stay far from equilibrium ?

   From single-particle to many-body.

2. An 'effective temperature' for *certain* out of equilibrium systems.

   LFC, J. Kurchan, L. Peliti 97

   – Measurement and properties.

   – Relation to entropy : Edwards' measure.

   – Fluctuation theorems.

3. Quantum quenches. **L. Foini, LFC & A. Gambassi 11**

4. Conclusions.

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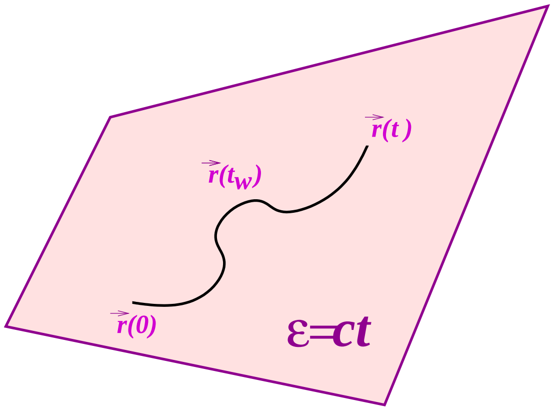
Temperature

Statistical mechanics definition



- Isolated system $\Rightarrow$ conserved energy $\mathcal{E}$
- Ergodic hypothesis

$$S = k_B \ln \mathcal{N} \qquad\qquad \beta \equiv \frac{1}{k_B T} = \left.\frac{\partial S}{\partial \mathcal{E}}\right|_{\mathcal{E}}$$

Microcanonical definition

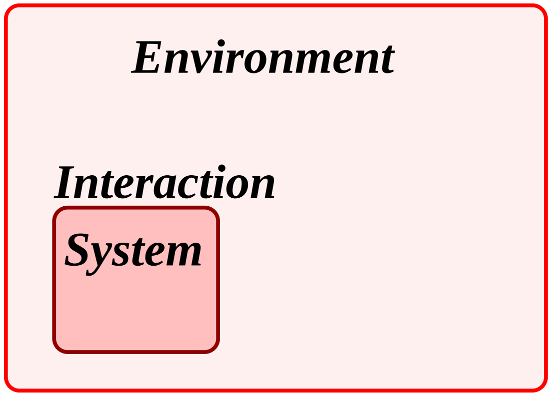
$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range int.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env}$$

$$p_{eq}(\mathcal{E}_{syst}) = g(\mathcal{E}_{syst}) e^{-\beta \mathcal{E}_{syst}} / Z$$

Canonical ensemble



Properties & measurement

Connection with thermodynamics

– Relation to entropy.

– Control of heat-flows : ΔQ follows ΔT.

– Partial equilibration – transitivity :

$T_A = T_B, T_B = T_C \Rightarrow T_A = T_C$.

thermometers for systems in

good thermal contact (ΔQ)

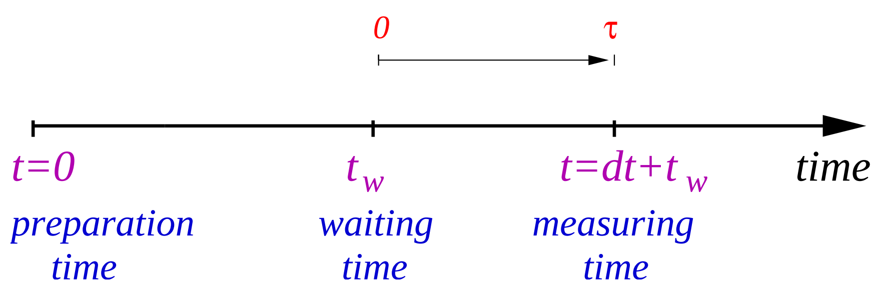
Whatever we identify with a temperature should have these properties

In and out of equilibrium

Take a **mechanical point of view** and call $\{\vec{r}_i\}(t)$ the variables

e.g. the particles' coordinates $\{\vec{x}_i(t)\}$ and momenta $\{\vec{p}_i(t)\}$

Choose an initial condition $\{\vec{r}_i\}(0)$ and let the system evolve.



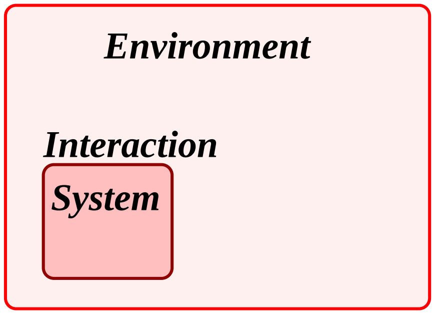
- For $t_w > t_{eq}$: $\{\vec{r}_i\}(t)$ reach the equilibrium pdf and thermodynamics and statistical mechanics apply. **Temperature** is a well-defined concept.
- For $t_w < t_{eq}$: the system remains out of equilibrium and thermodynamics and (Boltzmann) statistical mechanics **do not** apply.

Is there a quantity to be associated with a temperature ?

Dynamics in equilibrium

Conditions

Take an open system coupled to an environment



Necessary :

– The bath should be in equilibrium

same origin of noise and friction.

– Deterministic force :

conservative forces only, $\vec{F} = -\vec{\nabla} V$.

– Either the initial condition is taken from the equilibrium pdf, or the latter should be reached after an equilibration time t_{eq} :

$$\boxed{P_{eq}(v,x) \propto e^{-\beta(\frac{mv^2}{2}+V)}}$$

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Out of equilibrium

How can a classical system stay out of equilibrium ?

- The equilibration time goes beyond the experimentally accessible times.

$$\boxed{t_{eq} \gg t}$$

Microscopic system with no confining potential, $\boxed{t_{eq_x} = \infty}$

e.g., **Diffusion processes**.

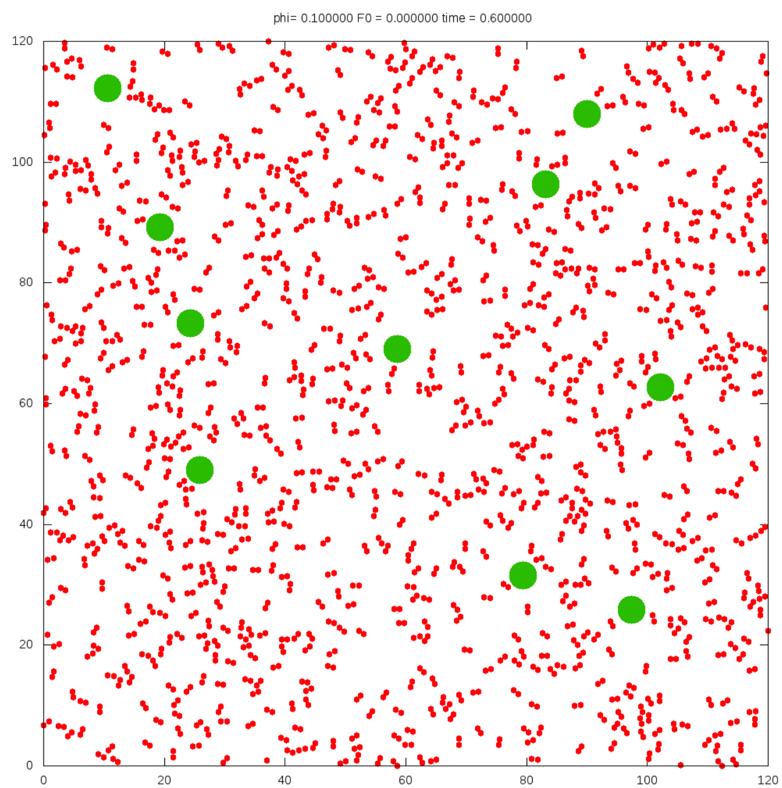
Macroscopic systems in which the equilibration time grows with the system size, $\boxed{\lim_{N \gg 1} t_{eq}(N) \gg t}$

e.g., **Critical dynamics, coarsening, glassy physics**.

- **Driven systems** $\boxed{\vec{F} \neq -\vec{\nabla} V(\vec{r})}$

e.g., **Sheared liquids, vibrated powders, active matter**.

Brownian motion



First example of dynamics of an *open system*

The system : the Brownian particle

The bath : the liquid

Interaction : collisional or potential

Canonical setting

A few Brownian particles or tracers ● imbedded in, say, a molecular liquid.

Late XIX, early XX (Brown, Einstein, Langevin)

Langevin approach

Stochastic Markov dynamics

From Newton's equation $\vec{F} = m\vec{a} = m\dot{\vec{v}}$ and $\vec{v} = \dot{\vec{x}}$

$$\boxed{m\dot{v}_a = -\gamma_0 v_a + \xi_a}$$

with $a = 1, \ldots, d$ (the dimension of space), m the particle mass,
γ_0 the friction coefficient,
and $\vec{\xi}$ the time-dependent thermal noise with Gaussian statistics,
zero average $\langle \xi_a(t) \rangle = 0$ at all times t,
and delta-correlations $\langle \xi_a(t)\xi_b(t') \rangle = 2\,\gamma_0 k_B T\,\delta_{ab}\,\delta(t-t')$.

Dissipation

for $\gamma_0 > 0$ the averaged energy is not conserved,

$$2\langle E(t) \rangle = m\langle v^2(t) \rangle \neq 0.$$

Brownian motion

Markov normal diffusion

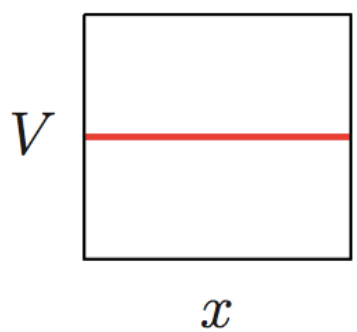
For simplicity, take a one dimensional system, $d = 1$.

The relation between friction coefficient γ_0 and amplitude of the noise correlation $2\gamma_0 k_B T$ ensures $\boxed{\text{equipartition}}$ for the velocity variable

$$\boxed{m\langle v^2(t)\rangle \to k_B T} \quad \text{for} \quad t \gg t_r^v \equiv \frac{m}{\gamma_0}$$

Langevin 1908

But the position variable x $\boxed{\text{diffuses}}$ and $e^{-\beta V}$ is not normalizable.



$$\boxed{\langle x^2(t)\rangle \to 2D\, t} \quad (t \gg t_r^v = m/\gamma_0)$$

$$\boxed{D = k_B T/\gamma_0}$$ diffusion constant.

The particle is out of equilibrium !

Brownian motion

Markov normal diffusion

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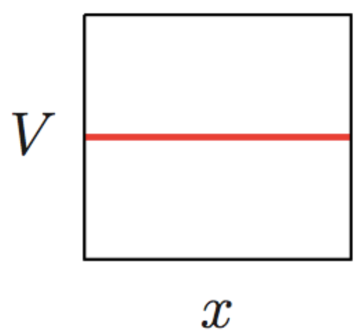

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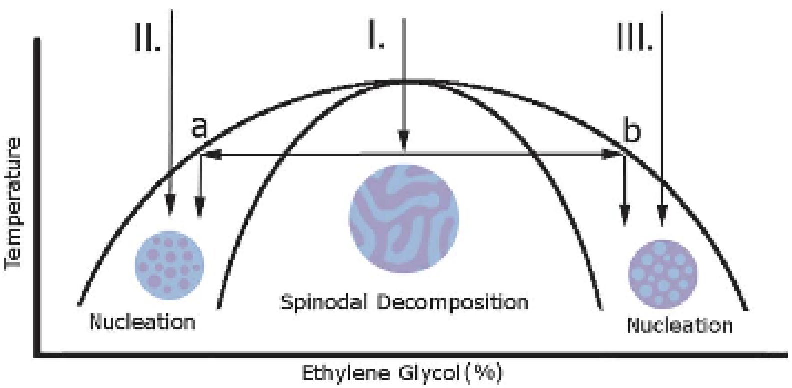
Coexistence of equilibrium (v) and out of equilibrium (x) variables

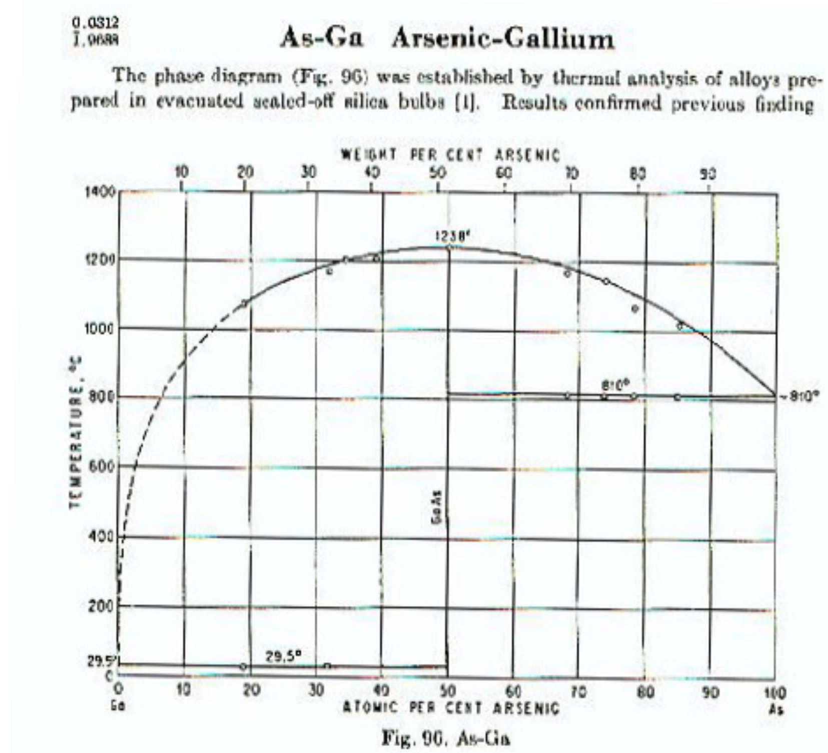
Demixing transitions

Many-body interacting system

Two species ● and ●, repulsive interactions between them.



Sketch

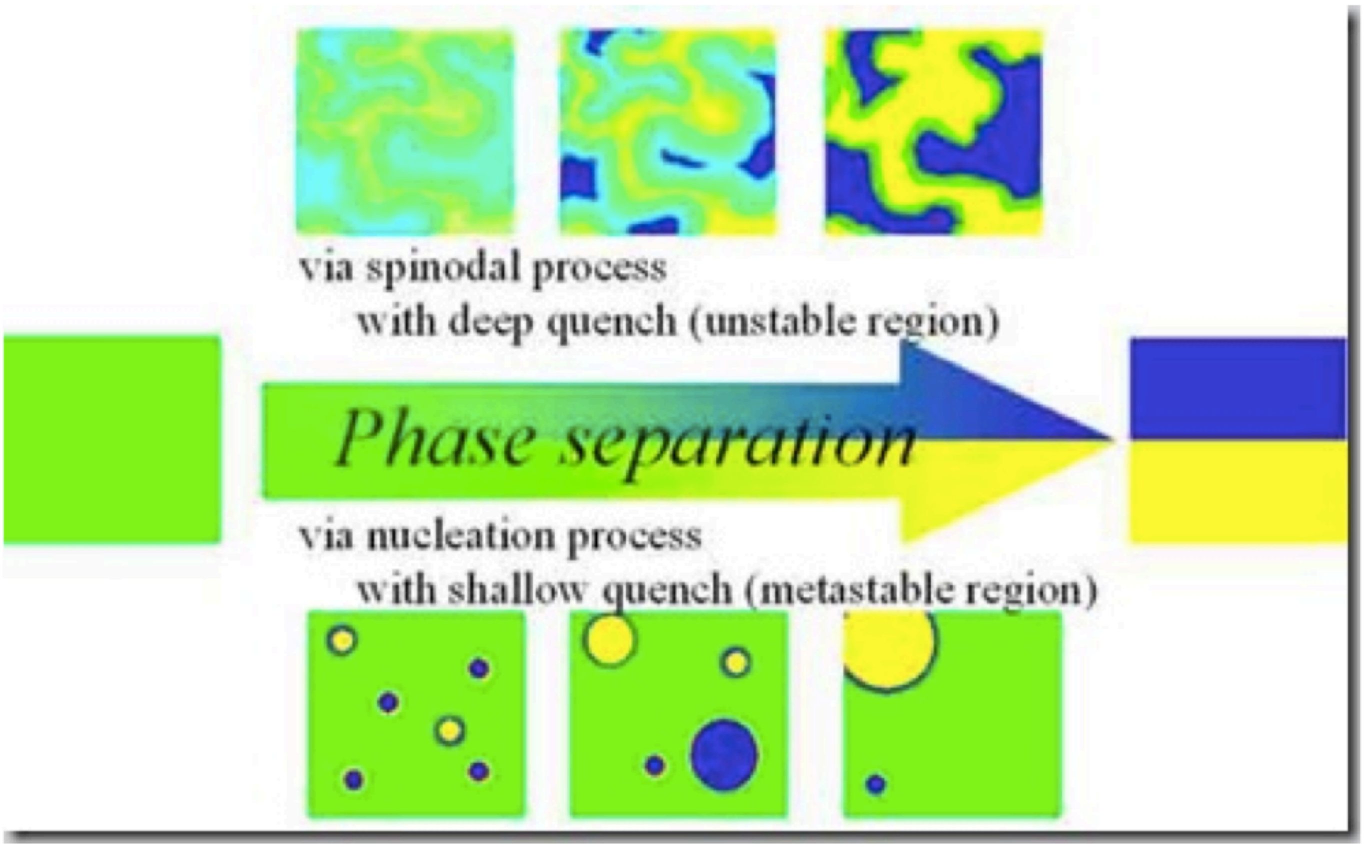


Experimental phase diagram

Binary alloy, **Hansen & Anderko, 54**

Phase separation

Phase ordering kinetics



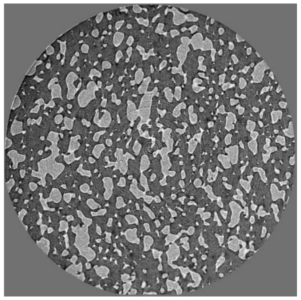
Phase ordering kinetics

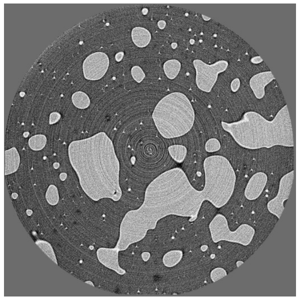
Are these quench dynamics fast processes ? Can we simply forget what happens during the transient, t_{eq}, and focus on the subsequent *equilibrium* behaviour ?

$\boxed{\textbf{No !}}$

It turns out that this is a very slow regime. Its duration grows with the size of the system and it diverges in the thermodynamic limit $\mathcal{V} \to \infty$.

We understand the mechanisms for relaxation : interface local curvature driven dynamics and matter diffusion.



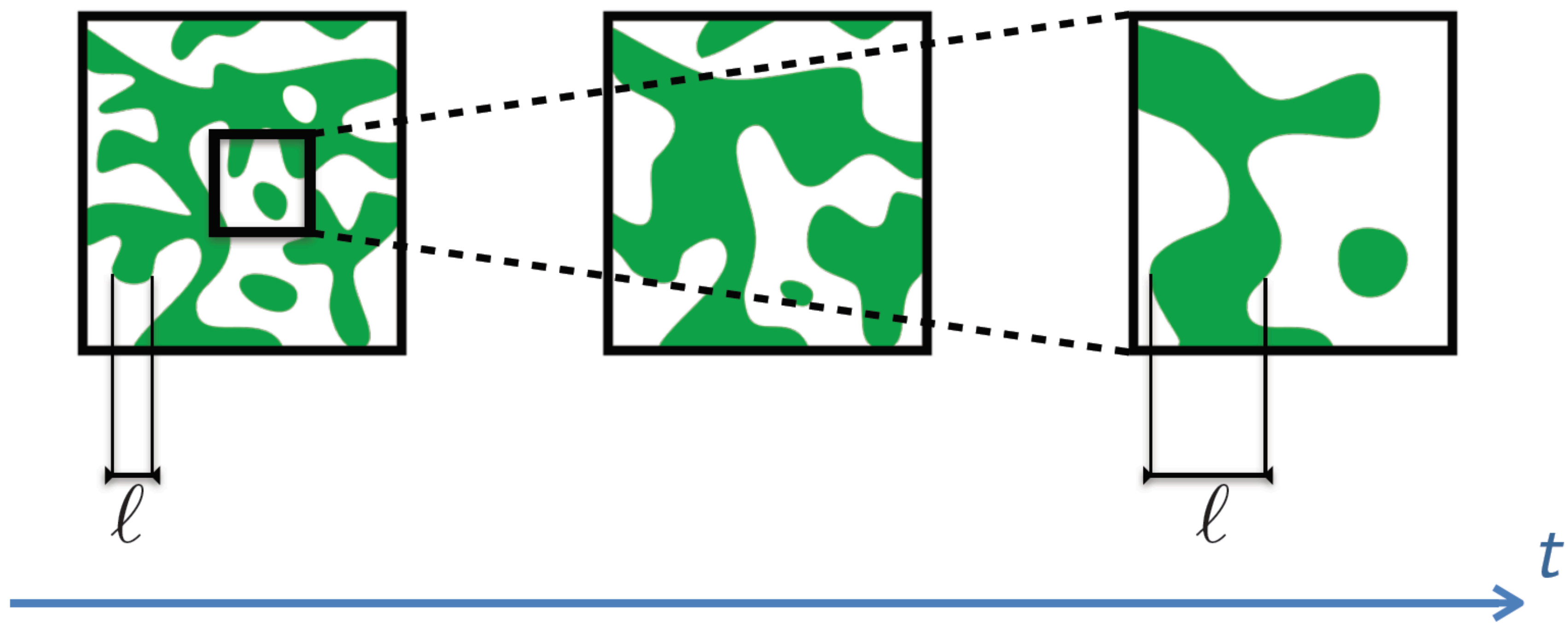


The domains get rounder

The regions get darker and lighter

Dynamic scaling

in phase ordering kinetics



Growing length $\ell(t)$ and equilibrium reached for $\ell(t_{eq}) \simeq L$

Typically $\ell(t) \simeq t^{1/z}$ and $t_{eq} \simeq L^z$

Excess energy w.r.t. the equilibrium one stored in the domain walls ; $\Delta\mathcal{E}(t) \simeq \ell^{-a}(t)$

Stochastic field theory

Formalism to treat open macroscopic systems

- Noise, fluctuations : stochastic calculus
- Dissipation, breakdown of time-reversal invariance : irreversibility.
- Non-linear Langevin equations for the order parameter, say $\vec{\phi}$

$$\underbrace{m\ddot{\vec{\phi}}(\vec{x},t)}_{\text{Inertia}} + \underbrace{\gamma_0\dot{\vec{\phi}}(\vec{x},t)}_{\text{Dissipation}} = \underbrace{\vec{F}(\vec{\phi})}_{\text{Deterministic}} + \underbrace{\vec{\xi}(\vec{x},t)}_{\text{Noise}}$$

- *e.g.* time-dependent Ginzburg-Landau symmetry-broken $\lambda\phi^4$ in its non-perturbative regime.

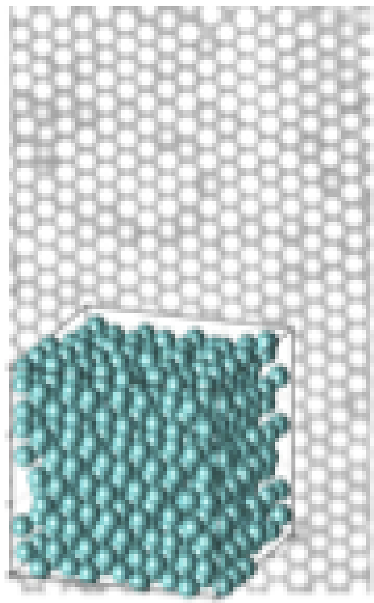
Out of equilibrium non-linear field theory ; no good t-RG

Review : **A. J. Bray 94**

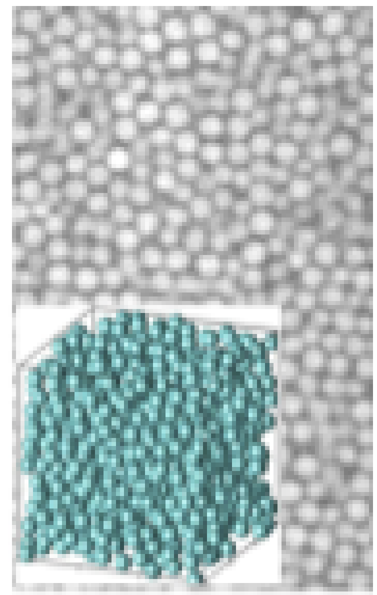
Glasses

***e.g.*, colloidal ensembles**

Micrometric spheres immersed in a fluid



Crystal



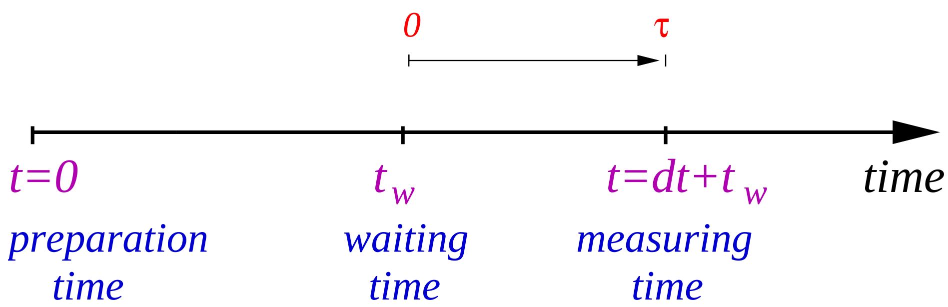
Glass

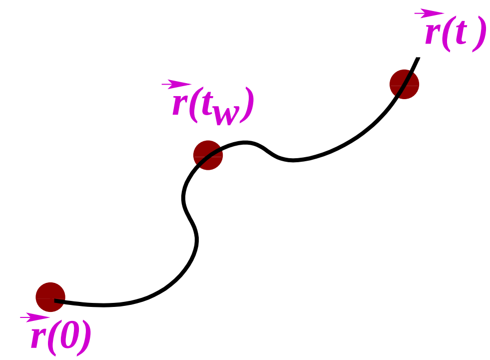
In the glass : no obvious growth of order, slow dynamics with, however, scaling properties.

What drives the slowing down ?

Two-time observables

Correlations





t_w not necessarily longer than t_{eq}.

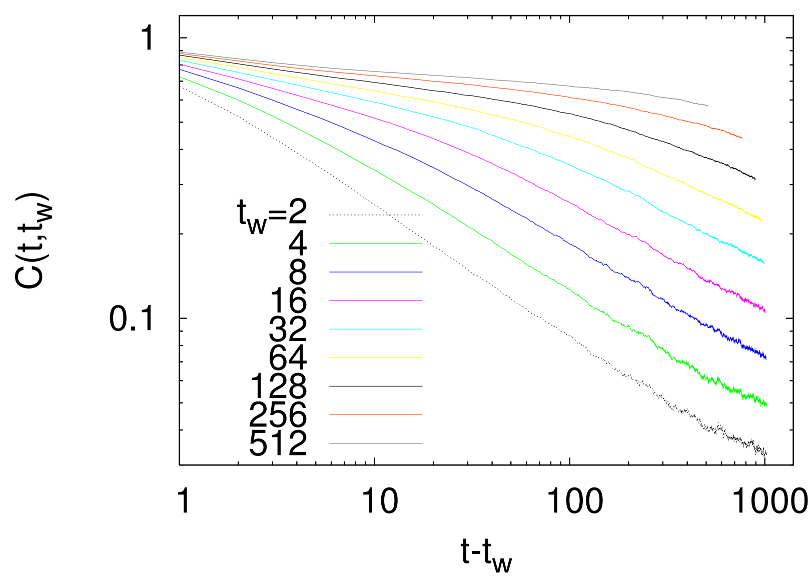
The two-time correlation between $A[\vec{x}(t)]$ and $B[\vec{x}(t_w)]$ is

$$C_{AB}(t, t_w) \equiv \langle\, A[\vec{x}(t)] B[\vec{x}(t_w)] \,\rangle$$

average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise in Langevin dynamics, etc.)

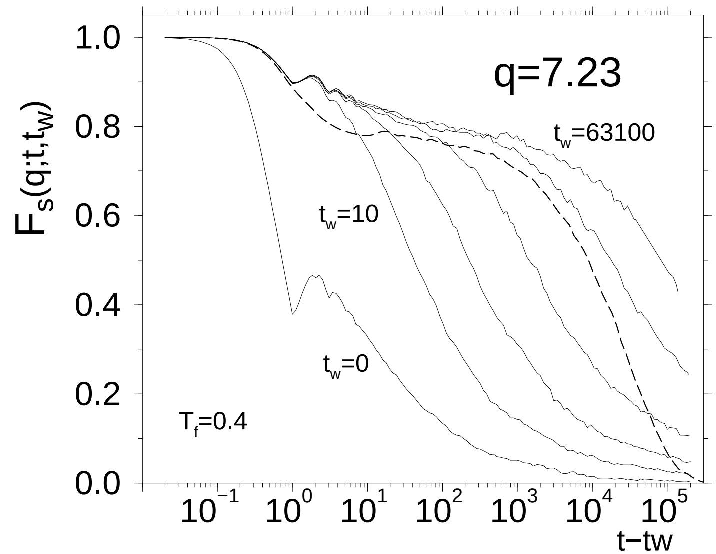
Two-time self-correlation

Same observable, quasi-stationary & ageing regimes



2d Ising model - spin-spin

Sicilia *et al.* **07**



Lennard-Jones - density-density

Kob & Barrat 99

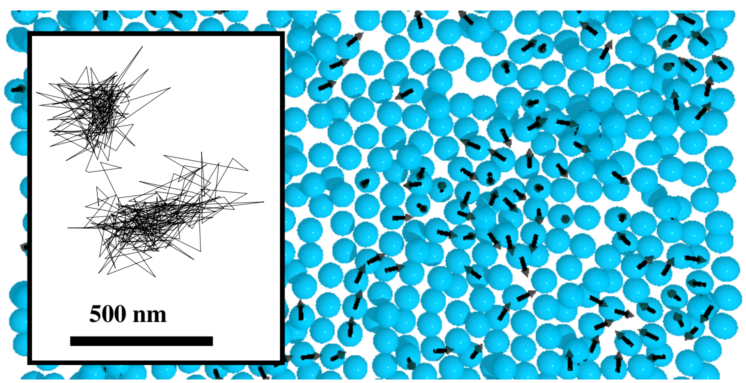
One correlation can exhibit stationary and non stationary relaxation

in different two-time regimes

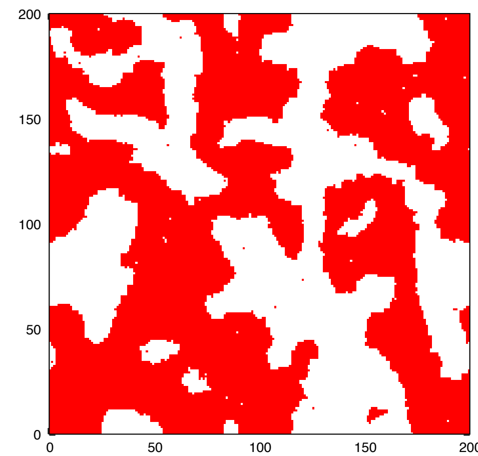
Different two-time regimes

Interpretation

- In particle systems, rattling within cages *vs.* structural relaxation.
- In phase ordering kinetics, thermal fluctuations within domains *vs.* domain wall motion.



Cages in colloidal suspensions



Phase separation in the 2d Ising model.

Challenges

in classical non-equilibrium macroscopic systems

- Coarsening

  The systems are taken across *usual phase transitions*.

  The *dynamic mechanisms* are well-understood :

  competition between equilibrium phases & topological defect annihilation.

  The difficulty lies in the calculation of observables in a time-dependent non-linear field theory.

- Glasses

  Are there phase transitions ?

  The dynamic mechanisms are not understood.

  The difficulty is conceptual (also computational).

- General question

  Do these, as well as sheared liquids or active matter, enjoy some kind of thermodynamic properties ?

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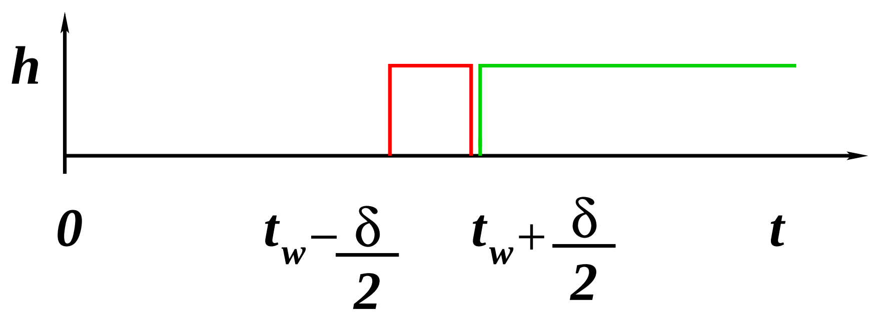
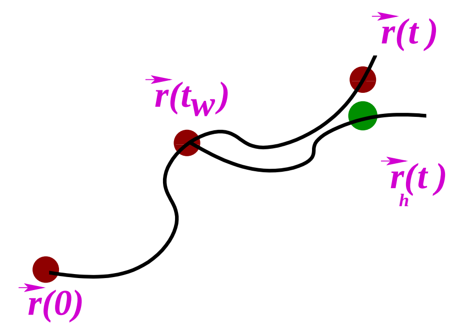
Two-time observables

Linear response





The perturbation couples linearly to the observable $B[\vec{x}(t_w)]$

$$E \quad \to \quad E - hB[\vec{x}(t_w)]$$

The linear instantaneous response of another observable $A[\vec{x}(t)]$ is

$$R_{AB}(t, t_w) \equiv \left. \frac{\delta \langle A[\vec{x}(t)] \rangle_h}{\delta h(t_w)} \right|_{h=0}$$

The linear integrated response is

$$\boxed{\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t')}$$

Rue de Fossés St. Jacques et rue St. Jacques

Paris 5ème Arrondissement.

LFC

Fluctuation-dissipation

In equilibrium

$$\boxed{P(\vec{r}, t_w) = P_{eq}(\vec{r})}$$

- The dynamics are stationary

$$C_{AB} \to C_{AB}(t - t_w) \text{ and } R_{AB} \to R_{AB}(t - t_w)$$

- The fluctuation-dissipation theorem between spontaneous (C_{AB}) and induced (R_{AB}) fluctuations

$$R_{AB}(t - t_w) = -\frac{1}{k_B T} \frac{\partial C_{AB}(t - t_w)}{\partial t} \, \theta(t - t_w)$$

holds and implies ($k_B = 1$ henceforth)

$$\boxed{\chi_{AB}(t - t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t') = \frac{1}{T}[C_{AB}(0) - C_{AB}(t - t_w)]}$$

Fluctuation-dissipation

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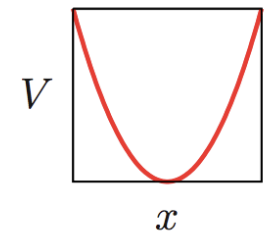
Fluctuation-dissipation

A simple example: the dissipative harmonic oscillator

$$\boxed{\dot{x}(t) = -kx(t) + h(t) + \xi(t)}$$



Solution : $x(t) = x_0 e^{-kt} + \int_0^t dt'\, e^{-k(t-t')}[\xi(t') + h(t')]$

$\langle x(t)x(t_w)\rangle_{h=0} = [x_0^2 - \frac{T}{k}]e^{-k(t+t_w)} + \frac{T}{k}e^{-k(t-t_w)}$ correlation

linear response $\left.\frac{\delta\langle x(t)\rangle_h}{\delta h(t_w)}\right|_{h=0} = e^{-k(t-t_w)}\,\theta(t-t_w)$ $\quad \boxed{t_{eq} = k^{-1}}$

If $k \neq 0$ $\quad TR_{xx}(t,t_w) = \partial_{t_w} C_{xx}(t,t_w)\,\theta(t-t_w)$ $\quad$ FDT holds ✔

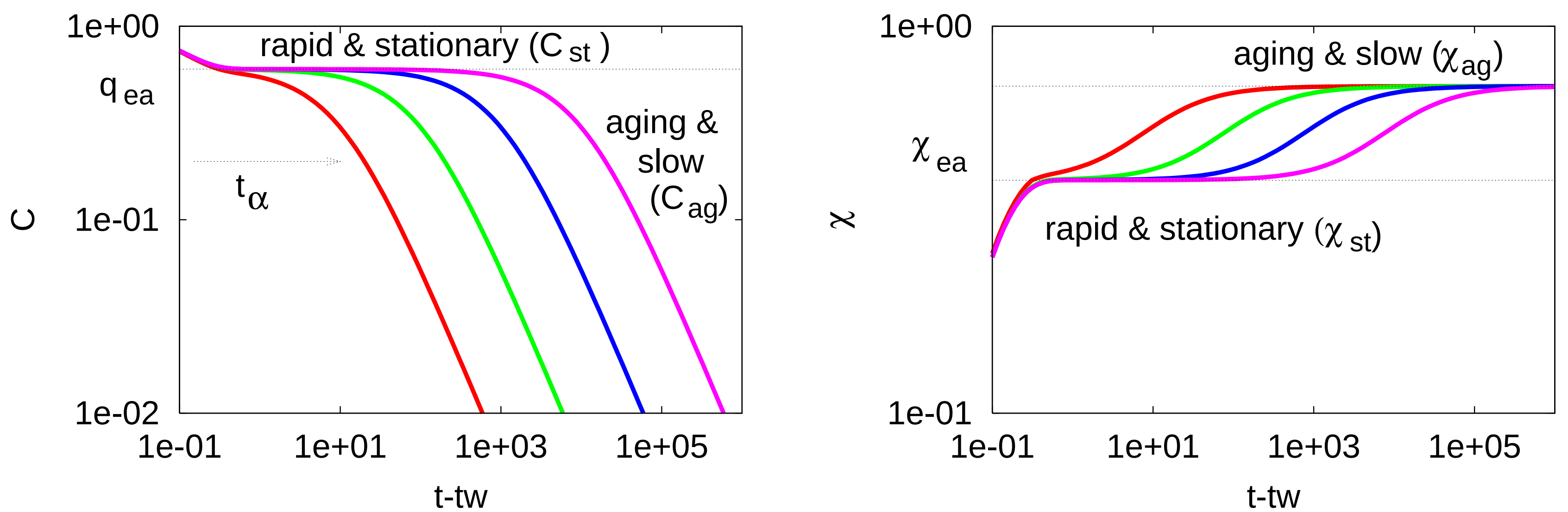
If $k \to 0$ $\quad TR_{xx}(t,t_w) \neq \partial_{t_w} C_{xx}(t,t_w)\,\theta(t-t_w)$ $\quad$ FDT does not hold

Fluctuation-dissipation

Solvable glasses: p spin-models & mode-coupling theory

- Stochastic dynamics of a particle in an *infinite dimensional* space under the effect of a quenched random potential.

- A fully-connected (Curie approximation) spin model with as many ferromagnetic as antiferromagnetic couplings.

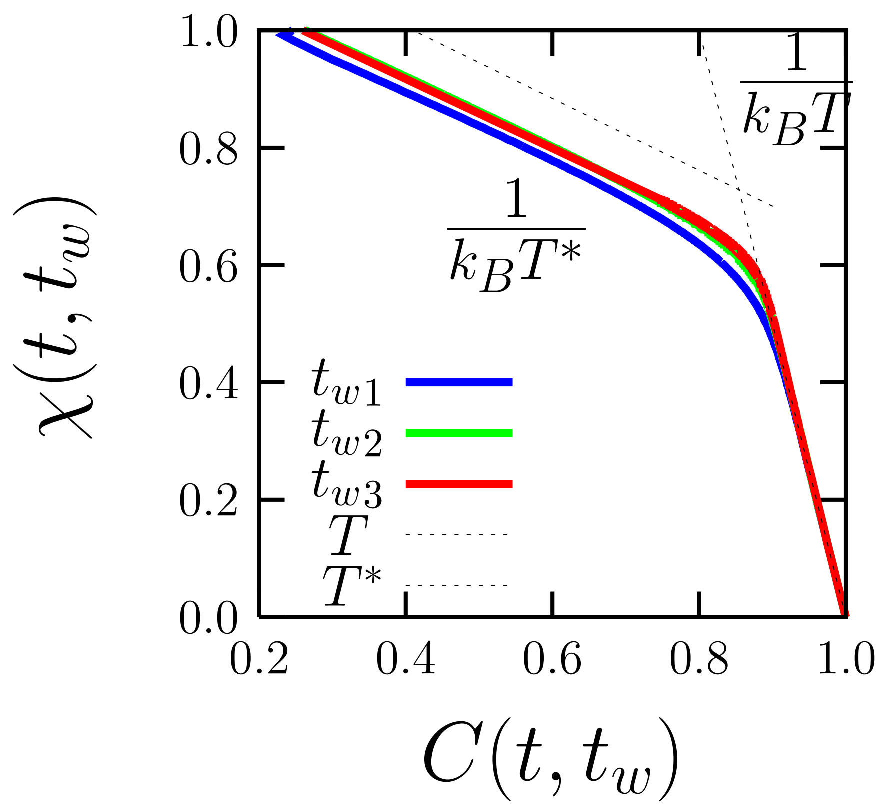


Sketch of the separation of time-scales in the out of equilibrium relaxation

Fluctuation-dissipation

Solvable glasses: p spin-models & mode-coupling theory

A quench from $T_0 \to \infty$ (gas) to $T < T_g$ (glass)



Parametric construction

t_w fixed

$\boxed{t_{w_1} < t_{w_2} < t_{w_3}}$

t : $t_w \to \infty$ or

$\tau \equiv t - t_w : 0 \to \infty$

used as a parameter

Note that $\boxed{T^* > T}$

LFC & Kurchan 93

Fluctuation-dissipation

Proposal

For non-equilibrium systems, relaxing slowly towards an **asymptotic** limit (*cfr.* threshold in p spin models) such that **one-time quantities** [*e.g.* the energy-density $\mathcal{E}(t)$] **approach a finite value**

$$\boxed{\lim_{\substack{\mathbf{t_w \to \infty} \\ C(t,t_w)=C}} \chi(t, t_w) = f_\chi(C)}$$

For weakly forced non-equilibrium systems in the limit of **small work**

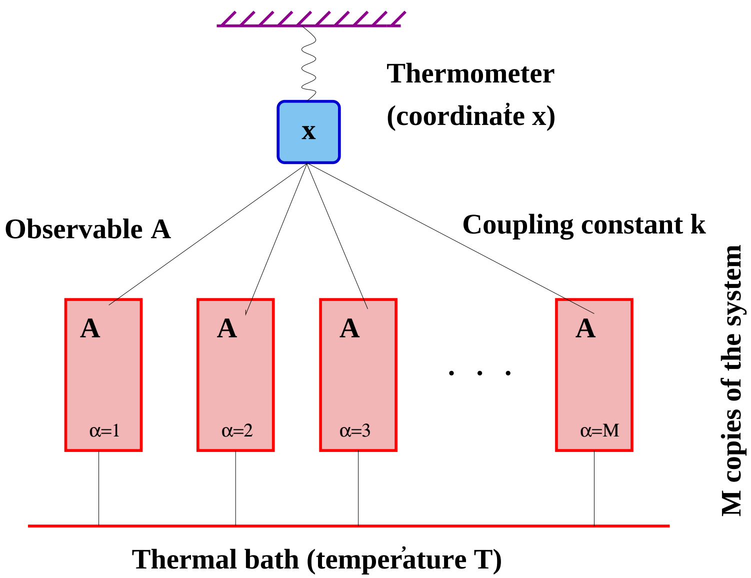
$$\lim_{\substack{\epsilon \to \mathbf{0} \\ C(t,t_w)=C}} \chi(t, t_w) = f_\chi(C)$$

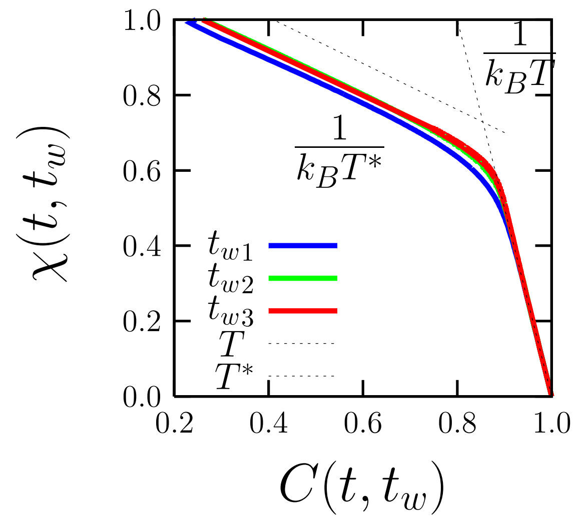
And the effective temperature is $\boxed{-\frac{1}{T_{\rm eff}} \equiv \frac{\partial \chi}{\partial C}}$

LFC & Kurchan 94

FDT & effective temperatures

Can one interpret the slope as a temperature ?





(1) Measurement with a **thermometer** with

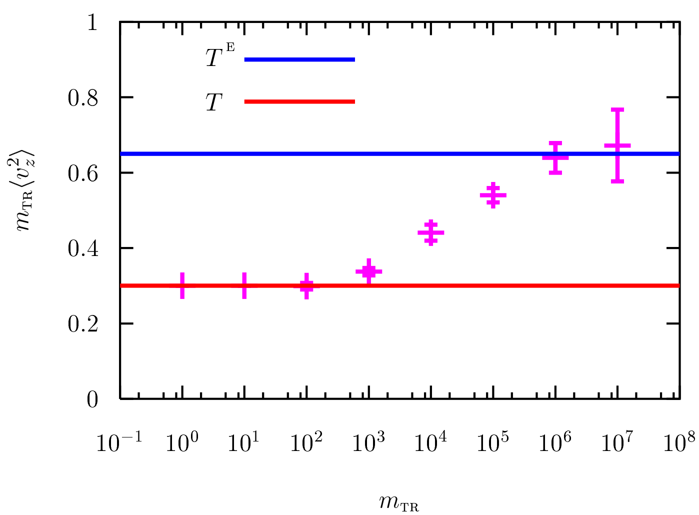
- Short internal time scale τ_0, fast dynamics is tested and T is recorded.
- Long internal time scale τ_0, slow dynamics is tested and T^* is recorded.

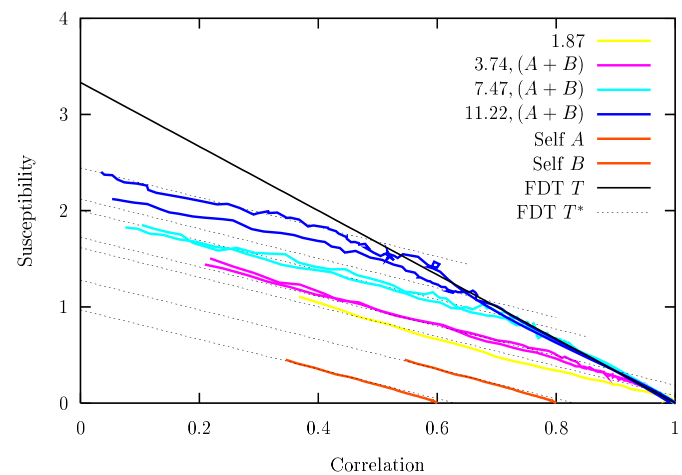
(2) **Partial equilibration** (3) **Direction of heat-flow**

LFC, Kurchan & Peliti 97

FDT & effective temperatures

Sheared binary Lennard-Jones mixture





Left: the kinetic energy of a tracer particle (the thermometer) as a function of its mass ($\tau_0 \propto \sqrt{m_{tr}}$) $\boxed{\frac{1}{2} m_{tr} \langle v_z^2 \rangle = \frac{1}{2} k_B T_{\text{eff}}}$

Right: $\boxed{\chi_k(C_k)}$ plot for different wave-vectors k, partial equilibrations.

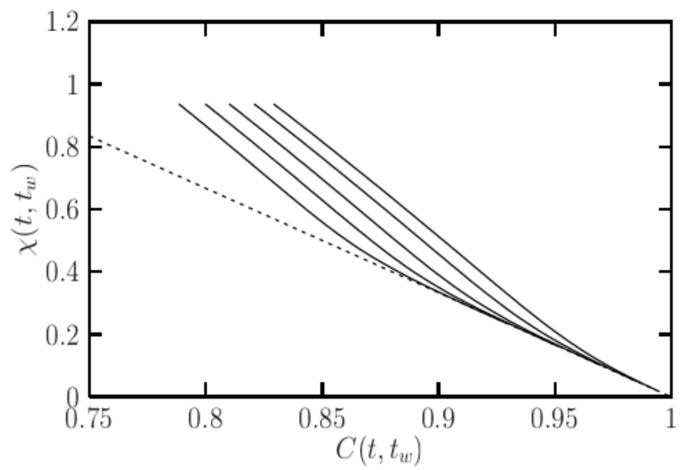
J-L Barrat & Berthier 00

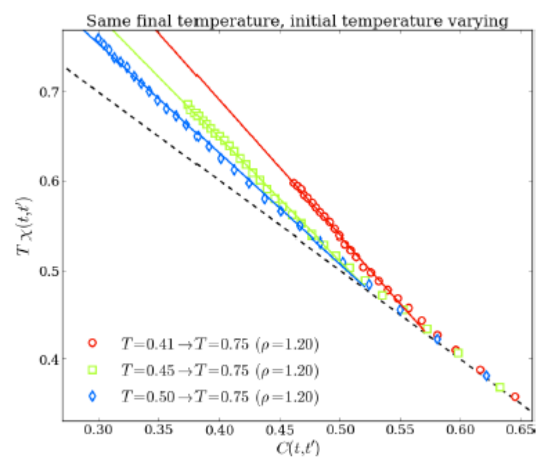
FDT & effective temperatures

Role of initial conditions

$T^* > T$ found for quenches from the disordered into the glassy phase

(Inverse) quench from an ordered initial state, $\boxed{T^* < T}$





2d XY model or O(2) field theory

Binary Lennard-Jones mixture

Berthier, Holdsworth & Sellitto 01

Gnan, Maggi, Parisi & Sciortino 13

Summary

- T_{eff} definition from the analysis of **fluctuation-dissipation relations** .
- Discussion of **thermodynamic meaning**.

  Shown for *mean-field models* – large N, large d or, in other words, within the mode-coupling approach to glassy systems.
- **Numerical evidence** *Lennard-Jones silica, soft particles ; vortex glasses granular matter ; thin magnetic films, active matter*, etc.
- Other evidence : extended Arrhenius law for activation (**Ilg & J-L Barrat**), fluctuation theorems (**Zamponi** *et al*), ratchets (**Gradenigo** *et al*), etc.
- **Experimental results** are less clear

  glycerol, laponite, spin-glasses, etc. (**Jabbari-Bonn, Abou-Gallet, Ciliberto** *et al.*, **Bartlett** *et al*, **Hérisson & Ocio**, etc.).

Summary

classical context

- The energy density approaches the equilibrium one, typically as $\Delta E \simeq t^{-b}$.
- The correlation and linear response functions have highly non-trivial time-dependencies (aging effects, non-exponential relaxations)
- There is an extended time-regime in which correlation and linear response vary "macroscopically" but the effective temperature $T_{\mathrm{eff}} = T^*$ is constant.
- T^* can be related to the variation of a configurational entropy with respect to the energy-density (*à la* micro-canonic.)
- T^* has intuitive properties : hotter for more disordered, colder for more ordered.

Cases in which this does not hold were exhibited by, *e.g.*, **Sollich et al**

in models with unbounded energy or artificial (emerging ?) dynamic rules.

Plan

1. Equilibrium *vs.* out of equilibrium classical systems.

   How can a classical system stay far from equilibrium ?

   From single-particle to many-body.

2. An ‘effective temperature’ for *certain* out of equilibrium systems.

   LFC, J. Kurchan, L. Peliti 97

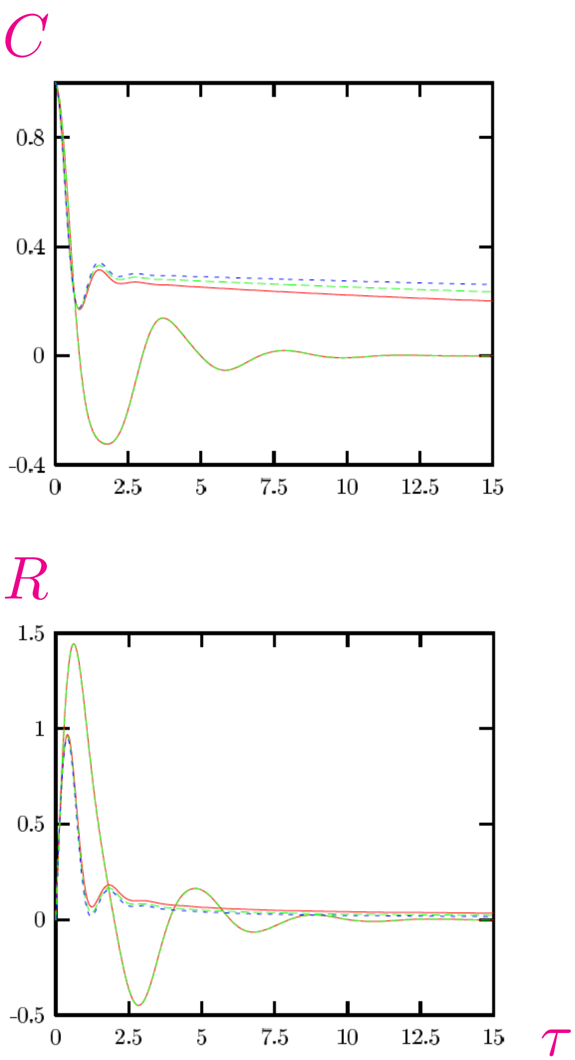
   – Measurement and properties.
   – Relation to entropy : Edwards’ measure.
   – Fluctuation theorems.

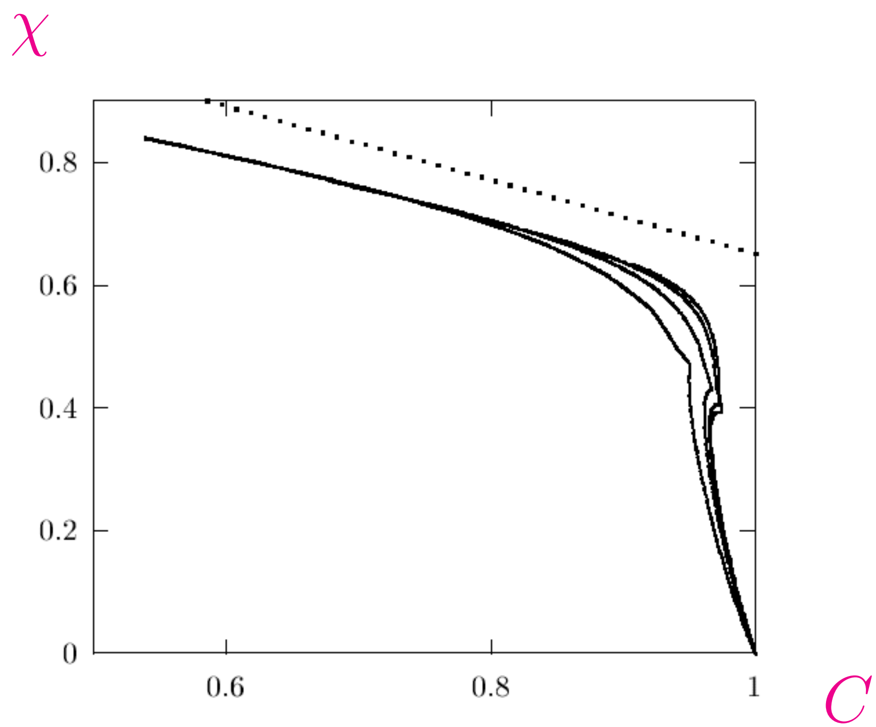
3. Quantum quenches. **L. Foini, LFC & A. Gambassi 11**

4. Conclusions.

Dissipative quantum glasses

Quantum p-spin coupled to a bath of harmonic oscillators





Out of equilibrium decoherence

LFC & Lozano 98

Isolated quantum systems

Quantum quenches

- Take an isolated quantum system with Hamiltonian H_0
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of H_0.
- Unitary time-evolution with $U = e^{-\frac{i}{\hbar}Ht}$ with a Hamiltonian H.

Does the system reach some steady state ?

Note that it is the ergodic theory question posed in the quantum context.

Motivated by cold-atom experiments & exact solutions of $1d$ quantum models.

Are at least some observables described by thermal ones?

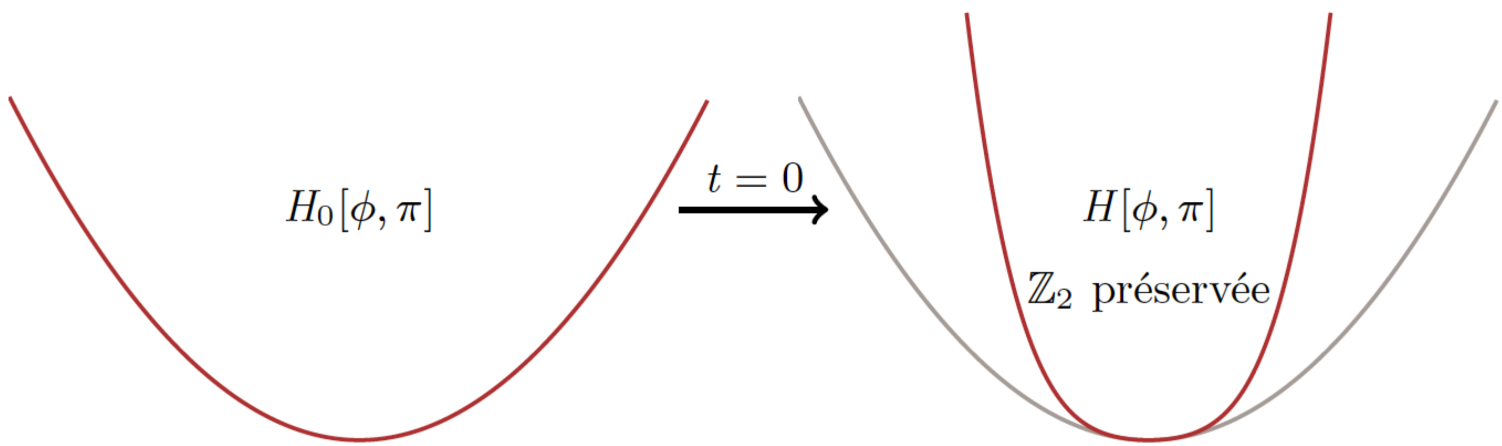
When, how, which?

Quantum quench

Setting

Take a closed system, H_0, in a given state, $|\psi_0\rangle$, and suddenly change a parameter, H. The unitary evolution is ruled by H.

e.g. $H = \int d^dx \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + r\phi^2 + \lambda\phi^4 \right\}$



$\lambda_0 \quad \to \quad \lambda$

Quantum quench

Setting

Take a closed system, H_0, in a given state, $|\psi_0\rangle$, and suddenly change a parameter, H. The unitary evolution is ruled by H.

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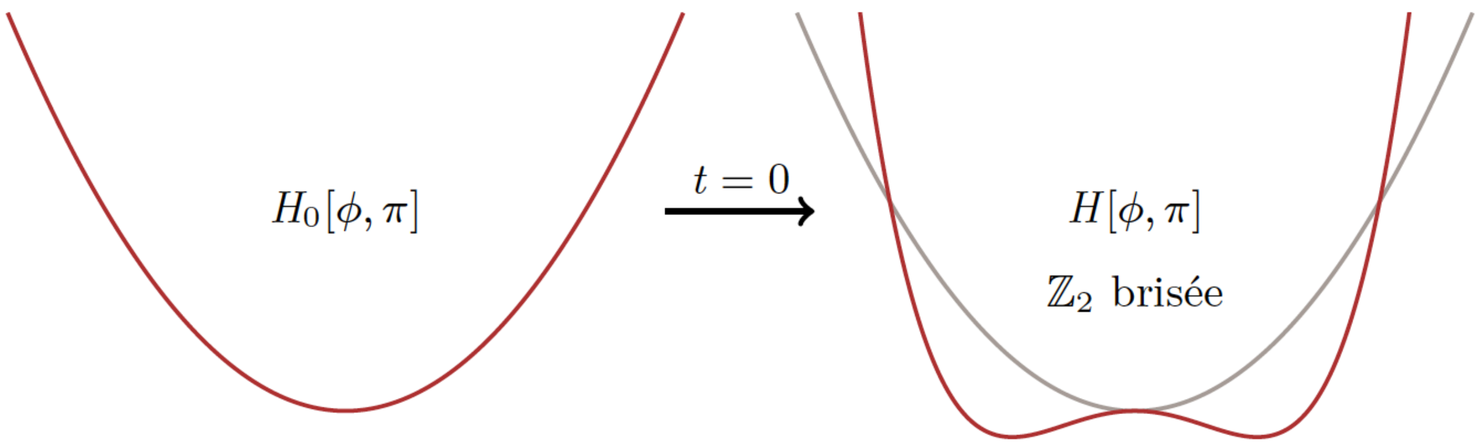


$r > 0 \quad \to \quad r < 0$

Quantum quenches

Questions

Does the system reach a thermal equilibrium density matrix ?

Under which conditions ?

non-integrable vs integrable systems ; role of initial states ; non critical vs. critical quenches

- Definition of T_e from $\langle\psi_0|H|\psi_0\rangle = \langle H\rangle_{T_e} = \text{Tr } He^{-\beta_e H}$

Just one number, it can always be done

- Comparison of dynamic and thermal correlation functions, *e. g.*

$C(r,t) \equiv \langle\psi_0|\phi(\vec{x},t)\phi(\vec{y},t)|\psi_0\rangle$ vs. $C(r) \equiv \langle\phi(\vec{x})\phi(\vec{y})\rangle_{T_e}$.

Calabrese & Cardy ; Rigol et al ; Cazalilla & Iucci ; Silva et al, etc.

Proposal : put qFDT to the test to check whether $T_{\text{eff}} = T_e$ exists

Fluctuation-dissipation theorem

Classical dynamics in equilibrium

The classical FDT for a stationary system with $\tau \equiv t - t_w$ reads

$$\boxed{\chi(\tau) = \int_0^\tau dt'\, R(t') = -\beta[C(\tau) - C(0)] = \beta[1 - C(\tau)]}$$

choosing $C(0) = 1$.

Linear relation between χ and C

Quantum dynamics in equilibrium

The quantum FDT reads

$$\boxed{\chi(\tau) = \int_0^\tau d\tau'\, R(\tau') = \int_0^\tau d\tau' \int_{-\infty}^{\infty} \frac{id\omega}{\pi\hbar}\, e^{-i\omega\tau'} \tanh\left(\frac{\beta\hbar\omega}{2}\right) C(\omega)}$$

Complicated relation between χ and C

Fluctuation-dissipation theorem

Quantum SU(2) Ising chain

The initial Hamiltonian $$\boxed{H_{\Gamma_0} = -\sum_i \sigma_i^x \sigma_{i+1}^x + \Gamma_0 \sum_i \sigma_i^z}$$

The initial state $|\psi_0\rangle$ ground state of H_0

Instantaneous quench in the transverse field $\Gamma_0 \to \Gamma$

Evolution with H_Γ.

Iglói & Rieger 00

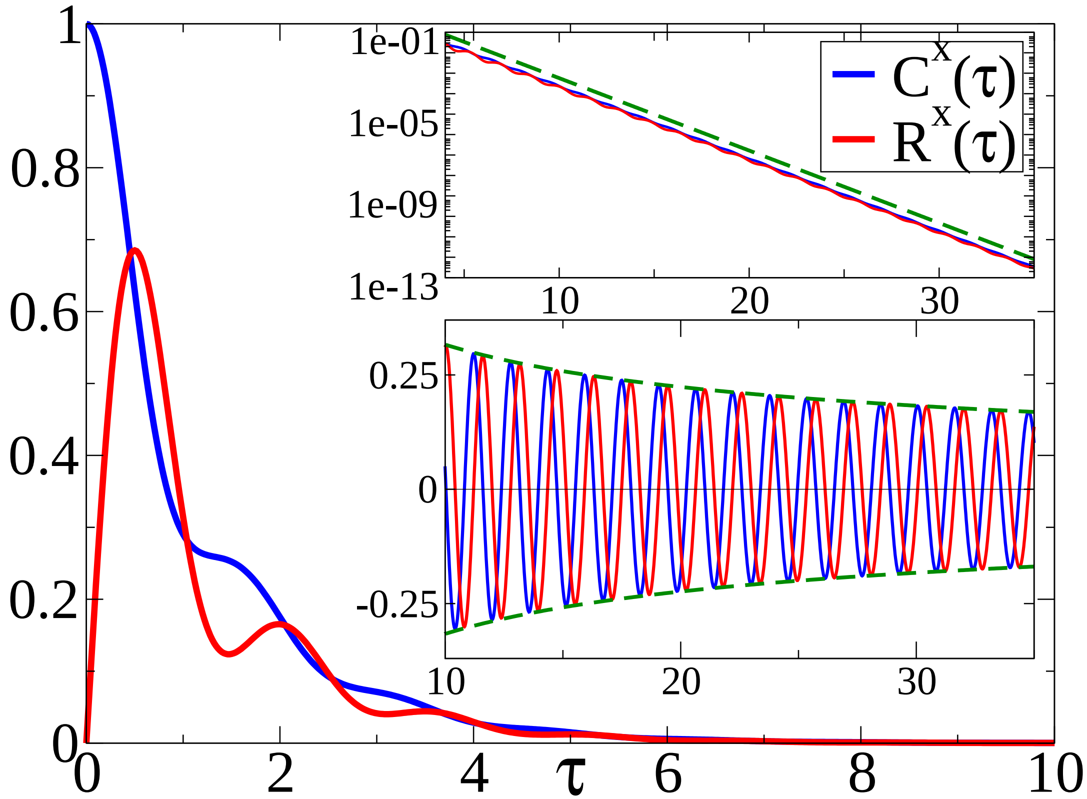
Reviews : **Karevski 06 ; Polkovnikov et al. 10 ; Dziarmaga 10**

Observables : correlation and linear response of local longitudinal and transverse spin, etc.

Specially interesting case $\Gamma_c = 1$ the critical point. **Rossini et al. 09**

Quantum quench

$T_{\rm eff}$ from FDT ? Longitudinal spin



Insets

$e^{-\tau/\tau_C}$

$\tau^{-2}\sin(4\tau+\phi)$

$$C^x(\tau) \simeq A_C e^{-\tau/\tau_C}[1 - a_C\tau^{-2}\sin(4\tau+\phi_C)]$$
$$R^x(\tau) \simeq A_R e^{-\tau/\tau_R}[1 - a_R\tau^{-2}\sin(4\tau+\phi_R)]$$

Foini, LFC & Gambassi

Quantum quench

T_{eff} from FDT ?

For sufficiently long-times such that one drops the power-law correction

$$\boxed{-\beta^x_{\text{eff}} \simeq \frac{R^x(\tau)}{d_\tau C^x_+(\tau)} \simeq -\frac{\tau_C A_R}{A_C}}$$

A constant consistent with a classical limit but

$$\boxed{T^x_{\text{eff}}(\Gamma_0) \neq T_e(\Gamma_0)}$$

Morever, a complete study in the full time and frequency domains confirms that $T^x_{\text{eff}}(\Gamma_0, \omega) \neq T^z_{\text{eff}}(\Gamma_0, \omega) \neq T_e(\Gamma_0)$ (though the values are close).

Fluctuation-dissipation relations as a probe to test thermal equilibration

No equilibration for generic Γ_0

Summary

$T_{\rm eff}$ from FDT

- $T_{\rm eff}$ from the analysis of **fluctuation-dissipation relations** in classical and quantum systems, closed or open.

  $T_{\rm eff}$ calculated for dissipative classical and quantum *mean-field* models – large N, large d or with self-consistent closure approximations.

  A *finite dimensional* solvable model with the phenomenology discussed is missing. (This is probably the same as finding a solvable glassy model !)

- Discussion of the **thermodynamic meaning** of $T_{\rm eff}$.

  A better understanding of the microscopic origin of $T_{\rm eff}$ is lacking.

- Use of **fluctuation-dissipation relations** to check for Boltzmann equilibrium (application to quantum quenches).

Fluctuation-dissipation

A proof

The generic **Langevin equation** for a particle in $1d$ is

$$\boxed{m\ddot{x}(t) + M'[x(t)]\int_{-\mathcal{T}}^{t} dt'\, \Gamma(t-t')M'[x(t')]\dot{x}(t') = F(t) + \xi(t)M'[x(t)]}$$

with the coloured noise $\boxed{\langle \xi(t)\xi(t')\rangle = T\,\Gamma(t-t')}$

The dynamic generating functional is a path-integral

$$\boxed{\mathcal{Z}_{dyn}[\eta] = \int dx_{-\mathcal{T}} d\dot{x}_{-\mathcal{T}} \int \mathcal{D}x\mathcal{D}i\hat{x}\; e^{-S[x,i\hat{x};\eta]}}$$

with $i\hat{x}(t)$ the ‘response’ variable.

$x_{-\mathcal{T}}$ and $\dot{x}_{-\mathcal{T}}$ are the initial conditions at time $-\mathcal{T}$.

Martin-Siggia-Rose-Jenssen-deDominicis formalism

Fluctuation-dissipation

A proof

The action has a deterministic part (Newton) that includes the initial condition and a dissipative part that depends upon Γ : $\boxed{S = S_{det} + S_{diss}}$

The transformation

$$\boxed{x(t) \to x(-t) \qquad\qquad i\hat{x}(t) \to i\hat{x}(-t) + \beta\dot{x}(-t)}$$

leaves S_{diss} and the path-integral measure invariant.

S_{det} is also invariant if $P(x_{-\mathcal{T}}, \dot{x}_{-\mathcal{T}}) = P_{eq}(x_{-\mathcal{T}}, \dot{x}_{-\mathcal{T}})$, and $F = V'[x]$

The **FDT** valid for Newton or Langevin dynamics

$$\boxed{R_{AB}(t, t_w) + R_{AB}(t_w, t) = \beta\partial_{t_w} C_{AB}(t, t_w)}$$

and higher-order extensions are Ward identities of this symmetry.

The fluctuation theorems can also be proven in this way.

Fluctuation theorems

Take a system *in equilibrium* and drive it into a

non-equilibrium steady state

with a perturbing force. The **fluctuations of 'entropy production rate'**

$$p \equiv (\tau\sigma_+)^{-1} \int_{-\tau/2}^{\tau/2} dt\, W(S_t)/T$$

where S_t is the trajectory of the system in phase space,

T is the temperature of the equilibrated unperturbed system,

$W(S_t)$ is the work done by the external forces, and

$T\sigma_+ \equiv \int dx P_{st}(x) W(x) \sim \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/s}^{\tau/s} dt\, W(t)$ is an

average over the steady state distribution, **satisfy**

$\xi(p) - \xi(-p) \;=\; p\sigma_+$ with $\xi(p) \;\equiv\; \lim_{\tau\to\infty} \frac{1}{\tau} \ln \pi_\tau(p)$

and π_τ the probability density of p.

Cohen, Morriss & Evans 90 ; Gallavoti & Cohen 93

Fluctuation theorems

Take a glass *out of equilibrium* and take it into a

driven steady glassy state

with a perturbing force.

For which entropy production rate does a fluctuation theorem hold ?

Since there is no meaning to T but there is to $T_{\rm eff}$ the proposal is to replace

$$\int_{-\tau/2}^{\tau/2} dt \ \frac{W(t)}{T} \quad \to \quad \int_{-\tau/2}^{\tau/2} dt \ \frac{W(t)}{T_{\rm eff}(t)}$$

with $T_{\rm eff}(t)$ the effective temperature as measured from

the fluctuation-dissipation relation of the *unperturbed* relaxing system

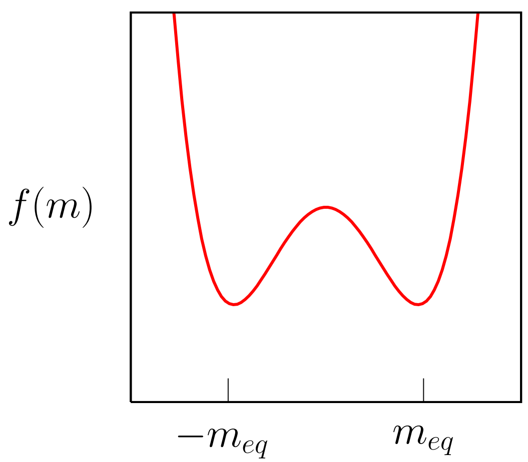
with its two values T and T^*

Zamponi, Bonetto, LFC & Kurchan 05

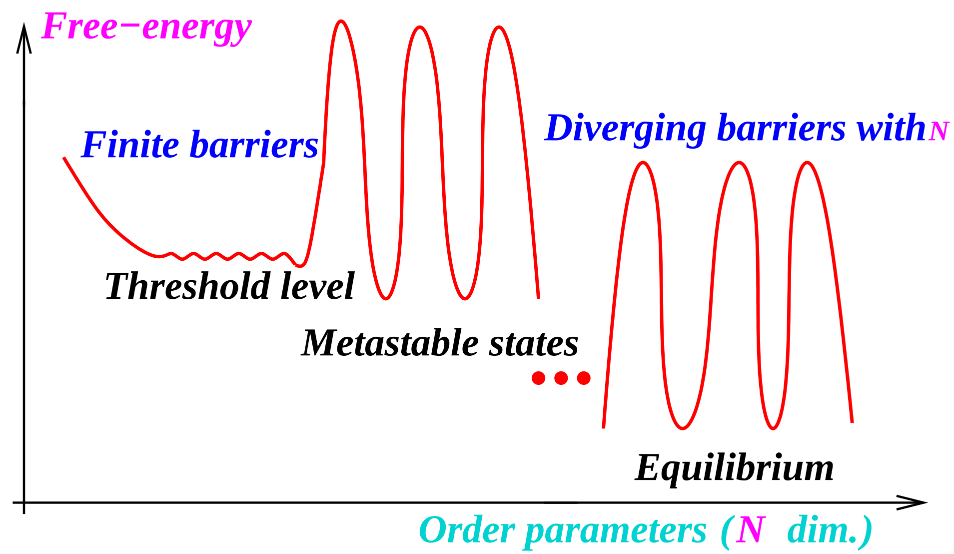
Is T_{eff} related to an entropy ?

Configurational entropy

An exponentially large number of metastable states is reached dynamically



Curie-Weiss (ferro)



Sketch of free-energy landscape

Threshold level is reached asymptotically

e.g. $\lim_{t_w \to \infty} \mathcal{E}(t) = \mathcal{E}_\infty > \mathcal{E}_{eq}$.

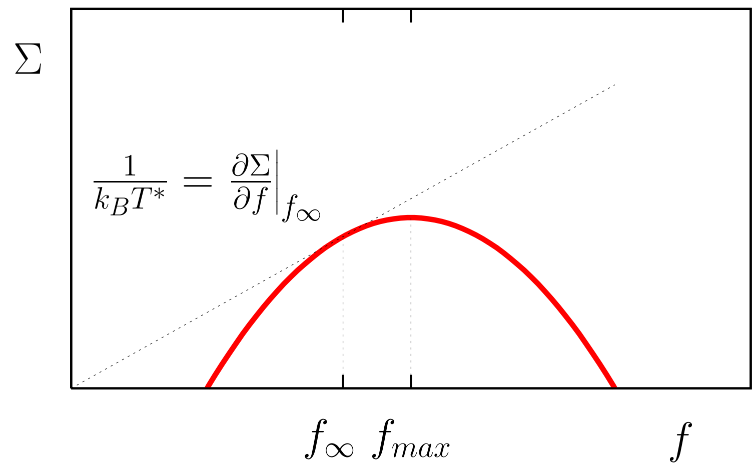
Well-understood in mean-field models with the

Thouless-Anderson-Palmer technique

Is $T_{\rm eff}$ related to an entropy ?

Configurational entropy

$$\Sigma(f) = k_B \ln \mathcal{N}(f) \qquad \Rightarrow \qquad \frac{1}{k_B T^*} = \left.\frac{\partial \Sigma(f)}{\partial f}\right|_{f_\infty}$$



NB $f_{max} \neq f_\infty \Rightarrow$ failure of ‘maximum entropy principles’.

Edwards & Oakshott 89, Monasson 95, Nieuwenhuizen 98

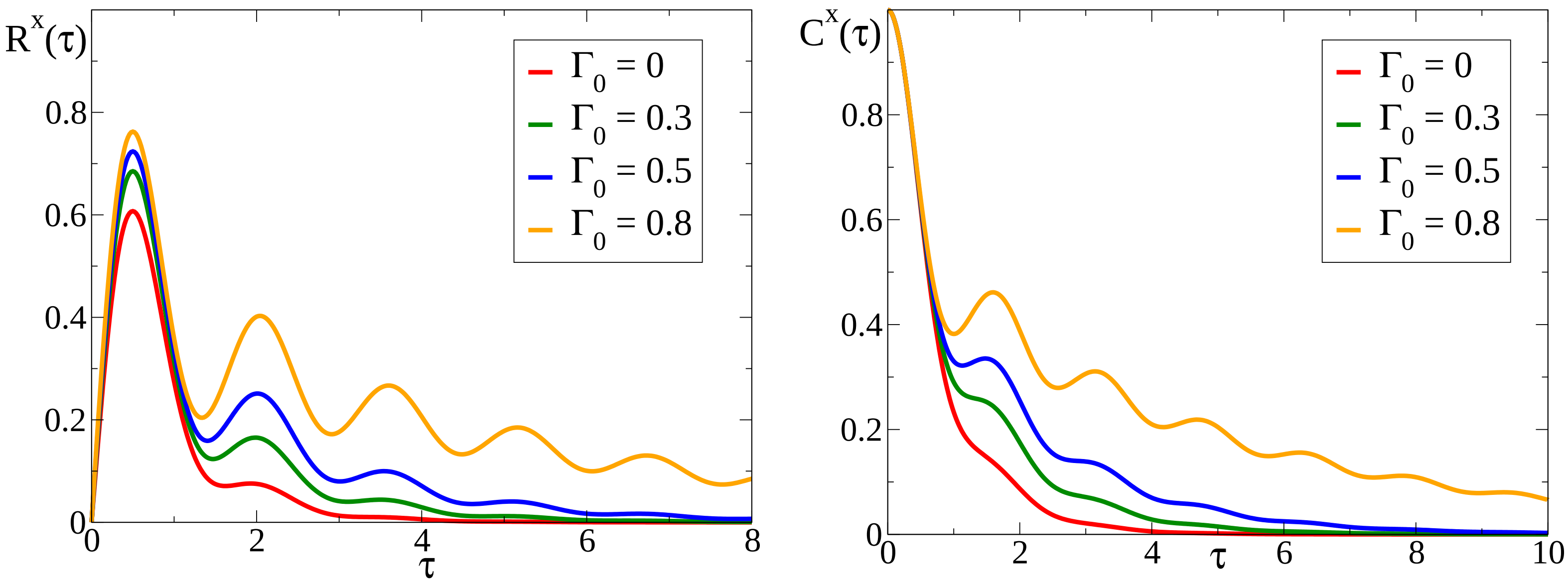
Very sketchy view : many amorphous solid configurations ($\Sigma \Leftrightarrow T^*$) and vibrations around them ($f \Leftrightarrow T$).

Quantum quench

T_{eff} from FDT ? Longitudinal spin

A quantum quench $\Gamma_0 \to \Gamma$ of the **isolated Ising chain**

Here : to its critical point $\Gamma = 1$



Linear response and symmetrized correlation of σ^x

Foini, LFC & Gambassi 11