From Brownian motion to impurity dynamics in 1d quantum liquids

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Taipei, July 2013
Preamble

- Interest is the dynamics of complex out of equilibrium classical and quantum matter (glasses, active matter, spin-ice, etc.)
- One way to study some of these systems is to use tracers and follow their evolution to characterise the medium.
- Here: revert the problem, back to Brownian motion for relatively simple baths and interactions.
Plan

- **Classical dissipative dynamics**
  - Langevin’s approach to Brownian motion.
  - White/coloured noise.
  - General Langevin equations with additive noise.
  - Examples: protein dynamics, tracers in active matter.
  - Microscopic modelling: equilibrium unchanged, dynamics modified.

- **Quantum systems**
  - Quantum Brownian motion and quenches
  - Experimental realisation and theoretical description
  - Bath and interaction modelling
  - Polaron effect and potential renormalisation
  - Consequences & conclusions
Plan

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Langevin’s approach

A Brownian particle immersed in a liquid in thermal equilibrium

The effect of the collisions with the liquid particles is mimicked by two forces:

- a viscous drag that tends to slow down the tracer, \(\vec{f}(\vec{v}) = -\eta_0 \vec{v} + \ldots\)
  - the relevant parameter is the friction coefficient \(\eta_0\).

- a random noise \(\vec{\xi}\) that mimics thermal agitation
  - the relevant parameter is temperature \(T\).

If the collision time \(\tau_c\) satisfies \(\tau_c \ll t_0\) (a particle’s ‘microscopic’ time),

- the statistics of the noise is Gaussian (central limit theorem).

If \(\tau_c \ll t\) (the observation time)

- the noise is white and the dynamics are Markovian (no memory).
Langevin’s approach

Stochastic Markov dynamics

From Newton’s equation \( \vec{F} = m \vec{a} = m \vec{\ddot{v}} \) and \( \vec{v} = \vec{\dot{r}} \)

\[
m\dot{v}_a = -\eta_0 v_a + \xi_a \quad (\vec{F}_{\text{ext}} = \vec{0})
\]

with \( a = 1, \ldots, d \) (the dimension of space),
\( m \) the particle mass,
\( \eta_0 \) the friction coefficient,
and \( \vec{\xi} \) the time-dependent thermal noise with Gaussian statistics,
zero average \( \langle \xi_a(t) \rangle = 0 \) at all times \( t \),
and delta-correlations \( \langle \xi_a(t) \xi_b(t') \rangle = 2 \eta_0 k_B T \delta_{ab} \delta(t - t') \).

Dissipation: for \( \eta_0 > 0 \) the averaged energy is not conserved,
\[
2\langle E(t) \rangle = m \langle v^2(t) \rangle \neq ct.
\]

The relation between friction coefficient & noise amplitude ensures

\[
\text{equipartition} \quad m \langle v^2(t) \rangle \rightarrow k_B T \quad \text{for} \quad t \gg t_r \equiv \frac{m}{\eta_0}
\]
Langevin’s approach

Markov normal diffusion

For simplicity: take a one dimensional system, \( d = 1 \).

\[
\langle x^2(t) \rangle \rightarrow 2D t^{2H}
\]

(for \( t \gg t^v_r = m/\eta_o \))

with \( D \) the diffusion coefficient,

\[
D = k_B T/\eta_o
\]

the Einstein relation,

\( H \) the Hurst exponent and

\[
H = 1/2
\]

i.e. normal diffusion

for a white bath (no correlations).
**Langevin’s approach**

Markov 1d relaxation in a harmonic potential

\[ F = -\frac{dV(x)}{dx} \quad \text{with} \quad V(x) = \frac{1}{2} \kappa x^2 \]

**Underdamped dynamics** for parameters such that \( \kappa > \frac{\eta_0^2}{m} \)

\[ \langle x^2(t) \rangle \rightarrow \frac{k_B T}{\kappa} + ct \sin \Omega t e^{-t/t_r} \]

with

\[ \Omega \propto \sqrt{\frac{\kappa}{m} - \frac{\eta_0^2}{m^2}} \quad \eta_0 \ll m \kappa \quad \sqrt{\frac{\kappa}{m}} \]

Damped \( (\eta_0 \neq 0 \text{ but small}) \) oscillations in a confining harmonic potential

Asymptotic equilibrium (equipartition)
Langevin’s approach

Markov 1d relaxation in a harmonic potential

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Overdamped dynamics for parameters such that \( \kappa < \frac{\eta_0^2}{m} \)

\[ \langle x^2(t) \rangle \rightarrow \frac{k_B T}{\kappa} + ct e^{-t/t_r^x} \]

for \( t \gg t_r^v = m/\eta_o \),
and \( t_r^x \propto \eta_o/\kappa \),
the position-relaxation time.

Exponential relaxation (\( \eta_o \) large) in a confining potential
Asymptotic equilibrium (equipartition)
The system, \( \{r_a\} \), with \( a = 1, \ldots, d \), coupled to a general environment in equilibrium evolves according to the non-Markov eq.

\[
\begin{align*}
 m \ddot{v}_a(t) + \int_{t_0}^{t} dt' \sum^K_B(t - t') v_a(t') &= -\frac{\delta V(\{\vec{r}\})}{\delta r_a(t)} + \xi_a(t) \\
 \text{Inertia} & \quad \text{friction} & \quad \text{deterministic force} & \quad \text{noise}
\end{align*}
\]

Coloured noise with correlation \( \langle \xi_a(t)\xi_b(t') \rangle = k_B T \delta_{ab} \Sigma^K_B(t - t') \) and zero mean.

\( T \) is the temperature of the bath and \( k_B \) the Boltzmann constant.

The friction kernel is \( \Sigma^K_B(t - t') \) [generalizes \( 2\eta_0 \delta(t - t') \)].
Colored noise

Equilibrium unaltered, dynamics altered

Generic: Most of the exact fluctuation-dissipation relations in and out of equilibrium remain unaltered for generic $\Sigma^K_B$, e.g. the fluctuation-dissipation theorem, fluctuation theorems, etc. Equilibrium at $V(\vec{r})$ is reached.

Particular: The observables’ time-dependent functional form depends on the noise, i.e. on $\Sigma^K_B$ or its Fourier transform $S(\nu)/\nu$

$$S(\nu) = \eta_0 \left( \frac{\nu}{\omega_o} \right)^\alpha e^{-\nu/\omega_c}$$

with the spectral density

for $\alpha \leq 1$ the high-frequency cut-off $\omega_c$ can be set to infinity and

$$\Sigma^K_B(t) = \frac{\eta_0}{\omega_o^{\alpha - 1} \Gamma_E(1 - \alpha)} t^{-\alpha} \quad \text{with} \quad 0 < \alpha \leq 1$$

with $\eta_0$ the ‘friction coefficient’ and $\Gamma_E$ the Euler-function.

for $\alpha > 1$ a $\omega_c < +\infty$ is needed and

$$\Sigma^K_B(t) = t^{-\alpha} g(\omega_c t)$$
Langevin’s approach

sub-Ohmic 1d over-damped relaxation in a harmonic potential

\[ F = -\frac{dV}{dx} \quad \text{with} \quad V(x) = \frac{1}{2} \kappa x^2 \quad \& \quad \text{bath with} \quad \alpha \leq 1 \]

\[ \langle x^2(t) \rangle \to \frac{k_B T}{\kappa} + ct E_{\alpha,1} \left( \frac{t}{t_r^x} \right) \]

for \( t \gg t_r^v = m/\eta_o \),

and \( t_r^x \propto \eta_o/\kappa \)

the position-relaxation time.

with \( E_{\alpha,1}(z) = \sum_{k=0} \frac{z^k}{\Gamma_E(\alpha k + 1)} \) the Mittag-Leffler function.

Only for an Ohmic bath \( \alpha = 1 \) the relaxation is exponential \( E_{1,1}(z) = e^z \)

sub-Ohmic bath \( \alpha < 1 \) : \( E_{\alpha,1}(z) \to z^{-1} \) for \( z \to -\infty \) algebraic decay.
Protein dynamics

Questions: what are the potential and the bath?

$x(t)$ distance between Tyr and FAD

$\alpha = 0.51 \pm 0.07$

Yang et al 03; Min, Luo, Cherayil, Kou & Xie 05
Active matter

Monitoring tracers’ diffusion to characterise the environment

Liquid-like regime

Clustering regime

Langevin description?

White/colored, additive/multiplicative out of equilibrium noise?

Work in progress w/ G. Gonnella, G. Laghezza, A. Lamura (Bari) & A. Sarracino (Roma/Paris)
Microscopic modelling

From deterministic to stochastic

In order to derive an effective Langevin equation for a classical system coupled to a classical bath one writes down:

the Hamiltonian of the ensemble

$$\mathcal{H} = \mathcal{H}_{\text{syst}} + \mathcal{H}_{\text{env}} + \mathcal{H}_{\text{int}}$$

The dynamics of all variables are given by Newton rules.

One has to give the initial \( \{\mathbf{q}_n(0), \mathbf{p}_n(0); \mathbf{r}(0), \mathbf{p}(0)\} \).

**Dissipative case**: if \( \mathcal{H}_{\text{int}} \neq 0 \) the total energy is conserved, \( E = ct \), but each contribution is not; in particular, \( E_{\text{syst}} \neq ct \) and we’ll take \( E_{\text{syst}} \ll E_{\text{env}} \).
Microscopic modelling

From deterministic to stochastic

E.g., an ensemble of 1d harmonic oscillators and coupling \( \sum_n c_n q_n f(x) \)

\[
\mathcal{H}_{\text{env}} + \mathcal{H}_{\text{int}} = \sum_{n=1}^{N} \left[ \frac{\pi_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} \left( \frac{c_n}{m_n \omega_n^2} f(x) - q_n \right)^2 \right]
\]

Classically, one can solve Newton’s equations for the oscillator variables. Assuming that the initial conditions are taken from a p.d.f. \( \varrho(t_0) \), that the environment is coupled to the sample at \( t_0 \),

\[
\varrho(t_0) = \varrho_{\text{syst}}(t_0) \varrho_{\text{env}}(t_0)
\]

and that its variables are characterized by a Gibbs-Boltzmann distribution at inverse temperature \( \beta \),

\[
\varrho_{\text{env}}(t_0) \propto e^{-\beta (\mathcal{H}_{\text{env}} + \mathcal{H}_{\text{int}})}
\]

one finds:

- a Langevin equation with multiplicative/additive colored-noise (e.o.m.)
- or a reduced dynamic generating functional \( Z_{\text{red}} \) (functional formalism).
General Langevin equation

Recall

The system, \{ r_a \}, with \( a = 1, \ldots, d \), coupled to a general environment in equilibrium evolves according to the non-Markov eq.

\[
\dot{m}v_a(t) + \int_{t_0}^{t} dt' \Sigma_B^{K}(t - t') v_a(t') = -\frac{\delta V(\{ \vec{r} \})}{\delta r_a(t)} + \xi_a(t)
\]

Inertia \hspace{2cm} friction \hspace{2cm} deterministic force \hspace{2cm} noise

Coloured noise with correlation \( \langle \xi_a(t)\xi_b(t') \rangle = k_B T \delta_{ab} \Sigma_B^{K}(t - t') \) and zero mean.

The friction kernel is \( \Sigma_B^{K}(t - t') \) [generalizes \( 2\eta_0 \delta(t - t') \)].

One can derive Langevin eqs. with multiplicative coloured noise as well.

Correlated initial conditions can be treated in the functional formalism.
Obtain the generating functional

\[ Z_{\text{red}}[\zeta] = \int D\text{variables} \ e^{-S[\zeta]} \]

with the action given by

\[ S = S_{\text{det}} + S_{\text{init}} + S_{\text{diss}} + S_{\text{sour}}[\zeta] \]

where \( S_{\text{det}} \) characterises the deterministic evolution, \( S_{\text{init}} \) the initial distribution, \( S_{\text{diss}} \) the dissipative and fluctuating effects due to the bath, and \( S_{\text{sour}} \) the terms containing the sources \( \zeta \).

Correlations between the particle and the bath at the initial time \( t_0 = 0 \) are taken into account via \( g(t_0) \) and then \( S_{\text{init}} \).

Once written in this way, the usual field-theoretical tools can be used. In particular, the minimal action path contains all information on the dynamics of quadratic theories.
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A quantum impurity

in a one dimensional harmonic trap

K atom : the impurity \((1.4 \text{ on average per tube})\)

Rb atoms : the bath \((180 \text{ on average per tube})\)

all confined in one dimensional tubes

\[ T \simeq 350 \text{ nK} \]

\[ \hbar \beta \sqrt{\kappa_0/m} \simeq 0.1 \]

Catani et al. 12 (Firenze)
A quantum impurity in a one dimensional harmonic trap

One atom trapped by a laser beam

\[ \hat{H}_{syst}^0 = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} \kappa_0 \hat{x}^2 \]

in contact with a bath made by a different species \( \hat{H}_{env} \).

Hamiltonian of the coupled system includes an interaction term

\[ \hat{H}_0 = \hat{H}_{syst}^0 + \hat{H}_{env} + \hat{H}_{int} \]

All atoms are within a wider (\( \kappa \) small) one-dimensional harmonic trap (not shown).
Experimental protocol

A quench of the system

Initial equilibrium of the coupled system:

\[ \tilde{\rho}(t_0) \propto e^{-\beta \hat{H}_0} \]

with

\[ \hat{H}_0 = \hat{H}^0_{syst} + \hat{H}_{env} + \hat{H}_{int} \]

and

\[ \hat{H}^0_{syst} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} \kappa_0 \hat{x}^2 \]

At time \( t_0 = 0 \) the impurity is released, the laser blade is switched-off and the atom only feels the wide confining harmonic potential \( \kappa_0 \rightarrow \kappa \)

subsequently, as well as the bath made by the other species.

Question: what are the subsequent dynamics of the particle?

Catani et al. 12
Equal-times correlation

Experiment: breathing mode

\[ \sigma^2(t) = C_x(t, t) \equiv \langle \hat{x}^2(t) \rangle \]

Under damped oscillations

For four values of the coupling to the bath, \( \eta \propto w/\omega_L \)

(interaction strength impurity-bath / interaction strength bath-bath).

Catani et al. 12
Reduced system

Model the environment and the interaction

E.g., an ensemble of quantum harmonic oscillators and \( \sum_n c_n \hat{q}_n f(\hat{x}) \)

\[
\hat{H}_{env} + \hat{H}_{int} = \sum_{n=1}^{N} \left[ \frac{\hat{\pi}_n^2}{2m_n} + \frac{m_n \omega_n^2}{2} \left( \frac{c_n}{m_n \omega_n^2} f(\hat{x}) - \hat{q}_n \right)^2 \right]
\]

Quantum mechanically, one can solve Heisenberg’s equations for the oscillator operators.

an operator Langevin equation with a force that depends on the oscillator’s initial values and is an operator

\( c \)-valued approximations are wrong; it is not obvious how to handle generic \( \hat{\varrho}(t_0) \) with this approach.

A reduced dynamic generating functional \( \mathcal{Z}_{red} \)

is a much more powerful technique.
Functional formalism

Influence functional

Feynman-Vernon 63, Caldeira-Leggett 84

Obtain the generating functional

\[ Z_{\text{red}}[\zeta] = \int \mathcal{D}\text{variables} \ e^{\frac{i}{\hbar} S[\zeta]} \]

with the action given by

\[ S = S_{\text{det}} + S_{\text{init}} + S_{\text{diss}} + S_{\text{sour}}[\zeta] \]

where \( S_{\text{det}} \) characterises the deterministic evolution, \( S_{\text{init}} \) the initial density matrix, \( S_{\text{diss}} \) the dissipative and fluctuating effects due to the bath, and \( S_{\text{sour}} \) the terms containing the sources \( \zeta \).

Correlations between the particle and the bath at the initial time \( t_0 = 0 \) are taken into account via \( \hat{\rho}(t_0) \) and then \( S_{\text{init}} \).

Once written in this way, the usual field-theoretical tools can be used. In particular, the minimal action path contains all information on the dynamics of quadratic theories.
Quantum dynamics

Some non-trivial effects under quantum Ohmic dissipation ($\alpha = 1$)

$P_{tunn} \rightarrow 0$

Suppression of tunnelling or

Localisation in a double well potential

at $k_B T = 0$ for $\tilde{\eta}_o > 1$

Bray & Moore 82, Leggett et al 87

\begin{equation}
\langle x^2(t) \rangle \rightarrow \begin{cases}
\frac{2k_B T}{\eta_o} t & \text{Classical } k_B T \neq 0 \\
\frac{\hbar}{\eta_o} \ln t & \text{Quantum } k_B T = 0
\end{cases}
\end{equation}

Schramm-Grabert 87
The model

The bath in the experiment

The environment is made of interacting bosons in one dimension that we model as a Luttinger liquid.

The local density operator is \( \hat{\rho}(x) = \rho_0 - \frac{1}{\pi} \frac{d}{dx} \hat{\phi}(x) \).

A canonical conjugate momentum-like operator \( \hat{\Pi}(x) \) is identified.

One argues

\[
\hat{H}_{env} = \frac{\hbar}{2\pi} \int dx \left[ \frac{u}{K} \left( \frac{d\hat{\phi}(x)}{dx} \right)^2 + \frac{uK\pi^2}{\hbar^2} \hat{\Pi}^2(x) \right]
\]

The sound velocity \( u \) and LL parameter \( K \) are determined by the microscopic parameters in the theory. For, e.g., the Lieb-Liniger model of bosons with contact potential \( \hbar\omega_L \sum_{i<j} \delta(\hat{x}_i - \hat{x}_j) \), one finds \( u(\gamma)K(\gamma) = \hbar\pi\rho_0/m_b \) and an expression for \( K(\gamma) \) with \( \gamma = m_b\omega_L/(\hbar\rho_0) \).

\( \gamma_{\text{exp}} \approx 1 \) Catani et al. 12

\( t \)-DMRG of Bose-Hubbard model confirmation for \( \hbar w \) small and \( \hbar\omega_L \) large

S. Peotta, M. Polini, D. Rossini, F. Minardi & R. Fazio 13
The model

The interaction in the experiment

- The interaction is $\hat{H}_{\text{int}} = \int drdr' U(|r - r'|) \delta(\hat{x} - r') \hat{\rho}(r)$ with $\tilde{U}(p) = \hbar we^{-p/p_c}$, quantized wave-vectors $p \rightarrow p_n = \pi \hbar n/L$, and $L$ the ‘length’ of the tube.

- After a transformation to ladder operators $\hat{b}_n, \hat{b}^\dagger_n$ for the bath, the coupling $\hat{H}_{\text{int}}$ becomes $\hat{H}_{\text{int}} \propto \sum p_n i p_n \tilde{U}(p_n) e^{-i p_n \hat{x}/\hbar} \hat{b}_{p_n} + \text{h.c.}$

- One constructs the Schwinger-Keldysh path-integral for this problem.

- Low-energy expansion: $e^{i p_n \hat{x}/\hbar}$ to quadratic order, the action becomes the one of a particle coupled to a bath of harmonic oscillators with coupling constants determined by $p_n$. The spectral density $S(\nu)/\nu$ is fixed. A further approximation, $L \rightarrow \infty$, is to be lifted later.
The model

Schwinger-Keldysh generating functional

The effective action has delayed quadratic interactions (both dissipative and noise effects) mediated by

\[ \Sigma^K_B(t - t') = 2 \int_0^\infty d\nu \frac{S(\nu)}{\nu} \cos[\nu(t - t')] \]

with the (Abraham-Lorentz) spectral density (\(\hbar = 1\))

\[ S(\nu) = \frac{\pi}{2L} \sum_{p_n} \frac{K}{2\pi} \left| p_n \right|^3 \left| \tilde{U}(p_n) \right|^2 \delta(\nu - u|p_n|) \]

\[ \rightarrow \eta_o \left( \frac{\nu}{\omega_c} \right)^3 e^{-\nu/\omega_c} \]

continuum limit for \( L \to \infty \)

\[ \eta_o = K \omega^2 \omega_c^3 / u^4 \] with \( \omega_c = up_c \)  

Super-Ohmic diss.  \( \alpha = 3 \)

\( K \) LL parameter, \( u \) LL sound velocity, \( \hbar w \) strength of coupling to bath, \( \omega_c \) high-freq. cut-off
The model

Schwinger-Keldysh generating functional

The action is quadratic in all the impurity variables.

The generating functional of all expectation values and correlation functions can be computed by the stationary phase method (exact in this case) as explained in, e.g., Grabert & Ingold’s review

with some extra features: rôle of initial condition, quench in harmonic trap, non-Ohmic spectral density, possible interest in many-time correlation functions.

The equal-times correlation \( C_x(t, t) = \langle \hat{x}^2(t) \rangle \) is thus calculated, ignoring for the moment the polaron effect (mass renormalisation) and the potential renormalisation due to the fact that the bath itself is confined.
Equal-times correlation

Experiment: breathing mode

\[ \sigma^2(t) = C_x(t, t) \equiv \langle \hat{x}^2(t) \rangle \]

Damped oscillations for four values of the coupling to the bath

around the same asymptotic value, that is independent of \( \eta = 0, 1, 4 \).
(For \( \eta > 4 \) the system is no longer 1d)

Catani et al. 12

NB independence of the width of breathing mode, \( \lim_{t \to \infty} \sigma^2(t) \) on \( \eta \)
for relatively small \( \eta \). Also seen in lattice numerics Peotta et al. 13.
Equal-times correlation

Theory

\[ \sigma^2(t) = C_x(t, t) \equiv \langle \hat{x}^2(t) \rangle \]

Damped oscillations

For two values of the coupling to the bath (to be made precise below).

Bonart & LFC 12
The frequency $\Omega$ increases with the coupling to the bath $\eta_0$ for sufficiently narrow (large $\sqrt{\kappa/m \omega_c^{-1}}$) harmonic traps.

The super-Ohmic $S(\nu)$ is responsible for this ‘classical’ feature.

The height of the peak depends on $\sqrt{\kappa/m \omega_c^{-1}}$ with $\omega_c$ the cut-off of the bath spectral function. The order of magnitude of this tiny effect (1%) is similar to the one measured, but the errorbars are larger (5%).

Bonart & LFC 12
Beyond

Polaron & potential renormalisation

Another point of view:

\[ \Omega \approx \sqrt{\frac{\kappa}{m}} \text{ for all } \eta \leq 4 \]

• The coupled Hamiltonian resembles strongly Frölich Hamiltonian and an impurity mass renormalisation \( m \to m^*(\eta) \) should be expected.

• However, the constancy of \( \Omega \) with \( \eta \) suggests that the harmonic potential spring constant should also be renormalised \( \kappa \to \kappa^*(\eta) \) to counteract this change and keep the natural frequency \( \sqrt{\frac{\kappa^*}{m^*}} \) (roughly) constant.

However, this is inconsistent with \( \langle \hat{x}^2(t) \rangle \to \frac{k_B T}{\kappa^*(\eta)} \)

since one does not observe an \( \eta \)-dependence of the asymptotic cloud width (for \( \eta \leq 4 \)).
around the same asymptotic value, that is independent of $\eta = 0, 1, 4$.
(For $\eta > 4$ the system is no longer $1d$)

Catani et al. 12

Independence of the width of breathing mode, $\lim_{t \to \infty} \sigma^2(t)$ on $\eta$ for relatively small $\eta$. Also seen in lattice numerics Peotta et al. 13.
A way out

**Polaron.** The bath density profile follows the impurity creating a dressed impurity with renormalised mass, \( m^* = (1 + \mu(v))m \),

with \( \mu(v) \) estimated from the kinetic energy gained by the impurity after rapid acceleration from \( v_o \) to \( v \) due to injection of energy that goes partially into a wave excitation, dynamic analysis of e.o.m.

**Potential renormalisation.** The impurity needs more energy to create a bath excitation (e.g., bosons have to climb the potential). The potential felt by the impurity gets renormalised \( \kappa^* = (1 + \tilde{\mu}(v))\kappa \),

with \( \tilde{\mu}(v) \) estimated from the force acting on the impurity & density cloud.

**Experimentally** : the dynamics feel these two effects but the asymptotic statics does not: \( \lim_{t \to \infty} \sigma^2(t) \to k_B T/\kappa \).

**Repeat Schwinger-Keldysh formalism & Gaussian approximation analysis**

with \( m^* \) and \( \kappa^* \) for the dynamics.

**Asymptotic behaviour cannot be described with the Gaussian approximation**; an interpolation is proposed.  

Bonart & LFC EPL 13
Potential renormalisation

Theory vs. experiment

\[
\frac{\sigma_p(\eta \neq 0)}{\sigma_p(\eta = 0)} \quad \simeq \quad \sqrt{\frac{\kappa}{\kappa^*}}
\]

\[2E_0 \simeq k_B T \]
\[\simeq \kappa^* \sigma_p^2(\eta) \]
\[\simeq \kappa \sigma_p^2(\eta = 0)\]

\[\eta \propto \omega / \omega_L\]

Experimental data points estimated from amplitude of the first oscillation

\[
\sigma_p(\eta \neq 0) / \sigma_p(\eta = 0)
\]

Triangles: equilibrium variational theory for \(\sqrt{m/m^*}\)  

Dotted lines: \(\sqrt{\kappa/\kappa^*}\) for \(\gamma = 0.2, 0.35, 0.5\)  

Catani et al. 12

Bonart & LFC EPL 13
Breathing mode

Theory vs. experiment

Dynamics with $m^*$ and $\kappa^*$, interpolation to $\lim_{t \to \infty} \sigma^2(t) \to k_B T / \kappa$:

$$
\sigma^2(t) = \frac{\hbar^2 \kappa_0}{4k_B T} \mathcal{R}(t) - \frac{\kappa^*}{k_B T} C_{eq}^2(t) + \frac{k_B T}{\kappa^*} + (1 - e^{-\Gamma t}) \left( \frac{k_B T}{\kappa} - \frac{k_B T}{\kappa^*} \right)
$$

$w/\omega_L = 1$

$w/\omega_L = 4$
Summary

- **Classical and quantum dynamics**
  
  technically very similar once in the path-integral formalism.

- **Classical systems**
  
  Single particle: non-Markovian environments are very popular in bio-physics; they are usually the ‘unknown’

- **Quantum systems**
  
  Quantum Brownian motion and quenches: a rather simple problem with non-trivial consequences of the coupling to the bath.
Breathing mode w/TDMRG

Bose-Hubbard model for the bath & interaction

Flat trap with length \( L = 250 \)

\( N_b = 22 \) bosons

\( n_i = \langle \hat{b}_i^\dagger \hat{b}_i \rangle \sim c t \lesssim 0.1 \) in \( 2L/3 \)

Model \( \simeq \) Lieb-Liniger

Coupling \( \mathcal{H}_{\text{int}} = u_{\text{int}} \sum_i \hat{n}_i \hat{N}_i \)

Mass difference mimicked by \( J_2/J_1 = 2 \).

Study of \( \Omega \) yields approximate independence of \( u_{\text{int}} \) for \( u_b \gtrsim 0.5 \) (Tonks-Girardeau limit).

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Breathing mode w/TDMRG

Bose-Hubbard model for the bath & interaction

Flat trap with length $L = 250$

$N_b = 22$ bosons

$n_i = \langle \hat{b}_i^\dagger \hat{b}_i \rangle \sim ct \lesssim 0.1 \text{ in } 2L/3$

Model $\simeq$ Lieb-Liniger

Coupling $\mathcal{H}_{\text{int}} = u_{\text{int}} \sum_i \hat{n}_i \hat{N}_i$

Mass difference mimicked by $J_2/J_1 = 2$.

Study of $\Omega$ yields approximate independence of $u_{\text{int}}$ for $u_b \gtrsim 0.5$ (Tonks-Girardeau limit).

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Mean occupation numbers

TDRG data

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A way out: details

- **Polaron.** Dressed impurity with renormalised mass

  \[ m^* = (1 + \mu(v))m, \]

  with \( \mu \propto \omega_c w^2 K/(mu^4) f(v_o, v) \), estimated from the kinetic energy gained by the impurity after rapid acceleration from \( v_o \) to \( v \) due to injection of energy that goes partially into a wave excitation.

- **Potential renormalisation.** The potential felt by the impurity gets renormalised

  \[ \kappa^* = (1 + \tilde{\mu}(v))\kappa, \]

  with \( \tilde{\mu}(v) = Kwug(v, u) \) estimated from sum of the force felt by the impurity \( -\kappa\dot{q} \rightarrow -\kappa \) plus the one felt by the cloud around it

  \[ -\kappa \int_{-L/2}^{L/2} dx x \hat{\rho}(x) \rightarrow -\kappa\tilde{\mu}(v) \]

- **Dynamics with** \( m^* \) and \( \kappa^* \), interpolation to \( \lim_{t \rightarrow \infty} \sigma^2(t) \rightarrow k_BT/\kappa \):

\[
\sigma^2(t) = \frac{\hbar^2 \kappa_0}{4k_BT} R(t) - \frac{\kappa^*}{k_BT} C_{eq}^2(t) + \frac{k_BT}{\kappa^*} + (1 - e^{-\Gamma t}) \left( \frac{k_BT}{\kappa} - \frac{k_BT}{\kappa^*} \right)
\]

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Breathing mode

Amplitude of first oscillation $\sigma_p$

- The impurity is initially in equilibrium with the bath at a ‘high’ temperature $T$ (the thermal energy is order 10 times the potential one).

  Its mean energy per d.o.f. is $2E_0 \simeq k_B T$.

- At the first peak the amplitude can be estimated as $\kappa^* \sigma_p^2(\eta) \simeq k_B T$ (dissipation only affects $\kappa^*$). Therefore,

\[
\frac{\sigma_p(\eta \neq 0)}{\sigma_p(\eta = 0)} = \sqrt{\frac{\kappa}{\kappa^*}}
\]

- As the breathing frequency is approximately $\eta$-independent $\Omega \simeq \Omega^*$:

\[
\sqrt{\frac{\kappa}{\kappa^*}} \simeq \sqrt{\frac{m}{m^*}}
\]
Breathing mode

Theory

\[ \sigma^2(t) = C_x(t, t)(m^*, \kappa^*) + \left(1 - e^{-\Gamma t}\right) \left(\frac{k_B T}{\kappa} - \frac{k_B T}{\kappa^*}\right) \]

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