1. Prove the static **FDT theorem**

$$\chi = \frac{\beta}{N} \sum_{ij} C_{ij}^{\text{conn}} \tag{1}$$

   that we used in the lectures, which is valid for all $\mathcal{H}$ and at all temperatures T, if the system is assumed to be in canonical equilibrium.

   Generalize it for

$$\chi_{AB} = \left. \frac{\partial \langle A \rangle}{\partial h_B} \right|_{h_B=0} = \beta C_{AB}^{\text{conn}} \tag{2}$$

   where the perturbation if such that $\mathcal{H} \mapsto \mathcal{H} - h_B B$.

2. **Short and long range interactions**
   Take a point-like particle of mass m located at the origin in interaction with matter with uniform density ρ within a D dimensional sphere with internal radius ϵ and external radius R. The interaction potential density is

$$v(r) = \frac{J\rho}{r^\alpha} \tag{3}$$

   (a) Analyze the R dependence of the total interaction energy between the particle and the mass in the sphere. Distinguish, short, marginal and long-range interaction depending on this energy approaching 0, $\ln R$ or infinity.

   (b) How do you classify Newtonian classical gravitation?

3. **Typical vs. average values of random variables**
   Consider a random variable z that takes only two values $z_1 = e^{\alpha\sqrt{N}}$ and $z_2 = e^{\beta N}$, with α and β two positive and finite numbers with α unconstrained and $\beta > 1$. The probabilities of the two events are $p_1 = 1 - e^{-N}$ and $p_2 = e^{-N}$.

   (a) Confirm that these probabilities are normalised.

   (b) Notice how these probabilities scale for N large and call a rare event the one with almost vanishing probability.

   (c) Compute the average $\langle z \rangle$, where the angular brackets indicate average with the probabilities p_1, p_2, and evaluate it in the limit $N \to \infty$. Calculate the most probable value taken by z, that we call z_{typ}, for typical (indeed, if we were to draw the variable we would typically get this value). Compare. In which of the two calculations does the rate event have an effect?

(d) Now, let us study the behaviour of $\ln z$ which is also a random variable. Compute its average. By which value of z is it determined? Does $\langle \ln z \rangle = (\ln z)_{\text{typ}}$ in the large N limit?

(e) Is $\langle \ln z \rangle = \ln \langle z \rangle$? The last result demonstrates the difference between what are called quenched and annealed averages. Which value is larger? Does the comparison comply with Jensen's inequality?

4. **Quenched vs. annealed averages**
Take the partition function $\mathcal{Z}_J = e^{\beta J N}$ with J a quenched random parameter, with can take two positive values according to

$$J = \begin{cases} J_1 \\ J_2 \end{cases} \qquad \text{with probability} \qquad p = \begin{cases} 1 - e^{-cN} \\ e^{-cN} \end{cases} \tag{4}$$

with $c > 0$. N plays the role of the "system size" and we will take it to infinity.

(a) Calculate the annealed and quenched free-energy densities

$$-\beta N f_{\text{annealed}} = \ln[\mathcal{Z}_J] \,, \qquad\qquad -\beta N f_{\text{quenched}} = [\ln \mathcal{Z}_J] \,, \tag{5}$$

where $[\dots]$ represent averages over J with the bimodal distribution above.

(b) Consider $c > \beta(J_2 - J_1)$, $c < \beta(J_2 - J_1)$ and $c = \beta(J_2 - J_1)$ separately, In which case(s) are f_{annealed} and f_{quenched} equal?

(c) When they are not, is Jensen's inequality satisfied?

(d) Conclude about the effect of the rare event J_2 (with vanishing probability). When does it contribute?

(e) In which case does it effect disappear?

A similar example is given by the dilute Ising chain

$$\mathcal{H} = -\sum_i J_i s_i s_{i+1} \tag{6}$$

with J_i i.i.d. quenched random couplings drawn from

$$p(J_i) = p\delta_{J_i,J_0} + (1-p)\delta_{J_i,0} \tag{7}$$

with $p \in (0,1)$.

(a) Calculate the annealed free-energy density

(b) Calculate the quenched free-energy density.

(c) Notice that for $\beta \to 0$ the two tend to coincide.

5. **Saddle-point & average**
Prove that the saddle-point u^* equals the average $\langle u \rangle$ in the integral

$$\int_{-\infty}^{\infty} du\, e^{-\beta N f(u)} \tag{8}$$

assuming that $f(u)$ has a single absolute minimum.

6. **Imry-Ma argument**
Determine the lower critical dimension D_l below which long-range ferromagnetic order in the standard D dimensional Ising model is destroyed by arbitrarily weak quenched random fields.
Consider a random field Ising model,

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle ij \rangle} s_i s_j - \sum_i h_i s_i \tag{9}$$

where $J > 0$ and the double sum runs over nearest neighbours on a D dimensional lattice. h_i are quenched i.i.d. random fields taken from a probability distribution $P(\{h_i\}) = \prod_i p(h_i)$ with $[h_i] = 0$ and $h_i^2] = h^2$.

Take a saturated ferromagnetic configuration with $m = -1$, say, and place a spherical domain with radius R of the degenerate ground state $m = 1$. Estimate the energy difference: $\Delta E(R) = E_{\text{with domain}}(R) - E_{\text{ferro}}$ and show that in $D > 1$ $\Delta E(R) < 0$ is a non-monotonic function of R with a maximum at $R_{\max}$ and zeroes at $R = 0$ and $R_0 > R_{\max}$.

Use $\Delta E(R)$ to argue that the zero temperature ferromagnetic phase is killed by the random fields in $D \leq 2$.

7. **TAP free-energy landscape**
Take the SK model. Assume that the joint pdf of the spin configurations is given by the factorized form

$$P(\{s_i\}) = \prod_i p(s_i) \qquad p(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1} \tag{10}$$

(a) Prove that this pdf is normalized.

(b) Show that $\langle s_i \rangle = m_i$.

(c) Use this form to derive the naive TAP free-energy density starting from the thermodynamic expression for the free-energy $F = \langle \mathcal{H} \rangle - TS$ with $S = -k_B \langle \ln P(\{s_i\}) \rangle$.

(d) Get the Onsager reaction contribution to the free-energy and verify that it is of the same order as the other terms concerning powers of N.