Landau-Lifshitz-Gilbert-Brown

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Plan

- Macromagnetism: the stochastic Landau-Lifshitz-Gilbert-Brown (sLLGB) equation (precession, dissipation, noise and torque).
- Discretisation schemes for multiplicative white noise (Markov) stochastic differential equations. Asymptotic measure and drift term.
- Back to the sLLGB equation, analytic approach:
  - drift term & functional formalism in Cartesian and spherical coordinates; exact results.
- Back to the sLLGB equation, numerical simulations:
  - analysis of a benchmark.
- Future work.
Macromagnetism: the stochastic Landau-Lifshitz-Gilbert-Brown (sLLGB) equation (precession, dissipation, noise and torque).

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Future work.
Magnetisation precession

Bloch equation

Evolution of the time-dependent 3d magnetisation density per unit volume, \( \mathbf{M} = (M_x, M_y, M_z) \), with constant modulus \( M_s = |\mathbf{M}| \)

\[
\frac{d}{dt} \mathbf{M} = -\gamma_0 \mathbf{M} \wedge \mathbf{H}_{\text{eff}}
\]

\( \gamma_0 \equiv \gamma\mu_0 \) is the product of \( \gamma = \mu_B g/\hbar \), the gyromagnetic ratio, and \( \mu_0 \), the vacuum permeability constant (\( \mu_B \) Bohr’s magneton and \( g \) Lande’s \( g \)-factor)

For the initial condition \( \mathbf{M}(t_i) = \mathbf{M}_i \) the magnetisation precesses around \( \mathbf{H}_{\text{eff}} \)

with \( 2\mathbf{M} \cdot \frac{d}{dt} \mathbf{M} = \frac{d}{dt}|\mathbf{M}|^2 = 0 \)

and \( \frac{d}{dt}(\mathbf{M} \cdot \mathbf{H}_{\text{eff}}) = 0 \) (if \( \mathbf{H}_{\text{eff}} = ct \))

Bloch 32
Dissipative effects

Landau-Lifshitz & Gilbert equations

\[
d_t M = -\frac{\gamma_0}{1 + \eta^2 \gamma_0^2} M \wedge \left[ H_{\text{eff}} + \frac{\eta \gamma_0}{M_s} (M \wedge H_{\text{eff}}) \right]
\]

\[
d_t M = -\gamma_0 M \wedge \left( H_{\text{eff}} - \frac{\eta}{M_s} d_t M \right)
\]

2nd terms in RHS: dissipative mechanisms slow down the precession and push \( M \) towards \( H_{\text{eff}} \)

with \( 2M \cdot d_t M = d_t |M|^2 = 0 \)

and \( d_t (M \cdot H_{\text{eff}}) > 0 \)
Magnetic field

Conservative & non-conservative contributions

Local magnetic field

\[ \mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{eff}}^c + \mathbf{H}_{\text{eff}}^{nc} \]

Conservative contribution

\[ \mathbf{H}_{\text{eff}}^c = -\mu_0^{-1} \nabla M U \]

with \( U \) the energy per unit volume, can originate from an external (possibly time-dependent) magnetic field \( \mathbf{H}_{\text{ext}} \), and a crystal field \( \mathbf{H}_{\text{ani}} \) associated to the crystalline anisotropy,

\[ U = -\mu_0 M \cdot \mathbf{H}_{\text{ext}} + V_{\text{ani}}(M) \]

\( V_{\text{ani}}(M) \) is the anisotropy potential (per unit volume), e.g.

\[ V_{\text{ani}}(M) = K \sum_{i \neq j} M_i^2 M_j^2 \] (cubic crystalline structure)

\[ V_{\text{ani}}(M) = K(M_s^2 - M_z^2) \] (uniaxial symmetry)

Bertotti, Mayergoyz & Serpico 09 ; Coffey & Kamykov 12
Magnetic field

Non-conservative component

Local magnetic field $H_{\text{eff}} = H_{\text{eff}}^c + H_{\text{eff}}^{nc}$

Non-conservative contribution $H_{\text{eff}}^{nc} \neq -\mu_0^{-1} \nabla_M U$

Spin-torque term in Gilbert equation

$$H_{\text{eff}}^{nc} = -\frac{g\mu_B \hbar J(t) \mathcal{P}}{2M_s^2 d e} (M \wedge \mathbf{p})$$

$J(t)$ the current per unit area,

$\mathcal{P}$ the (dimensionless) polarisation function of the fixed layer,

$\mathbf{p}$ is a unit vector in the direction of the current,

$d$ the interlayer separation

$e$ is the electric charge of the carriers.

Berger 96; Slonczewski 96
Thermal fluctuations

à la Langevin in Gilbert’s formulation

\[ d_t M = -\gamma_0 M \wedge \left( H_{\text{eff}} + H - \frac{\eta}{M_s} d_t M \right) \]

\( H \) is a white random noise, with zero mean \( \langle H_i(t) \rangle = 0 \) and correlations

\[ \langle H_i(t)H_j(t') \rangle = 2D \delta_{ij} \delta(t - t') \]

For the moment, the (diffusion) parameter \( D \) is free.

The noise \( H \) multiplies the magnetic moment \( M \).

This is the Markov stochastic Landau-Lifshitz-Gilbert-Brown (sLLGB) multiplicative white noise stochastic differential equation.

Subtleties of Markov multiplicative noise processes are now posed.

Brown 63
Set up a path-integral formalism.

Set up a numerical integrator.

Use them.
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Stochastic calculus

Discretization prescriptions

\[ \mathrm{d}_t x(t) = f[x(t)] + g[x(t)] \, H(t) \]

with \( \langle H(t) \rangle = 0 \) and \( \langle H(t)H(t') \rangle = 2D \, \delta(t-t') \) means

\[ x(t + \mathrm{d}t) = x(t) + f[x(t)] \, \mathrm{d}t + g[x(t)] \, H(t) \mathrm{d}t \]

with

\[ \overline{x}(t) = \alpha x(t + \mathrm{d}t) + (1 - \alpha) x(t) \]

and \( 0 \leq \alpha \leq 1 \). Particular cases are \( \alpha = 0 \) Itô; \( \alpha = 1/2 \) Stratonovich.

The **chain rule** for the time-derivative is

\[ \mathrm{d}_t Y(x) = \mathrm{d}_t x \, \mathrm{d}_x Y(x) + D(1 - 2\alpha) \, g^2(x) \, \mathrm{d}_x^2 Y(x) \]

For \( \alpha = 1/2 \) (Stratonovich) one recovers the usual expression.

Stratonovich 67; Gardiner 96; Øksendal 00; van Kampen 07
The Fokker-Planck equation

\[ \partial_t P(x, t) = -\partial_x \left[ (f(x) + 2D\alpha g(x)d_x g(x))P(x, t) \right] + D \partial_x^2 [g^2(x)P(x, t)] \]

has the stationary measure

\[ P_{st}(x) = Z^{-1} [g(x)]^{2(\alpha - 1)} e^{\frac{1}{D} \int^x f(x') \frac{f(x')}{g^2(x')}} = Z^{-1} e^{-\frac{1}{D} U_{\text{eff}}(x)} \]

with

\[ U_{\text{eff}}(x) = - \int^x \frac{f(x')}{g^2(x')} + 2D(1 - \alpha) \ln g(x) \]

Remark: the potential \( U_{\text{eff}}(x) \) depends upon \( \alpha \) and \( g(x) \).

Noise induced phase transitions

Stratonovich 67; Sagués, Sancho & García-Ojalvo 07
Stochastic calculus

Fokker-Planck & stationary measure

e.g. $U_{\text{eff}}(x) = V(x) + 2D(1 - \alpha) \ln g(x)$

$x^2 + 2D(1 - \alpha) \ln x$

$g(x) = x$

$x^2 + 2D(1 - \alpha) \ln(1 - x^2)$

$g(x) = (1 - x^2)$
The **Gibbs-Boltzmann equilibrium**

\[ P_{\text{GB}}(x) = Z^{-1} e^{-\beta U(x)} \]

is approached if

\[ f(x) \leftrightarrow -g^2(x)dx U(x) - 2D\alpha g(x) dx g(x) \]

**Remark:** the drift is also needed for the Stratonovich mid-point scheme.

**Important choice:** if one wants the dynamics to approach thermal equilibrium independently of \( \alpha \) and \( g \) the drift term has to be added.
Stochastic calculus

Path-integral representation

The initial state at time $-\mathcal{T}$ is drawn from a probability distribution $P_i(x_{-\mathcal{T}})$.

The noise generates random trajectories with probability density $P(x; \alpha)$.

$$P(x; \alpha) = \langle \prod_t \delta(x_t - x_{t}^{\text{sol}}) \rangle$$

where $x_{t}^{\text{sol}}$ is a solution to the Langevin equation and depends on the noise $H$.

The constraint can be written $J \delta(\prod_t \text{Eq}_t[x, H; \alpha])$ with the Jacobian

$$J = \det_{tt'} \left[ \frac{\delta \text{Eq}_t[x, H; \alpha]}{\delta x_{t'}} \right]$$

Using the exponential representation of the delta $\delta(y) \propto \int d\hat{y} \ e^{i\hat{y}y}$

We write the probability density of a trajectory $x_t$ from $t = -\mathcal{T}$ to $t = \mathcal{T}$

$$P(x; \alpha) = P_i(x_{-\mathcal{T}}) \langle J \int \mathcal{D}[\hat{x}] \ e^{\int_{-\mathcal{T}}^{\mathcal{T}} dt \ i\hat{x}_t \ \text{Eq}_t[x, H; \alpha]} \rangle$$
Stochastic calculus

Path-integral representation

\[ P(x; \lambda, \alpha) \propto \int \mathcal{D}[\hat{x}] \, P(x, \hat{x}; \lambda, \alpha) = \int \mathcal{D}[\hat{x}] \, e^{S(x, i\hat{x}; \lambda, \alpha)} \]

and the Martin-Siggia-Rose-Janssen 79 action

\[
S(x, i\hat{x}; \lambda, \alpha) \equiv \ln P_i(x_{-\tau}, \lambda_{-\tau}) + \int \left[ i\hat{x}_t (d_t x_t - f_t + 2D\alpha g_t d_x g_t) + D(i\hat{x}_t)^2 g_t^2 - \alpha d_x f_t \right]
\]

From the Jacobian and the integration over the noise

cfr. Langouche, Roekaerts & Tirapegui 79; Lau & Lubensky 07

Remark: The action depends on \( \alpha \) and \( g \).

Observable averages can now be calculated as

\[
\langle A(x_t) \rangle = \int \mathcal{D}[x] \, P(x; \lambda, \alpha) A(x_t) \quad \text{(and do not depend on \( \alpha \))}
\]
Path probabilities

Path-integral representation

\[
P(\mathcal{T}x, \mathcal{T}\hat{x}; \bar{\lambda}, \bar{\alpha}) = e^{\Delta S(x, \hat{x}; \lambda, \alpha)}
\]

where \( \mathcal{T}x \) and \( \mathcal{T}\hat{x} \) are transformed trajectories,
\( \bar{\lambda} \) the transformed parameter in the potential,
\( \bar{\alpha} \) a different discretization parameter;

and from here obtain relations between observables by averaging this relation: equilibrium fluctuation dissipation (\( \Delta S = 0 \)), or out of equilibrium theorems (\( \Delta S \neq 0 \)).

Jarzinsky 97, Crooks 00, & many others
Let us define

\[ d_t^{(\alpha)} x_t \equiv d_t x_t - 2\beta^{-1}(1 - 2\alpha)g_t d_x g_t \]

and group two terms in the action due to the coupling to the bath

\[ S_{\text{diss}}[x, i\hat{x}] = \int i\hat{x}_t \left[ d_t^{(\alpha)} x_t + \beta^{-1}i\hat{x}_t g_t^2 \right] \]

This expression suggests to use the generalized transformation on the time-dependent variables \( \{x_t, i\hat{x}_t\} \)

\[ \mathcal{T}_c = \begin{cases} 
  x_t & \mapsto x_{-t}, \\
  i\hat{x}_t & \mapsto i\hat{x}_{-t} - \beta g_{-t}^{-2} d_t^{(\alpha)} x_{-t}, \end{cases} \]
For initial conditions drawn from \( P_i(x) = Z^{-1} e^{-\beta U(x)} \) and

\[
f(x) = -g^2(x) \frac{d}{dx} U(x) + 2D\alpha g(x) \frac{d}{dx} g(x)
\]

one proves

\[
S_{\text{det+jac}}[\{ c i \hat{x}, c x; \alpha \}] = S_{\text{det+jac}}[\{ i \hat{x}, x; \alpha \}]
\]

that implies

\[
P[\{ c i \hat{x}, c x; \alpha \}] = P[\{ i \hat{x}, x; \alpha \}]
\]

From this result we can prove exact equilibrium relations such as the fluctuation-dissipation theorem

\[
R(t, t') = \left. \frac{\delta \langle x_t \rangle}{\delta h_{t'}} \right|_{h=0} = \beta \partial_{t'} \langle x_t x_{t'} \rangle \theta(t - t')
\]
Stochastic calculus

Consequences of the transformation

For initial conditions drawn from $P_i(x) = Z^{-1}e^{-\beta U(x)}$ and

$$f(x, \lambda_t) = -g^2(x)\partial_x U(x, \lambda_t) + 2D\alpha g(x)d_x g(x)$$

one proves

$$P[T_c x; \alpha, \bar{\lambda}_t] = e^{\beta W - \beta \Delta F}$$

with

$$W = \int dt \, d_t \lambda_t \partial_{\lambda} U(x, \lambda)$$

$$\Delta F = \ln Z(\lambda_{\tau}) - \ln Z(\lambda_{-\tau})$$

the work, and free-energy difference between initial and fictitious final states.

Exact out of equilibrium relations such as the Jarzinsky relation follow

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$
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Magnetisation dynamics

Conservation of magnetisation modulus

\[ d_t M = -\gamma_0 M \wedge (\ldots) \quad \rightarrow \quad M \cdot d_t M = 0 \]

but

\[ d_t |M|^2 = d_t M_s^2 = 4D(1 - 2\alpha) \frac{\gamma_0^2}{1 + \eta^2 \gamma_0^2} M_s^2 \neq 0 \]

because of the \( \alpha \)-dependent chain rule associated to the Markov Landau-Lifshitz-Gilbert-Brown stochastic equation.

The sLLGB equation has to be modified to ensure the modulus conservation. This is achieved by replacing the derivative by the operator

\[ d_t \mapsto D_t^{(\alpha)} = d_t + 2D(1 - 2\alpha) \frac{\gamma_0^2}{1 + \eta^2 \gamma_0^2} \]
Magnetisation dynamics

Fokker-Planck equation and Gibbs-Boltzmann equilibrium

With this change, the probability density $P(M, t)$ is determined by an \( \alpha \)-independent Fokker-Planck equation.

If only conservative fields are applied $H_{\text{eff}} = \mu_0^{-1} \nabla_M U$, one proves that its asymptotic solution is of the Gibbs-Boltzmann form

$$P_{\text{GB}}(M) = Z^{-1} e^{-\beta VU(M)}$$

with $V$ the sample volume and $U$ the potential energy per unit volume, \textit{provided} the diffusion coefficient in the noise-noise correlation be chosen to satisfy

$$D = \frac{\eta k_B T}{M_s V \mu_0}$$
Formalism

Choices

Path-integral construction:

Landau-Lifshitz & Gilbert formalism; Cartesian & spherical coordinates.

Aron, Barci, LFC, González-Arenas & Lozano 14

Generic results on equilibrium and out of equilibrium theorems:

We preferred to use the Gilbert formalism in Cartesian coordinates.

Aron, Barci, LFC, González-Arenas & Lozano 14

Numerical simulations.

We preferred to use the Landau-Lifshitz formalism in spherical coordinates.

Romá, LFC & Lozano 14
The action in the Gilbert formalism in Cartesian coordinates

\[ S_G = S_{G,\text{det}} + S_{G,\text{diss}} + S_{G,\text{jac}} \]

with

\[ S_{G,\text{det}} = \ln P_i[\mathbf{M}(t_i), \mathbf{H}_{\text{eff}}(t_i)] + \int i\hat{\mathbf{M}}_\parallel \cdot D_t^{(\alpha)} \mathbf{M} \]
\[ + \int i\hat{\mathbf{M}}_\perp \cdot \left( M_s^{-2} D_t^{(\alpha)} \mathbf{M} \wedge \mathbf{M} + \gamma_0 \mathbf{H}_{\text{eff}} \right) \]

\[ S_{G,\text{diss}} = \int \gamma_0 i\hat{\mathbf{M}}_\perp \cdot \left( D \gamma_0 i\hat{\mathbf{M}}_\perp - \frac{\eta}{M_s} D_t^{(\alpha)} \mathbf{M} \right) \]

\[ S_{G,\text{jac}} = \frac{\alpha \gamma_0}{1 + \eta^2 \gamma_0^2 / M_s} \frac{1}{M_s} \int \left[ 2\eta \gamma_0 \mathbf{M} \cdot \mathbf{H}_{\text{eff}} + M_s \epsilon_{ijk} M_k \partial_j H_{\text{eff},i}^{nc} \right. \]
\[ - \eta \gamma_0 (M_s^2 \delta_{ij} - M_i M_j) \partial_j H_{\text{eff},i} \]
The transformation

in the Gilbert formalism in Cartesian coordinates

\[ \mathcal{T}_c = \begin{cases} 
M_t & \mapsto -M_{-t} \\
\gamma_0 \, i\hat{M}_t & \mapsto -\gamma_0 \, i\hat{M}_{-t} - \beta V \mu_0 \, d_t M_{-t} \\
\hat{M}_{\parallel} & \mapsto \hat{M}_{\parallel} \\
i\hat{M}_{\perp} & \mapsto i\hat{M}_{\perp} 
\end{cases} \]

and

\[ \bar{\alpha} = 1 - \alpha \]

if one simultaneously changes the sign of the effective field

\[ H_{\text{eff}}^c_t \mapsto -H_{\text{eff}}^c_{-t}. \]

This change is a consequence of the transformation \( M_t \mapsto -M_{-t} \) when the effective field derives from a potential \( H_{\text{eff}} = -\mu_0^{-1} \nabla_M U \) with \( U \) an even function of \( M \).
The transformation in the Gilbert formalism in Cartesian coordinates

With this formalism one proves

the equilibrium fluctuation-dissipation theorem.

out of equilibrium fluctuation relations.

One could also apply these ideas in the Lifshitz-Landau formulation and in spherical coordinates.

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Aron, Barci, LFC, González-Arenas & Lozano 14

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Aron, Barci, LFC, González-Arenas & Lozano 14

Numerical simulations.

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Romá, LFC & Lozano 14
The equations in spherical coordinates

The vector $\mathbf{M}$ defines the local basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ with $\mathbf{M} \equiv M_s \mathbf{e}_r(\theta, \phi)$ and the Cartesian components $M_x(t) = M_s \sin \theta(t) \sin \phi(t)$, $M_y(t) = M_s \sin \theta(t) \cos \phi(t)$ and $M_z(t) = M_s \cos \theta(t)$.

The sLLGB equation in this system of coordinates becomes $d_t M_s = 0,

$$
\begin{align*}
   d_t \theta &= \frac{D(1 - 2\alpha)\gamma_0^2}{1 + \eta^2 \gamma_0^2} \cot \theta + \frac{\gamma_0}{1 + \eta^2 \gamma_0^2} \left[ H_{\text{eff},\phi} + H_\phi \right. \\
   & \quad \left. + \eta \gamma_0 (H_{\text{eff},\theta} + H_\theta) \right], \\
   \sin \theta \ d_t \phi &= \frac{\gamma_0}{1 + \eta^2 \gamma_0^2} \left[ \eta \gamma_0 (H_{\text{eff},\phi} + H_\phi) - (H_{\text{eff},\theta} + H_\theta) \right],
\end{align*}
$$

with $H_\theta = H_x \cos \theta \cos \phi + H_y \cos \theta \sin \phi - H_z \sin \theta$ and $H_\phi = -H_x \sin \phi + H_y \cos \phi$. Similarly for $\mathbf{H}_{\text{eff}}$. 
Magnetisation dynamics

Adimensionalisation and \( \alpha \)-points

We introduce \( m = M / M_s, \ h_{\text{eff}} = H_{\text{eff}} / M_s, \ h = H / M_s \)
\[ \tau = \gamma_0 M_s t, \ \eta_0 = \eta \gamma_0, \ \text{and} \ D_0 = D \gamma_0 / M_s \]

We define the \( \alpha \)-prescription angular variables
\[ \phi^\alpha_T \equiv \alpha \phi(\tau + \Delta \tau) + (1 - \alpha) \phi(\tau) \]
\[ \phi^\alpha \equiv \alpha \phi(\tau + \Delta \tau) + (1 - \alpha) \phi(\tau) \]

with \( 0 \leq \alpha \leq 1 \). The effective fields at the \( \alpha \)-point are

\[ h^\alpha_{\text{eff}, \theta} \equiv h_{\text{eff}, \theta}(\phi^\alpha_T, \phi^\alpha) \]
\[ h^\alpha_{\text{eff}, \phi} \equiv h_{\text{eff}, \phi}(\theta^\alpha_T, \phi^\alpha) \]

We first draw the Cartesian components of the fields as \( \Delta W_i = h_i \Delta \tau = \omega_i \sqrt{2D_0 \Delta \tau} \) where the \( \omega_i \) are Gaussian random numbers with mean zero and variance one, and we then calculate \( \Delta W_\phi = h_\phi \Delta \tau \) and \( \Delta W_\theta = h_\theta \Delta \tau \).
Magnetisation dynamics

The equations in spherical coordinates

The discretized dynamic equations now read $F_\theta = 0$ and $F_\phi = 0$ with

\[
F_\theta \equiv - \left( \theta_{\tau+\Delta\tau} - \theta_\tau \right) + D_0 \Delta\tau \frac{(1 - 2\alpha)}{(1 + \eta_0^2)} \cot \theta_\tau
\]
\[
+ \frac{\Delta\tau}{1 + \eta_0^2} \left[ h_{\text{eff,}\phi}^\alpha + \eta_0 h_{\text{eff,}\theta}^\alpha \right] + \frac{1}{1 + \eta_0^2} \left[ \Delta W_\phi + \eta_0 \Delta W_\theta \right]
\]

\[
F_\phi \equiv - \left( \phi_{\tau+\Delta\tau} - \phi_\tau \right)
\]
\[
+ \frac{\Delta\tau}{1 + \eta_0^2} \left[ \eta_0 \frac{h_{\text{eff,}\phi}^\alpha - h_{\text{eff,}\theta}^\alpha}{\sin \theta_\tau^\alpha} \right] + \frac{1}{1 + \eta_0^2} \left[ \frac{\eta_0 \Delta W_\phi - \Delta W_\theta}{\sin \theta_\tau^\alpha} \right]
\]

We used a Newton-Raphson routine and we imposed $F_\theta^2 + F_\phi^2 < 10^{-10}$ to find $\phi_{\tau+\Delta\tau}$ and $\theta_{\tau+\Delta\tau}$.

To avoid singular behavior when the magnetization gets too close to the $z$ axis we apply a $\pi/2$ rotation of the coordinate system around the $y$ axis.
Magnetisation dynamics

The benchmark

Uniformly magnetised ellipsoid with volume \( V = 6.702 \times 10^{-26} \) m\(^3\).

The potential energy per unit volume is

\[
U(M) = -\mu_0 \mathbf{M} \cdot \mathbf{H}_{\text{ext}} + \frac{\mu_0}{2} (d_x^2 M_x^2 + d_y^2 M_y^2 + d_z^2 M_z^2)
\]

We set \( \mathbf{H}_{\text{ext}} = 0 \) and \( d_x = d_y = 0.4132 \) and \( d_z = 0.0946 \).

After adimensionalisation

the friction coefficient becomes \( \eta_0 = \eta \gamma_0 \ll 1 \)

the time-step \( \Delta \tau = 1 \) corresponds to \( \Delta t = 3.2 \) ps

We set the temperature to \( T = 300 \) K; then \[
\frac{k_B T}{V \Delta U} \simeq 0.153
\]
Magnetisation dynamics

Typical trajectories of the three magnetisation components

(a) $m_x$

(b) $m_y$

(c) $m_z$
Magnetisation dynamics

Decay of the averaged $m_z$ component

Stratonovich scheme
Magnetisation dynamics

Distribution of the $m_z$ component

Stratonovich scheme, comparison to the analytic form (solid line)
Magnetisation dynamics

Decay of the $m_z$ component

Dependence on $\Delta \tau$; $\alpha = 1/2$ (main) and $\alpha = 0$ (inset)

cfr. Kloeden & Platen *Numerical solutions of stochastic differential equations*
Magnetisation dynamics

Distribution of the $m_z$ component, Itō scheme

For $\alpha = 0$, $\Delta \tau = 0.05$ is needed to get close to the equilibrium pdf.
Conclusions

Numerics

- All discretisation schemes converge to Gibbs-Boltzmann equilibrium

\[ P(M) = Z^{-1} e^{-\beta VU(M)} \]

once the correct drift term has been added.

- The magnetisation modulus is conserved by the numerical integration
  with no need for artificial rescaling.

- The Stratonovich convention is the most efficient one in the sense that
  one can use larger values of \( \Delta \tau \) and stay close to the continuous
  time limit.

We will study the dependence of the algorithms’ precision, for different
\( \alpha \), in the future.
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• Future work.
Related & future work

**Some projects en route**

Analytics: Dean-Kawasaki stochastic equation for the spatial magnetisation density of magnetic moments in pair interaction and controlled by the sLLGB microscopic equation.

LFC, Déjardin, Lozano & van Wijland, arXiv:1412.6497

Numerics: spin-torque effect, experimentally relevant situations.

Better understanding of the algorithms’ precision.

Romá, LFC & Lozano
Dissipative effects

Landau-Lifshitz & Gilbert equations

\[ d_t \mathbf{M} = - \frac{\gamma_0}{1 + \eta^2 \gamma_0^2} \mathbf{M} \wedge \left[ \mathbf{H}_{\text{eff}} + \frac{\eta \gamma_0}{M_s} (\mathbf{M} \wedge \mathbf{H}_{\text{eff}}) \right] \]

2nd terms in RHS: dissipative mechanisms slow down the precession and push \( \mathbf{M} \) towards \( \mathbf{H}_{\text{eff}} \)

with \( 2\mathbf{M} \cdot d_t \mathbf{M} = d_t |\mathbf{M}|^2 = 0 \)

and \( d_t (\mathbf{M} \cdot \mathbf{H}_{\text{eff}}) > 0 \)

Proof: take the scalar product with \( \mathbf{H}_{\text{eff}} \) and use \( (\mathbf{M} \cdot \mathbf{H}_{\text{eff}})^2 < M_s^2 H_{\text{eff}}^2 \)
Magnetisation dynamics

Distribution of the $m_x$ component

Stratonovich scheme, Gaussian pdf
Magnetisation dynamics

Decay of the average $m_z$ component

Stratonovich scheme

dependence on the friction coefficient $\eta_0$ and the time-step $\Delta \tau$
Magnetisation dynamics

$\langle m_z \rangle$ under an applied field $H_{\text{ext}} \neq 0$

For $\alpha = 0.5$, little dependence on $\Delta \tau = 0.05$ vs $\Delta \tau = 0.5$

Note that a very large value $\Delta \tau = 0.5$ can be used.
Magnetisation dynamics

\[ \langle m_z \rangle \text{ under an applied field } H_{\text{ext}} \neq 0 \]

Dependence on \( \alpha \) cured by using smaller \( \Delta \tau \):
poorer precision for \( \alpha \neq 0.5 \) confirmed.