(Non) equilibrium dynamics: a (broken) symmetry

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Spoiler

Thermal equilibrium dynamics

$\Updownarrow$

Symmetry of action

$$\psi \mapsto \mathcal{T}_\beta[\psi]$$

$$S[\psi] = \int \mathrm{d}x\, \mathcal{L}(\psi(x), \partial_\mu \psi(x); x) \mapsto S[\psi]$$

Ward-Takahashi identies

$$\langle \mathcal{O}[\psi(x)] \ldots \mathcal{O}[\psi(x')] \rangle = \langle \mathcal{O}[\mathcal{T}_\beta \psi(x)] \ldots \mathcal{O}[\mathcal{T}_\beta \psi(x')] \rangle$$

Aron Biroli Cugliandolo (2010)
Sieberer Chiocchehetta Gambassi Tauber Diehl (2015)
Aron Biroli Cugliandolo (2017)

Spoiler

~~Thermal equilibrium~~ dynamics

Non-equilibrium

↕

~~Symmetry~~ of action

Symmetry breaking

$$\psi \mapsto \mathcal{T}_\beta[\psi]$$

$$S[\psi] = \int \mathrm{d}x\, \mathcal{L}(\psi(x), \partial_\mu \psi(x); x) \mapsto S[\psi] + \Delta S[\psi]$$

Entropy production

Fluctuation theorems

~~Ward-Takahashi identies~~

$$\langle \mathcal{O}[\psi(x)] \ldots \mathcal{O}[\psi(x')] \rangle = \langle \mathcal{O}[\mathcal{T}_\beta \psi(x)] \ldots \mathcal{O}[\mathcal{T}_\beta \psi(x')]\, \mathrm{e}^{\Delta S} \rangle$$

Aron Biroli Cugliandolo (2017)

Spoiler

1 ~~Thermal equilibrium~~ dynamics

2 **Non-equilibrium**

5 ~~Symmetry~~ of action

6 **Symmetry breaking**

$$\psi \mapsto \mathcal{T}_\beta[\psi]$$

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3 **Entropy production**

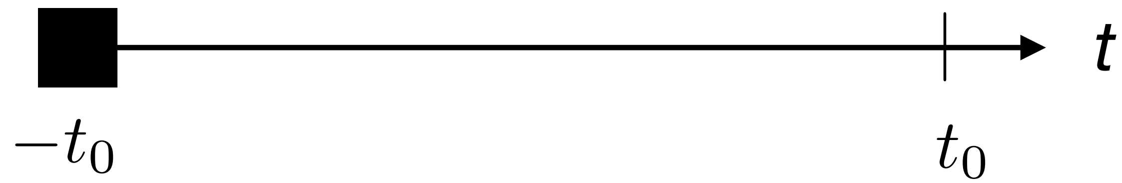
4 **Fluctuation theorems**

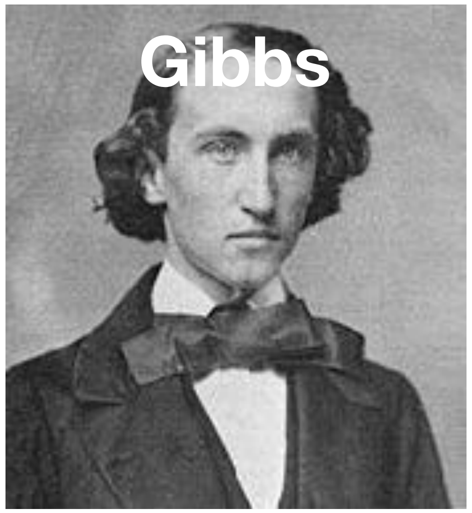
~~Ward-Takahashi identies~~

$$\langle \mathcal{O}[\psi(x)] \ldots \mathcal{O}[\psi(x')] \rangle = \langle \mathcal{O}[\mathcal{T}_\beta \psi(x)] \ldots \mathcal{O}[\mathcal{T}_\beta \psi(x')]\, \mathrm{e}^{\Delta S} \rangle$$

Aron Biroli Cugliandolo (2017)

Equilibrium dynamics





	preparation	evolution
Classical	$P_{\rm GB}(x,\dot{x}) \sim \mathrm{e}^{-\beta\mathcal{E}(x,\dot{x})}$	$m\ddot{x} = -V'(x)$
Quantum	$\hat{\rho}_{\rm GB} \sim \mathrm{e}^{-\beta\hat{H}}$	$\hat{O}_{\rm H}(t) = \mathrm{e}^{\mathrm{i}\hat{H}t}\,\hat{O}\,\mathrm{e}^{-\mathrm{i}\hat{H}t}$

Equilibrium conditions

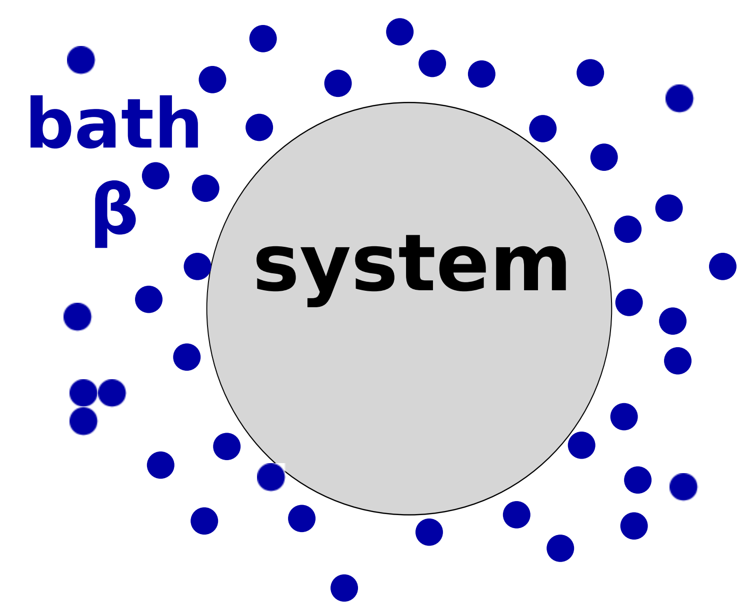
- prepare in equilibrium at temperature β^{-1} wrt $\hat{H}$
- evolve with same $\hat{H}$
- (coupled to bath at β^{-1})

Prepare

- Isolated (thermalization)
- With thermostat

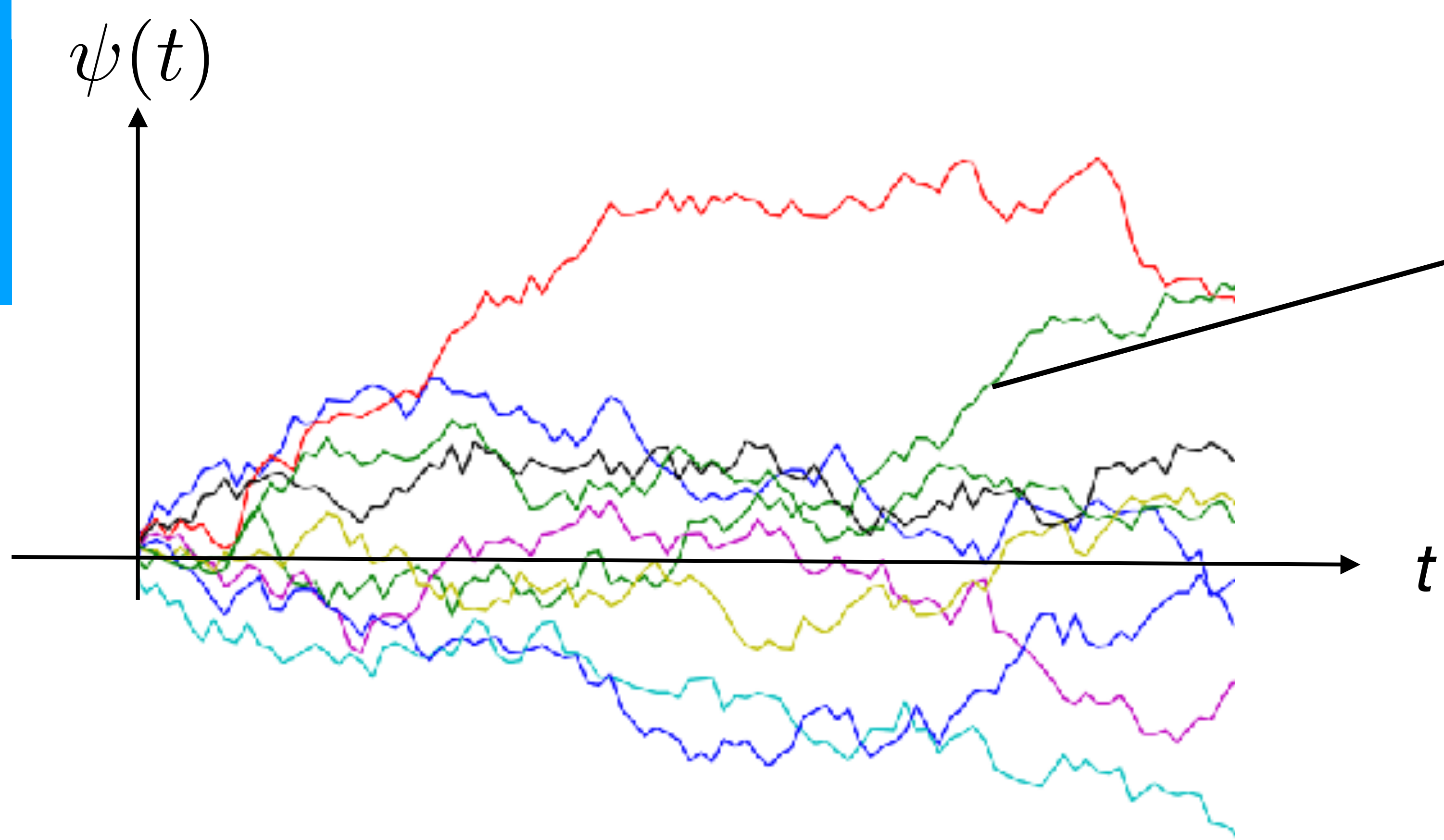
ex: $m\ddot{x} = -V'(x) - \eta\dot{x} + \xi(t)$

$$\langle \xi(t)\xi(t')\rangle = 2\eta\beta^{-1}\,\delta(t-t')$$



Stochastic thermodynamics

classical



single trajectory

- work $\mathcal{W}[\psi]$
- heat $\mathcal{Q}[\psi]$
- entropy $\mathcal{S}^{\mathrm{irr}}[\psi]$

First law

$$\Delta\mathcal{E} = \mathcal{W}[\psi] + \mathcal{Q}[\psi]$$

Second law

$$\Delta S = \beta\mathcal{Q}[\psi] + \underbrace{\mathcal{S}^{\mathrm{irr}}[\psi]}$$

in average $\langle \mathcal{S}^{\mathrm{irr}}[\psi]\rangle \geq 0$

Fluctuation Theorem

$$P(\mathcal{S}^{\mathrm{irr}}) = P_{\mathrm{r}}(-\mathcal{S}^{\mathrm{irr}})\,\mathrm{e}^{\mathcal{S}^{\mathrm{irr}}}$$

Gallavoti Cohen (1995)

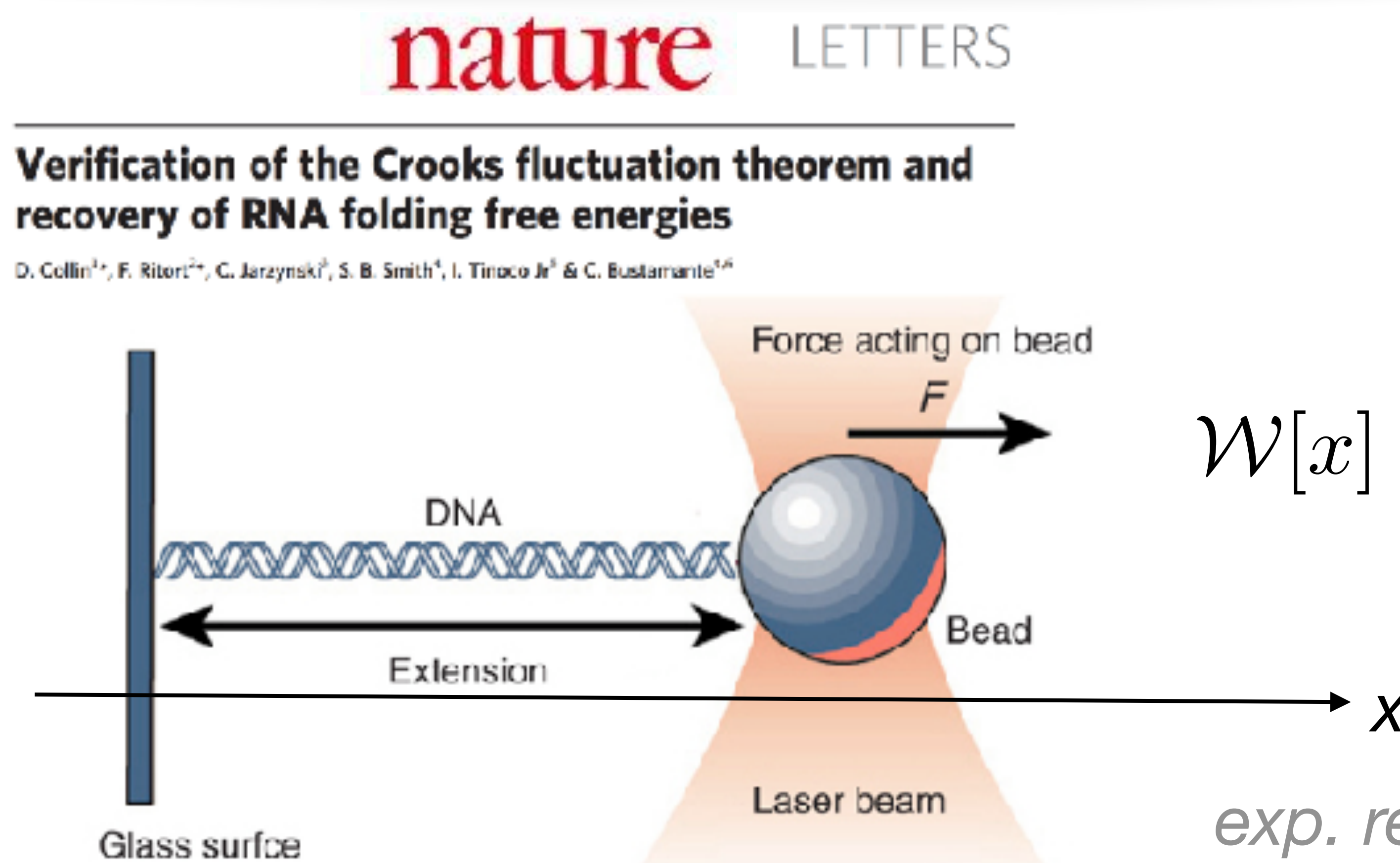
classical

Fluctuation theorems

Work fluctuation theorem

$$P(\mathcal{W}) = P_{\mathrm{r}}(-\mathcal{W})\, \mathrm{e}^{\beta(\mathcal{W}-\Delta\mathcal{F})}$$

Crooks (1999)



$$\mathcal{W}[x] = \sum_i F_i\, \Delta x_i$$

Jarzynski et al. (2005)

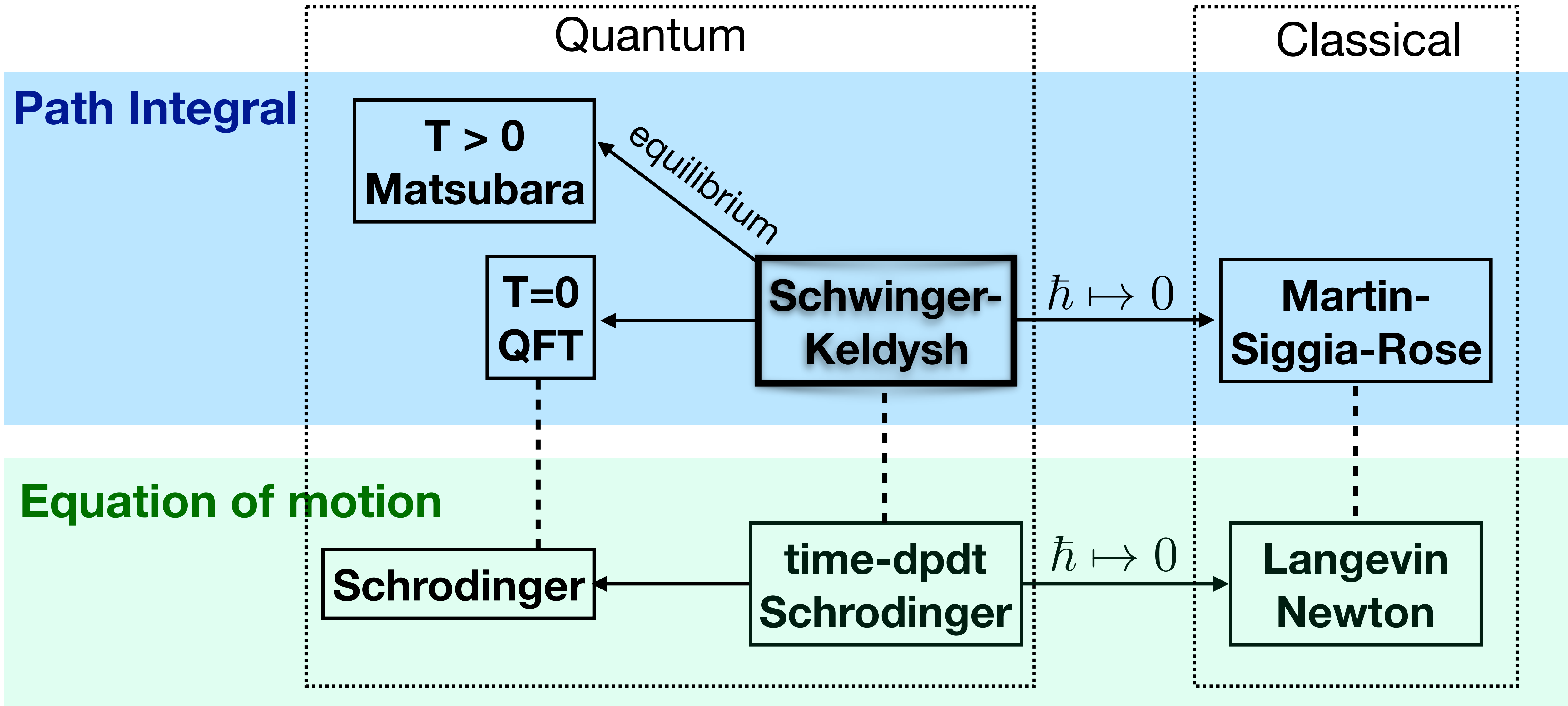
exp. review: Ciliberto (2017)

quantum

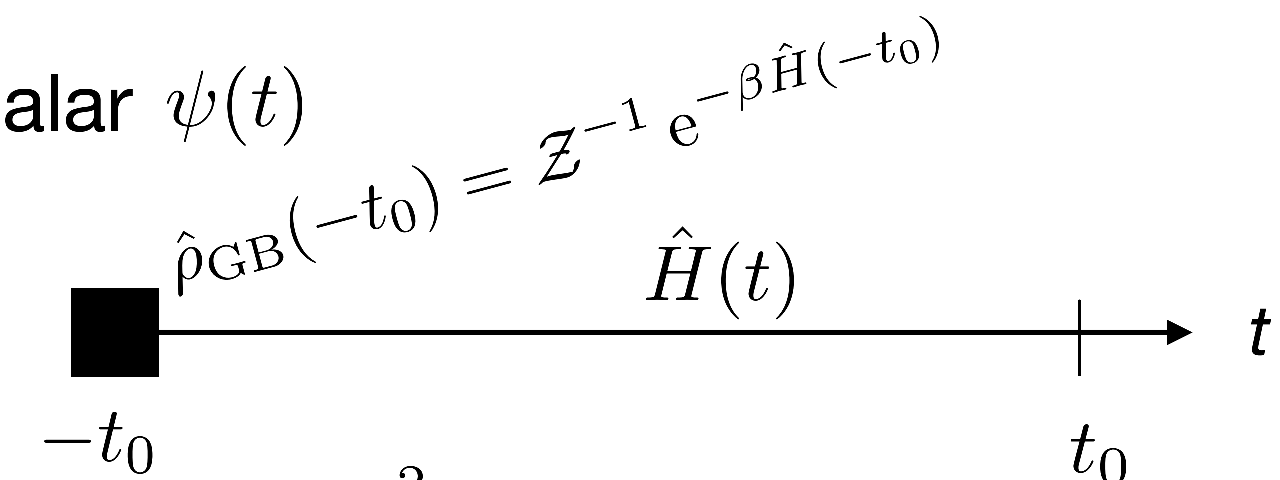
Quantum trajectories

- ☑ Work fluctuation theorem — *Kurchan (2000)*
  cQED: Murch, Lutz et al. (2017)
- ☐ Entropy production, $\mathcal{S}^{\mathrm{irr}}[\psi]$?
- ☐ Entropy fluctuation theorem ?

Formalisms

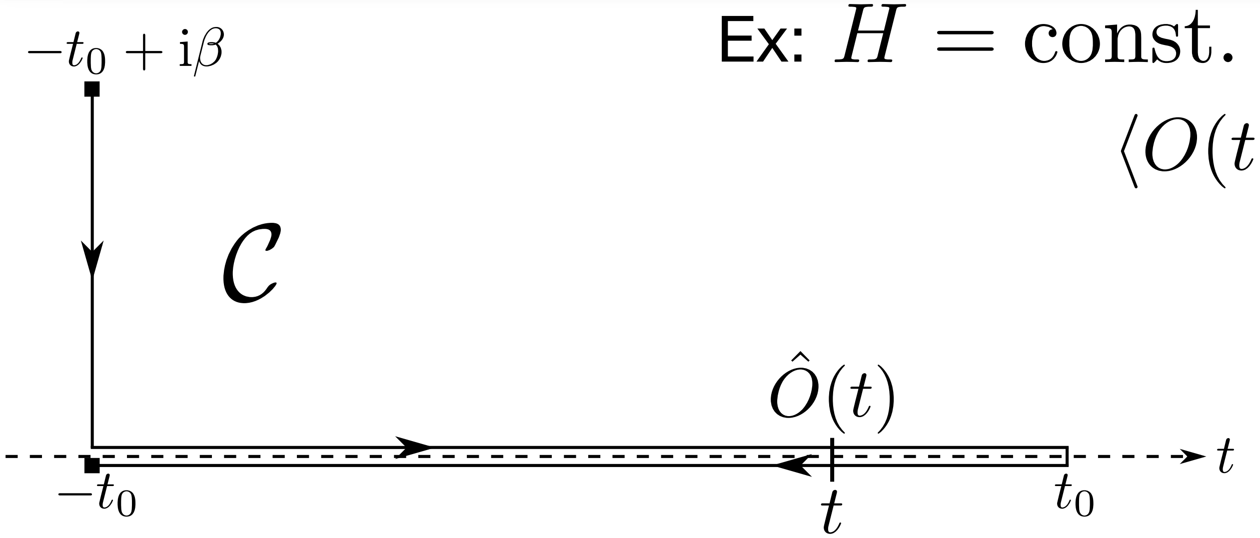


non-relativistic real scalar $\psi(t)$



ex: $\hat{H}(t) = \dfrac{\pi^2}{2m} + V(\psi; t)$

Schwinger-Keldysh formalism



Ex: $\hat{H} = \text{const.}$

$$\langle O(t)\rangle = \mathcal{Z}^{-1}\,\mathrm{Tr}\big[\mathrm{e}^{\mathrm{i}\hat{H}t}\,\hat{O}(t)\,\mathrm{e}^{-\mathrm{i}\hat{H}t}\,\mathrm{e}^{-\beta\hat{H}}\big]$$

$$= \mathcal{Z}^{-1}\,\mathrm{Tr}\big[\mathrm{T}_{\mathcal{C}}\,\mathrm{e}^{\mathrm{i}\int_{\mathcal{C}}\mathrm{d}\tau\,\hat{H}}\,\hat{O}(t)\big]$$

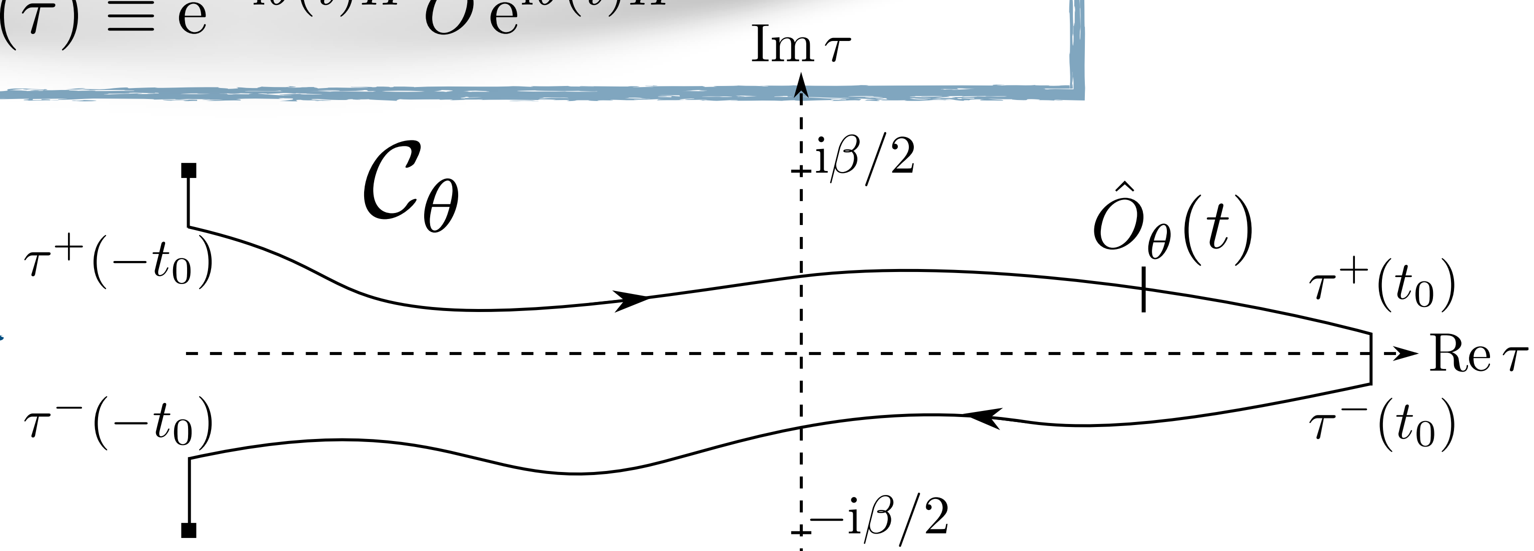
$$= \mathcal{Z}^{-1}\int\mathcal{D}[\psi]\,\mathrm{e}^{\mathrm{i}\int_{\mathcal{C}}\mathrm{d}\tau\,\mathcal{L}[\psi(\tau)]}\,\langle\psi(t)|\hat{O}(t)|\psi(t)\rangle$$

$$\mathbb{I} = \int\mathrm{d}\psi(t)\;\mathrm{e}^{+\mathrm{i}\theta(t)\hat{H}}\,|\psi(t)\rangle\langle\psi(t)|\,\mathrm{e}^{-\mathrm{i}\theta(t)\hat{H}}$$

Novel formal degree of freedom

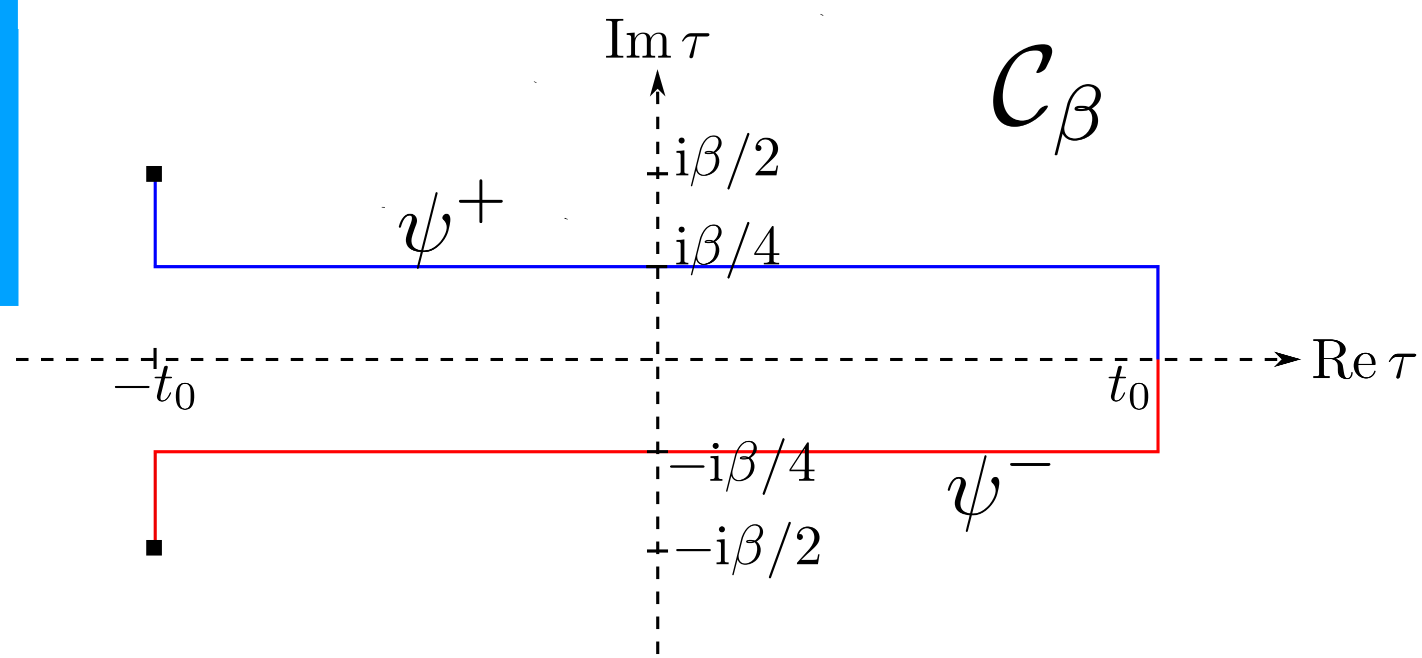
$$t \mapsto \tau = t + \theta(t) \qquad\qquad \psi(t) \mapsto \psi(\tau)$$

$$\hat{O} \mapsto \hat{O}_\theta(\tau) \equiv \mathrm{e}^{-\mathrm{i}\theta(t)\hat{H}}\,\hat{O}\,\mathrm{e}^{\mathrm{i}\theta(t)\hat{H}}$$



$$\langle O(t)\rangle = \mathcal{Z}^{-1}\int\mathcal{D}[\psi]\,\mathrm{e}^{\mathrm{i}\int_{\mathcal{C}_\theta}\mathrm{d}\tau\,\mathcal{L}[\psi(\tau)]}\,\langle\psi(\tau)|\hat{O}_\theta(\tau)|\psi(\tau)\rangle$$

Symmetry of equilibrium



$$\hat{H} = \text{const.}$$

$$S[\psi] = \int_{\mathcal{C}_\beta} \mathrm{d}\tau\, \mathcal{L}[\psi(\tau)]$$

Field transformation

$$\mathcal{T}_\beta : \ \psi^\pm(\tau) \mapsto \psi^\pm(-\tau \pm \mathrm{i}\beta/2)$$

$$\mathcal{T}_\beta \circ \mathcal{T}_\beta = \mathrm{Id}$$

Invariance of the action

$$\beta\mathcal{F}(-t_0) + \mathrm{i}S[\psi] \overset{\mathcal{T}_\beta}{\mapsto} \beta\mathcal{F}(-t_0) + \mathrm{i}S[\psi]$$

$$\mathcal{Z} = \mathrm{e}^{-\beta\mathcal{F}}$$

Consequences of the symmetry

Ward-Takahashi identies

$$\langle \psi^a(\tau)\, \psi^b(\tau') \ldots \rangle_{\mathcal{S}_\beta} = \langle \mathcal{T}_\beta \psi^a(\tau)\, \mathcal{T}_\beta \psi^b(\tau') \ldots \rangle_{\mathcal{S}_\beta}$$

Keldysh Green's functions

$$\mathrm{i}G^{ab}(t,t') = \langle \psi^a(t)\, \psi^b(t') \rangle \quad a, b = +, -$$

Fluctuation-dissipation theorem

$$G^{-+}(t,t') \stackrel{\mathcal{T}_\beta}{=} \mathrm{e}^{\mathrm{i}\beta/2\,(\partial_t - \partial_{t'})}\, G^{+-}(-t', -t)$$

Classical limit

$$\left.\frac{\delta \langle \psi(t) \rangle}{\delta f(t')}\right|_{f=0} = -\beta\, \partial_t \, \langle \psi(t) \psi(t') \rangle \quad \text{if } t > t'$$

$$= 0 \quad \text{if } t < t'$$

Symmetry breaking

Time-dependent drive: $\hat{H} \mapsto \hat{H}(t)$

$$S[\psi] = \int_{\mathcal{C}_\beta} \mathrm{d}\tau \; \mathcal{L}[\psi(\tau); t(\tau)] + \int_{\mathcal{C}_\beta} \mathrm{d}t \; \widetilde{\mathcal{L}}[\psi(\tau)); t(\tau)]$$

New !

Variation of the action

$$\beta\mathcal{F}(-t_0) + \mathrm{i}S[\psi] \overset{\mathcal{T}_\beta}{\mapsto} \beta\mathcal{F}_\mathrm{r}(-t_0) + \mathrm{i}S_\mathrm{r}[\psi] + \beta\Delta\mathcal{F}_\mathrm{r} - \Sigma_\mathrm{r}[\psi]$$

New !

$\mathcal{S}_\mathrm{r}^\mathrm{irr}[\psi]$

with

$$\Sigma[\psi] = \int_{\mathcal{C}} \mathrm{d}t \; \langle\psi(t)| \, \mathrm{e}^{\beta/4\,\hat{H}(t)} \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\beta/2\,\hat{H}(t)}\right] \mathrm{e}^{\beta/4\,\hat{H}(t)} \, |\psi(t)\rangle$$

$\xrightarrow{\hbar \mapsto 0} \beta\mathcal{W}[\psi]$

Symmetry breaking

Measure of irreversibility

$$\mathcal{S}^{\mathrm{irr}}[\psi] = \beta\Delta\mathcal{F} - \Sigma[\psi]$$

$$\Sigma[\psi] = \int_{\mathcal{C}} \mathrm{d}t \, \langle\psi(t)| \mathrm{e}^{\beta/4\,\hat{H}(t)} \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\beta/2\,\hat{H}(t)}\right] \mathrm{e}^{\beta/4\,\hat{H}(t)} |\psi(t)\rangle$$

- ☑ $\langle \mathcal{S}^{\mathrm{irr}}[\psi] \rangle \geq 0$
- ☑ Classical limit
- ☑ Reversible process

☑ Fluctuation Theorem

$$P(\mathcal{S}^{\mathrm{irr}}) = P_{\mathrm{r}}(-\mathcal{S}^{\mathrm{irr}})\,\mathrm{e}^{\mathcal{S}^{\mathrm{irr}}}$$

Entropy production operator ?

Recap

Non-equilibrium

~~Thermal equilibrium~~ dynamics

↕

~~Symmetry~~ of action

Symmetry breaking

$$S[\psi] \xmapsto{\mathcal{T}_\beta} S[\psi] + \mathcal{S}^{\mathrm{irr}}[\psi]$$

Entropy production

Fluctuation theorems

~~Ward-Takahashi identies~~

$$\langle \mathcal{O}[\psi(\tau)] \ldots \mathcal{O}[\psi(\tau')] \rangle = \langle \mathcal{O}[\mathcal{T}_\beta \psi(\tau)] \ldots \mathcal{O}[\mathcal{T}_\beta \psi(\tau')] \, \mathrm{e}^{\mathcal{S}_{\mathrm{r}}^{\mathrm{irr}}[\psi]} \rangle_{\mathrm{r}}$$