Slow dynamics :

aging, weak long-term memory & time reparametrization invariance

Leticia F. Cugliandolo

Sorbonne Université & Institut Universitaire de France

leticia@lpthe.jussieu.fr

www.lpthe.jussieu.fr/~leticia

SYK models : from strongly correlated systems to quantum gravity

Solvay Conference, Bruxelles, 2023

Plan

Apologies for talking about rather old work

— The simplest aging example :

domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation :

self-correlation and linear response

— Fluctuation-dissipation relations :

effective temperatures

— Mean-field modeling :

separation of time scales

— Reparametrization invariance :

sigma model

— Fluctuations :

local two-time observables

Plan

Schematic

— **The simplest aging example :**

domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation :

self-correlation and linear response

— Fluctuation-dissipation relations :

effective temperatures

— Mean-field modeling :

separation of time scales

— Reparametrization invariance :

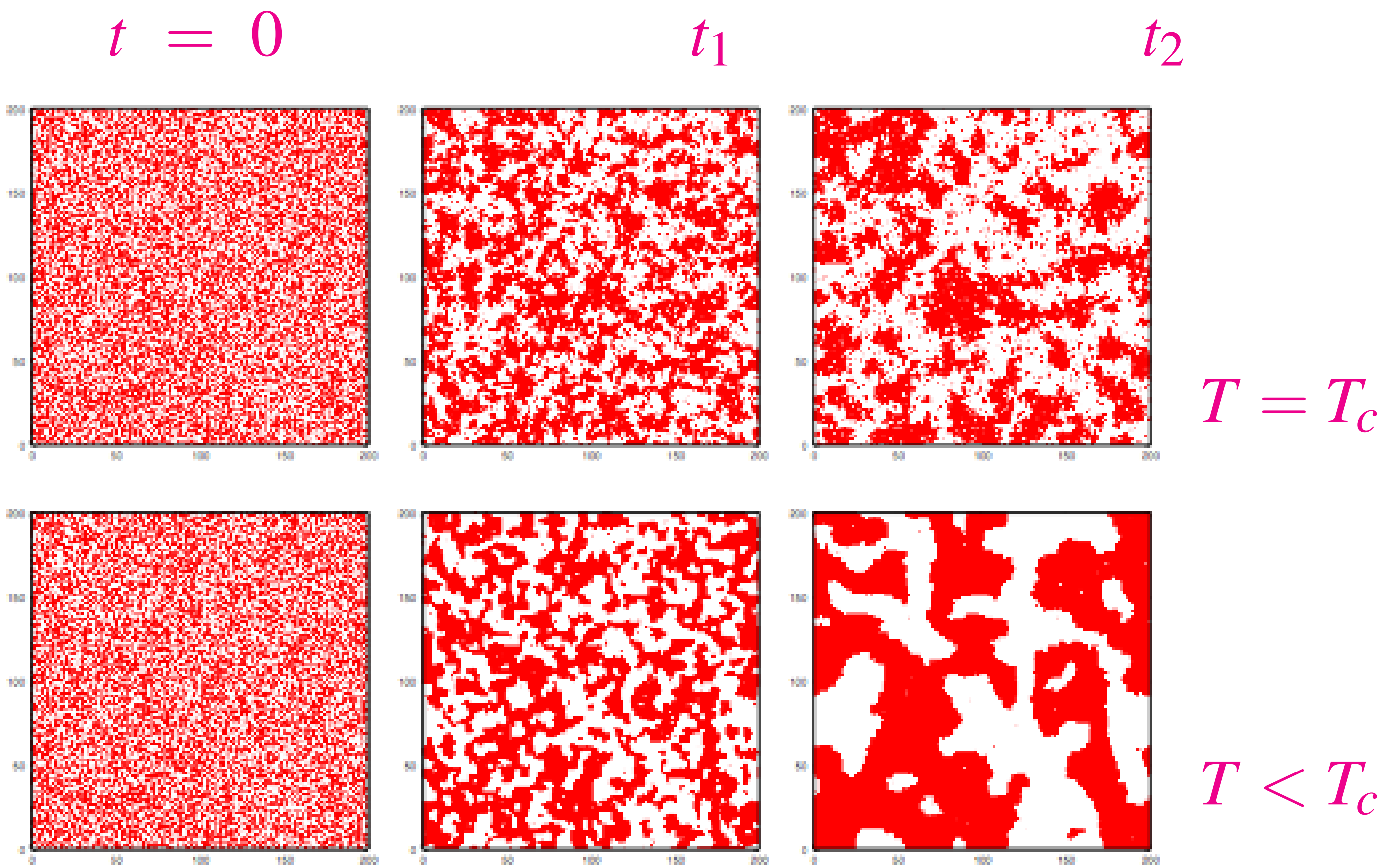
sigma model

— Fluctuations :

local two-time observables

$2d$ Ising model

Snapshots after an instantaneous quench from $T_0 \to \infty$ to $T \leq T_c$



At $T = T_c$ **critical dynamics** At $T < T_c$ **coarsening**

A certain number of **interfaces** or **domain walls** in the last snapshots.

Phenomenon

In both cases one sees the growth of ‘red and white’ patches and **interfaces** surrounding such geometric domains.

Spatial regions of local equilibrium (with vanishing, at T_c, or non-vanishing, at $T < T_c$, order parameter) grow in time and

a single **growing length** $\mathcal{R}(t, T/J)$ can be identified and will be at the heart of dynamic scaling.

Aging

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

$\mathcal{R}$ grows but $\dot{\mathcal{R}}$ decreases in time

e.g. curvature driven dynamics* $\dfrac{d\mathcal{R}^2}{dt} = \text{cts}$ and then

$$\dot{\mathcal{R}} \propto \mathcal{R}^{-1} \qquad \Longrightarrow \qquad \mathcal{R} \propto t^{1/2}$$

*From scalar t-dependent Ginzburg-Landau $\lambda\phi^4$ – Allen-Cahn late 70s

Plan

Schematic

— The simplest aging example :

domain growth coarsening & the growing length

— **Spontaneous and perturbed global relaxation :**

self-correlation and linear response

— **Fluctuation-dissipation relations :**

effective temperatures

— Mean-field modeling :

separation of time scales

— Reparametrization invariance :

sigma model

— Fluctuations :

local two-time observables

Two-time dependencies

Self-correlation and linear response

Self correlation and integrated linear response

$$C(t,t_w) \equiv \frac{1}{N}\sum_i [\langle s_i(t) s_i(t_w)\rangle]$$

$$\chi(t,t_w) \equiv \frac{1}{N}\sum_i \int_{t_w}^{t} dt'\, R(t,t') = \frac{1}{N}\sum_i \int_{t_w}^{t} dt' \; [\left.\frac{\delta\langle s_i(t)\rangle_h}{\delta h_i(t')}\right|_{h=0}]$$

Extend the notion of order parameter

They are not related by FDT out of equilibrium

Magnetic notation but general

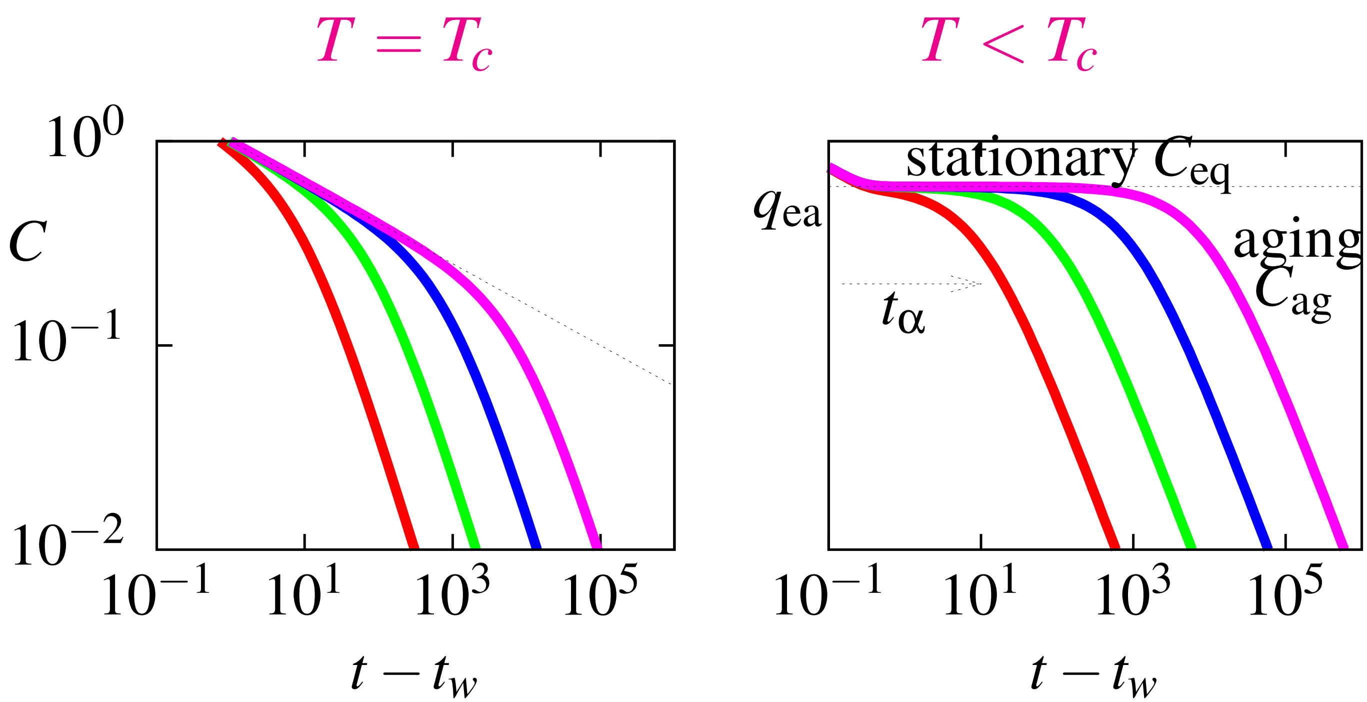
The averages are thermal (and over initial conditions) $\langle \ldots \rangle$

and over quenched randomness $[\ldots]$ (if present)

t_w waiting-time and t measuring time

Two-time self-correlation

Comparison of critical and subcritical



$t_{w1} < t_{w2} < t_{w3} < t_{w4} \Rightarrow$ older relax more slowly

Separation of time-scales

Multiplicative

$$C_{eq}(t - t_w) C_{ag}(t, t_w)$$

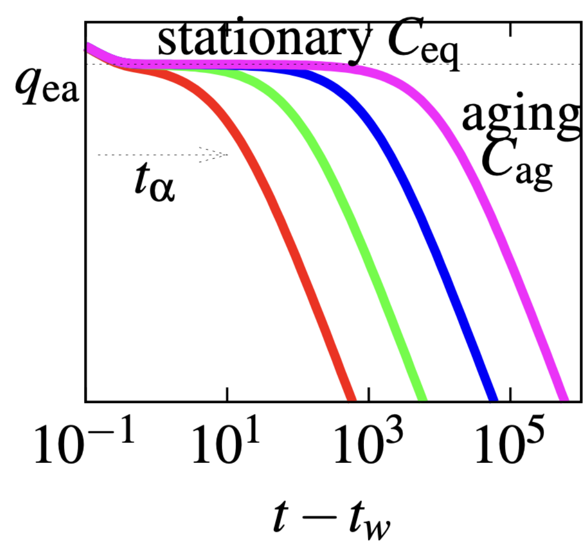
Additive

$$C_{eq}(t - t_w) + C_{ag}(t, t_w)$$

Two-time self-correlation

Focus on subcritical

$T < T_c$



Two scales $C_{\rm eq}(t-t_w) + C_{\rm ag}(t,t_w)$

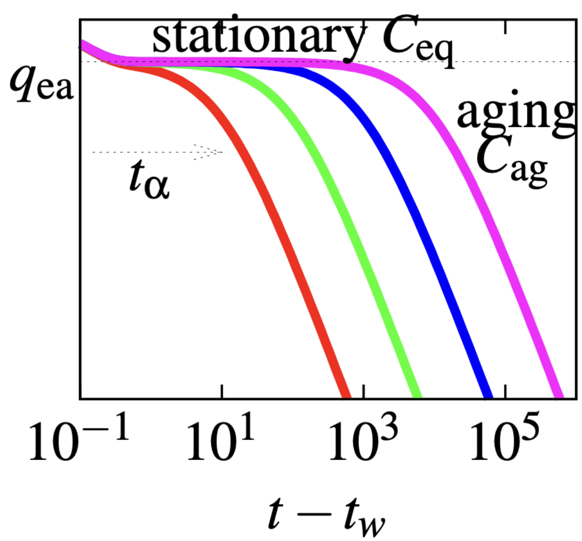
$$C_{\rm eq}(t-t_w) \sim f_{\rm eq}\left(\frac{e^{-t/t_{\rm eq}}}{e^{-t_w/t_{\rm eq}}}\right) \qquad C_{\rm ag}(t,t_w) \sim f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

The dependence of $\mathcal{R}(t)$ on the control parameters, T/J or others, is not important

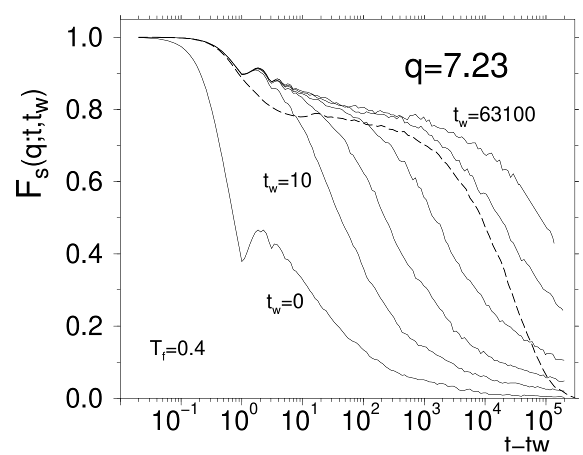
Two-time self-correlation

Focus on subcritical

$T < T_c$



Lennard-Jones mixtures **Kob & Barrat 97**



Two scales $C_{eq}(t-t_w) + C_{ag}(t,t_w)$

$$C_{eq}(t-t_w) \sim f_{eq}\left(\frac{e^{-t/t_{eq}}}{e^{-t_w/t_{eq}}}\right) \qquad C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

Also found in glassy systems for which there is no clear visualization of $\mathcal{R}$

Aging

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

Breakdown of stationarity of the self-correlation

$$C(t,t_w) \neq C(t-t_w)$$

In each regime, equilibrium and aging, scaling*

$$C(t,t_w) = C\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

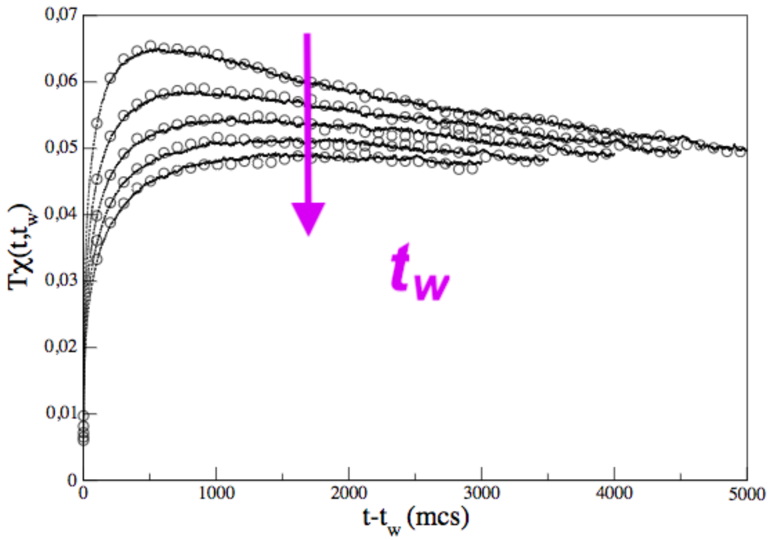
*the scaling form can be proven from general properties of temporal correlation functions

No obvious interpretation of $\mathcal{R}(t)$ in aging **glassy** systems

Linear response

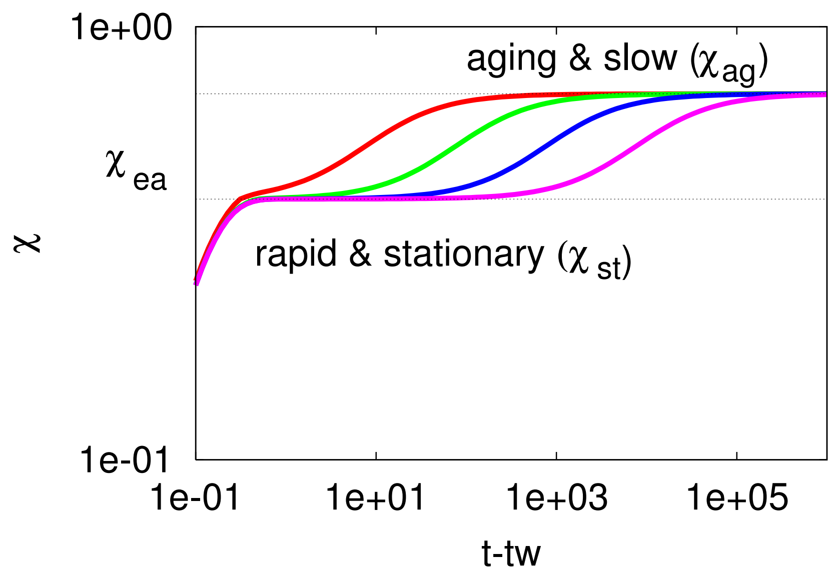
An important difference

Coarsening



Lippiello, Corberi & Zannetti 05

Glassy



Sketch **Chamon & LFC 07**

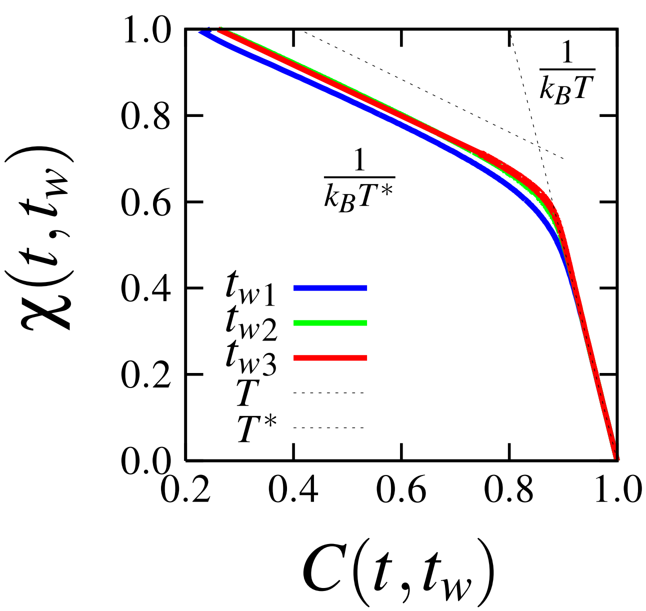
Weak long-term memory in the glassy but not in the coarsening problem.

Just the stationary part will remain asymptotically, contrary to the sketch on the right valid for glasses & spin-glasses.

Fluctuation-dissipation

Induced vs. spontaneous fluctuations in glasses

A quench from $T_0 \to \infty$ to $T < T_c$



Parametric construction

t_w fixed

$t_{w_1} < t_{w_2} < t_{w_3}$

$t - t_w : 0 \to \infty$

used as a parameter

Note that $T^* > T$

Breakdown of the equilibrium FDT $k_B T \chi = C$

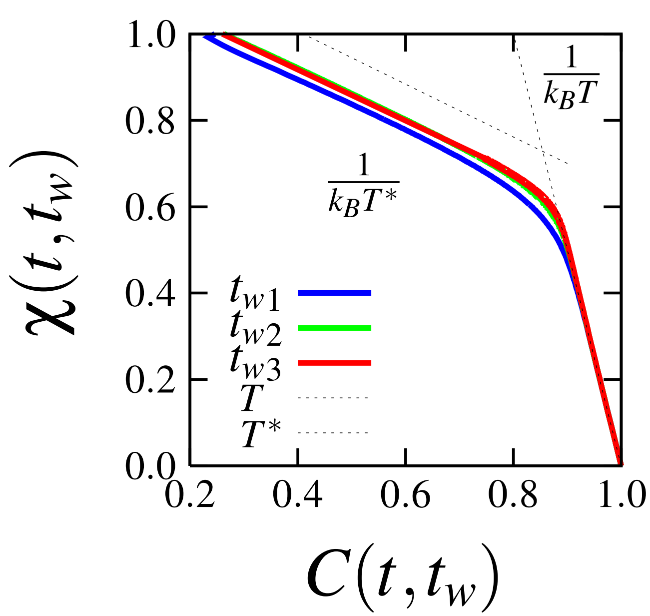
Convergence to $k_B T \chi(C)$, two linear relations for $C \lessgtr q_{\rm ea}$

Mean-field models **LFC & Kurchan 93** & effective temperature interpretation **LFC, Kurchan & Peliti 97**

Fluctuation-dissipation

Correlation scales

A quench from $T_0 \to \infty$ to $T < T_c$



Parametric construction

t_w fixed

$t_{w_1} < t_{w_2} < t_{w_3}$

$t - t_w : 0 \to \infty$

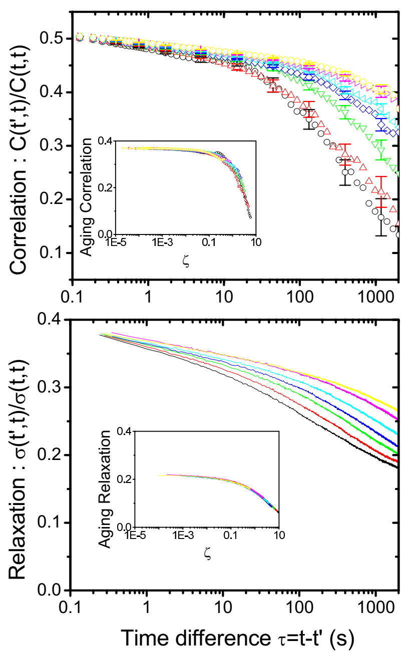
used as a parameter

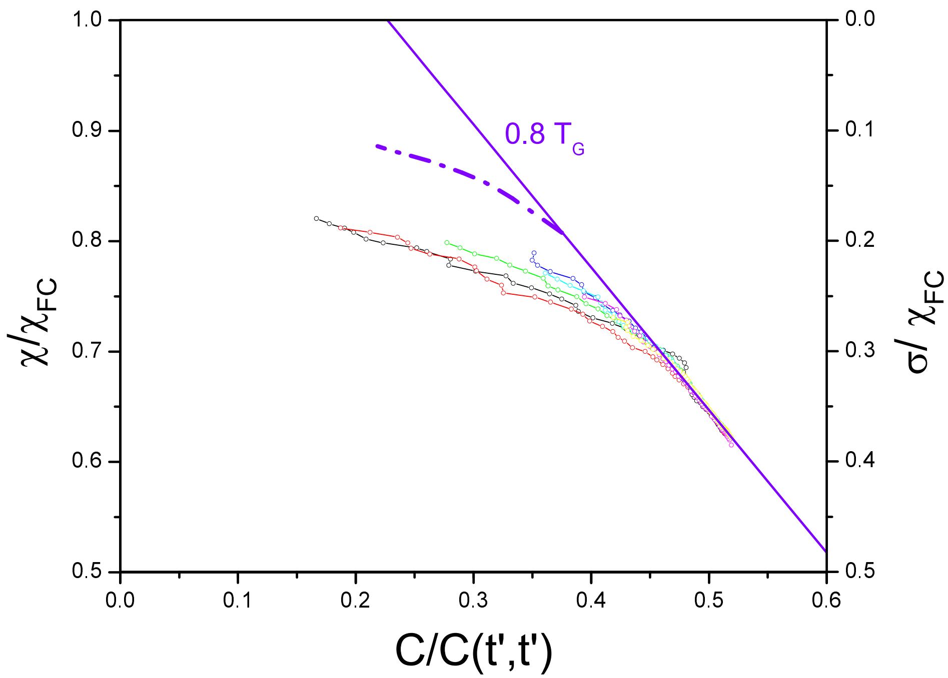
Note that $T^* > T$

Physical picture : each scale evolves with its own "clock" $\mathcal{R}(t)$ (e^{-t/t_0}, t/t_0 or other) and its temperature (T the bath temperature, or T^*)

Experiments in SGs

Correlations, responses and fluctuation dissipation relations





Correlation & response **Parametric construction**

Experiments Hérisson & Ocio 02-04 **Results on SK** LFC & Kurchan 94

Plan

Schematic

— The simplest aging example :

domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation :

self-correlation and linear response

— Fluctuation-dissipation relations :

effective temperatures

— **Mean-field modeling :**

separation of time scales

— **Reparametrization invariance :**

sigma model

— Fluctuations :

local two-time observables

Microscopic models

Classical p-spin spherical

Potential energy

$$V_J[\{s_i\}] = -\sum_{i_1 \neq \cdots \neq i_p} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p}$$

quenched random couplings $J_{i_1 \dots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \dots i_p}\}]$

(over-damped) **Langevin dynamics** (coupling to a bath)

$$\frac{ds_i}{dt} = -\frac{\delta V_J}{\delta s_i} + z_t s_i + \xi_i$$

z_t is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} s_i^2 = N$

$p = 2$ mean-field **domain growth**

$p \geq 3$ RFOT modelling of **fragile glasses**

Dynamic equations

Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t,t_w) = \int dt' \left[\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t') \right] + 2k_B T R(t_w,t)$$

$$(\partial_t - z_t) R(t,t_w) = \int dt' \, \Sigma(t,t') R(t',t_w) + \delta(t - t_w)$$

where the self-energy and vertex depend on C and R. For the p spin models

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \qquad \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t') R(t,t')$$

The Lagrange multiplier z_t is fixed by $C(t,t) = 1$. Random initial conditions.

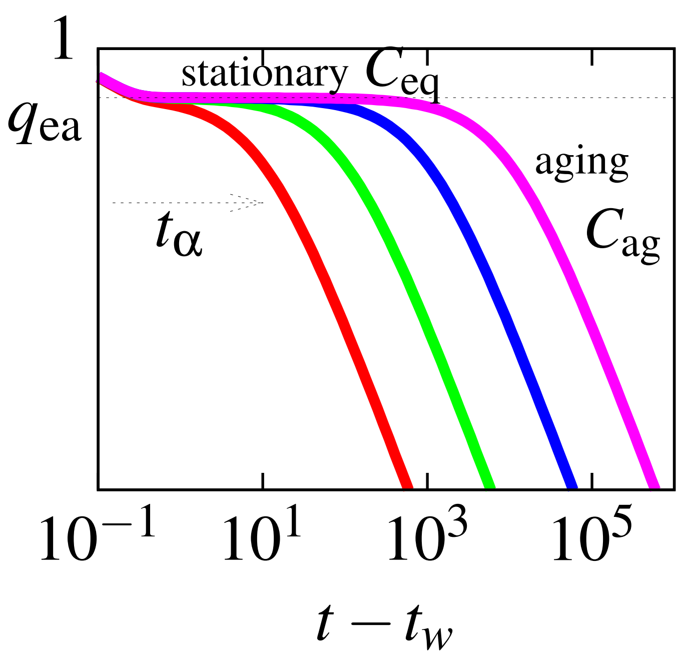
(Average over randomness already taken ; later, interest in noise-induced fluctuations)

See **Sompolinsky & Zippelius 82, LFC & Kurchan 93**

Separation of time-scales

In the long t_w limit

Fast $t - t_w \ll t_w$



The aging part is slow

Slow $\mathcal{R}(t)/\mathcal{R}(t_w) = O(1)$

$$C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t,t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t,t_w) \ll C_{ag}(t,t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{ea}$:

Approx. asymptotic time-reparametization invariance $\boxed{t \to h(t)}$

Time-reparametrization

Example : the equation $(\partial_t - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w)$

- Focus on times such that $z_t \to z_\infty$, $C \sim C_{\rm ag}$ and $R \sim R_{\rm ag}$

- Separation of time-scales (drop $\partial_t R$ and approximate the integral) :

$$-z_\infty R_{\rm ag}(t,t_w) \sim \int dt' \, D'[C_{\rm ag}(t,t')] R_{\rm ag}(t,t') \, R_{\rm ag}(t',t_w) \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \qquad \begin{cases} C_{\rm ag}(t,t_w) \to C_{\rm ag}(h_t,h_{t_w}) \\ R_{\rm ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} R_{\rm ag}(h_t,h_{t_w}) \end{cases}$$

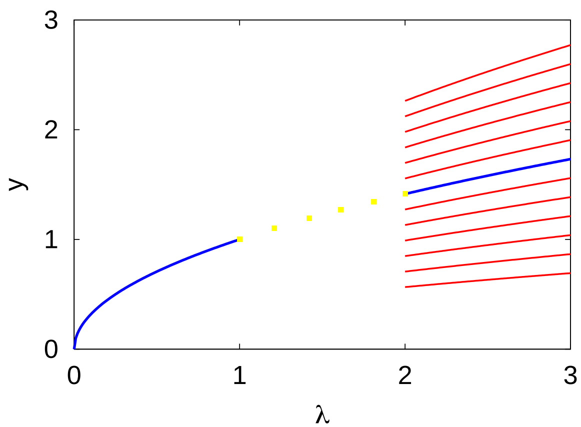
with h_t positive and monotonic leaves eq. (1) **invariant** :

$$-z_\infty R_{\rm ag}(h_t,h_{t_w}) \sim \int dh_{t'} \, D'[C_{\rm ag}(h_t,h_{t'})] R_{\rm ag}(h_t,h_{t'}) \, R_{\rm ag}(h_{t'},h_{t_w})$$

Time reparametrization

A nuisance

Similar to the **matching problem** in non-linear diff. eqs.



$$\frac{dy}{d\lambda} = g[y(\lambda)]$$

Many asymptotic solutions if one sets $\frac{dy}{d\lambda} = 0$ for large λ

One is selected by the small λ behavior

A problem which is still open for the p-spin Schwinger-Dyson equations

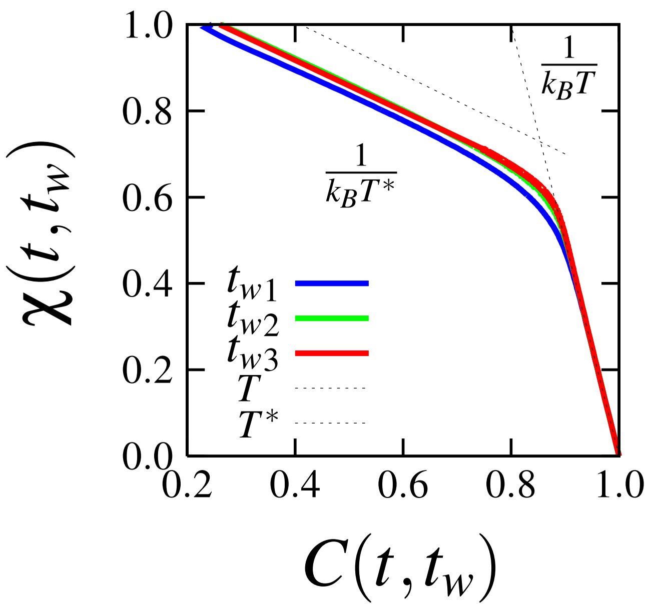
Time reparametrization

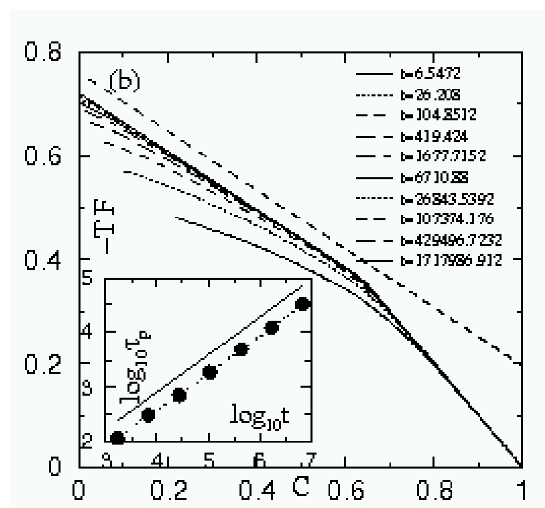
One can compute analytically $f_{\rm ag}$ and $\chi_{\rm ag}(C_{\rm ag})$

for times t and t_w such that $C_{\rm ag}(t,t_w) \sim f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$, *e.g.*

$$\chi_{\rm ag}(t,t_w) \equiv \int_{t_w}^{t} dt'\, R(t,t') \sim \frac{1-q_{\rm ea}}{T} + \frac{1}{T^*}\left[q_{\rm ea} - C_{\rm ag}(t,t_w)\right]$$

but not the 'clock' $\mathcal{R}(t)$





Kim & Latz 00 very precise numerical solution

Remarks

Symmetry breaking terms $\partial_t C(t, t_w)$, etc.

vanish in the long $t_w \to \infty$ and $t - t_w \to \infty$ limits

Ultra soft mode

One can modify the actual $h(t)$ very easily by, *e.g.*,

– weak shearing $\Rightarrow$ stationary

– weak periodic shaking $\Rightarrow$ periodic but stroboscopic aging

– coupling to various non-Markovian baths $\Rightarrow$

apply a thermal bath with a characteristic time-scale on one end and a different thermal bath with a different characteristic time-scale on the opposite end and see how a time-reparametrization flow establishes in the model

Plan

Schematic

— The simplest aging example :

domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation :

self-correlation and linear response

— Fluctuation-dissipation relations :

effective temperatures

— Mean-field modeling :

separation of time scales

— **Reparametrization invariance :**

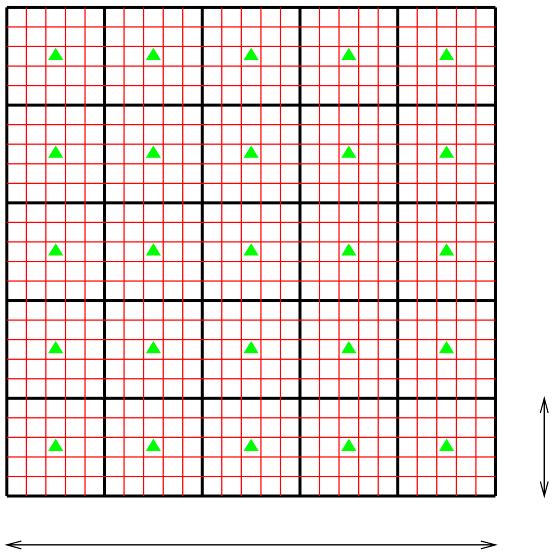
sigma model

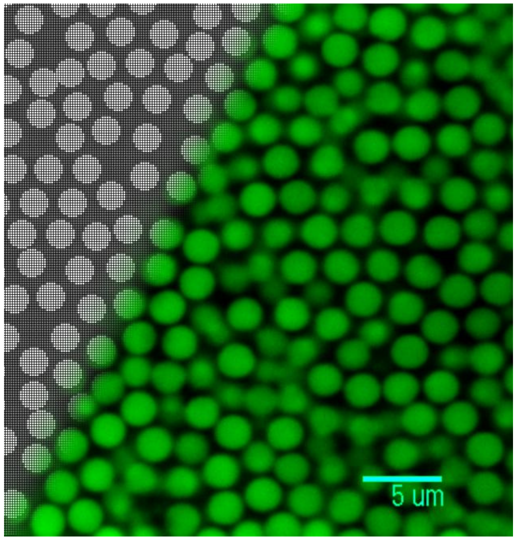
— **Fluctuations :**

local two-time observables

Turn it useful

Characterize the spatial fluctuations





$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i(t) s_i(t_w)$$

$$\chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} \int_{t_w}^{t} dt' \left. \frac{\delta s_i^{(h)}(t)}{\delta h_i(t')} \right|_{h=0}$$

Review **Chamon & LFC 07**

Consequences

Characterize the spatial fluctuations

- There is an approximate dynamic symmetry :

  global time reparametrization invariance

- There is a **soft/massless** dynamic mode associated to it,

  with a **two-time diverging correlation length** $\xi(t,t_w)$

  Extract it from, *e.g*

$$C_4(r,t,t_w) = \frac{1}{N} \sum_{i,j/|\vec{r}_i - \vec{r}_j| = r} \langle s_i(t) s_i(t_w) s_j(t) s_j(t_w) \rangle_c$$

- Characterize **dynamic fluctuations - heterogeneities**

  $C_{\vec{r}}(t,t_w;\ell,\xi)$, $\rho(C_{\vec{r}},\chi_{\vec{r}};t,t_w;\ell,\xi)$, multi-time functions, *etc.*

- Disentangle simple **dynamic scaling** implications from **time reparametrization invariance** ones.

Consequences

Characterize the spatial fluctuations

In the **scaling limit**

lattice spacing $\ll$ coarse-graining length $\ll$ correlation length $\ll$ system size

$$a \ll \ell \ll \xi(t, t_w) \ll L$$

- The **'clock'** $h_{\vec{r}}(t)$ is **local** (analogy : angle - soft mode - in a Mexican hat potential)
- The **scaling functions** ($f_{\rm ag}$, $\chi_{\rm ag}(C_{\rm ag})$) **do not fluctuate** (modulus)

$\Rightarrow T_{\rm eff}$ **does not fluctuate**

In **practice (simulations, experiments)**

$$a \lesssim \ell \lesssim \xi(t, t_w) \ll L$$

- $\ell/\xi(t, t_w)$ is an additional scaling variable.

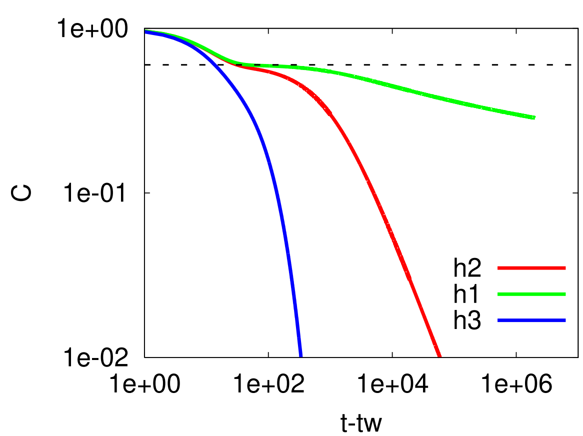
Leading fluctuations

Global to local correlations

$$C_{\rm ag}(t,t_w) \approx f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow$ $C^{\rm ag}_{\vec r}(t,t_w) \sim f_{\rm ag}\left(\frac{h_{\vec r}(t)}{h_{\vec r}(t_w)}\right)$

Ex. $h_{\vec r_1} = \frac{t}{t_0}$, $h_{\vec r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec r_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions



Same t_w, slower and faster decays

Castillo, Chamon, LFC, Iguain, Kennett 02, 03

Chamon, Charbonneau, LFC, Reichman, Sellitto 04

Jaubert, Chamon, LFC, Picco 07

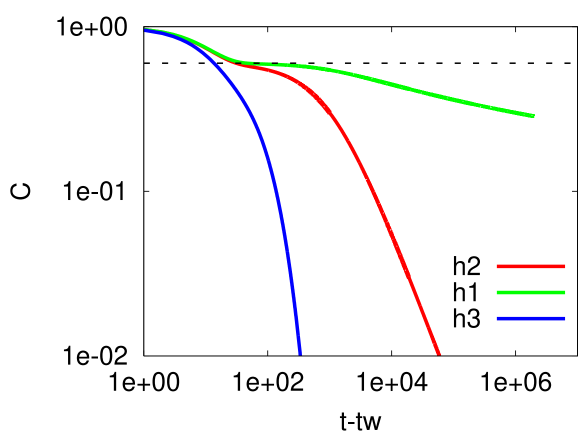
Leading fluctuations

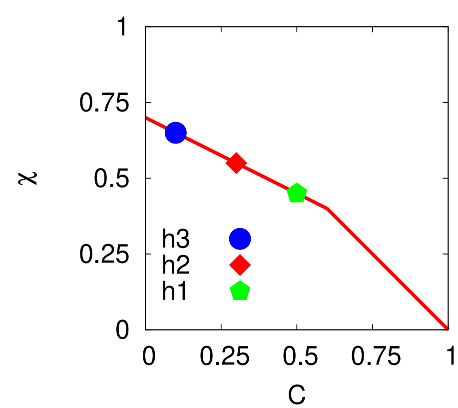
Global to local correlations & responses

$$C_{\rm ag}(t,t_w) \approx f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow$ $C^{\rm ag}_{\vec r}(t,t_w) \sim f_{\rm ag}\left(\frac{h_{\vec r}(t)}{h_{\vec r}(t_w)}\right)$

Ex. $h_{\vec r_1} = \frac{t}{t_0}$, $h_{\vec r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec r_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions

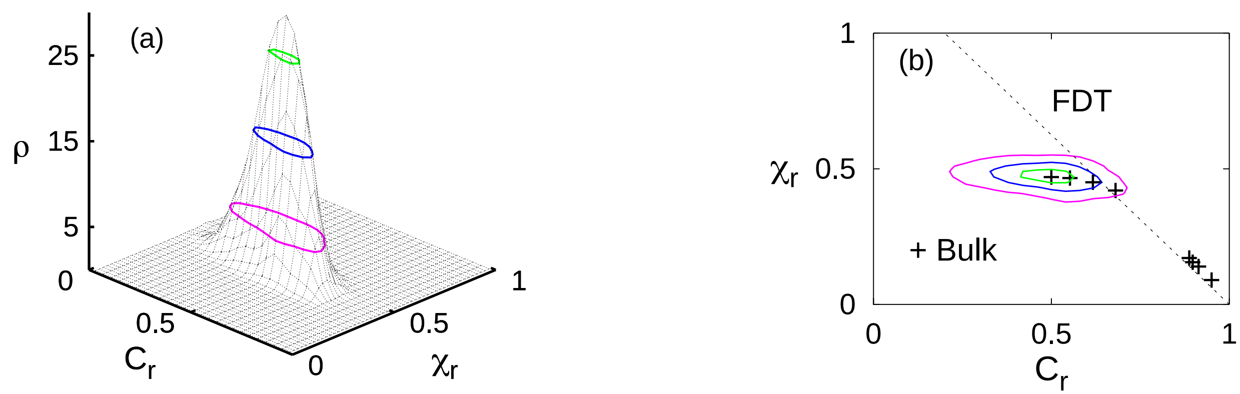




Local correlations & responses

$3d$ **Edwards-Anderson spin-glass**

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} s_i(t)s_i(t_w) \; , \quad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} \int_{t_w}^{t} dt' \; \left.\frac{\delta s_i(t)}{\delta h_i(t')}\right|_{h=0}$$



+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 $t \; (> t_w)$

ρ corresponds to the maximum t yielding the smallest C (left-most +)

Castillo, Chamon, LFC, Iguain, Kennett 02

Kinetically constrained models + Charbonneau, Reichman & Sellitto 04

Sigma Model

Conditions & expression

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{ag}(\vec{r},t,t_w) = f_{ag}\left(e^{-\int_{t_w}^{t} dt'\, \partial_{t'}\varphi(\vec{r},t')}\right)$$

i. The action must be invariant under a global time reparametrization $t \to h(t)$.

ii. If our interest is in short-ranged problems, the action must be written using local terms. The action can thus contain products evaluated at a single time and point in space of terms such as $\varphi(\vec{r},t)$, $\partial_t\varphi(\vec{r},t)$, $\nabla\varphi(\vec{r},t)$, $\nabla\partial_t\varphi(\vec{r},t)$, and similar derivatives.

iii. The scaling form in eq. (29) is invariant under $\varphi(\vec{r},t) \to \varphi(\vec{r},t) + \Phi(\vec{r})$, with $\Phi(\vec{r})$ independent of time. Thus, the action must also have this symmetry.

iv. The action must be positive definite.

These requirements largely restrict the possible actions. The one with the smallest number of spatial derivatives (most relevant terms) is

$$\mathcal{S}[\varphi] = \int d^d r \int dt \left[K \, \frac{(\nabla\partial_t\varphi(\vec{r},t))^2}{\partial_t\varphi(\vec{r},t)} \right] , \qquad (30)$$

Chamon & LFC 07

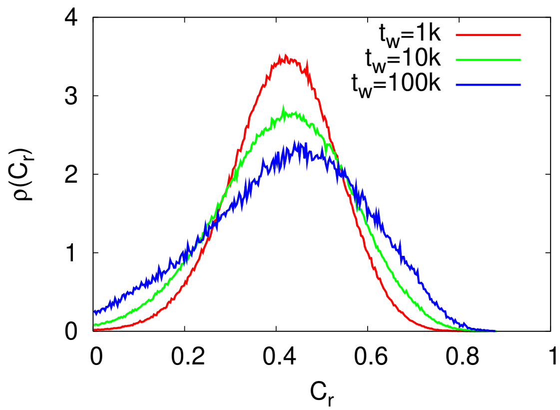
Sigma Model

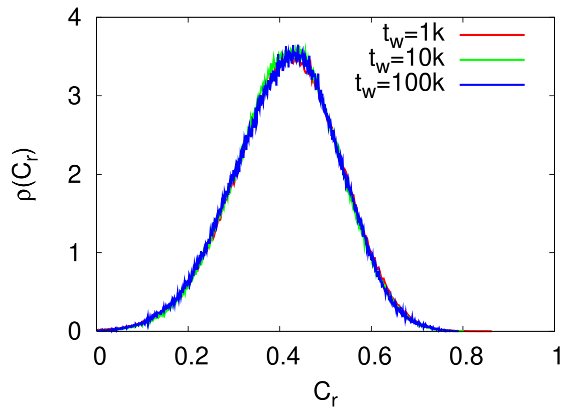
Some consequences - $3d$ Edwards Anderson model

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{\text{ag}}(\vec{r},t,t_w) = f_{\text{ag}}(e^{-\int_{t_w}^{t} dt'\, \partial_{t'}\varphi(\vec{r},t')})$$

Distribution of local correlations depends on times t, t_w only through C, ξ

$$\rho(C_{\vec{r}};\, t, t_w, \ell, \xi(t,t_w)) \to \rho(C_{\vec{r}};\, C_{\text{ag}}(t,t_w), \ell/\xi(t,t_w))$$





t, t_w such that $C_{\text{ag}}(t,t_w) = C$ $\qquad$ ℓ such that $\ell/\xi = \text{cst}$ **Jaubert, Chamon, LFC, Picco 07**

predictions on the form of ρ derived from $S[\varphi]$ too

Tests in Lennard-Jones systems **Avila, Castillo, Mavimbela, Parsaeian 06-12**

How general is this ?

Coarsening & domain growth

e.g. the d-dimensional $O(N)$ model in the large N limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

N component field $\vec{\phi} = (\phi_1, \dots, \phi_N)$ with Langevin dynamics

$$\partial_t \phi_\alpha(\vec{r},t) = \nabla^2 \phi_\alpha(\vec{r},t) + \lambda |N^{-1}\phi^2(\vec{r},t) - 1|\, \phi_\alpha(\vec{r},t) + \xi_\alpha(\vec{r},t)$$

$\phi_\alpha(\vec{k},0)$ Gaussian distributed with variance Δ^2

> Time reparametrization invariance is reduced to time rescalings
>
> $$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$

Same in the $p = 2$ spherical model

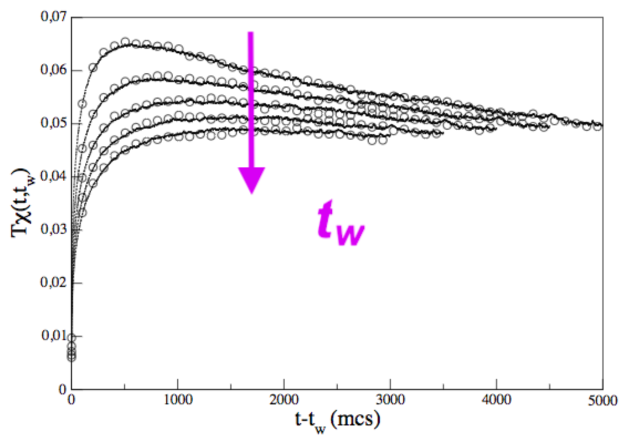
Chamon, LFC, Yoshino 06

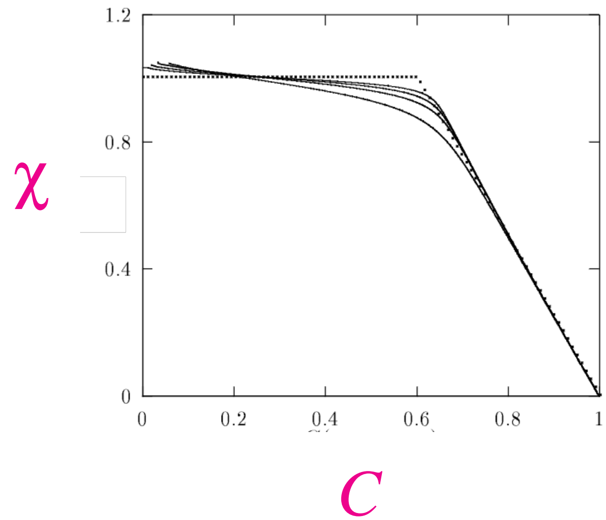
How general is this ?

Coarsening & domain growth

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$





Ising FM, $O(N)$ field theory, or $p = 2$ spherical model

Related to $T^* \to \infty$ and simplicity of free-energy landscape

Conclusions

(Annoying) global time-reparametrization invariance $t \to h(t)$ in models in which

- $C_{\rm ag}(t,t_w) \gg \partial_t C_{\rm ag}(t,t_w)$ (slow dynamics)
- $\chi_{\rm ag}(t,t_w) \gg \partial_t \chi_{\rm ag}(t,t_w)$ (weak long-term memory)

and finite effective temperature $T_{\rm eff} < +\infty$

Reason for the large dynamic fluctuations (heterogeneities) $h(\vec{r},t)$

Effective action for $\varphi(\vec{r},t)$ in $h(\vec{r},t) = e^{-\varphi(\vec{r},t)}$

Quantum : the rapid equilibrium regime is modified but the slow aging one is classical in nature controlled by a $T_{\rm eff} > 0$, then the same applies

Triangular relations

Scaling of the aging global correlation

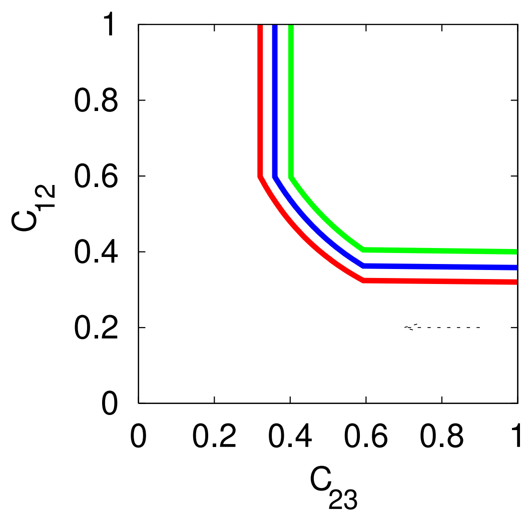
Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1,t_2),\ C(t_2,t_3),\ C(t_1,t_3)$$

If, in the aging regime $C^{ij}_{ag} \equiv C_{ag}(t_i,t_j) = f_{ag}\left(\frac{h(t_i)}{h(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{ag} = f_{ag}\left(\frac{h(t_1)}{h(t_3)}\frac{h(t_3)}{h(t_2)}\right) = f_{ag}\left(\frac{f^{-1}_{ag}(C^{13}_{ag})}{f^{-1}_{ag}(C^{23}_{ag})}\right)$$

q_{ea}



q_{ea}

choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12} = q_{ea} C^{13}/C^{23}$

Triangular relations

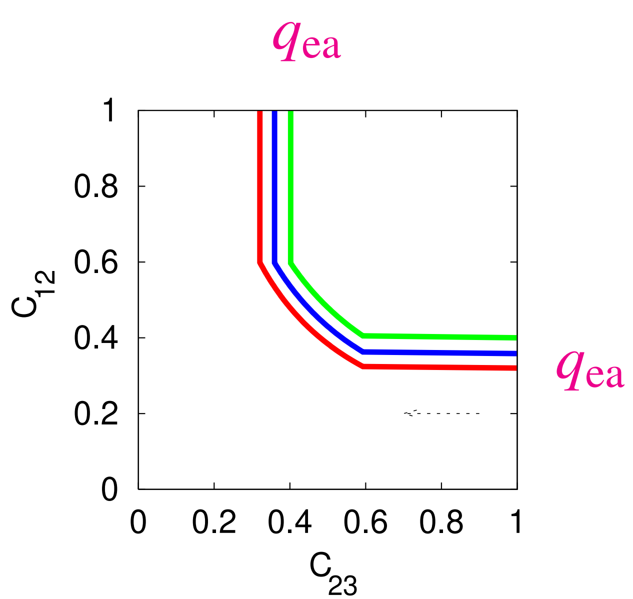
Scaling of the slow part of the global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec{r}}(t_1,t_2),\ C_{\vec{r}}(t_2,t_3),\ C_{\vec{r}}(t_1,t_3)$$

If, in the aging regime $C^{ij}_{\vec{r}} \equiv C_{\vec{r}}(t_i,t_j) = f_{\rm ag}\left(\frac{h_{\vec{r}}(t_i)}{h_{\vec{r}}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{\vec{r}} = f_{\rm ag}\left(\frac{f_{\rm ag}^{-1}(C^{13}_{\vec{r}})}{f_{\rm ag}^{-1}(C^{23}_{\vec{r}})}\right)$$



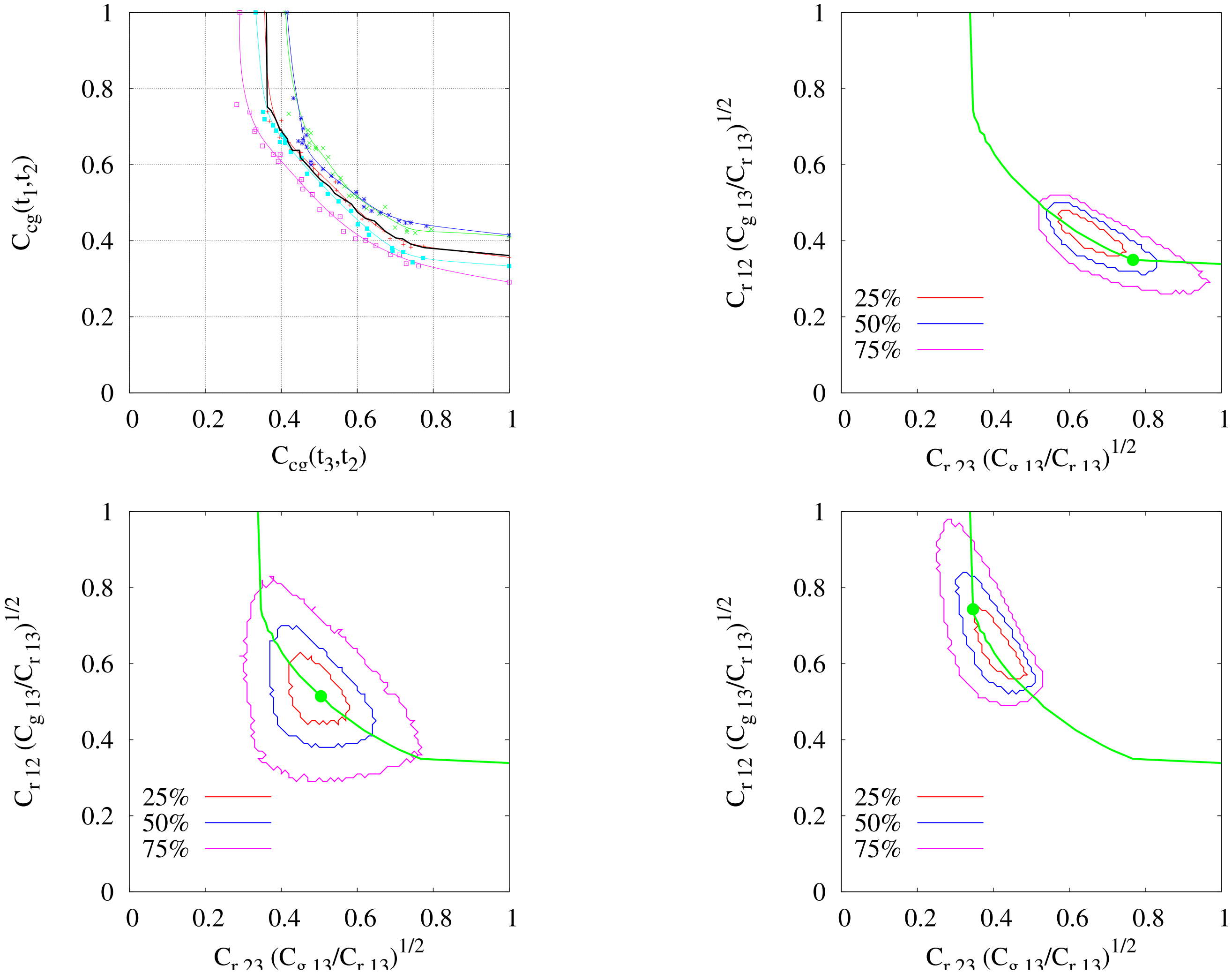
choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12}_{\vec{r}} = q_{\rm ea} C^{13}_{\vec{r}} / C^{23}_{\vec{r}}$.

Triangular relations

$3d$ Edwards-Anderson model



Jaubert, Chamon, LFC & Picco 07