Dynamics of the bidimensional Potts model in the large q limit

Marco Picco*
Sorbonne Université and CNRS
Laboratoire de Physique Théorique et Hautes Energies

14/03/2023

*In collaboration with F. Chippari and L. Cugliandolo (Sorbonne Université), F. Corberi, M. Esposito and O. Mazzarisi (Salerno University, Italy),

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

Introduction and Motivations

Introduction and Motivations

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- q state Potts model is a simple extension of the ferromagnetic Ising model.
- In two dimensions, $q > 4$ corresponds to a model with a first order transition with metastability.
- We study sub-critical quenches starting from a completely disordered configuration.
- Already many studies in the past, which observed different phenomenas : freezing or blocking at low temperature, multi nucleation, metastability, coarsening, etc.
- Most of these studies with small values of q with strong finite size corrections and finite q corrections !!

Phase diagram

Introduction and Motivations

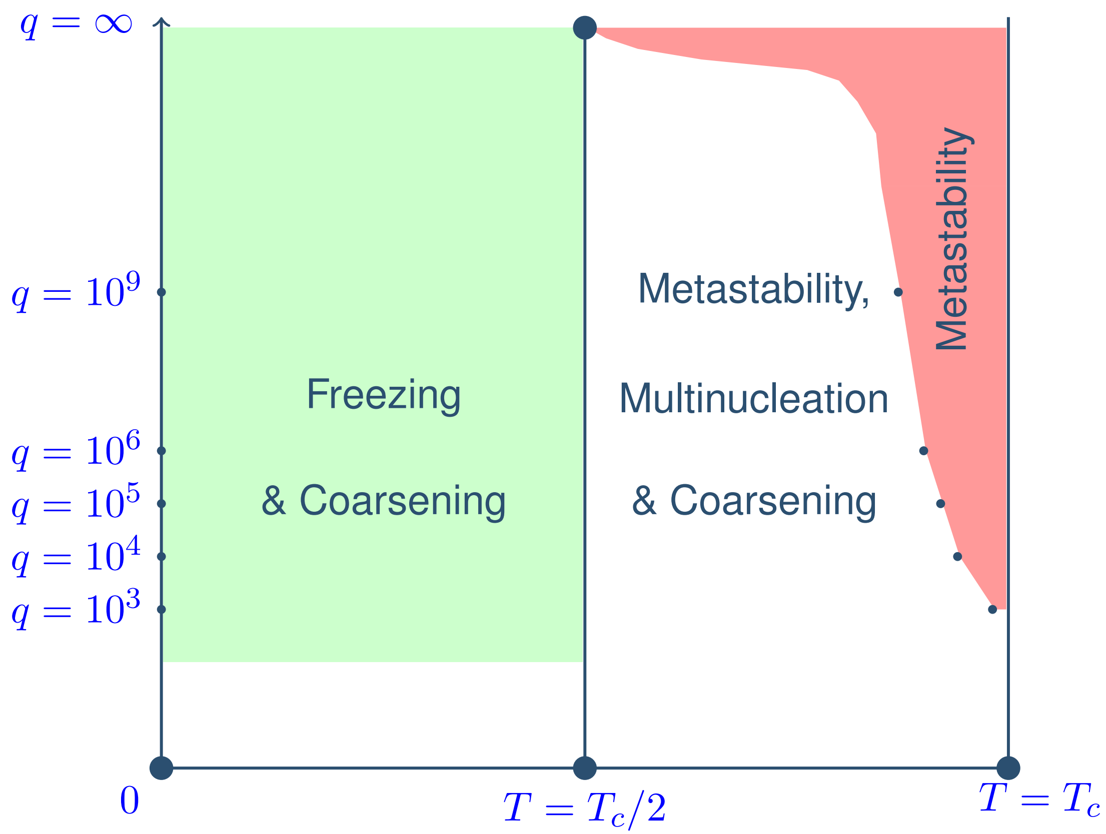
Low temperature

Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion



Phase diagram of the $2d$ Potts square-lattice model with the crossover lines between different types of dynamic behaviour.

Definitions

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- On a square lattice, with periodic boundary conditions, we consider the Hamiltonian defined by

$$H_J[\{s_i\}] = -J \sum_{<ij>} \delta_{s_i s_j} \ ,$$

with $< ij >$ the sum restricted to nearest-neighbours, δ_{ab} the Kronecker delta and s_i take integer values from 1 to q.

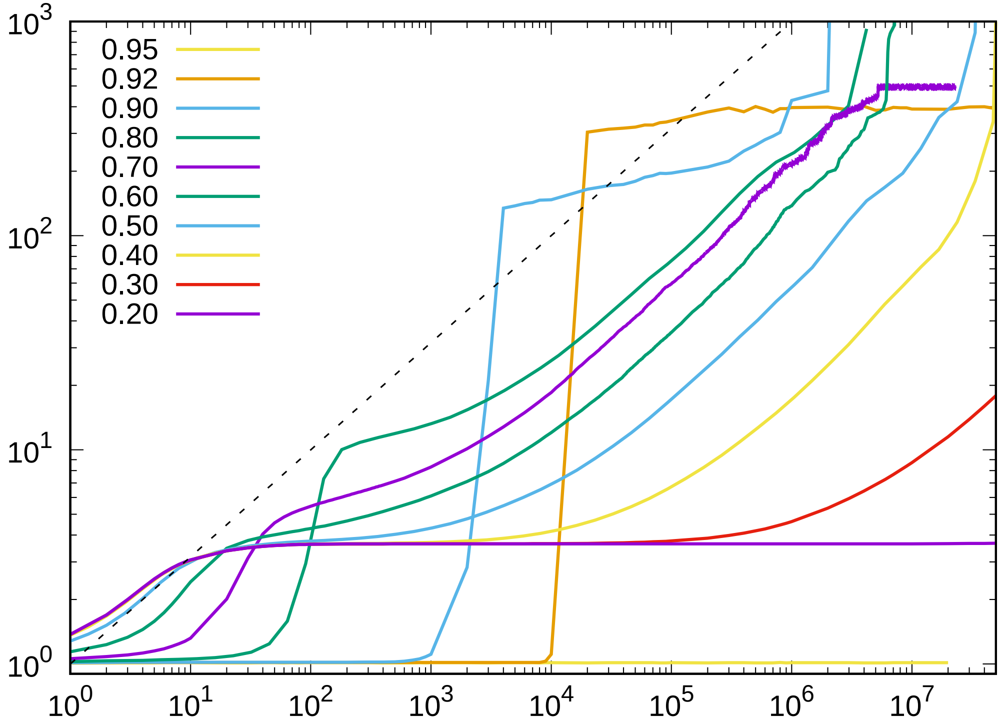
- Transition for $\beta_c = \log\left(1 + \sqrt{q}\right)$.
- Dynamics : Metropolis $\rightarrow$ Heat bath, which is much faster, by a factor q.
- Heat bath dynamics allows for a partial analytic treatment.

Growing length

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- $R(t,q,T/T_c) = \frac{e_0}{e_0 - e(t)}$; $e_0 =$ eq. energy.

$Q = 10^4$



- One observes : i) $T < T_c/2$ freezing and next coarsening ; ii) $T > T_c/2$ metastability then jump (multi nucleation) then coarsening ; iii) T close to T_c, only metastability.

Introduction and Motivations

Low temperature

Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion

Low temperature

Introduction and Motivations

Low temperature

Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion

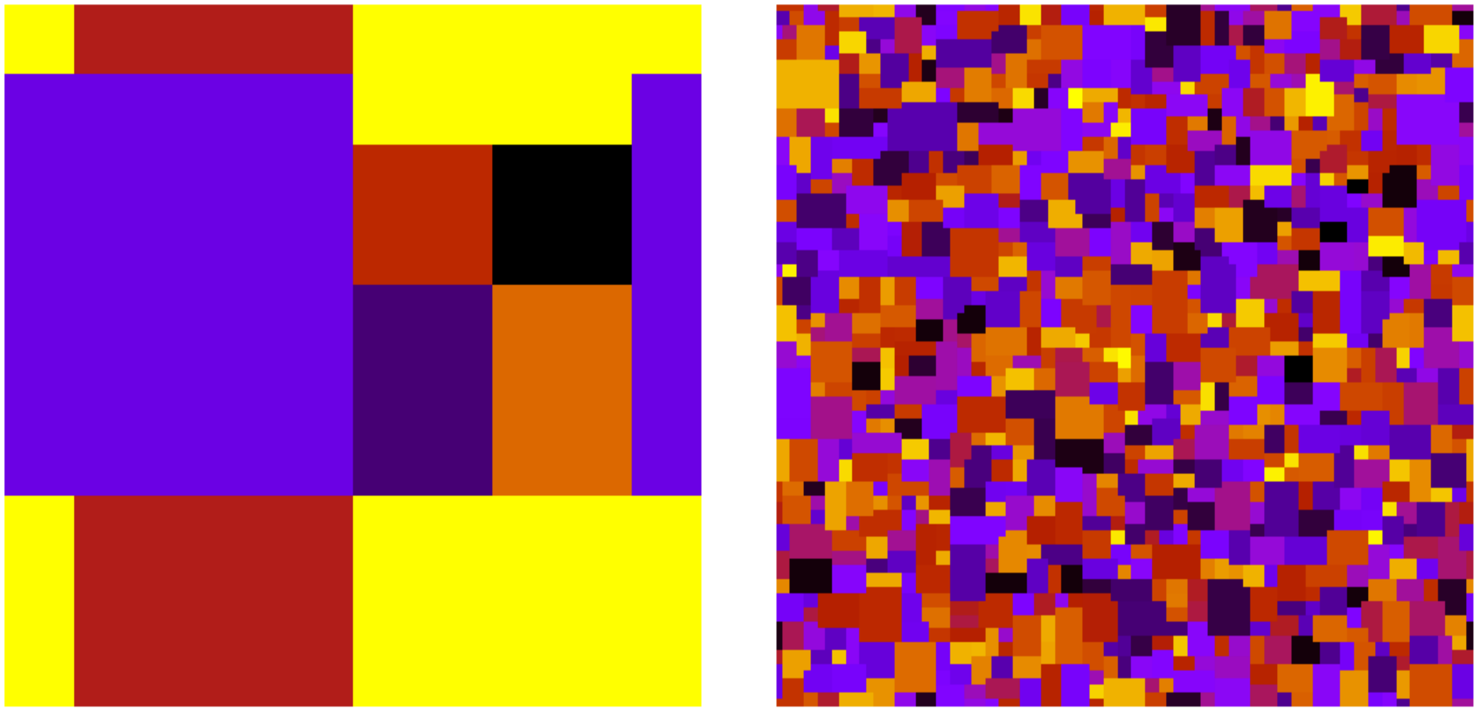


Figure 1: Configurations of the Potts model in the infinite q limit after a quench to $T < T_c/2$, $L = 10$ and $L = 10^2$.

Introduction and Motivations

Low temperature

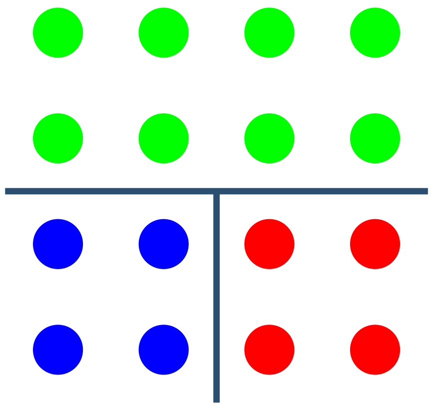
Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion

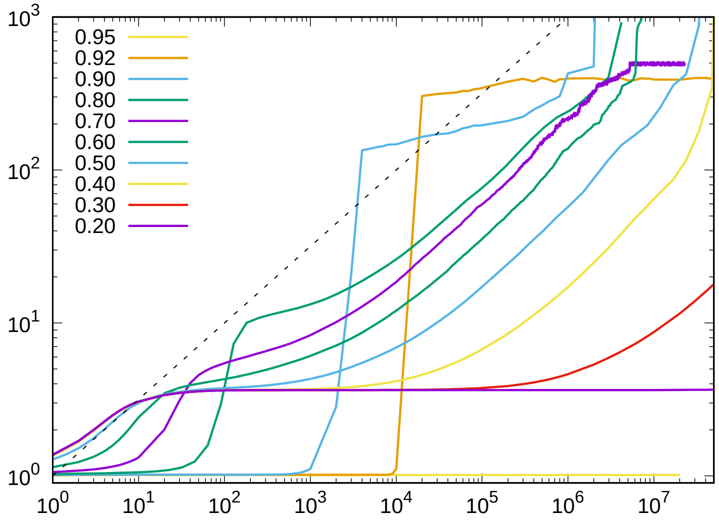
- Same structure at $T = 0$ for finite (and large) q and for infinite q and $T < T_c/2$.
- T blocking or freezing due to so called T-junctions for $q \geq 3$ at zero temperature (J. Glazier, M. Anderson and G. S. Grest, 1990, J. Olejarz, P. Krapivsky, and S. Redner, 2013)

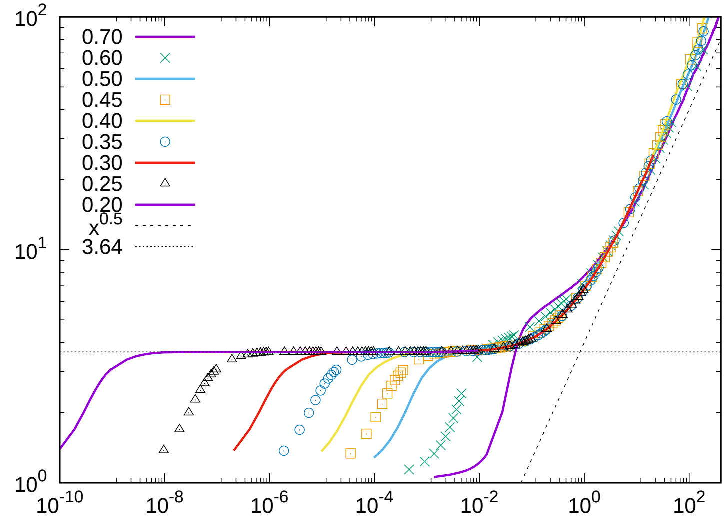


- Needs to reverse one corner, which costs

$$e^{-\Delta E} = e^{-1/T} \rightarrow t_s = e^{1/T}$$

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion





R vs. t (left) and t/t_s (right) for $q = 10^4$.

At late time, $R(t) \simeq t^{1/z}$ with $z = 2$: coarsening.

Introduction and Motivations

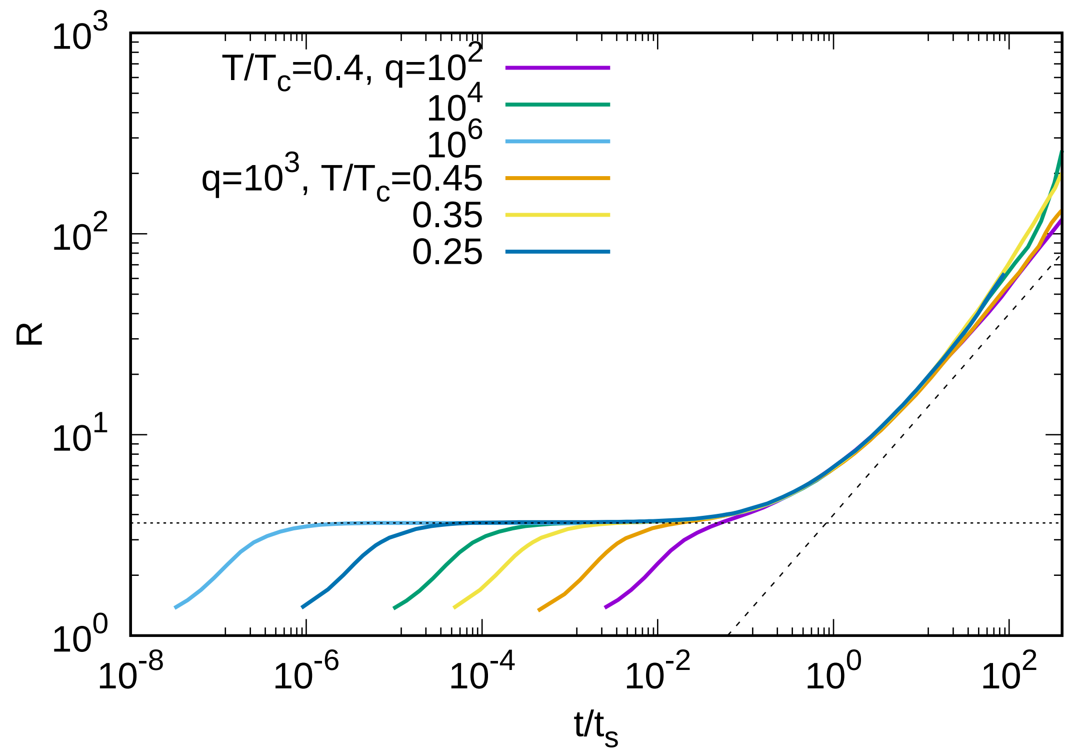
Low temperature

Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion



Growing length R vs. t/t_S for various T/T_c and q : Universality !!!

(F. Chippari, L. F. Cugliandolo, & M. P., 2021)

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

Metastability then coarsaning

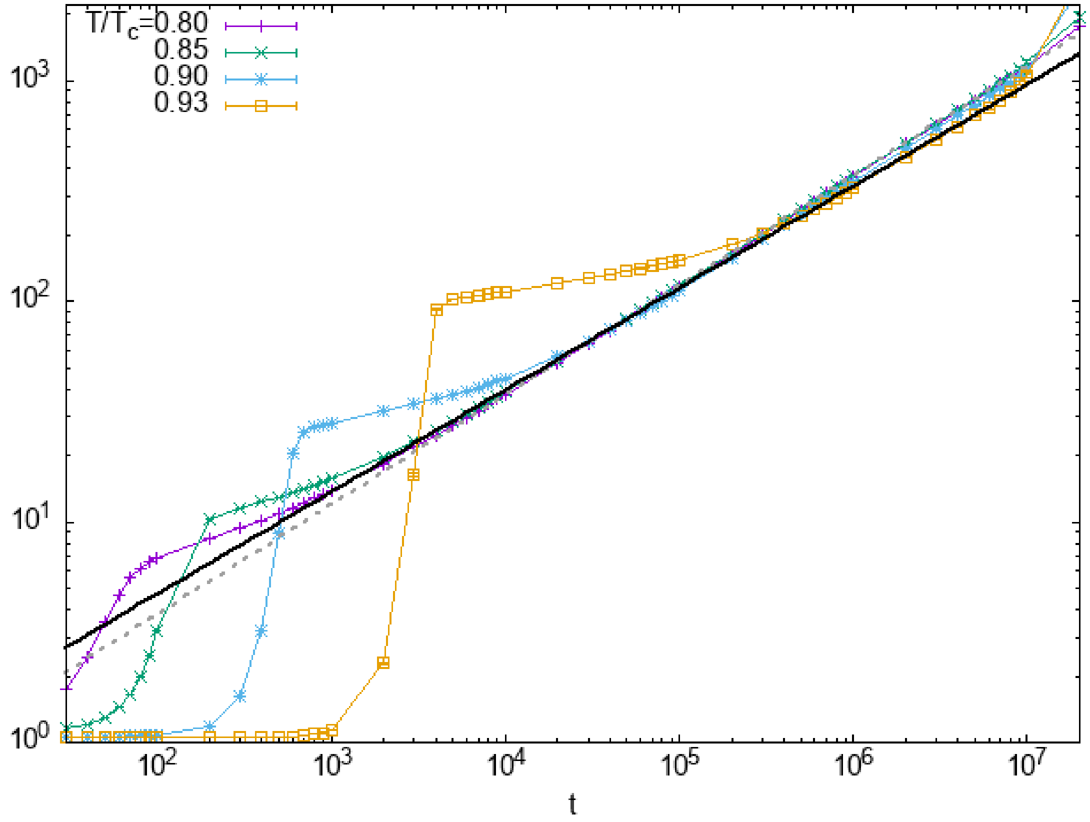
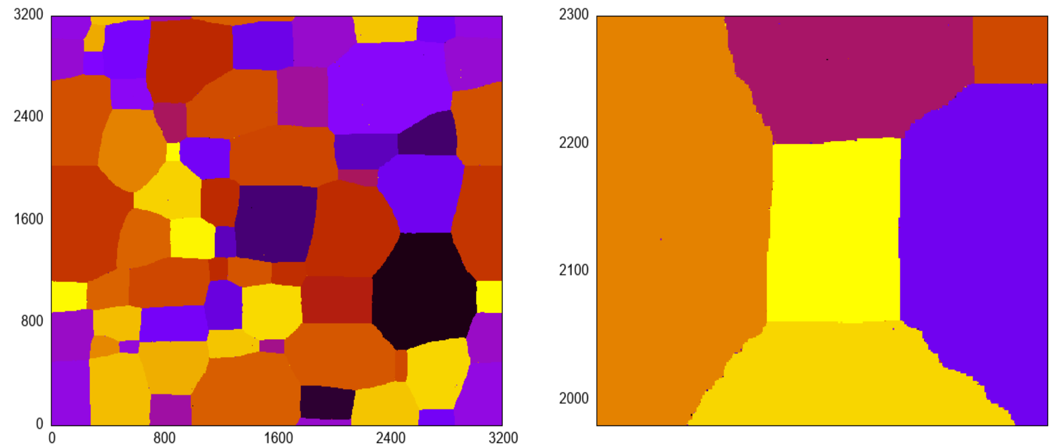


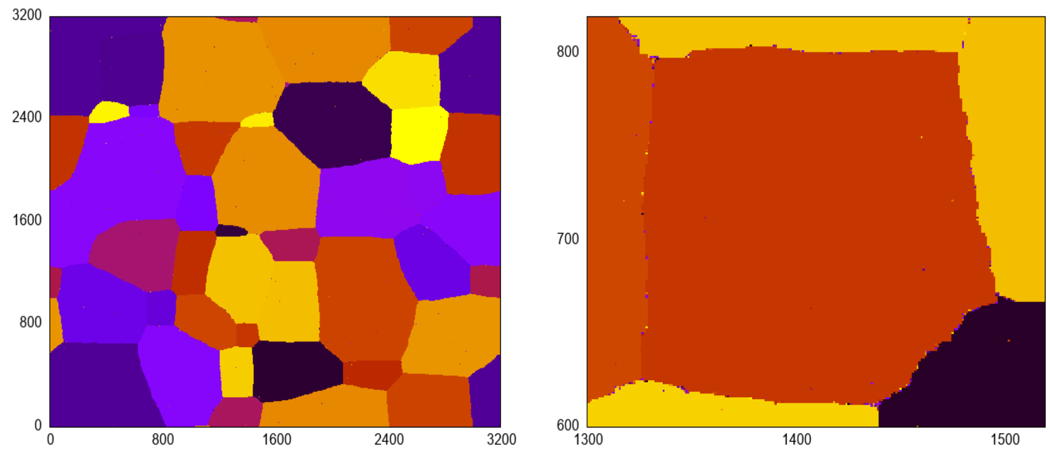
Figure 2: The growing length R vs. t for $q = 10^3$ and $L = 3200$. Dotted line is the $t^{1/2}$ law.

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

$\frac{T}{T_c} = 0.85$



$\frac{T}{T_c} = 0.93$



Snapshots at $t = 10^6$. One note the existence of ”sand” on the borders and in the bulk.

Introduction and Motivations

Low temperature

Metastability then coarsaning

Metastability close to T_c

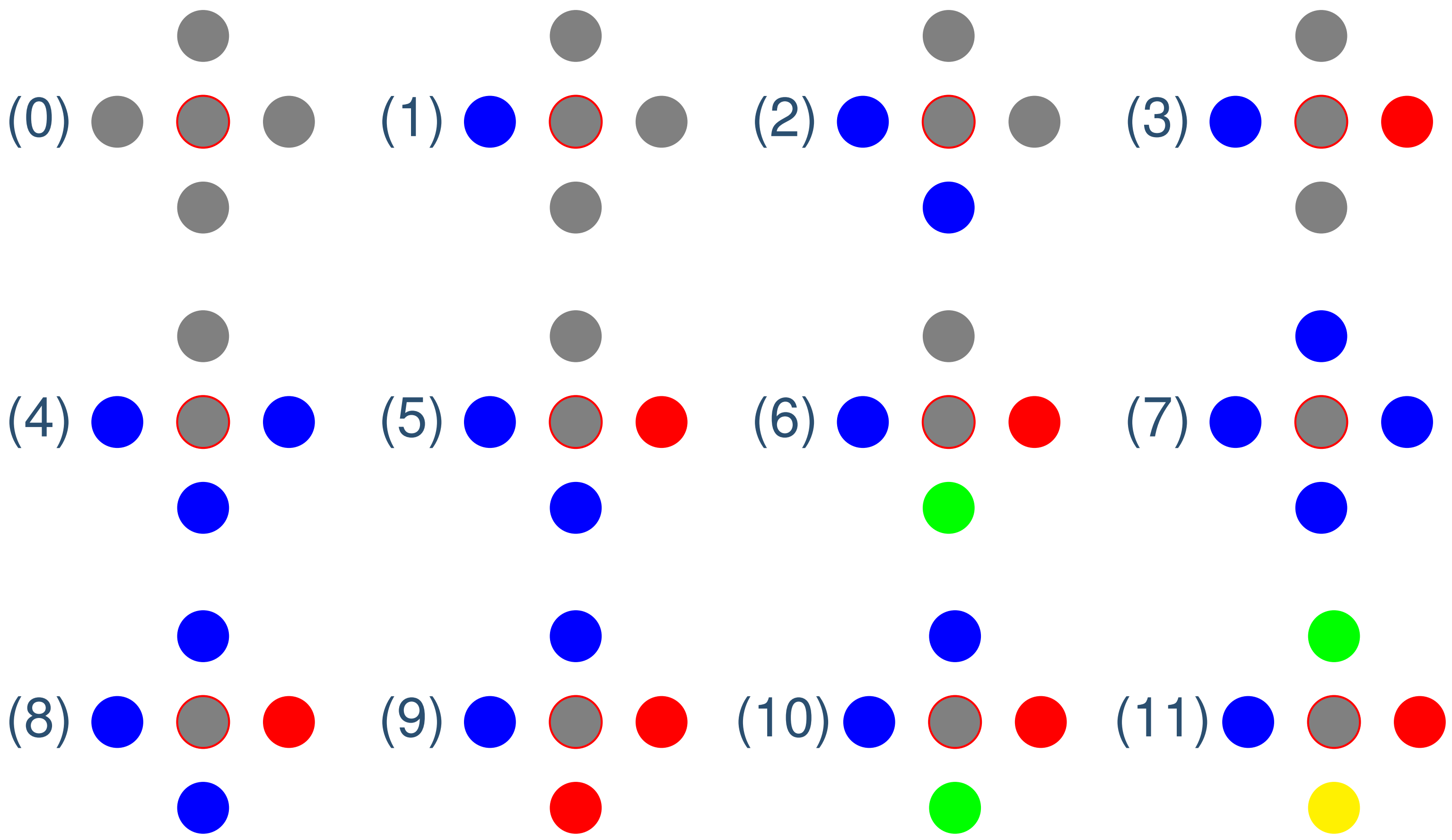
Extensions

Conclusion

Metastability close to T_c

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

Enumeration : all the states can be described as :



Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- The evolution among these states is simple to describe. For example

$$P_{11\to 11} = \frac{q-4}{4e^{\beta}+q-4}\ , P_{11\to 6} = \frac{4e^{\beta}}{4e^{\beta}+q-4} \tag{1}$$

- $\beta_c = \log(1+\sqrt{q})$, thus in the large q limit $e^{\beta_c} \simeq q^{1/2}$

$$P_{11\to 11} \simeq \frac{q}{4q^{T_c/2T}+q}\ , P_{11\to 6} \simeq \frac{4q^{T_c/2T}}{4q^{T_c/2T}+q} \tag{2}$$

- For $q \to \infty$, remains disordered forever for a quench to a final temperature $T_c/2 < T_f < T_c$.

$$P_{11\to 11} = 1\ \ ,\ \ P_{11\to 6} = 0 \tag{3}$$

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- For large (but finite) q, $P_{11\to 6} \simeq \dfrac{4q^{T_c/2T}}{4q^{T_c/2T}+q} \simeq 4q^{T_c/2T-1} \simeq p$, with p a small parameter.
- We consider the densities $N_0(t), N_1(t), \cdots, N_{11}(t)$. $N_{11}(t)$ corresponds to the number of spins in the (11) state, etc.
- In a paramagnetic state ($t=0$), $N_{11}(0) \simeq 1, N_6(0) \simeq 4/q$, $N_{10}(0) \simeq 6/q, \cdots$.
- Next, $\dot{N}_6(t) = 2N_{11}(t)p + \cdots\ ;\ \dot{N}_{11}(t) = -2N_{11}(t)p + \cdots$

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

One obtains (O. Mazzarisi, F. Corberi, L. F. Cugliandolo, & M. P., 2020)

$$\dot{N}_{11} = -\,12N_{11}p^2 - 2N_{11}p + 2N_6 - \frac{7}{4}N_6p - N_{3a} + N_{3b} + N_{10a} + N_{10b}\ ,$$

$$\dot{N}_6 = 2N_{11}p - 2N_6 + \frac{1}{2}N_6p + 3(N_{3a} + N_{3b}) + N_{3c} + N_{10a} + N_{10b} - N_{10c}\ ,$$

$$\dot{N}_{3a} = \frac{1}{4}N_6p - \frac{5}{2}N_{3a} + \frac{1}{2}N_{10a} + \frac{1}{2}(N_{3c} - N_{10c})\ ,$$

$$\dot{N}_{3b} = \frac{1}{2}N_6p - \frac{5}{2}N_{3b} + \frac{1}{2}N_{10b} + \frac{1}{2}(N_{3c} - N_{10c})\ ,$$

$$\dot{N}_{3c} = -\,2N_{3c} + 2N_{10c}\ ,$$

$$\dot{N}_{10a} = 4N_{11}p^2 - \frac{5}{2}N_{10a} + \frac{1}{2}N_{3a}\ ,$$

$$\dot{N}_{10b} = 8N_{11}p^2 - \frac{5}{2}N_{10b} + \frac{1}{2}N_{3b}\ ,$$

Next, we impose stationarity, $\dot{N}_a(t) = 0$, and solve in powers of p.

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

We end up with

$$N_{11} = 1 - p - \frac{605}{56}p^2, \qquad N_6 = p + \frac{95}{28}p^2,$$
$$N_{3a} = \frac{7}{16}p^2, \qquad N_{3b} = \frac{11}{14}p^2, \qquad N_{3c} = \frac{11}{14}p^2, \tag{4}$$
$$N_{10a} = \frac{27}{16}p^2, \qquad N_{10b} = \frac{41}{14}p^2, \qquad N_{10c} = \frac{11}{14}p^2.$$

T/T_c	p		N_{11}	N_6	$10^3 N_{3a}$	$10^3 N_{3b}$	$10^3 N_{3c}$	$10^3 N_{10a}$	$10^3 N_{10c}$
0.88	0.01017	numeric	0.9895816	0.0101646	0.0130	0.0260	0.1772	0.0020	0.0039
		analytic	0.9895916	0.0101674	0.0129	0.0259	0.1705	0.0020	0.0039
0.92	0.00725	numeric	0.9926679	0.0072481	0.0066	0.0132	0.0444	0.0020	0.0039
		analytic	0.9926690	0.0072485	0.0066	0.0131	0.0438	0.0020	0.0040
0.98	0.00459	numeric	0.9953845	0.0045892	0.0026	0.0053	0.0070	0.0020	0.0040
		analytic	0.9953847	0.0045892	0.0026	0.0053	0.0070	0.0020	0.0040
0.99	0.00428	numeric	0.9957020	0.0042752	0.0023	0.0046	0.0053	0.0020	0.0040
		analytic	0.9957023	0.0042751	0.0023	0.0046	0.0053	0.0020	0.0040

N_a for systems with linear size $L = 10^3$, $q = 10^6$

Summary of results

Introduction and Motivations

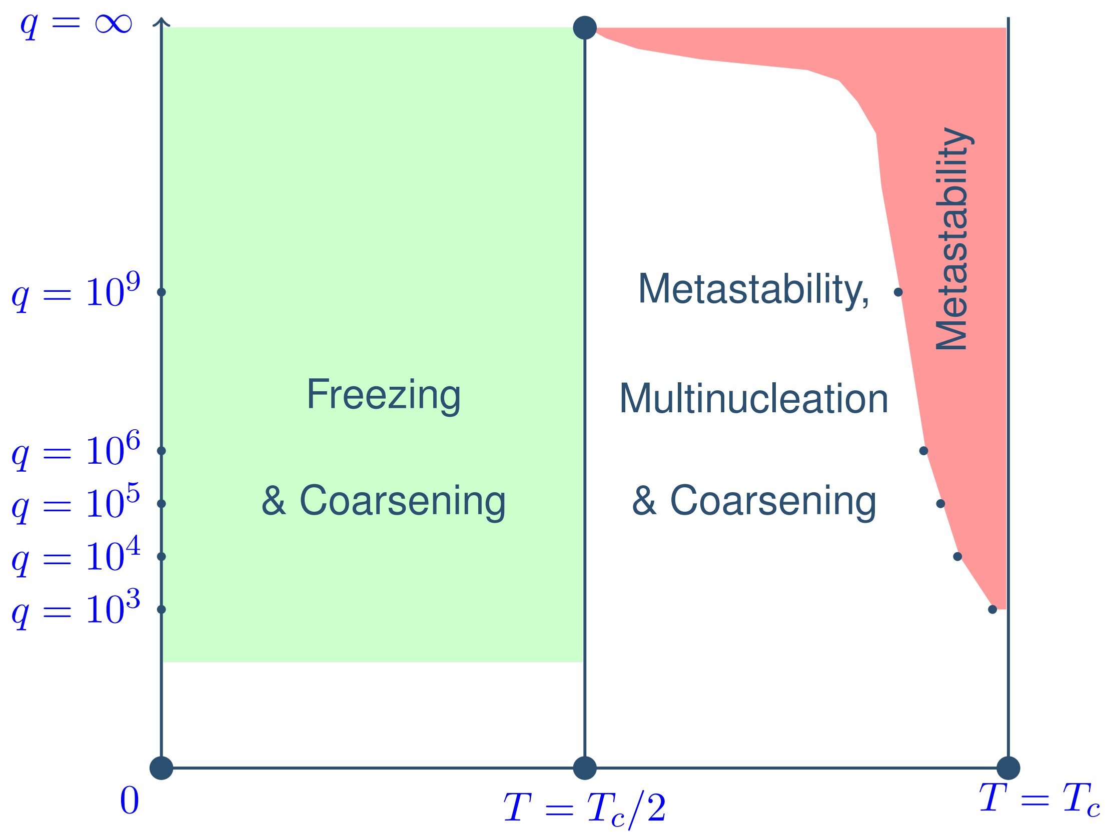
Low temperature

Metastability then coarsaning

Metastability close to T_c

Extensions

Conclusion



Phase diagram of the $2d$ Potts square-lattice model with the crossover lines between different types of dynamic behaviour.

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

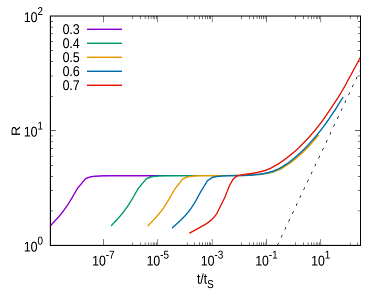
Extensions

Extensions

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- Similar results for the hexagonal lattice. Freezing at low temperature already for $q = 2$ (H. Takano, and S. Miyashita, 1993), with the same behaviour for $T \leq \frac{2}{3}T_c$ and for all $q \geq 2$.

$q = 10^4$



- No freezing for the Triangular lattice for $T \leq \frac{1}{3}T_c$, just coarsening (G. N. Hassold, and E. A, Holm, 1993).
- Freezing up to $t \simeq e^{1/T}$ for $T \leq \frac{1}{3}T_c$ for the cubic Potts model (F. Chippari and M. P, 2022)
- Disorder slows the dynamics but not other changes.

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

Conclusion

Conclusion

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- Universal behaviour in $2d$ and $3d$ as a function of $q >> 1$.
- For the square lattice in $2d$, we show
  - For $T < T_c/2$, freezing, then for $t > e^{1/T}$, coarsening.
  - For $T > T_c/2$ and not to close to T_c, metastability, multinucleation and coarsening.
  - For $T \simeq T_c$ and large q, metastable equilibrium that we can completely characterise.
- For other lattices, the spinodal temperature is $\frac{2}{z}T_c$ with z the coordination number.

Introduction and Motivations
Low temperature
Metastability then coarsaning
Metastability close to T_c
Extensions
Conclusion

- "Multinucleation in the first-order phase transition of the 2d Potts model", F. Corberi, L. F. Cugliandolo, M. Esposito, & M. Picco, J. Phys.: Conf. Ser. 1226 (2019) 012009.
- "Metastability in the Potts model: exact results in the large q limit", O. Mazzarisi, F. Corberi, L. F. Cugliandolo, & M. Picco, J. Stat. Mech. (2020) 063214.
- "Low-temperature universal dynamics of the bidimensional Potts model in the large q limit", F. Chippari, L. F. Cugliandolo, & M. Picco, J. Stat. Mech. (2021) 093201.
- "How many phases nucleate in the bidimensional Potts model?", F. Corberi, L. F. Cugliandolo, M. Esposito, O. Mazzarisi, & M. Picco, J. Stat. Mech. (2022) 073204.
- "Freezing vs. equilibration dynamics in the Potts model", F. Chippari, & M. Picco, J. Stat. Mech. (2023) 023201.