The $\pm J$ $2d$ Ising model

Quantum codes, network models & disorder

Focus on critical dynamics

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Plan

Schematic

— The $\pm J$ $2d$ Ising Model

Statistical physics perspective

Condensed-matter - quantum Hall systems

— The equilibrium phase diagram

Critical behaviour

The Nishimori line

Quantum computation - coding theory

— Out of equilibrium critical dynamics - **Universality ?**

Space-time correlations

Short-time dynamics

Space-time winding angle

Criticality and dynamic exponent - long crossovers

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$\pm J$ $2d$ Ising Model

Definition

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

$s_i = \pm 1$

$J_{ij} = \pm J$ **quenched randomness**

$$P(J_{ij}) = p \underbrace{\delta_{J_{ij},-J}}_{\text{AF}} + (1-p) \underbrace{\delta_{J_{ij},J}}_{\text{FM}}$$

$[J_{ij}] = (1-2p)J$

$[J_{ij}^2] = J^2$

p controls the level of **frustration**

$p = 0$	Ferromagnetic Ising Model
$p = 1/2$	(unbiased) Ising Spin-Glass
$p = 1$	Anti-Ferromagnetic Ising Model

symmetry $p \leftrightarrow 1-p$

Effects of disorder

Generic questions from the 70s-80s

– Does disorder kill the ordered phase ?

– If not, does it round the phase transitions ?

e.g., 1st order phase transitions transformed into 2nd order ones

– In continuous phase transitions,

does it change the critical properties ?

– Are there new kinds of ordered phases ?

e.g., spin-glass phases

Effects of disorder

Results for weak disorder

– **Does disorder kill the ordered phase ?** no but $T_c(p) \searrow$ for $p \nearrow$ expected

– **Effect on the phase transition**

Harris Criterium : the randomness is relevant (irrelevant) if the specific heat exponent α of the pure ($p = 0$) model is positive (negative)

A. B. Harris, J. Phys. C7, 1671 (1974) but for the $2d$ Ising Model $\alpha = 0$

– **Conformal field theory in $2d$?**

The $n = 0$ Gross-Neveu model (for not too large p, see below)

Vik. S. Dotsenko & Vl. S. Dotsenko, Sov. Phys. JETP Lett. 33, 37 (1981)

– **Do critical exponents change ?**

No, close to $T_{\rm Is}$ **Vl. S. Dotsenko, M. Picco & P. Pujol**, Nucl. Phys. 455, 701 (1995)

Random networks

Localization phenomena

Express the partition function as $Z \propto \mathrm{Tr} \prod_k \hat{T}_k$ a product of transfer matrices

All $\hat{T}_k$ are different since disorder-dependent, expressed in terms of $\hat{\sigma}_i^x$, $\hat{\sigma}_i^z$

Use Jordan-Wigner transformation to introduce fermions, then transform them to Dirac fermions (by doubling the model)

Network tight-binding Hamiltonian for free fermions with random hopping

paramagnet $\equiv$ insulator

Localization problem

ferromagnet $\equiv$ quantum Hall conductor

S. Cho and M. P. A. Fisher, PRB 55, 1025 (1997)

I. Gruzberg, N. Read, and A. Ludwig, PRB 63, 024404 (2001)

F. Merz and J. T. Chalker, PRB 65, 054425 (2002)

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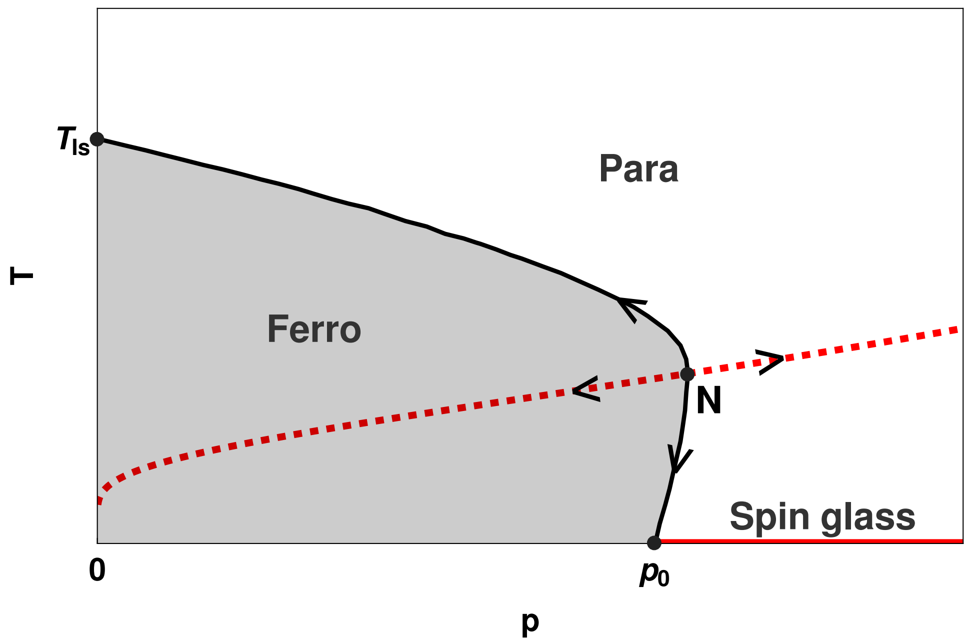
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram ($J = 1$)

Second order phase transition between FM & PM phases

$(T_{\rm Is} = 2.27,\ p = 0)$ **L. Onsager**, Phys. Rev. 65, 117 (1944)



Ferromagentic phase

Paramagnetic phase

Spin-glass phase

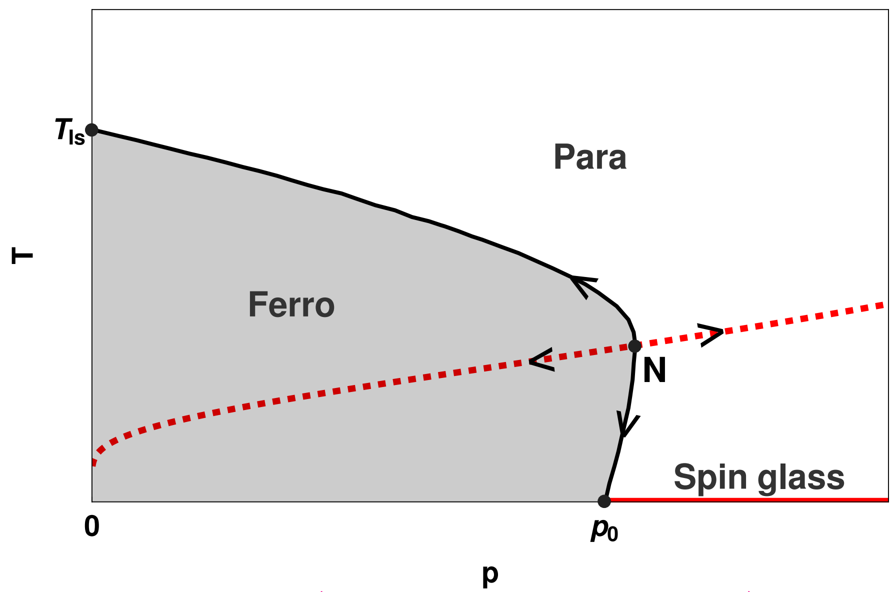
$(T_0 = 0,\ p_0 = 0.103)$

$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

Second order phase transition between FM & PM phases

$(T_{\rm Is} = 2.27,\, p = 0)$



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

enhanced symmetry properties

$(T_0 = 0,\, p_0 = 0.103)$ $(T_N = 0.95,\, p_N = 0.109)$

*__H. Nishimori__, Prog. Theor. Phys. 66, 1169 (1981)

The Nishimori line

Special features

Local gauge invariance : simultaneous spin and couplings transformation which leave the functional form of $\mathcal{H}$ invariant but change $P(J_{ij})$

On the **Nishimori line** $e^{-2\beta J} = \frac{p}{1-p}$: exact expression for $[\langle \mathcal{H} \rangle](p)$, etc.

The Nishimori line meets the FM-PM transition line at a **tri-critical point** (p_N, T_N)

Phase transition in the Kitaev's **quantum toric code**

A. Yu. Kitaev, Russian Math. Surveys 52, 1191 (1997)

Below p_N encoded information can be protected arbitrarily well

Above p_N it cannot

p is the qu-bit (independent) error probability, in the limit of a large code block

E. Dennis, A. Kitaev, A. Landahl & J. Preskill, J. Math. Phys. 43, 4452 (2002)

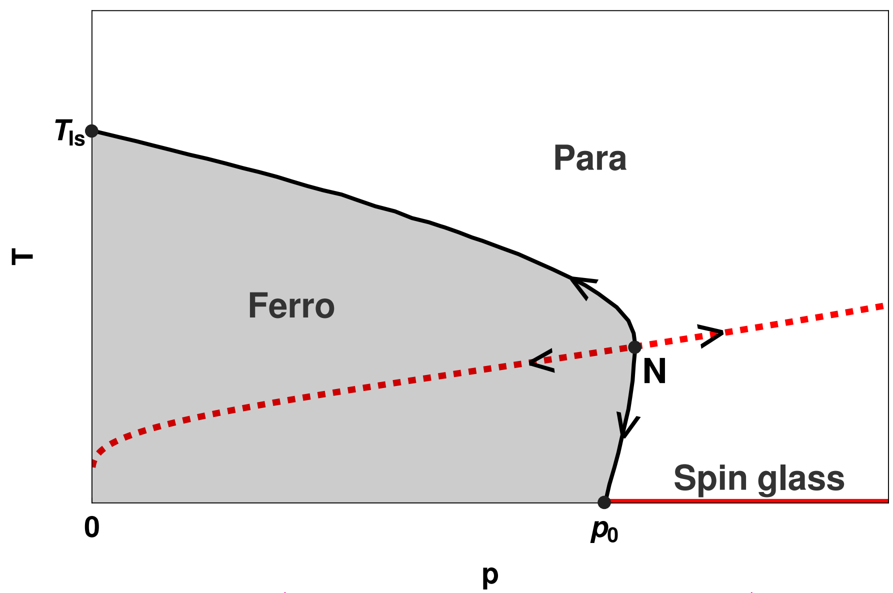
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

$T_{\rm Is} \geq T > T_N$ Disorder is marginally relevant $\Rightarrow$ $(T_{\rm Is}, p=0)$ **PM-FM Ising criticality**

Vik. Dotsenko and Vl. Dotsenko, Adv. Phys. 32, 129 (1983)

M. Picco, A. Honecker, and P. Pujol, J. Stat. Mech. P09006 (2006)



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

$(T_0 = 0,\ p_0 = 0.103)$ $(T_N = 0.95,\ p_N = 0.109)$

$0 \leq T < T_N$ Strong disorder $\Rightarrow$ $(T_0 = 0, p_0)$ **criticality**

F. Parisen Toldin, A Pelisetto, and E. Vicari, J Stat Phys 135, 1039 (2009)

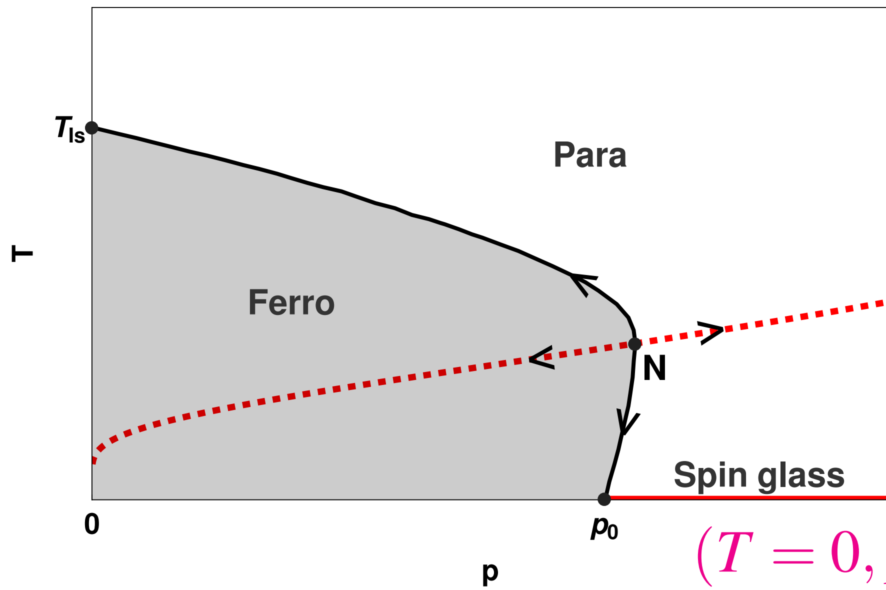
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

$T_{\rm Is} \geq T > T_N$ Disorder is marginally relevant $\Rightarrow$ $(T_{\rm Is}, p = 0)$ **PM-FM Ising criticality**

A. B. Harris, J. Phys. C : Sol. St. Phys. 7, 1671 (1974)

M. Picco, A. Honecker, and P. Pujol, J. Stat. Mech. P09006 (2006)



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

$(T = 0, p_0 < p < 1 - p_0)$ **spin-glass**

T. Jörg, J. Lukic, E. Marinari, and O. C. Martin, Phys. Rev. Lett. 96, 237205 (2006)

F. Parisen Toldin, A Pelisetto, and E. Vicari, Phys. Rev. E 82, 021106 (2010)

Critical points

Exponents & equilibrium universality classes

p_c	T_c	ν	η	κ^*	
0	$T_{\rm Is} = 2.29$	1	0.25	3	FM-PM Ising[1]
$p_N = 0.109$	$T_N = 0.95$	$4/1.5$	0.18	2.22	Bi-critical[2]
$p_0 = 0.103$	$T_0 = 0$	1.5	0.128	1.93	FM-SG[3]
$p_0 < p < 1 - p_0$	$T_{\rm SG} = 0$	∞	0.14	2.1	SG-PM[4]

[1] **L. Onsager**, Phys. Rev. 65, 117 (1944)

[*] **O. Schramm**, Isr. J. Math. 118, 221 (2000) **J. Cardy**, Ann. Phys. 318, 81 (2005)

[2] **W. L. Mc Millan**, PRB 29, 4026 (1984) **M. Hasenbusch** *et al.*, PRE 77, 051115 (2008)

[3] **F. Parisen Toldin, A. Pelissetto, and E. Vicari**, J. Stat. Phys 135, 1039 (2009)

[4] **H. Katzgraber, L. W. Lee, and I. A. Campbell**, PRB 75, 014412 (2007)

J. Poulter and J. A. Blackman, Phys. Rev. B 72, 104422 (2005).

$2 - 1/\nu = (6-\kappa)/\kappa$ works at $T > T_N$ and also on the SG if $\kappa = 2$

Critical dynamics of the $\pm J$ $2d$ Ising Model?

What is the Conformal Field Theory for the N point (what is κ)?

What happens in $3d$?

(more interesting from the quantum codes viewpoint)

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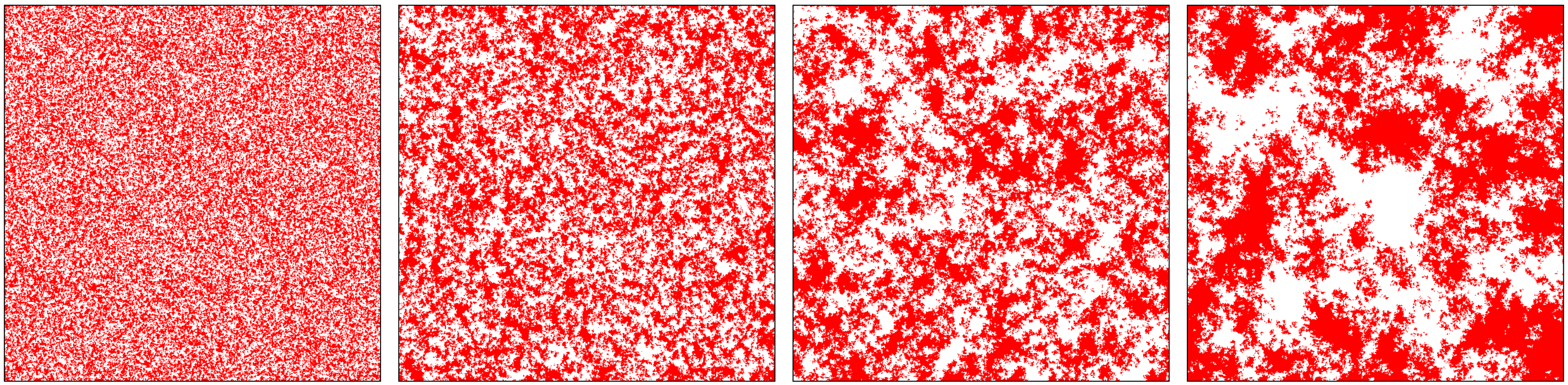
$2d$ FM Ising Model

$p = 0$ critical dynamics

Instantaneous quench to the Ising FM-PM critical point from $T_i \to \infty$



Progressive growth of critical structures

Typical length scale of critical patches growing algebraically

$$\xi(t) \sim t^{1/z_c}$$

Similar phenomenology expected on the full critical FM-PM line

How to measure z_c ?

Space-time correlations

of simultaneous fluctuations

$$C(r,t) = [\langle s_i(t) s_j(t) \rangle] - [\langle s_i(t) \rangle][\langle s_j(t) \rangle] \quad \text{for} \quad \vec{r}_i - \vec{r}_j = r$$

Scaling for the infinite size $L \to \infty$ system

$$C(r,t) = r^{-\eta} f\left(\frac{r}{\xi(t)}\right)$$

Effective dynamic exponent tends to Dynamic critical exponent

$$\frac{1}{z_{\text{eff}}(t)} = \frac{d \ln \xi(t)}{d \ln t} \quad \Rightarrow \quad z_c = \lim_{t \to \infty} z_{\text{eff}}(t)$$

$z_c = 2.17$ at the $p = 0$ FM $2d$ case

from Monte Carlo numerical simulations, but also RG, high temperature series expansions, damage spreading, etc

Short-time dynamics

at a critical point

$$m_2(t) = \left[\left\langle \left(\frac{1}{N} \sum_{i=1}^{N} s_i(t) \right)^2 \right\rangle \right]$$

for $R_{\min} \ll \xi(t) \ll \xi_{eq}, L$

Increase right after the quench from $T_i \to \infty$ with (similar to *initial slip* exponent)

$$m_2(t) \sim t^{\zeta} \quad \text{with} \quad \zeta = \frac{1}{z_c} \left(d - \frac{2\beta}{\nu} \right)$$

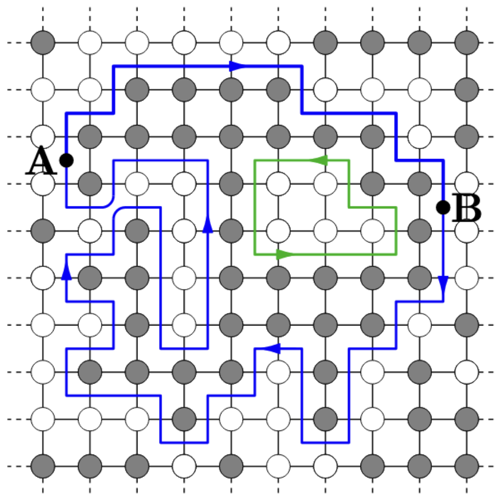
H. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B Cond. Matt. 73, 539 (1989)

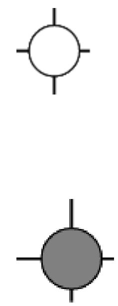
E. V. Albano *et al.*, Rep. Prog. Phys. 74, 026501 (2011)

Winding angle

Definition - critical curves

$$[\langle \theta^2(r,t) \rangle]$$





In equilibrium at a critical point $$[\langle \theta^2(r) \rangle] = c + \frac{4\kappa}{8+\kappa} \ln\left(\frac{r}{a}\right)$$

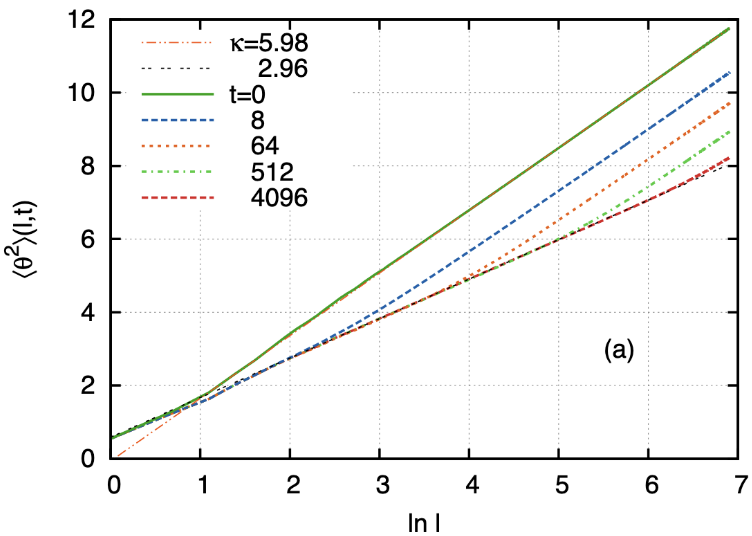
$d_f = 1 + \kappa/8$ $\kappa = 3$ Critical Ising $\kappa = 6$ Critical percolation

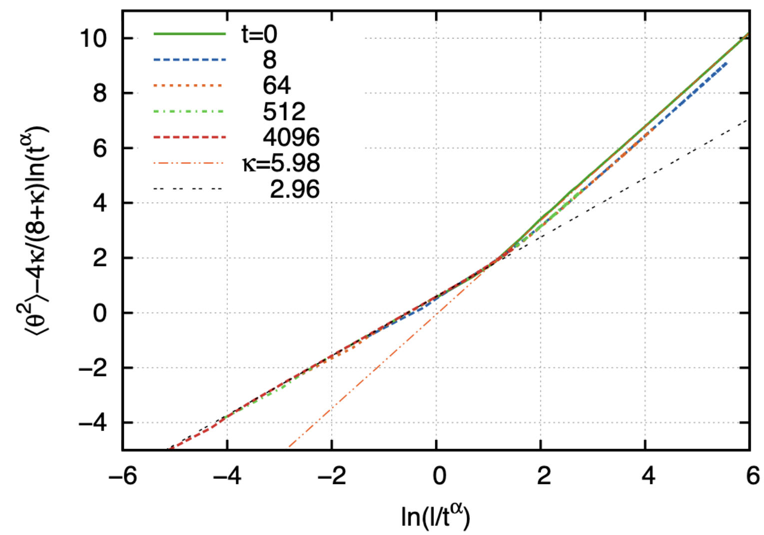
Out of equilibrium $[\langle \theta^2(r,t) \rangle] \sim \dfrac{4\kappa}{8+\kappa} \ln\left(\dfrac{r}{\xi^{d_f}(t)}\right)$

Blanchard, LFC, Picco & Tartaglia, 2012-2018

Winding angle

$2d$ FM Ising Model quenched from $T_i \to \infty$ to T_c





Out of equilibrium $[\langle \theta^2(r,t)\rangle] \sim \dfrac{4\kappa}{8+\kappa} \ln\left(\dfrac{r}{\xi^{d_f}(t)}\right)$ $\qquad \alpha = d_f/z_c$

$\kappa = 6$ & $d_f = 7/4$ Critical percolation $r > \xi(t)$ & $t > t_p$

$\kappa = 3$ & $d_f = 11/8$ Critical Ising $r < \xi(t)$

Blanchard, LFC & Picco, J. Stat. Mech. P05026 (2012)

$\pm J$ $2d$ Ising Model

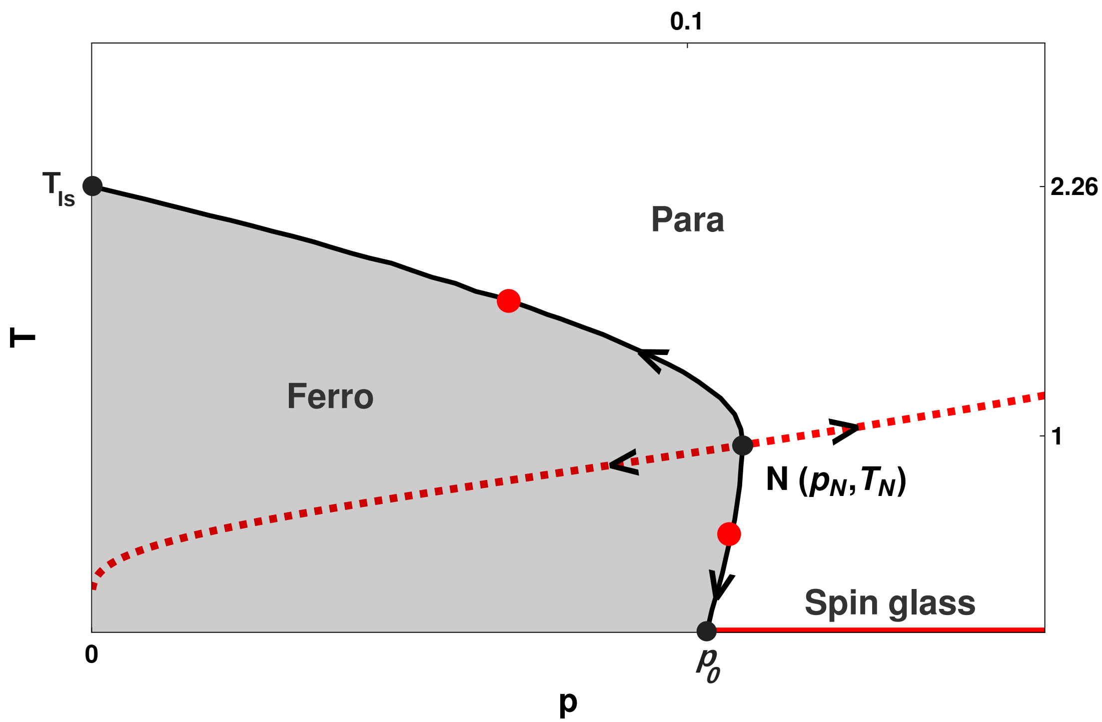
Use these tools to characterize the critical dynamics

$\pm J$ $2d$ Ising Model

Simulation parameters

Second order phase transition between FM & PM phases

$(T_{\mathrm{Is}} = 2.27,\, p = 0)$



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

enhanced symmetry properties

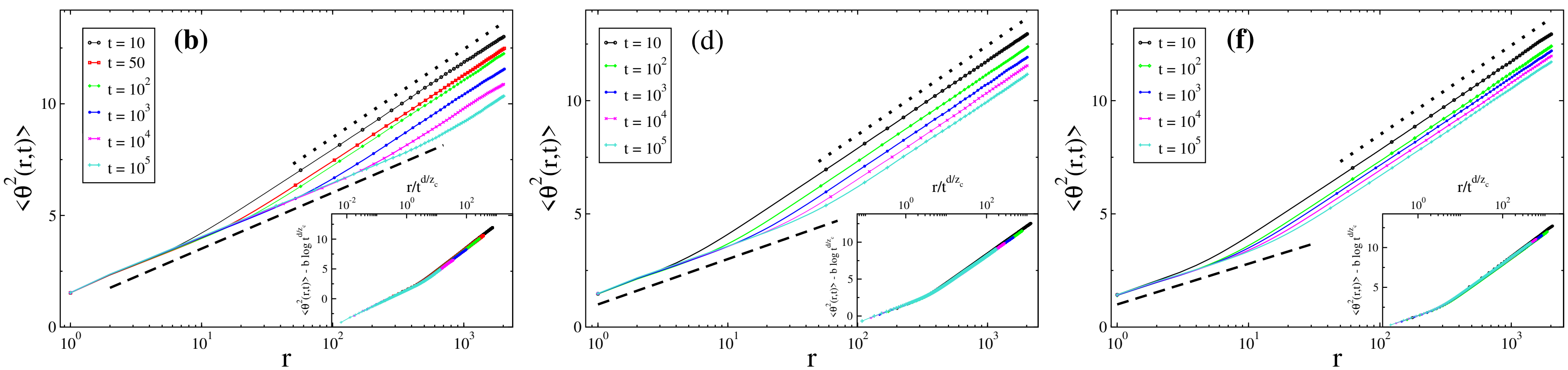
$(T_0 = 0,\, p_0 = 0.103)$ $(T_N = 0.95,\, p_N = 0.109)$

Results

Quenches from $T_i \to \infty$ to T

$\forall T \qquad r > \xi(t) \qquad \kappa = 6$ **Critical percolation**

. . . . lines



$T = T_{\rm Is} \qquad T = T_N \qquad T < T_N$

$r < \xi(t)$ — — — — — — lines

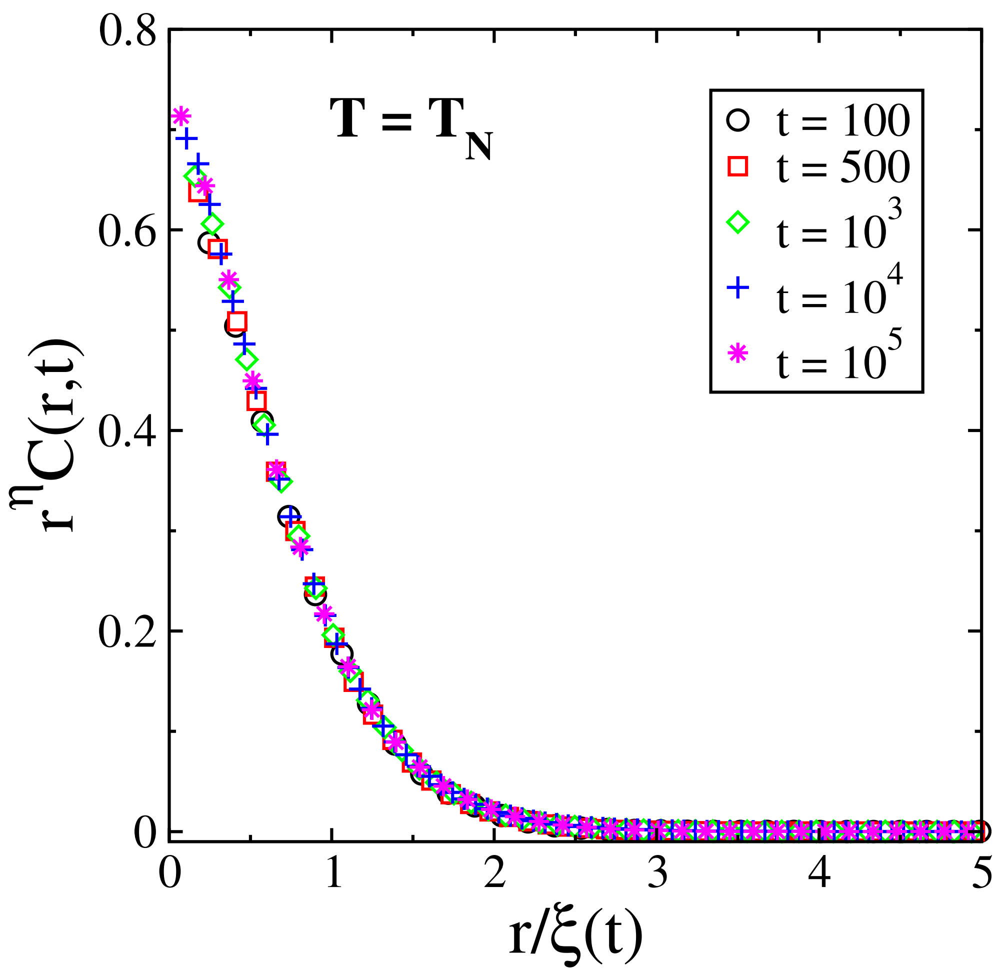
$\kappa = 3 \qquad \kappa = 2.2 \qquad \kappa = 1.93$

Critical Ising **? ? ?**

Results

Dynamic scaling of the space time correlation $\xi(t)$

$T = T_N \qquad \eta = 0.18$



Results

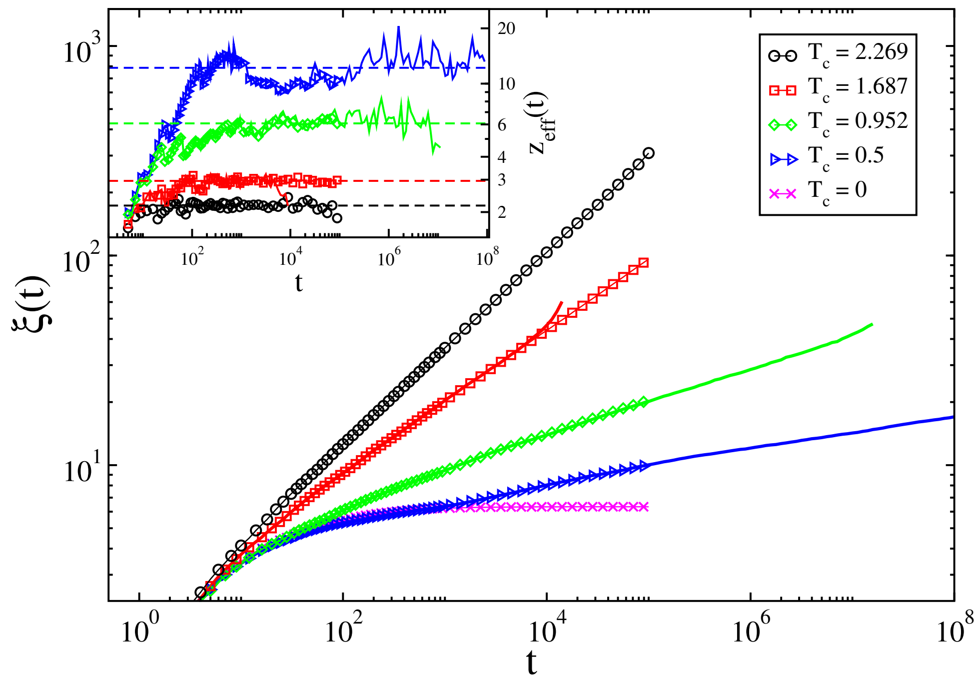
Pre-asymptotic dynamic critical exponent

$T < T_N$

$T = T_N$

$T > T_N$

$T = T_{\rm Is}$



Data-points

$L = 1024$

Solid lines

$L = 128$

No visible finite size effects

FM-PM Ising critical point $z_c \sim 2.17$ **OK**

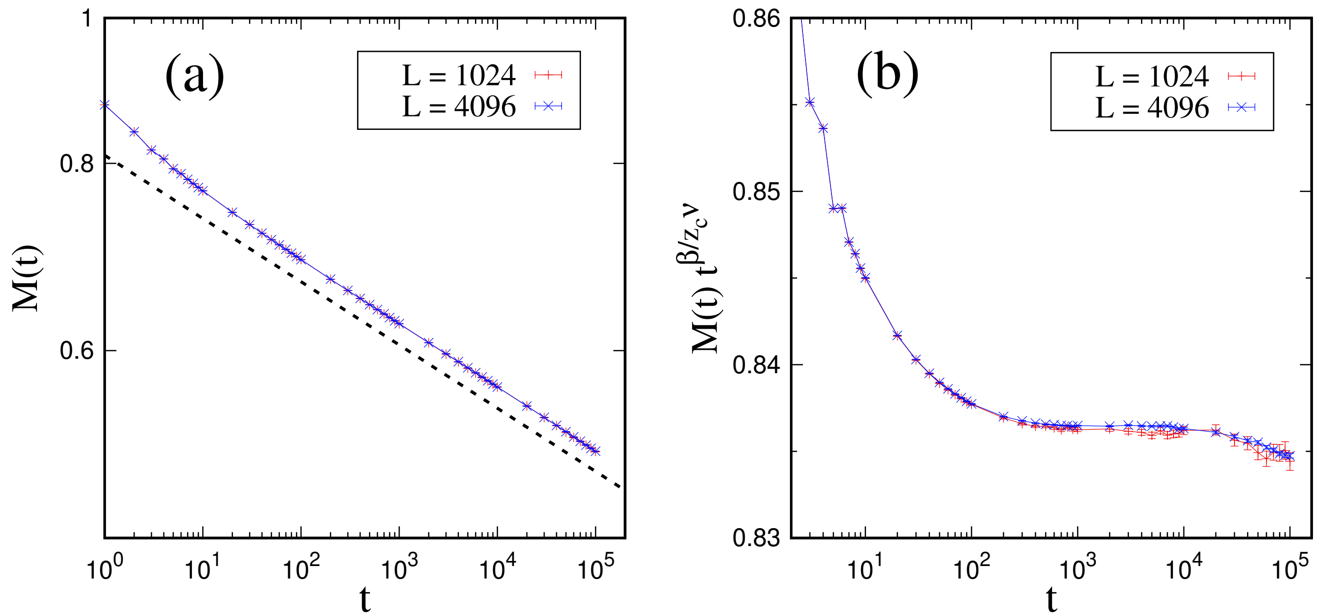
Then, **disorder dependent dynamic critical exponent ?**

Should not be...

Results

Decay from a magnetized initial condition $M(t) \sim t^{-\beta/(\nu z_c)}$

$T > T_N$



$\beta/\nu = 0.125$ the Ising critical value and $z_c = 2.96$ from the space-time correlation

Crossover at an L independent time $t_{\mathrm{cross}} \sim 10^4$ presumably fixed by the disorder strength p

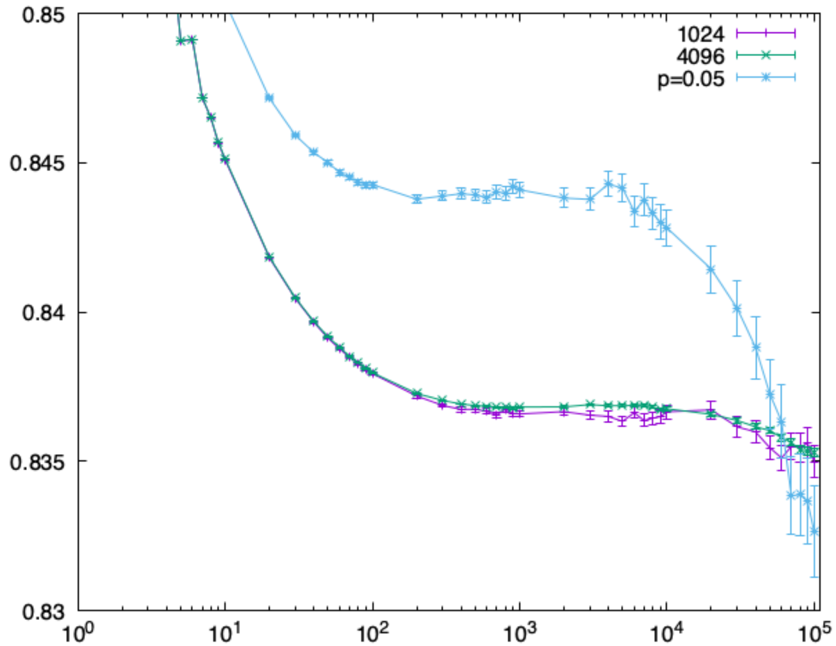
very weak drift $z_c \searrow$ after t_{cross}

It should converge to $z_c = 2.17$, the critical Ising value

Results

Decay from a magnetized initial condition $M(t) \sim t^{-\beta/(\nu z_c)}$

$T > T_N$



$z_c = 2.56$ for $p = 0.05$

$z_c = 2.96$ for $p = 0.07$

$\beta/\nu = 0.125$ the Ising critical value and z_c from the space-time correlation

$t_{\text{cross}} \sim 7 \times 10^3$ for $p = 0.05$ $<$ $t_{\text{cross}} \sim 2 \times 10^4$ for $p = 0.07$

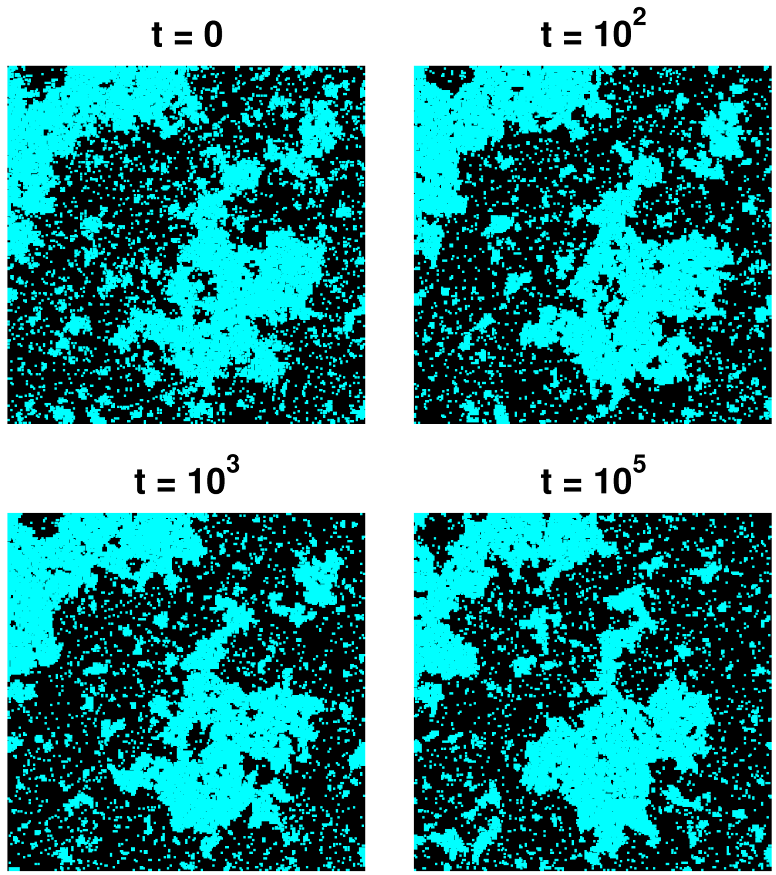
L independent t_{cross} (being checked)

drift $z_c \searrow$ after t_{cross}

For $p = 0.05$, z_c has already reached 2.2 at $t = 10^5$ (not far from $z_c = 2.17$)

Ultra slow dynamics at p_N, T_N

Quench from $T_i = T_{\mathrm{Is}}$ to T_N



A portion of the system

The overall structure changes very little over a long time span

Conclusions

Hard to get strong quantitative results

– $T_N < T \leq T_{\rm Is}$ static universality class of the Ising critical point

Most probably also the same dynamic universality class

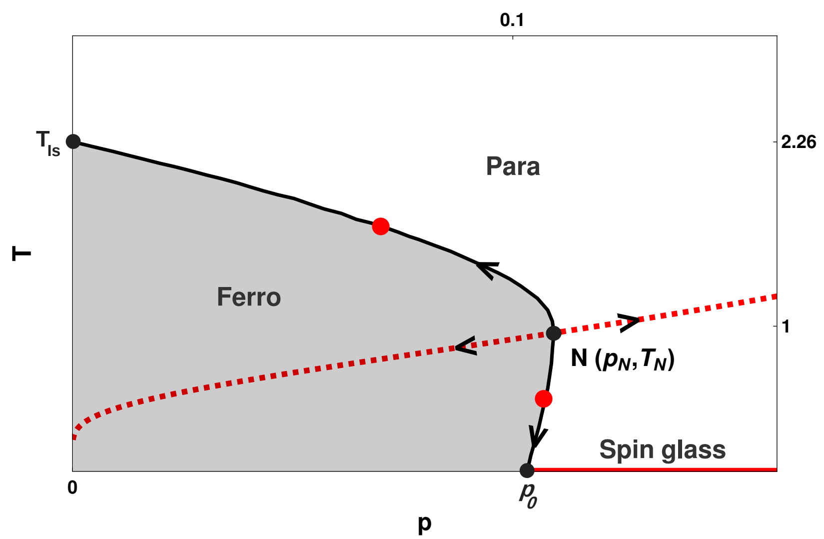
– $T = T_N$ new static & dynamic universality classes $\kappa \sim 2.2$

– $T_0 \leq T < T_N$ strong disorder static universality class, but κ?

The low T dynamics is way

too slow to conclude

$3d$ case next



Appendices

Details

Codes

Definition

During the transmission of information, errors may occur

The aim is to minimize their number/strength

Idea, code the message and uncode it at the end

Quantum Toric Codes

Definition

A **qu-bit** is a two-state quantum variable, $|\psi_i\rangle = a_i|\uparrow\rangle + b_i|\downarrow\rangle$

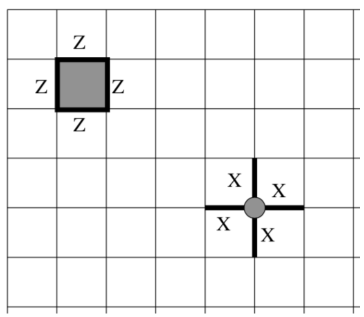
Flip errors, $\hat{\sigma}_x|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$ & phase errors, $\hat{\sigma}_z|\uparrow\rangle = \pm|\uparrow\rangle$ occur independently with probability p

Place **qu-bits** on the links of a square lattice defined on a $2d$ surface with non trivial topology, e.g. a torus $\prod_i \otimes |\psi_i\rangle$

Check local operators : plaquette or link operators, tensor product of four Pauli operators acting on the four qu-bits on the links times identities on all other links

Check operators commute

Measurements of check operators yield $+1$ no error, or -1 error.



Quantum Toric Codes

Definition

Stabilizer group G a set of n check operators which applied to a basis state of the quantum error correction code have eigenvalue one, $P_k|\psi_j\rangle = |\psi_j\rangle$ for any kth element in the group and any jth element of the basis. Abelian group

Particular case : product of $\hat{\sigma}_x$ or product of $\hat{\sigma}_z$ operators.

$\prod$ of neighbouring plaquette operators : loop on the lattice.

$\prod$ of neighbouring vertex operators : loop on the dual lattice.

Error operators $E|\psi\rangle = |\psi'\rangle$

String of flip errors on the lattice : vertex operators on the ends yield -1

Correction operators E' such that $E'E \in G$

Another string with the same end points so as to close the loop

Error correction

Optimal toric code decoder threshold

Call p the (independent) probability of a qu-bit error

What is the maximal p such that code can be corrected ?

Probability of a string E' on the lattice that corrects another string of errors E

$$P(E') = (1-p)^N \prod_k \left(\frac{p}{1-p}\right)^{n'^E_k} = e^{\beta \sum_{\langle ij \rangle} J_{ij} s_i s_j}$$

$J_{ij} = \pm J$ with probability $1-p, p$ and $p/(1-p) \equiv e^{-2\beta J}$ (Nishimori)

Have to study the sum over all paths E'

$$Z = \sum_{E'/EE' \in G} e^{\beta \sum_{\langle ij \rangle} J_{ij} s_i s_j}$$

Mapping to the classical $\pm J$ $2d$ Ising model on the Nishimori line

p_N is the optimal decoding threshold

Local Gauge invariance

Ising disordered spin models

Transform the Ising spins $s_i = \pm 1$ into new Ising spins $\sigma_i = \eta_i s_i = \pm 1$

Transform the couplings $J_{ij} = \pm J$ into new ones $\bar{J}_{ij} = \eta_i \eta_j J_{ij} = \pm J$

with $\eta_i = \pm 1$ so that $\eta_i^2 = 1$ for all i

The Hamiltonian of the system remains unchanged

$$\mathcal{H}_{\bar{J}_{ij}}[\{\sigma_i\}] = -\sum_{\langle ij \rangle} \bar{J}_{ij} \sigma_i \sigma_j = -\sum_{\langle ij \rangle} J_{ij} s_i s_j = \mathcal{H}_{J_{ij}}[\{s_i\}]$$

but the distribution of couplings may change depending on the η_is

$$P(J_{ij}) \mapsto \overline{P}(\bar{J}_{ij})$$

Valid $\forall$ Ising models with two-body couplings on any lattice/graph

The Nishimori line

Special features

The bimodal distribution of couplings can be rewritten as

$$P(J_{ij}) = (1-p)\,\delta_{J_{ij},J} + p\,\delta_{J_{ij},-J} = \frac{e^{K_p J_{ij}/J}}{2\cosh K_p}$$

with $e^{2K_p} \equiv \dfrac{1-p}{p}$

It transforms according to $$P(J_{ij}) \mapsto \overline{P}(\overline{J}_{ij}) = \eta_i \eta_j \, \frac{e^{K_p \overline{J}_{ij} \eta_i \eta_j / J}}{2\cosh K_p}$$

The **Nishimori line** is defined by $\beta J = K_p = \frac{1}{2} \ln\left(\dfrac{1-p}{p}\right)$

with the limits $p = 0, T = 0$ and $p = 1/2, T \to \infty$

Several exact results can be derived on the Nishimori line

(p_N, T_N) is a **multi-critical point**, different from critical percolation