# Out of Equilibrium dynamics with a GGE description

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**Jérôme Dubail editor**

# Plan

— Introduction: equilibrium, non-equilibrium & the goal

— Hamiltonian dynamics of a (simple) integrable model in the thermodynamic limit and the GGE measure

— Stochastic evolution towards the GGE measure

— Summary

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# Equilibrium Statistical Physics

**Advantage**

**No need to solve the Newton dynamics!**

Under the *ergodic hypothesis*, after some *equilibration time* $t_{\rm eq}$, a *macroscopic observable* $A$ of a *macroscopic system* can be, on average, obtained with a *static calculation*, as an average over all configurations in phase space weighted with a probability distribution function $\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$

$$\langle A \rangle = \int \prod_i d\boldsymbol{p}_i d\boldsymbol{x}_i \; \rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \; A(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$$

$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \; A(\{\boldsymbol{p}_i(t'), \boldsymbol{x}_i(t')\})$

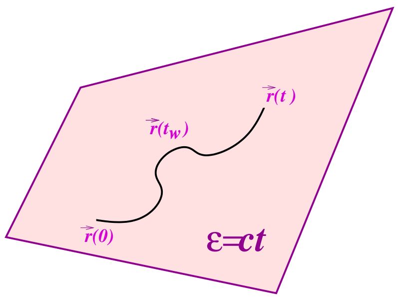
the *time average* typically measured experimentally

**Ergodicity**

**Boltzmann, late XIX**

# Equilibrium Statistical Physics

### Recipes for $\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$ according to circumstances




Isolated system

$$\mathcal{E} = \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct$$

## Microcanonical ensemble

$$\boxed{\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto \delta(\mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) - \mathcal{E})}$$

Flat probability density

$$S_{\mathcal{E}} = k_B \ln g(\mathcal{E}) \qquad \beta \equiv \frac{1}{k_B T} = \left.\frac{\partial S_{\mathcal{E}}}{\partial \mathcal{E}}\right|_{\mathcal{E}}$$

Entropy

Temperature

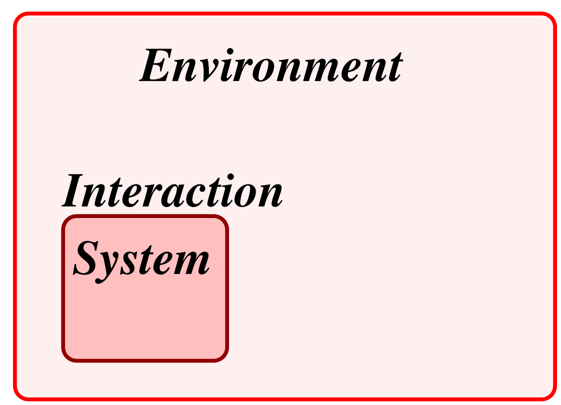
---

$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range interact.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env} \qquad \beta = \frac{\partial S_{\mathcal{E}_{env}}}{\partial \mathcal{E}_{env}}$$

$$\boxed{\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto e^{-\beta \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}}$$



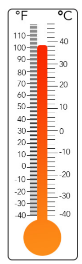

## Canonical ensemble

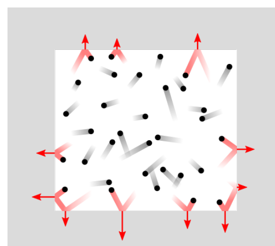
# Equilibrium Statistical Physics

## Accomplishments

- Microscopic definition & derivation of thermodynamic concepts

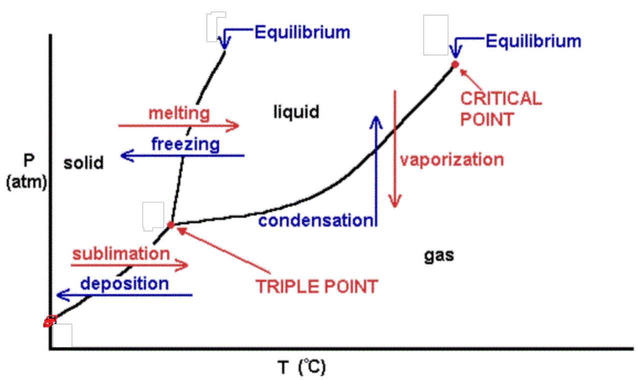
  (temperature, pressure, *etc.*) and laws (equations of state, *etc.*)





$$PV = nRT$$

- Theoretical understanding of collective effects $\Rightarrow$ phase diagrams




Phase transitions : sharp changes in the macroscopic behavior when an external (*e.g.* the temperature of the environment) or an internal (*e.g.* the interaction potential) parameter is changed

- Calculations can be difficult but the theoretical frame is set beyond doubt

# Out of equilibrium

## Three possible reasons

- The equilibration time goes beyond the experimentally accessible times in **macroscopic systems** in which $t_{\rm eq}$ grows with the system size,

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N) \gg t}$$

*e.g.*, **critical slowing down, coarsening, glassy physics**

---

- **Driven systems** Energy injection

$$\boxed{\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x})}$$

*e.g.*, **active matter**

---

- **Integrability**

$$\boxed{I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N}$$

Too many constants of motion inhibit equilibration to the Gibbs ensembles.

*e.g.*, **$1d$ bosonic gases**

---

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*e.g.*, **diffusion, critical slowing down, coarsening, glassy physics**

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- **Driven systems** Energy injection

$$\boxed{\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x})} \qquad \boxed{\Gamma_1 \neq \Gamma_2}$$

*e.g.*, **active matter**

---

- **Integrability** $\boxed{I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N}$

Too many constants of motion inhibit equilibration to the Gibbs ensembles

*e.g.*, **$1d$ bosonic gases**

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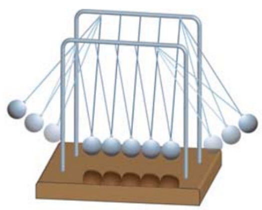
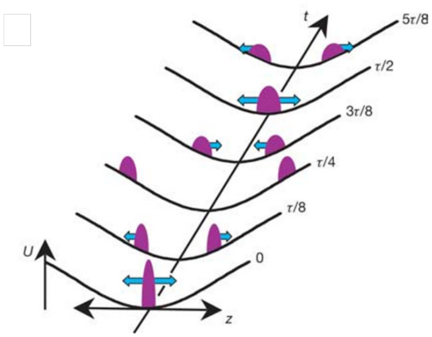
# Motivation

## Isolated quantum systems: experiments and theory $\sim$ 15y ago






A quantum Newton's cradle

**experiment**

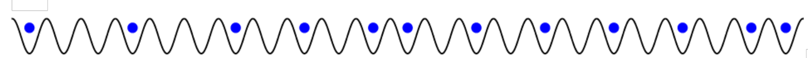
cold atoms in isolation

**Kinoshita, Wenger & Weiss 06**

(Conformal) field **theory** methods for quantum quenches

**Calabrese & Cardy 06**

**Numerical** study of lattice hard core bosons



**Rigol, Dunjko, Yurovsky & Olshanii 07**

Mostly 1d systems

**And many others**

# Classical quenches

## Definition

- Take a **classical mechanical system**

- Initialize it in configurations drawn from, *e.g.*, a standard equilibrium Gibbs-Boltzmann distribution

$$\boxed{\frac{1}{\mathcal{Z}(\beta_0)} \, e^{-\beta_0 \mathcal{H}_0(\{\boldsymbol{p}_i(0), \, \boldsymbol{x}_i(0)\})}}$$

Other choices are possible but this is a natural one. Corresponds to having equilibrated the system $\mathcal{H}_0$ with a thermal bath at inverse temperature $\beta_0$

- Evolve it in isolation with Newton dynamics with another Hamiltonian $\mathcal{H}$

Corresponds to switching off the coupling to the thermal bath at the initial time $t = 0$ and changing the Hamiltonian

# Classical quenches

## Questions

Does an $i = 1, \ldots, N \to \infty$ system reach a steady state ?

Is it a micro-canonical measure ?

Do some local observables behave as canonical ones, described by a thermal equilibrium probability $e^{-\beta\mathcal{H}}$ with some inverse temperature $\beta$ ? Old ergodicity issue

And for integrable models with as many integrals of motion as degrees of freedom ?

Generalisations of the measures ? New kind of ergodicity

# Steady state

## of a macroscopic system in isolation

- In a **non-integrable** system there are a few constants of motion

$$I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) \qquad\qquad \mu = 1,\dots,n$$

and the microcanonical measure is

$$\boxed{\rho \propto \prod_{\mu=1}^{n} \delta(I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) - \mathcal{I}_\mu)}$$

with $I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\})$ fixed by the initial conditions to $\mathcal{I}_\mu$ for all $\mu = 1,\dots,n$

typically, just the **energy**, and the usual **microcanonical** measure

---

- In an **integrable** system, there are $\mu = 1,\dots,N = \#$ d.o.f. constants $I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\})$

fixed by the initial conditions and

$$\boxed{\rho \propto \prod_{\mu=1}^{N} \delta(I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) - \mathcal{I}_\mu)}$$

**Yuzbashyan 18**

# The canonical version

## Open system

One hardly works with a microcanonical measure

Equivalence of ensembles $\Rightarrow$ canonical one,* but not obvious for integrable cases

**Are local observables characterised by a canonical measure ?**

**Is it a Generalized Gibbs Ensemble :**

$$\rho_{\rm GGE} \propto e^{-\beta \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}$$

**with $\beta\gamma_\mu$ fixed requiring $I_\mu(0^+) = I_\mu(t) = \langle I_\mu \rangle_{\rm GGE}$ ?**

* the proof of the equivalence cannot be simply applied in integrable cases

# Statistical Physics

## An ergodic hypothesis

**No need to solve the Newton dynamics!**

Under a *new ergodic hypothesis*, after some *characteristic time* $t_{\rm GGE}$, a *local observable* $A$ of a *macroscopic system* can be, on average, obtained with a *static calculation*, as an average over all configurations in phase space weighted with $\rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$

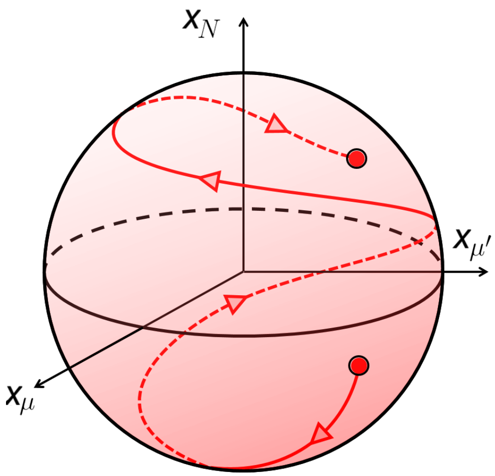
$$\langle A \rangle = \int \prod_i d\boldsymbol{p}_i d\boldsymbol{x}_i \; \rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \; A(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$$

$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \dfrac{1}{\tau} \displaystyle\int_{t_{\rm GGE}}^{t_{\rm GGE}+\tau} dt' \; A(\{\boldsymbol{p}_i(t'), \boldsymbol{x}_i(t')\})$

the *time average* typically measured experimentally

# Goal I: realize the GGE

**with a simple model**




The GGE measure

$$\rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto e^{-\beta \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}$$
$$= e^{-\beta \mathcal{H}_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}$$

$$\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$$

with the $\{\beta\gamma_\mu\}$ fixed by

the initial conditions and the quench

# Goal II: sample $\rho_{\rm GGE}$

## Stochastic dynamics: e.g., Langevin, Monte Carlo

The measure to sample is

$$\boxed{\rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto e^{-\beta \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i,\, \boldsymbol{x}_i\})} = e^{-\beta \mathcal{H}_{\rm GGE}(\{\boldsymbol{p}_i,\, \boldsymbol{x}_i\})}}$$

Assuming one knows the parameters $\{\beta\gamma_\mu\}$,

couple the system $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i,\; \boldsymbol{x}_i\})$ to an equilibrium bath at inverse temperature $\beta = {}^1\!/_{k_B T}$ and a Lagrange multiplier to stay on the sphere

Use, e.g., a white bath and Langevin stochastic dynamics

$$\boxed{\dot{x}_\mu = \frac{\delta \mathcal{H}^z_{\rm GGE}}{\delta p_\mu} \qquad\qquad \dot{p}_\mu(t) + \eta \dot{x}_\mu(t) = -\frac{\delta \mathcal{H}^z_{\rm GGE}}{\delta x_\mu(t)} + \xi_\mu(t)}$$

with $\langle \xi_\mu(t) \rangle = 0$ and $\langle \xi_\mu(t) \xi_\nu(t') \rangle = 2\eta k_B T \delta_{\mu\nu} \delta(t-t')$

2019-22 ⟷ 2025

| **Newton dynamics**<br>after a quench of $\mathcal{H}$,<br>the integrable model | “Ergodicity” & the **GGE ensemble**<br>$P_{\rm GGE} \propto e^{-\beta \mathcal{H}_{\rm GGE}}$<br>also the GB measure of $\mathcal{H}_{\rm GGE}$ | **Langevin relaxation**<br>at $\beta^{-1}$ of $\mathcal{H}_{\rm GGE}$<br>$\leftarrow$ approaches the GB measure |
|---|---|---|
| $\mathcal{H}$ | $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$ | $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$ |
| $I_\nu(t=0^+)$ | $\langle I_\nu \rangle_{\rm GGE}$ | $\lim_{t\to\infty} \langle I_\nu(t) \rangle_{\xi, i.c.}$ |
| Given by initial conditions | $=$ average with $P_{\rm GGE}$ | $=$ asymptotic limit |
| and the quench | fixes $\{\beta\gamma_\mu\}$ | |
| | to represent the quench of | |
| | the isolated $\mathcal{H}$ | |
| $\overline{\langle p_\mu^2(t \gg t_0) \rangle_{i.c.}}$ | $\langle p_\mu^2 \rangle_{\rm GGE}$ | $\lim_{t\to\infty} \langle p_\mu^2(t) \rangle_{\xi, i.c.}$ |
| $\overline{\langle x_\mu^2(t \gg t_0) \rangle_{i.c.}}$ | $\langle x_\mu^2 \rangle_{\rm GGE}$ | $\lim_{t\to\infty} \langle x_\mu^2(t) \rangle_{\xi, i.c.}$ |

# Plan

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# Strategy

**Choice of model**

– Choose a simple classical *integrable interacting* model with

(not just harmonic oscillators)

an interesting *phase diagram* to atudy different *initial conditions*

and *quenches* across the *phase transition(s)*

– Solve the dynamics after the quenches

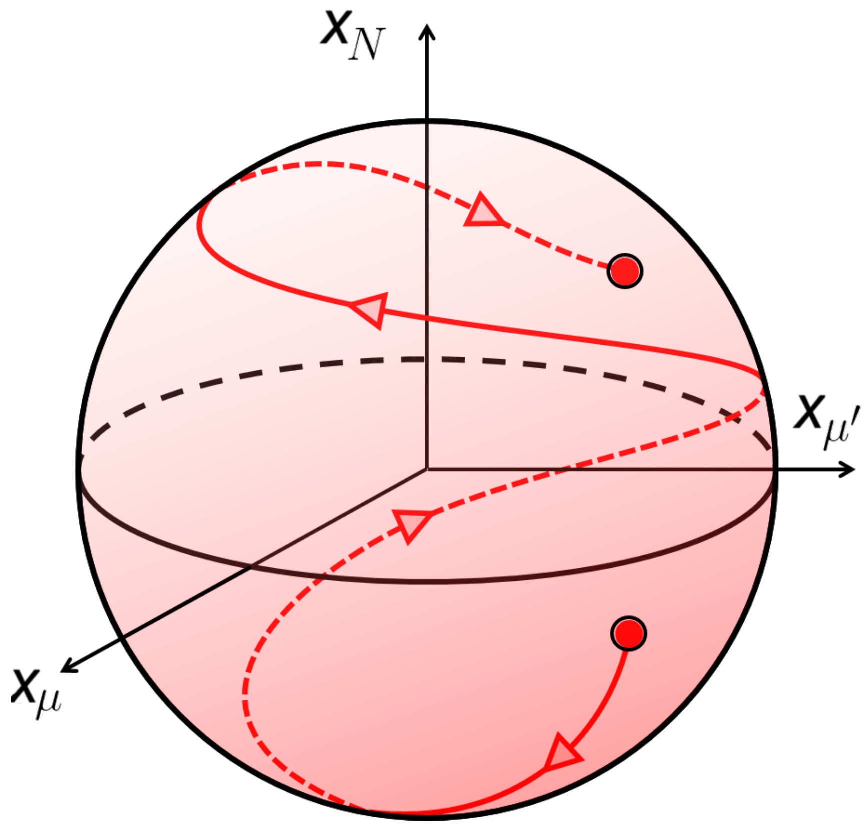
– Build a Generalised Gibbs Ensemble (GGE)

– Prove that the asymptotic limit of local observables is given by the GGE

**Not trivial ! First interacting problem with phase transitions**

# The model

## A particle moving on the sphere




Strict constraints

$$\phi : \sum_\mu x_\mu^2 - N = 0$$

$$\phi' : \sum_\mu x_\mu p_\mu = 0$$

$\mu = 1, \ldots, N$ label the coordinates

# The model

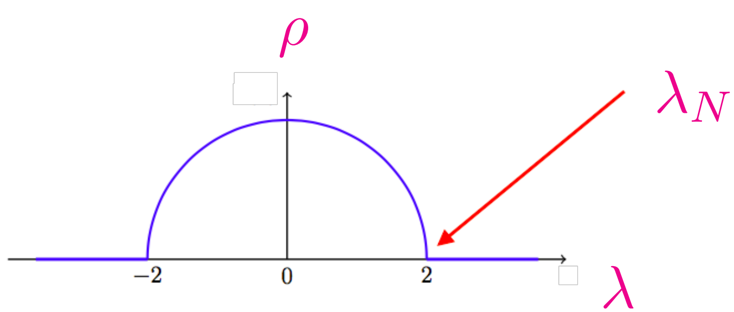
## The potential and kinetic energies

### Potential energy

$$V_J^{(z)}[\{x_\mu\}] = -\frac{1}{2}\sum_\mu \lambda_\mu \, x_\mu^2 + \frac{z(\boldsymbol{p},\ \boldsymbol{x})}{2}\left(\sum_\mu^N x_\mu^2 - N\right)$$

with $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ the eigenvalues of a matrix from the

GOE ensemble in the $N \to \infty$ limit : **Wigner's semi-circle** law*




and standard **kinetic energy** $K = \sum_\mu \frac{p_\mu^2}{2m}$ Newton dynamics

*handy choice, not essential. This model is also a **spin-glass** like problem $\Rightarrow$ Toolbox.

# Neumann’s model

**1859**

Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Als Fortsetzung des von

A. L. Crelle

gegründeten Journals

herausgegeben

unter Mitwirkung der Herren

Steiner, Schellbach, Kummer, Kronecker, Weierstrass

von

C. W. Borchardt.

Mit thätiger Beförderung hoher Königlich-Preufsischer Behörde

Sechs und funfzigster Band.

In vier Heften.

Berlin, 1859.

Druck und Verlag von Georg Reimer.

Newton dynamics on a sphere

under an anisotropic

harmonic potential

$$-\frac{1}{2}\sum_\mu \lambda_\mu x_\mu^2$$

with spring constants

$$\lambda_\mu \neq \lambda_\nu$$

De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur.

(Auctore *C. Neumann*, Hallae.)

§. 1.

*Problema proponitur.*

Sint puncti mobilis Coordinatae orthogonales $x, y, z$; sit

$$x^2+y^2+z^2 = 1$$

Journal of Pure & Applied Math.

Crelle Journal

# Neumann's model

**Integrable**

$N$ constants of motion in involution $\{I_\mu, I_\nu\} = 0$ fixed by the initial conditions

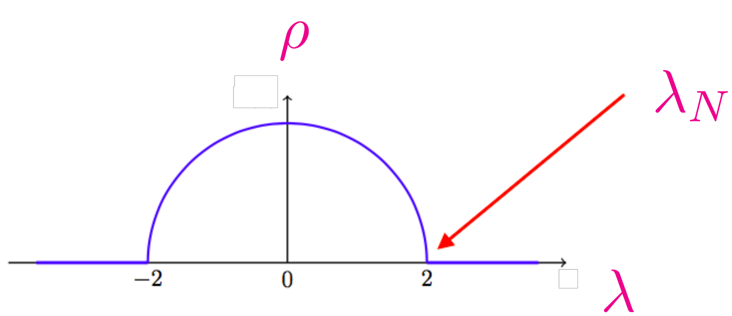
$$\boxed{I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}}$$

**K. Uhlenbeck 80s**

Studies by **Avan, Babelon and Talon 90s** and many others for $\boxed{\text{finite } N}$

$\boxed{\textbf{Thermodynamic } N \to \infty \textbf{ limit ?}}$

Draw the $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ from **Wigner's semi-circle** law




**Kosterlitz, Thouless & Jones 76**

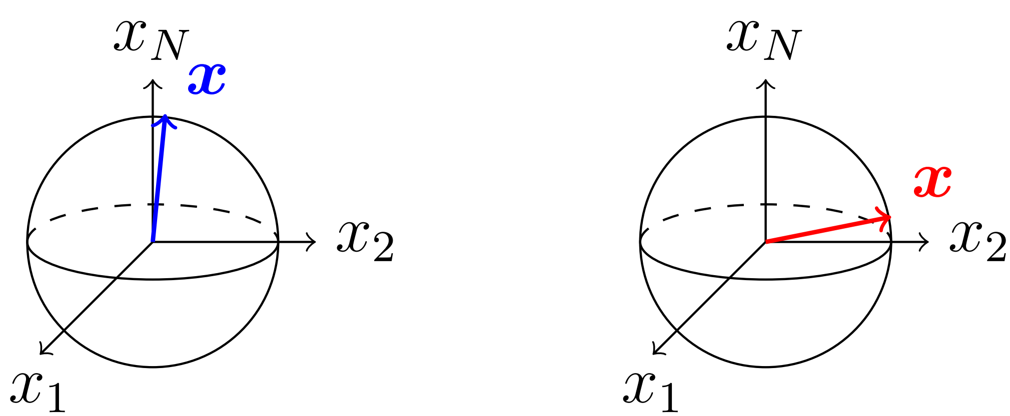
# Initial conditions

**Drawn from canonical equilibrium with $\lambda_\mu^{(0)}$ at $T_0$**

$\langle x_N^2 \rangle_{\rm eq} = qN$ $\qquad$ $\langle x_N^2 \rangle_{\rm eq} = \mathcal{O}(1)$

**condensed** $\qquad$ **extended** $\qquad$ $T_0/J_0$

$T_0/J_0 < 1$ $\qquad$ $T_0/J_0 > 1$




**Condensed** on $N$th direction, with largest $\lambda_\mu^{(0)}$ $\qquad$ **Extended**

Relation to BEC

**Kosterlitz, Thouless & Jones 76, Zannetti 15, Crisanti, Sarracino & Zannetti 19**

# Instantaneous quench

**Global rescaling of all spring constants**

At time $t = 0$

to keep some memory of the initial conditions

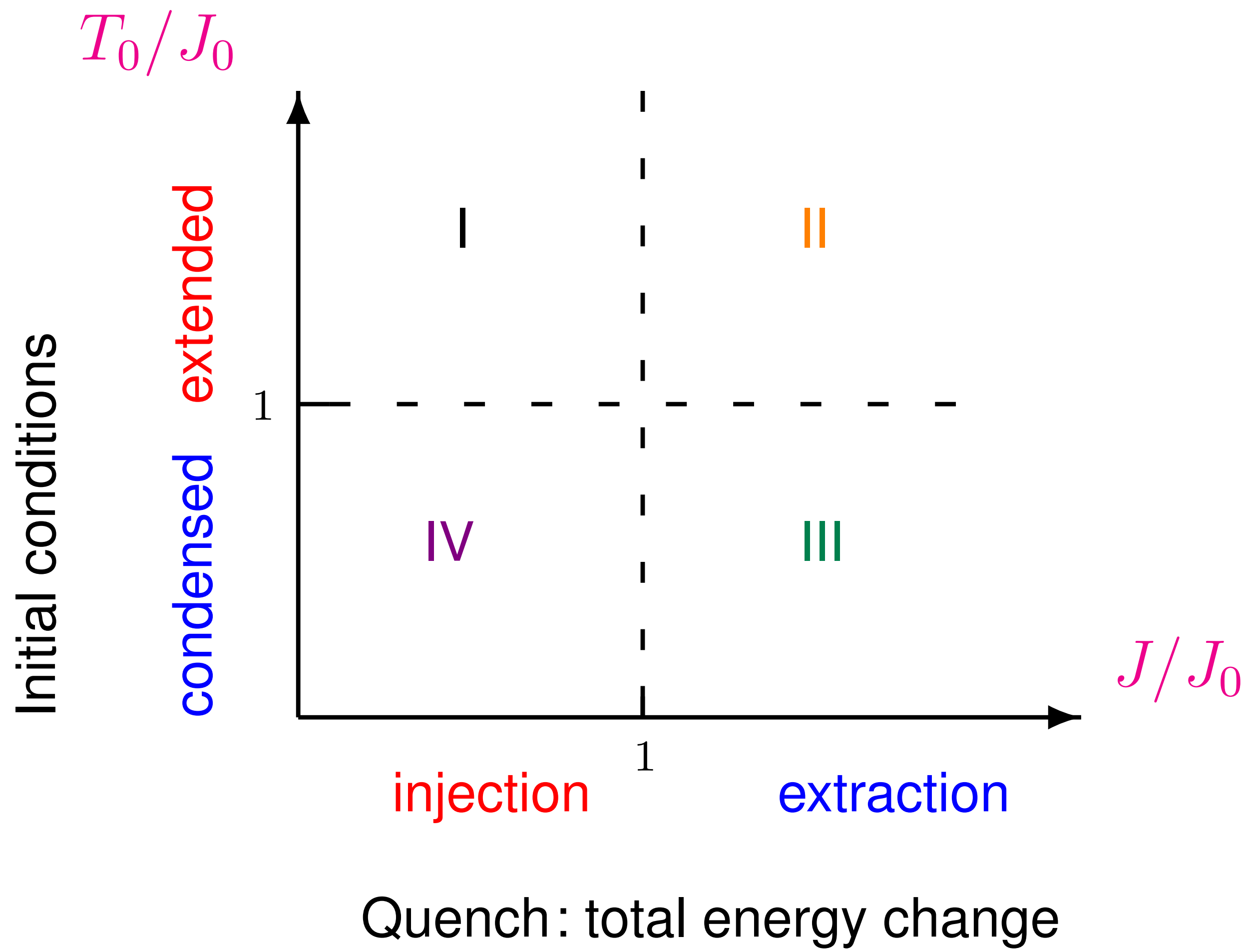
$$\boxed{\lambda_\mu^{(0)} \mapsto \lambda_\mu = \frac{J}{J_0}\lambda_\mu^{(0)}}$$

No change in configuration $\{x_\mu(0^-) = x_\mu(0^+),\ p_\mu(0^-) = p_\mu(0^+)\}$ but macroscopic energy change

$$\Delta E = \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{for} \quad \frac{J}{J_0} \begin{cases} < 1 & \textbf{Injection} \\ > 1 & \textbf{Extraction} \end{cases}$$

# Control parameters

**Total energy change & initial conditions**



# Classical quenches

## Strategy

Choose a sufficiently simple classical *integrable interacting* model

(not just harmonic oscillators)

with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

Solve the dynamics after a quench

Build a Generalised Gibbs Ensemble

Prove that the asymptotic limit of local observables is captured by the GGE

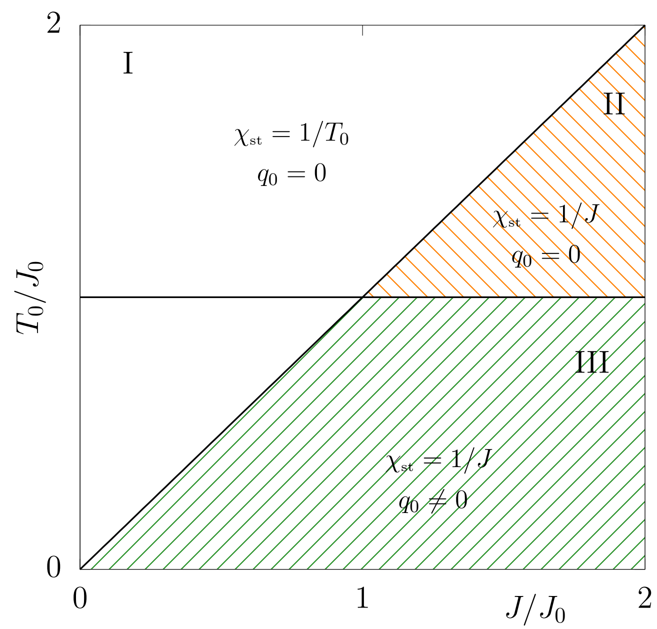
# The dynamic phase diagram

## Schwinger-Dyson eqs - DMFT - sphere dimension $N \to \infty$

$$\chi_{\rm st} = \lim_{t \gg t_0} \int_0^t dt' \, R(t,t')$$
$$z_f = \lim_{t \gg t_0} z(t)$$

extended

condensed




Initial conditions

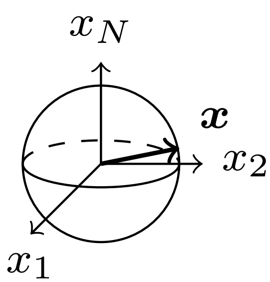
**LFC, Lozano, Nessi, Picco & Tartaglia 18**

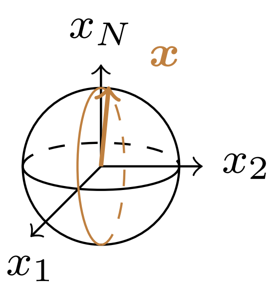
Injection Extraction

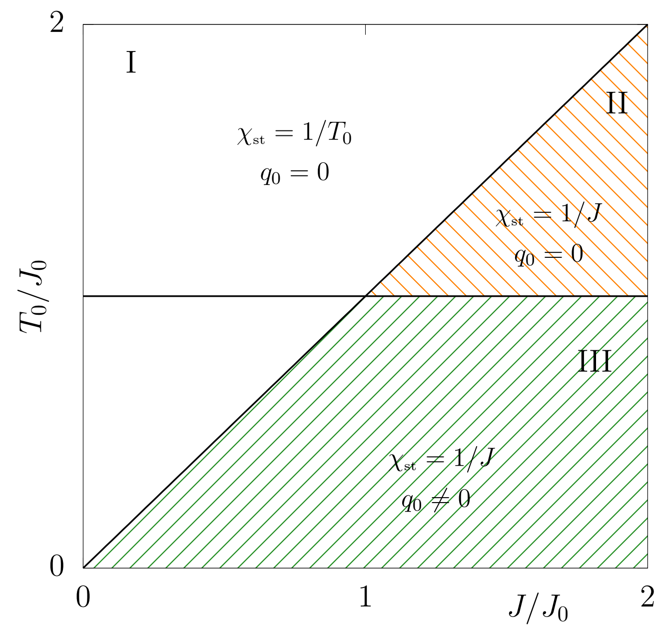
| | | | |
|---|---|---|---|
| I | $\chi_{\rm st} = 1/T_0$ | $z_f > \lambda_N = 2J$ | $\lim_{t \gg t_0} C(t,0) = q_0 = 0$ |
| II | $\chi_{\rm st} = 1/J$ | $z_f = \lambda_N = 2J$ | $\lim_{t \gg t_0} C(t,0) = q_0 = 0$ |
| III | $\chi_{\rm st} = 1/J$ | $z_f = \lambda_N = 2J$ | $\lim_{t \gg t_0} C(t,0) = q_0 > 0$ |

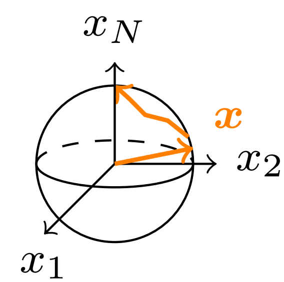
# The particle's motion

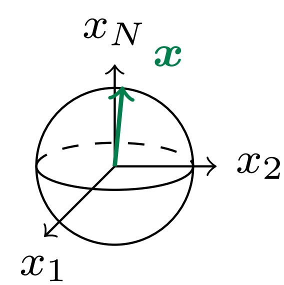
## Mode resolved dynamics




Distinguished by the scalings $\langle x_N^2 \rangle = \mathcal{O}(N^a)$ and $\langle p_N^2 \rangle = \mathcal{O}(N^b)$

# Asymptotic measure

**Is the Generalized Gibbs Ensemble the good one ?**

The GGE "canonical" measure is

$$\boxed{\rho_{\mathrm{GGE}}(\boldsymbol{p},\boldsymbol{x}) = \mathcal{Z}^{-1}(\{\beta\gamma_\mu\})\; e^{-\beta \sum\limits_{\mu=1}^{N} \gamma_\mu I_\mu(\boldsymbol{p},\boldsymbol{x})}}$$

with

$$\boxed{I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}} \qquad \mu = 1,\dots,N$$

(quartic & non-local) and we fix the $\gamma_\mu$ **on average** by imposing

$$\boxed{\langle I_\mu \rangle_{\mathrm{GGE}} = \langle I_\mu \rangle_{i.c.}} \qquad \forall\, \mu$$

NB in interacting quantum integrable models the charges are usually not known. But we do know them for this model !

# Dynamics vs GGE

**In the $N \to \infty$ limit**

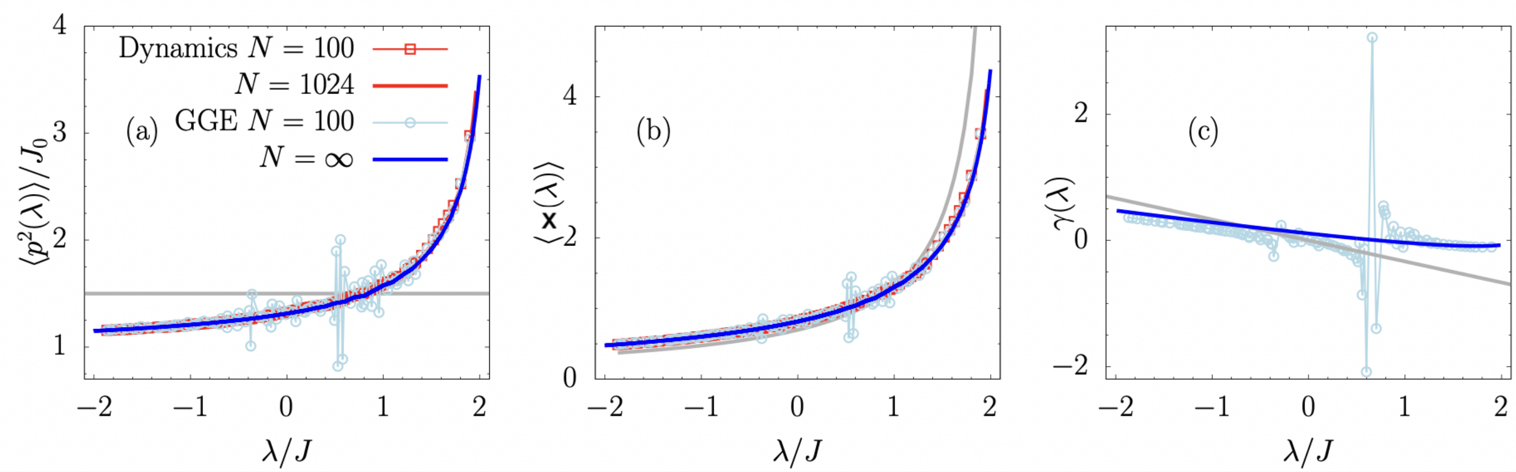
$$\langle x_\mu^2 \rangle_{\rm GGE} \stackrel{?}{=} \overline{\langle x_\mu^2(t) \rangle_{i.c.}} \qquad \text{and} \qquad \langle p_\mu^2 \rangle_{\rm GGE} \stackrel{?}{=} \overline{\langle p_\mu^2(t) \rangle_{i.c.}}$$

With all other modes acting as a bath on the selected $\mu$th ?

# Dynamics vs GGE

**e.g., comparison for quenches in Phase I**

$T(\lambda)$




In gray, the initial functions

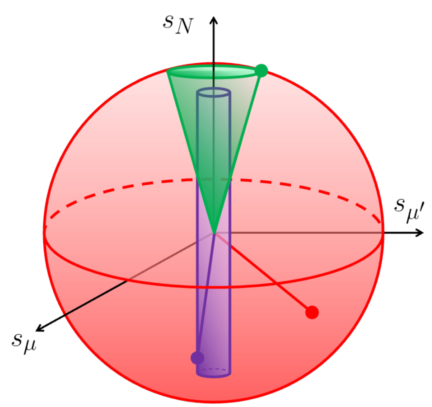
Similar coincidence in Phases II, III & IV

Interesting features linked to "fluctuations catastrophe" (BEC) in Phase IV

**Harmonic Ansatz $\equiv$ saddle-point evaluation of the GGE**

# Goals achieved

**In the late times limit taken after the large $N$ limit**




We solved

– the *global dynamics* with Schwinger-Dyson/ DMFT eqs.

– the *mode dynamics* with parametric oscillator techniques of the *(soft) Neumann model*

With the *GGE measure*

$$\rho_{\rm GGE}(\boldsymbol{p}, \boldsymbol{x}) = \mathcal{Z}^{-1}(\{\beta\gamma_\mu\})\, e^{-\beta \sum_\mu \gamma_\mu I_\mu(\boldsymbol{p}, \boldsymbol{x})}$$

– we calculated & proved

$$\langle x_\mu^2 \rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE} - \lambda_\mu} = \overline{\langle x_\mu^2(t) \rangle_{i.c.}}$$

$$\langle p_\mu^2 \rangle_{\rm GGE} = T_\mu = \overline{\langle p_\mu^2(t) \rangle_{i.c.}}$$

obtaining also $\{T_\mu, \beta\gamma_\mu\}$

**The canonical GGE describes the asymptotic dynamics of this non-trivial integrable model**

**Barbier, LFC, Lozano, Nessi 22**

# Plan

— Introduction : equilibrium, non-equilibrium & the goal

— Hamiltonian dynamics of a (simple) integrable model in the thermodynamic limit and the GGE measure

— *Stochastic evolution towards the GGE measure*

— Summary

# Langevin dynamics

**of the $\mathcal{H}^z_{\mathrm{GGE}}$ model**

$$\boxed{\dot{x}_\mu = \frac{\delta \mathcal{H}^z_{\mathrm{GGE}}}{\delta p_\mu} \qquad \dot{p}_\mu + \eta \dot{x}_\mu(t) = -\frac{\delta \mathcal{H}^z_{\mathrm{GGE}}}{\delta x_\mu(t)} + \xi_\mu(t)}$$

with white noise $\langle \xi_\mu(t) \rangle = 0$ and $\langle \xi_\mu(t) \xi_\nu(t') \rangle = 2\eta k_B T \delta_{\mu\nu} \delta(t - t')$

and the Hamiltonian $\mathcal{H}^z_{\mathrm{GGE}} = \sum_\mu \gamma_\mu I_\mu + \frac{z}{2}(x^2 - N)$

where $I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu (\neq \mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$ **note the weird form !**

Aim, check

$$\overline{x_\mu^2(t)} = \langle x_\mu^2 \rangle_{\mathrm{GGE}} \stackrel{?}{=} \langle x_\mu^2 \rangle_{\xi, i.c.}$$

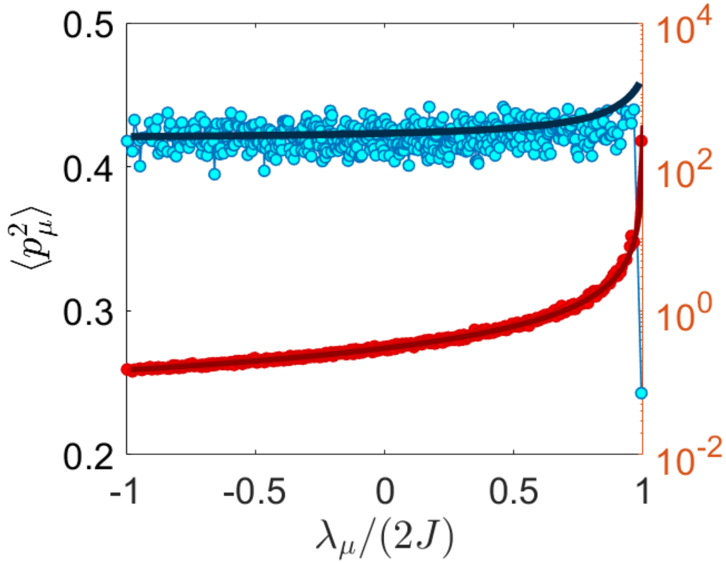
$$\overline{p_\mu^2(t)} = \langle p_\mu^2 \rangle_{\mathrm{GGE}} \stackrel{?}{=} \langle p_\mu^2 \rangle_{\xi, i.c.}$$

# Langevin dynamics

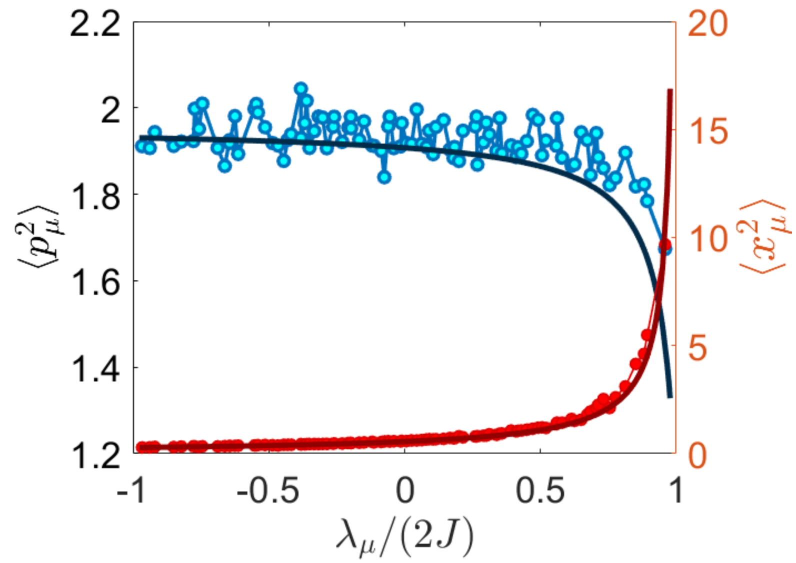
**Using the $\{\beta\gamma_\mu\}$ of the quench of the isolated model**

$$\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$$

Extended phase




Towards condensation




Data points $\langle \ldots \rangle_{\xi, i.c.}$ and solid lines $\langle \ldots \rangle_{\rm GGE}^{\mathcal{H}} = \langle \ldots \rangle_{\rm GB}^{\mathcal{H}_{\rm GGE}}$

**LFC, González-Albaladejo, Lozano & Stariolo 25**

# Plan

— Introduction : equilibrium, non-equilibrium & the goal

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— Stochastic evolution towards the GGE measure

— Summary

2019-22 $\longleftrightarrow$ 2025 $\longleftrightarrow$

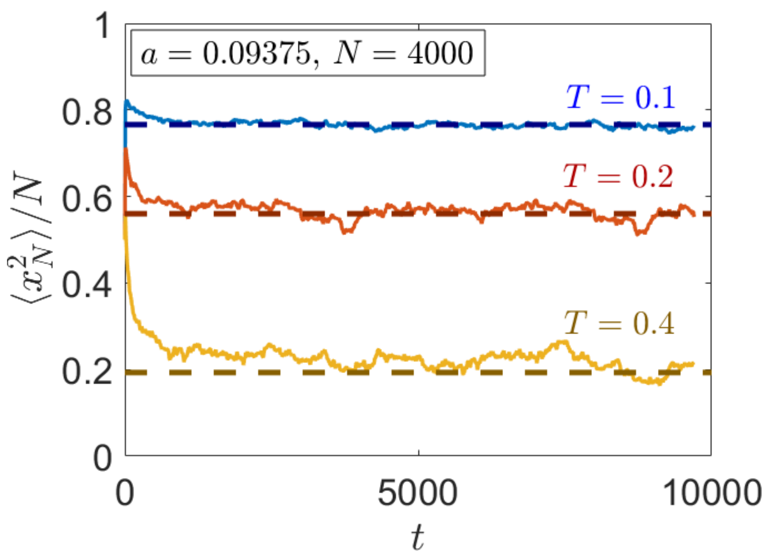
| **Newton dynamics**<br>after a quench of $\mathcal{H}$<br>the integrable model | "Ergodicity" & the **GGE ensemble**<br>$P_{\rm GGE} \propto e^{-\beta \mathcal{H}_{\rm GGE}}$<br>also the GB measure of $\mathcal{H}_{\rm GGE}$ | **Langevin relaxation** at $\beta^{-1}$<br>of $\mathcal{H}_{\rm GGE}(\{\gamma_\mu\})$<br>$\leftarrow$ approaches the GB measure |
|---|---|---|
| $\mathcal{H}$ | $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$ | $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$ |
| $\overline{\langle I_\nu(t=0^+)\rangle_{i.c.}}$ | $\langle I_\nu \rangle_{\rm GGE}$ | $\lim\limits_{t\to\infty} \langle I_\nu(t)\rangle$ |
| $\mathcal{I}_\nu\left(\frac{T_0}{J_0}, \frac{J}{J_0}\right)$ | $= I_\nu^{\rm GGE}\left(\{\beta\gamma_\mu\}\right)$<br>$\{\beta\gamma_\mu\}$ fixed from $\mathcal{I}_\nu = I_\nu^{\rm GGE}$<br>to represent the quench of $\mathcal{H}_{\rm N}$ | $= \lim\limits_{t\to\infty} \langle I_\nu(t)\rangle_{\xi,i.c.}$ |
| $\overline{\langle p_\mu^2(t \gg t_0)\rangle_{i.c.}}$ | $\langle p_\mu^2\rangle_{\rm GGE}$ | $\lim\limits_{t\to\infty} \langle p_\mu^2(t)\rangle_{\xi,i.c.}$ |
| $\overline{\langle x_\mu^2(t \gg t_0)\rangle_{i.c.}}$ | $\langle x_\mu^2\rangle_{\rm GGE}$ | $\lim\limits_{t\to\infty} \langle x_\mu^2(t)\rangle_{\xi,i.c.}$ |

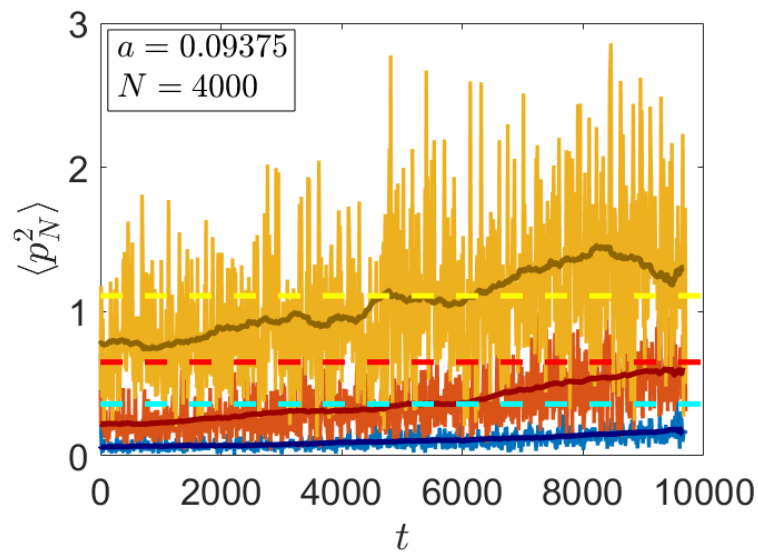
# Tests

**Low temperatures, $\beta > \beta_c$, condensation close to the North pole**

$$\mathcal{H}_{\mathrm{GGE}} = \sum_\mu \gamma_\mu I_\mu$$

Choice of parameters $\gamma_\mu = {}^a/_J\,\lambda_\mu^2 - {}^1/_2\,\lambda_\mu$




Data points $\langle \ldots \rangle_{\xi, i.c.}$ and dashed lines $\langle \ldots \rangle_{\mathrm{GGE}}^{\mathcal{H}} = \langle \ldots \rangle_{\mathrm{GB}}^{\mathcal{H}_{\mathrm{GGE}}}$

**LFC, González-Albaladejo, Lozano & Stariolo 25**