A classical integrable interacting model with a GGE description

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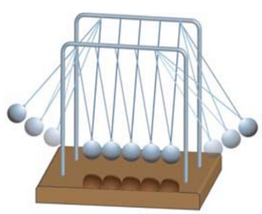
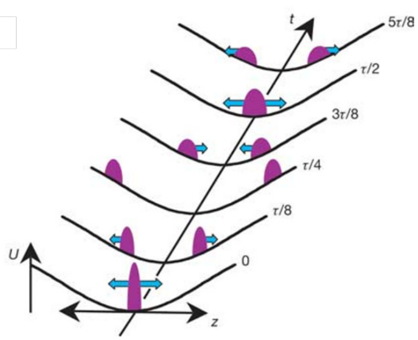
Motivation

Isolated quantum systems: experiments and theory $\sim$ 15y ago





A quantum Newton's cradle

experiments

cold atoms in isolation

Kinoshita, Wenger & Weiss 06

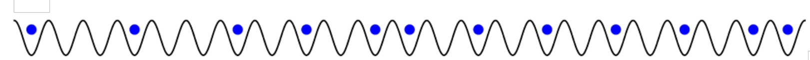
(Conformal) field **theory** methods for

quantum quenches

Calabrese & Cardy 06

Numerical study of

lattice hard core bosons



Rigol, Dunjko, Yurovsky & Olshanii 07

and many others

Mostly 1d systems

Questions

Does an isolated quantum system reach some kind of equilibrium ?

Boosted by recent interest in

— the dynamics after quantum quenches of cold atomic systems

rôle of interactions (integrable *vs.* non-integrable)

— many-body localisation

novel effects of quenched disorder

And, an isolated classical system ?

The (old) ergodicity question revisited

Our contribution **Barbier, LFC, Lozano, Nessi, Picco, Tartaglia 17-21**

Quantum quenches

Definition & questions

- Take an **isolated** quantum system with Hamiltonian $\hat{H}_0$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_0$ or any $\hat{\rho}(t_0)$
- Unitary time-evolution $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ with a Hamiltonian $\hat{H} \neq \hat{H}_0$.

Does the system reach (locally) a steady state ? (for $N \to \infty$)

Is it described by a thermal equilibrium density matrix $e^{-\beta\hat{H}}$?

Do at least some local observables behave as thermal ones?

Does the evolution occur as in equilibrium ?

If not, other kinds of density matrices ?

Classical quenches

Definition & questions

- Take an **isolated** classical system with Hamiltonian H_0, evolve with H
- Initialize it in, say, ψ_0 a configuration, *e.g.* $\{\vec{q}_i, \vec{p}_i\}_0$ for a particle system

  ψ_0 could be drawn from a probability distribution, *e.g.* $\mathcal{Z}^{-1}\, e^{-\beta_0 H_0(\psi_0)}$

Does the system reach a steady state ? (in the $N \to \infty$ limit)

Is it described by a thermal equilibrium probability $e^{-\beta H}$?

Do at least some local observables behave as thermal ones ?

Does the evolution occur as in equilibrium ?

If not, other kinds of probability distributions ?

Classical quenches

Definition & questions

In the steady state of a classical macroscopic ($N \to \infty$) model

Time averages $\overline{O(t)} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_{\rm st}}^{t_{\rm st}+\tau} dt' \; O(t')$

& statistical averages $\langle O \rangle \equiv \int \prod_i dq_i \prod dp_i \; O(s_i, p_i) \; \rho(s_i, p_i)$

should be equal $\overline{O(t)} = \langle O \rangle$ for a generalised micro-canonical measure ρ

in which, in integrable cases, all constants of motion are fixed **Yuzbashyan 18**

Are local observables characterised by a "canonical" measure ?

If yes, which one ?

Classical quenches

Interest in integrable models : strategy & goals

– Choose a sufficiently simple classical *integrable interacting* model with (not just harmonic oscillators)

an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

– Solve the *dynamics* after the quenches

– Build a *Generalised Gibbs Ensemble* (GGE)

– Prove that the asymptotic limit of *local observables* is captured by the GGE

Classical quenches

Strategy

Choose a sufficiently simple classical *integrable interacting* model

(not just harmonic oscillators)

with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

Solve the dynamics after a quench

Build a Generalised Gibbs Ensemble

Prove that the asymptotic limit of local observables is captured by the GGE

Model choice.

Inspiration from

- disordered systems,
- phase ordering kinetics

knowledge

A spin model with randomness

The spherical SK ($p = 2$) model — **Kosterlitz, Thouless & Jones 76**

$$V^{(z)}_{J_0}(\{s_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j + \frac{z(\vec{s})}{2}\left(\sum_i s_i^2 - N\right)$$

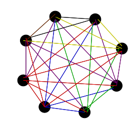
Fully connected interactions & $s_i \in \mathbb{R}$

Global spherical constraint $|\vec{s}\,|^2 = \sum_i s_i^2 = N$

imposed on average by a Lagrange multiplier $z(\vec{s})$

Gaussian distributed interaction strengths

$J_{ij} = J_{ji}$, $[J_{ij}] = 0$ & $[J_{ij}^2] = J_0^2/(2N)$



N eigenvalues $\lambda^{(0)}_\mu$ and

eigenvectors $\vec{v}_\mu$

$\rho(\lambda^{(0)}_\mu) \propto \sqrt{(2J_0)^2 - (\lambda^{(0)}_\mu)^2}$

Diagonalised effective potential in the basis of eigenvectors $s_\mu = \vec{v}_\mu \cdot \vec{s}$

$$V^{(z)}_{J_0}(\{s_\mu\}) = -\frac{1}{2}\sum_{\mu} \lambda^{(0)}_\mu s_\mu^2 + \frac{z(\vec{s})}{2}\left(\sum_\mu s_\mu^2 - N\right)$$

A spin model with randomness

The spherical SK ($p = 2$) model

We plan to choose *initial conditions* drawn from the *canonical Gibbs-Boltzmann* equilibrium measure.

Physically :

– the system is in thermal equilibrium with a bath at temperature T_0 until $t = 0^-$

– the coupling to the bath is switched off at this instant $t = 0$

– it further evolves in isolation at $t > 0$

A spin model with randomness

The canonical equilibrium of the spherical SK ($p = 2$) model

$$\mathcal{Z}(\beta_0 J_0) \propto \int dz \int \prod_\mu ds_\mu \; e^{-\beta_0 V^{(z)}_{J_0}(\{s_\mu\})}$$

Gaussian integrals yield **Kosterlitz, Thouless & Jones 76**

$$\langle s_\mu^2 \rangle_{\rm eq} = \frac{T_0}{z - \lambda_\mu^{(0)}}$$ with $\sum_\mu \langle s_\mu^2 \rangle_{\rm eq} \xrightarrow[N\to\infty]{} \int d\lambda \, \rho(\lambda) \, \frac{T_0}{z - \lambda_\mu^{(0)}} = N$ fix z

At $T_0 > T_c = J_0$ all modes $s_\mu = \vec{s} \cdot \vec{v}_\mu$ are **massive** $z - \lambda_N^{(0)} > 0$ & $\langle s_\mu^2 \rangle_{\rm eq} = \mathcal{O}(1)$

At $T_0 \leq T_c = J_0$ the Nth mode $s_N = \vec{s} \cdot \vec{v}_N$ is **massless** $\lim_{N\to\infty} (\lambda_N^{(0)} - z) = 0$

and **condenses** $\langle s_N^2 \rangle_{\rm eq} = q(T_0/J_0) N$

NB spherical constraint is imposed on average

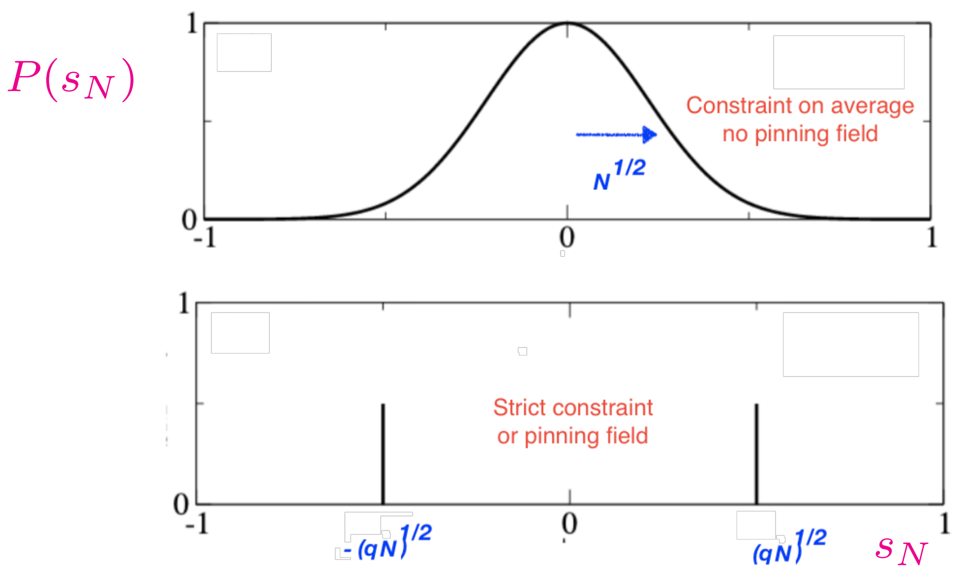
A spin model with randomness

The canonical equilibrium of the spherical SK ($p = 2$) model

$$\mathcal{Z}(\beta_0 J_0) \propto \int dz \int \prod_\mu ds_\mu \, e^{-\beta V^{(z)}_{J_0}(\{s_\mu\})}$$

The spherical constraint fixes $\langle s_N^2 \rangle_{\rm eq} = qN$ *via* $\int d\lambda^{(0)} \, \rho(\lambda^{(0)}) \, \frac{T_0}{2J_0 - \lambda^{(0)}_\mu} + \frac{\langle s_N^2 \rangle_{\rm eq}}{N} = 1$

Two possibilities



At $T_0 \leq T_c = J_0$ the mode

$s_N = \vec{s} \cdot \vec{v}_N$ is **massless**

$$\lim_{N\to\infty} (\lambda^{(0)}_N - z) = 0$$

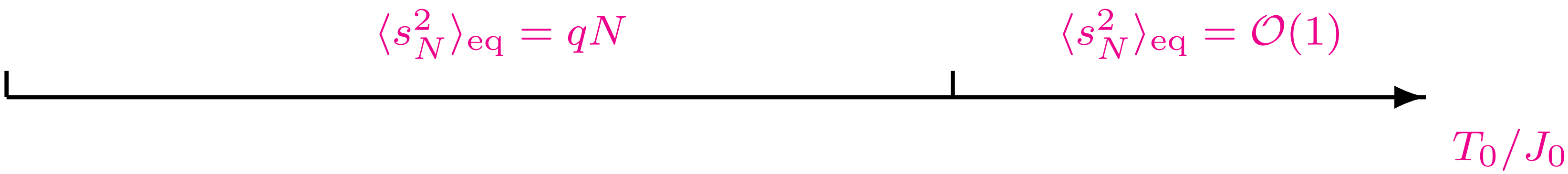
and **condenses**

$$\langle s_N^2 \rangle_{\rm eq} = q(T_0/J_0)N$$

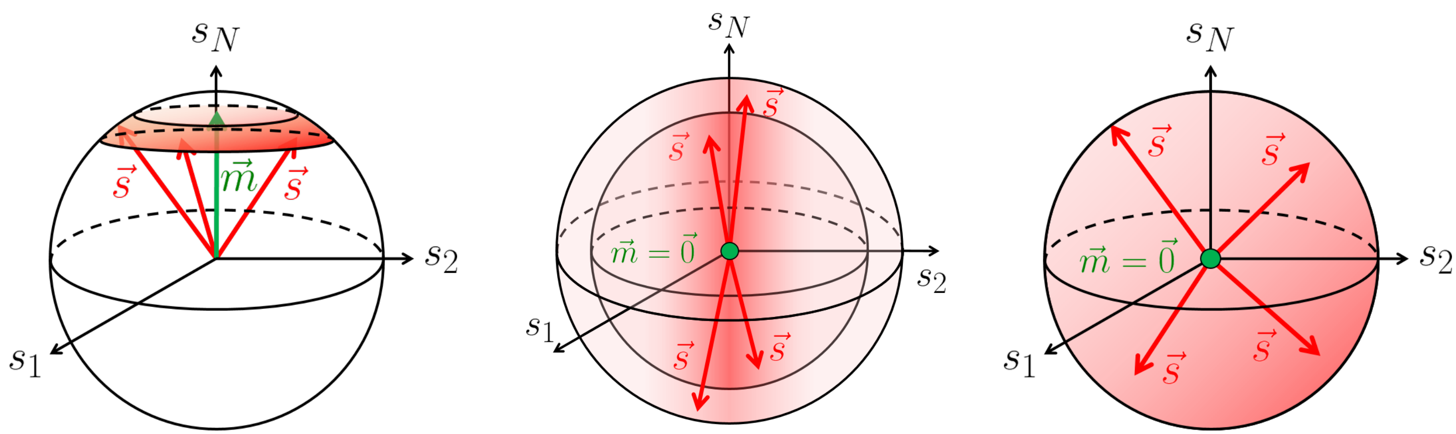
Kac & Thompson 71, Zannetti 15, Crisanti, Sarracino & Zannetti 19

A spin model with randomness

The canonical equilibrium of the spherical SK ($p = 2$) model



$T_0/J_0 < 1$ $\qquad$ $T_0/J_0 > 1$



Condensed symmetry broken or symmetric $\qquad$ **Extended**

$$\vec{m} \equiv \langle \vec{s} \rangle_{\text{eq}}$$

From statistical physics to classical mechanics

$t > 0$ evolution

Classical dynamics

From spins to a particle moving on the S_{N-1} sphere

Coordinate-momenta pairs $\{\vec{s},\vec{p}\}$ and Hamiltonian (const w/Lagrange mult.)

$$H_J^{(z)} = E_{\rm kin}(\vec{p}) + V_J(\vec{s}) + \frac{z(\vec{s},\vec{p})}{2}\sum_{\mu=1}^{N}(s_\mu^2 - N)$$

with the kinetic energy $E_{\rm kin}(\vec{p}) = \frac{1}{2m}\sum_{\mu=1}^{N} p_\mu^2$

Newton-Hamilton equations

$$\dot{s}_\mu = p_\mu/m \qquad\qquad \dot{p}_\mu = -\delta V_J(\vec{s})/\delta s_\mu - z(\vec{s},\vec{p})s_\mu$$

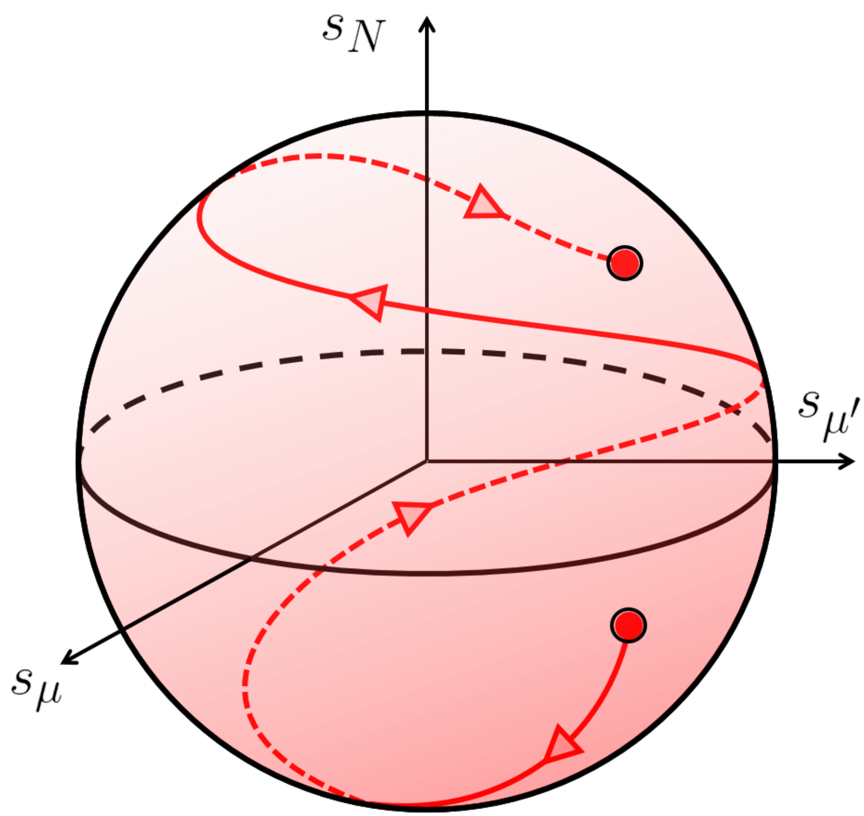
An anisotropic harmonic potential energy

$$V_J(\vec{s}) = -\tfrac{1}{2}\sum_{i\neq j} J_{ij}s_is_j = -\frac{1}{2}\sum_\mu \lambda_\mu s_\mu^2$$

but $V_J^{(z)}(\vec{s})$ is quartic due to $z(\vec{s},\vec{p})$

Classical dynamics

From spins to a particle moving on the S_{N-1} sphere



Initial conditions averaged constraints

$$\langle \phi \rangle_{i.c.} : \sum_\mu \langle s_\mu^2 \rangle_{i.c.} - N = 0$$

$$\langle \phi' \rangle_{i.c.} : \sum_\mu \langle s_\mu p_\mu \rangle_{i.c.} = 0$$

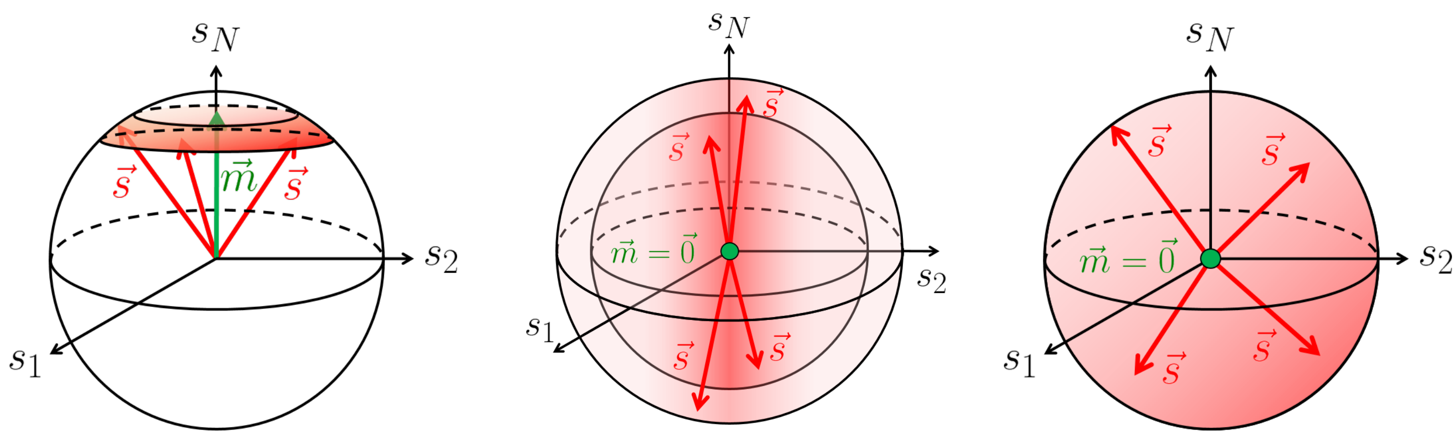
The initial conditions

The red arrows are different initial conditions

$\langle s_N^2 \rangle_{\rm eq} = qN$ $\langle s_N^2 \rangle_{\rm eq} = \mathcal{O}(1)$

T_0/J_0

$T_0/J_0 < 1$ $T_0/J_0 > 1$



Condensed symmetry broken or symmetric **Extended**

$\vec{m} \equiv \langle \vec{s} \rangle_{\rm eq}$

Focus on symmetry broken ones for $T_0/J_0 < 1$

Instantaneous quench

Global rescaling of all coupling constants

At time $t = 0$

$$J_{ij}^{(0)} \mapsto J_{ij} = \frac{J}{J_0} J_{ij}^{(0)}$$

to keep some memory of the initial conditions.

It is equivalent to

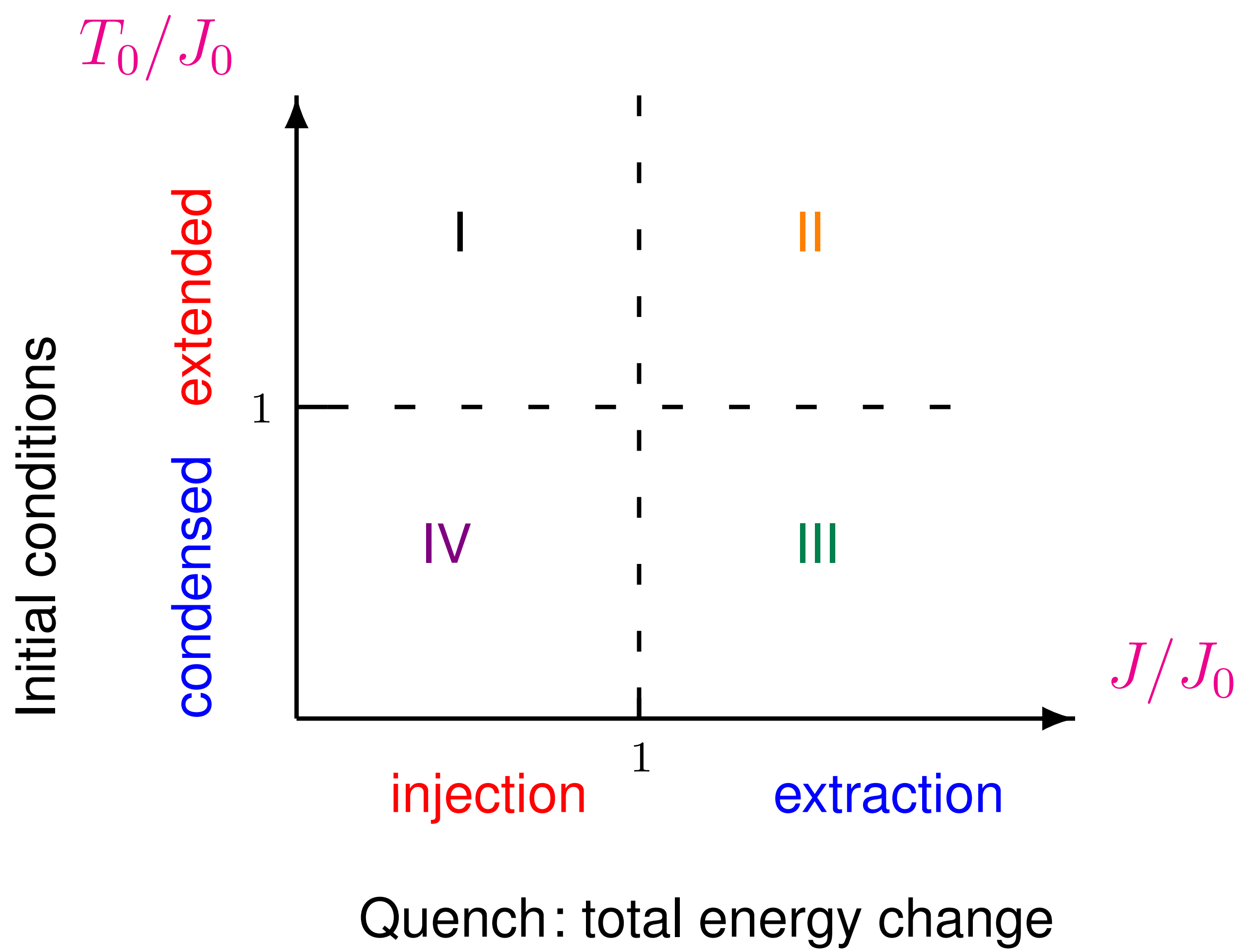
$$\lambda_\mu^{(0)} \mapsto \lambda_\mu = \frac{J}{J_0} \lambda_\mu^{(0)}$$

No change in configuration $\{s_\mu(0^-) = s_\mu(0^+), \; p_\mu(0^-) = p_\mu(0^+)\}$ but macroscopic energy change

$$\Delta E = \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{for} \quad \frac{J}{J_0} \begin{cases} < 1 & \textbf{Injection} \\ > 1 & \textbf{Extraction} \end{cases}$$

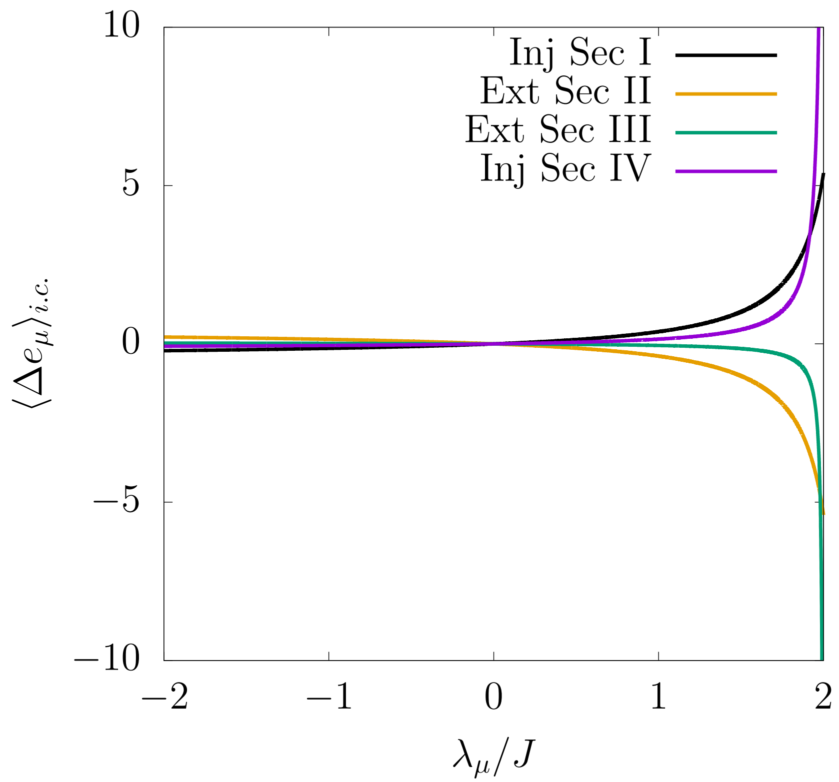
Control parameters

Total energy change & initial conditions

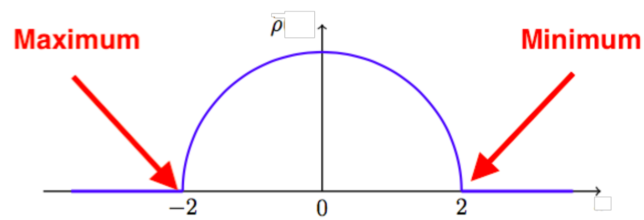


Instantaneous quench

Mode energy change under $J_0 \mapsto J$



Wigner density of eigenvalues $\rho(\lambda/J)$



Mode energy spectrum

$$e_\mu = \frac{1}{2m} p_\mu^2 - \frac{1}{2} \lambda_\mu s_\mu^2$$

Mode energy change

$$\langle \Delta e_\mu \rangle_{i.c.} = \langle e_\mu(0^+) - e_\mu(0^-) \rangle_{i.c.}$$

The energies of the modes at the **right edge** of the λ_μ spectrum are the more affected ones

These are the **softer modes**

Neumann's model

1859

Journal

für die

reine und angewandte Mathematik

In zwanglosen Heften.

Als Fortsetzung des von

A. L. Crelle

gegründeten Journals

herausgegeben

unter Mitwirkung der Herren

Steiner, Schellbach, Kummer, Kronecker, Weierstrass

von

C. W. Borchardt.

Mit thätiger Beförderung hoher Königlich-Preufsischer Behörden.

Sechs und funfzigster Band.

In vier Heften.

Berlin, 1859.

Druck und Verlag von Georg Reimer.

Journal of Pure & Applied Math.

Crelle Journal

A particle on a sphere

under harmonic potentials

De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur.

(Auctore *C. Neumann*, Hallae.)

§. 1.

Problema proponitur.

Sint puncti mobilis Coordinatae orthogonales x, y, z; sit

$$x^2+y^2+z^2 = 1$$

Just the same model with

$$\lambda_\mu \neq \lambda_\nu$$

in spherical SK model ensured by GOE

Thanks to **O. Babelon**

Strict constraint

Neumann's model

Integrability

N constants of motion in involution $\{I_\mu, I_\nu\} = 0$ fixed by the initial conditions

$$I_\mu = s_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(s_\mu p_\nu - s_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

K. Uhlenbeck 82

Modified angular momentum. The notation is such that $s_\mu = \vec{s} \cdot \vec{v}_\mu$ and $p_\mu = \vec{p} \cdot \vec{v}_\mu$

$H_J = E_{\rm kin} + V_J = -\frac{1}{2} \sum_\mu \lambda_\mu I_\mu$ and $N = \sum_\mu I_\mu$ (using $\sum_\mu s_\mu^2 = N$ & $\sum_\mu s_\mu p_\mu = 0$)

Studies by **Avan, Babelon and Talon 90s** for finite N

Thermodynamic $N \to \infty$ limit ?

No canonical GB equilibrium expected but Generalised Gibbs Ensemble

$$\rho_{\rm GGE}(\vec{s}, \vec{p}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\vec{s},\vec{p})}$$ **?**

How to study the large N dynamics ?

Firstly, identify the **constants of motion**

Barbier, LFC, Lozano, Nessi, Picco & Tartaglia 18-20

The constants of motion

$\langle I_\mu(0^+)\rangle_{i.c.}$ **averaged over the initial measure**

Extended

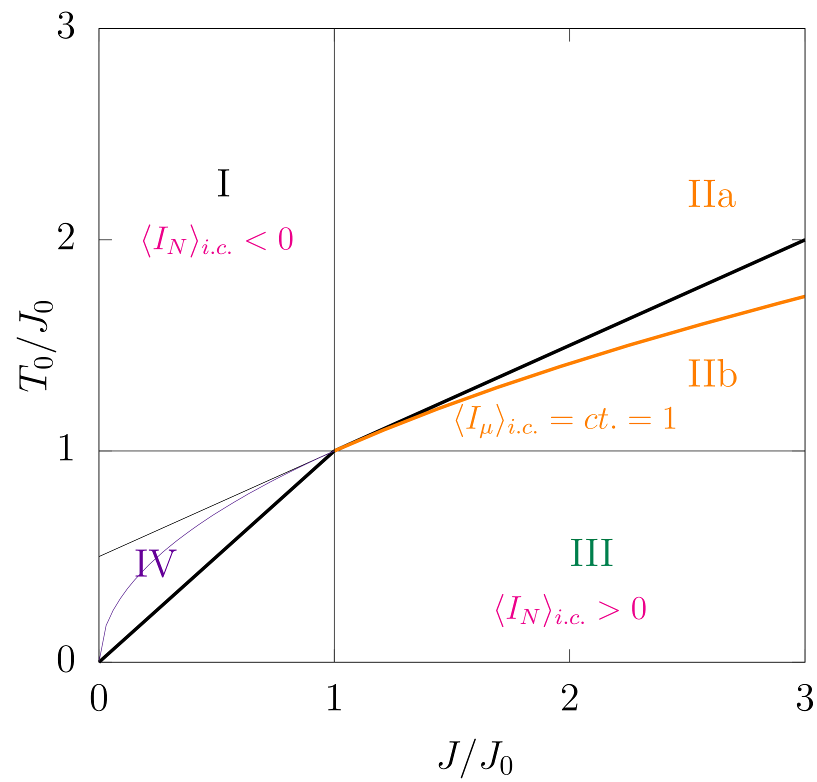
$$\frac{{T_0}^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{J(J_0+{T_0}^2/J_0)/T_0-\lambda_\mu}$$

Condensed & $\mu \neq N$

$$\frac{T_0^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{2J-\lambda_\mu}$$

Condensed & $\mu = N$

$$\left(1-\frac{T_0}{J_0}\right)\left(1-\frac{T_0}{J}\right)N+\mathcal{O}(1)$$



NB On the thick orange line the constants are all equal !

The constants of motion

$\langle I_\mu(0^+)\rangle_{i.c.}$ **averaged over the initial measure**

Extended

$$\frac{{T_0}^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{J(J_0+{T_0}^2/J_0)/T_0-\lambda_\mu}$$

Condensed & $\mu \neq N$

$$\frac{T_0^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{2J-\lambda_\mu}$$

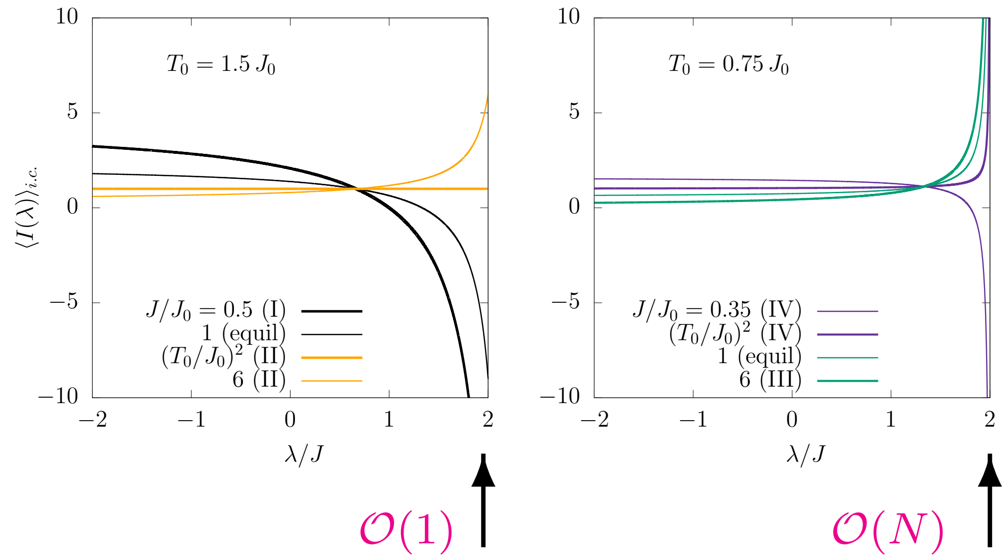
Condensed & $\mu = N$

$$\left(1-\frac{T_0}{J_0}\right)\left(1-\frac{T_0}{J}\right)N+\mathcal{O}(1)$$

Bulk constants

$T_0/J_0 > 1$ — **Extended**

$T_0/J_0 < 1$ — **Condensed**



NB for $T_0/J_0 > 1$ and $(T_0/J_0)^2 = J/J_0$ the constants are all equal

How to study the large N dynamics ?

Secondly, analysis of **global – macroscopic – observables**

Conservative dynamics

on average over randomness & the initial measure

In the $N \to \infty$ limit exact Schwinger-Dyson (DMFT) equations for the global self-correlation and linear response averaged over the $\{\lambda_\mu\}$, denoted $[\ldots]_J$, and the initial conditions, noted $\langle \ldots \rangle_{i.c.}$,

$$NC(t,t') = \sum_\mu [\langle s_\mu(t) s_\mu(t') \rangle_{i.c.}]_J$$ Self-correlation

$$NC(t,0) = \sum_\mu [\langle s_\mu(t) s_\mu(0) \rangle_{i.c.}]_J$$ "Fidelity"

$$NR(t,t') = \sum_\mu \, [\langle \frac{\delta s_\mu(t)}{\delta h_\mu(t')} \Big|_{\vec{h}=0} \rangle_{i.c.}]_J$$ Linear response

Coupled causal integro-differential equations

$$(m\partial_t^2 - z_t) R(t,t') = \int dt'' \, \Sigma(t,t'') R(t'',t') + \delta(t-t')$$

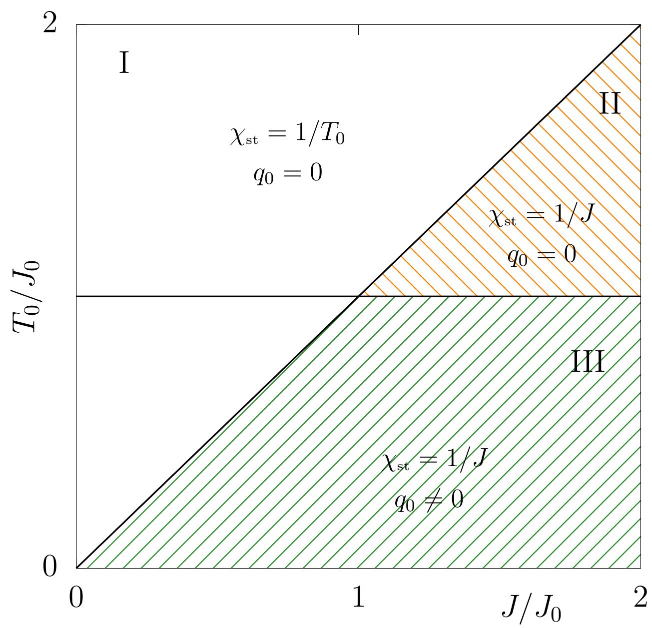
+ two other ones, with terms fixing the initial conditions

Solvable numerically & analytically at long times

The dynamic phase diagram

from Schwinger-Dyson equations

$$\chi_{\rm st} = \lim_{t \gg t_0} \int_0^t dt' \, R(t,t')$$
$$z_f = \lim_{t \gg t_0} z(t)$$



Initial conditions

Injection Extraction

I $\chi_{\rm st} = 1/T_0$ $z_f > \lambda_N = 2J$ and $\lim_{t\gg t_0} C(t,0) = q_0 = 0$

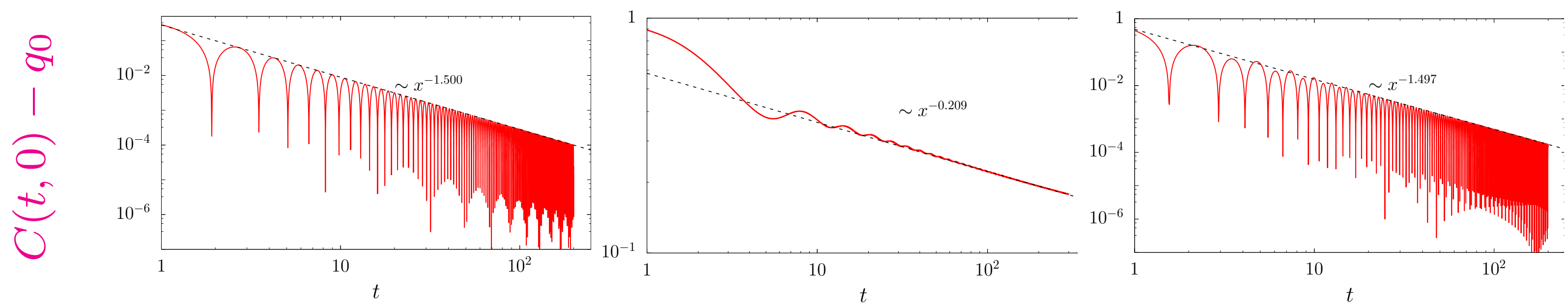
II $\chi_{\rm st} = 1/J$ $z_f = \lambda_N = 2J$ and $\lim_{t\gg t_0} C(t,0) = q_0 = 0$

III $\chi_{\rm st} = 1/J$ $z_f = \lambda_N = 2J$ and $\lim_{t\gg t_0} C(t,0) = q_0 > 0$

Asymptotic analysis

Algebraic approach to $q_0 = \lim_{t \gg t_0} C(t,0)$ **- fidelity**

condensed critical extended



$T_0/J_0 < 1 \quad J > J_0$

$C(t,0) = q_0 + c\, t^{-1.5} g(t)$

the exponent is independent

$T_0/J_0 > 1$

$C(t,0) = c\, t^{-0.2} g(t)$

dependent of parameters

Similar time-dependencies & asymptotics for $z(t)$

Stationary limit

of macroscopic – global – one-time quantities

The Lagrange multiplier approaches a constant,

$$z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)] \to z_f$$

so do the kinetic & potential energies,

$$e_{\rm kin}(t) \to e^f_{\rm kin} \quad \text{and} \quad e_{\rm pot}(t) \to e^f_{\rm pot}$$

The correlation with the initial condition as well

$$C(t,0) \to q_0$$

in all phases (q_0 vanishes in some)

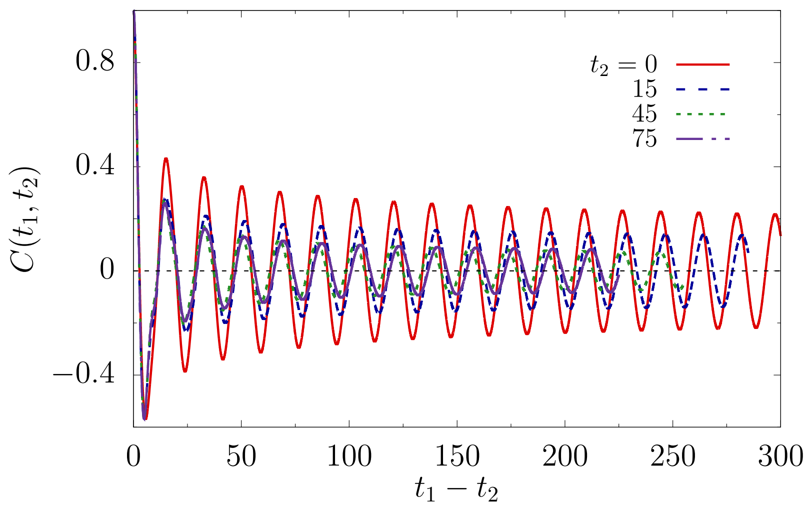
Non-conserved global one-time observables reach constants

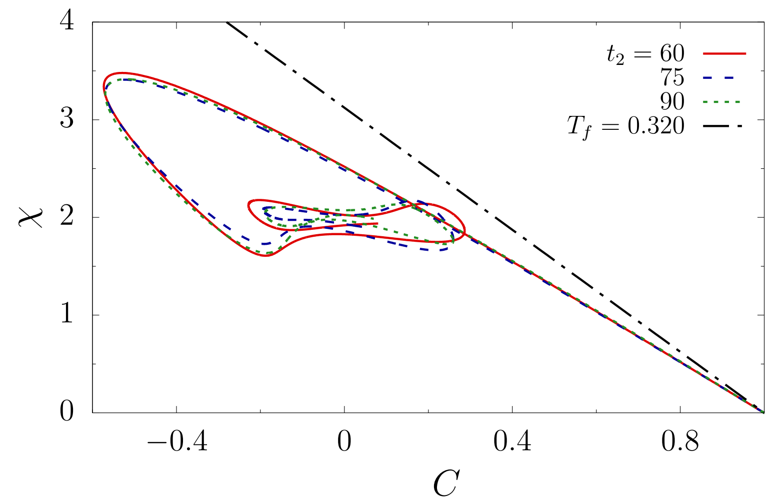
Stationary dynamics ? Is this GB equilibrium ?

No Gibbs-Boltzmann equilibrium

e.g. large energy injection on a condensed state (Sector IV)

$C(t,t') \to C_{\rm st}(t-t')$ but





$$\chi_{\rm st}(t-t') \equiv \int_{t'}^{t} dt'' \, R_{\rm st}(t,t'') \neq -\beta_f \, [C_{\rm st}(t-t') - C_{\rm st}(0)]$$

Stationary dynamics but no FDT at a single temperature **no GB equilibrium**

Not surprising since the model is integrable.

Thirdly, **dynamic single mode analysis**

to better understand the steady state

Mode dynamics

Non-linear coupling, no average over disorder, any N

The $s_\mu (= \vec{s} \cdot \vec{v}_\mu)$ with $\mu = 1, \ldots, N$ obey parametric oscillator equations

$$m\ddot{s}_\mu(t) = -[z(t) - \lambda_\mu] s_\mu(t)$$

with $z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)]$ & λ_μ the eigenvalues of J_{ij}.

The solution is

$$s_\mu(t) = s_\mu(0) \sqrt{\frac{\Omega_\mu(0)}{\Omega_\mu(t)}} \cos \int_0^t dt' \; \Omega_\mu(t') + \frac{\dot{s}_\mu(0)}{\Omega_\mu(0)\Omega_\mu(t)} \sin \int_0^t dt' \; \Omega_\mu(t')$$

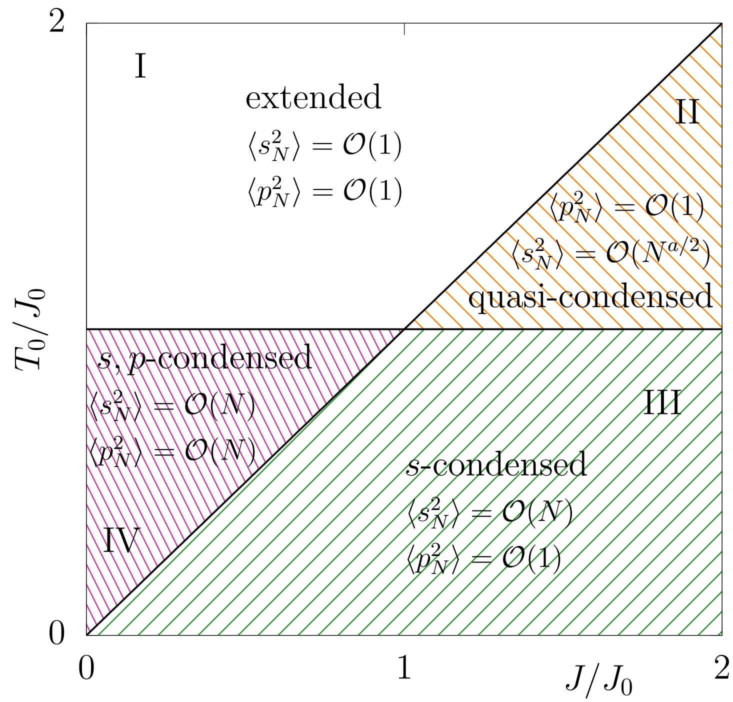
+ equations for the time-dependent frequencies $\Omega_\mu(t)$ and $z(t)$.

Similar to **Sotiriadis & Cardy 10** for the quantum O(N) model

Solvable numerically for any finite N

The dynamic phase diagram

Looking more carefully at the condensation phenomena



Initial conditions

Injection Extraction

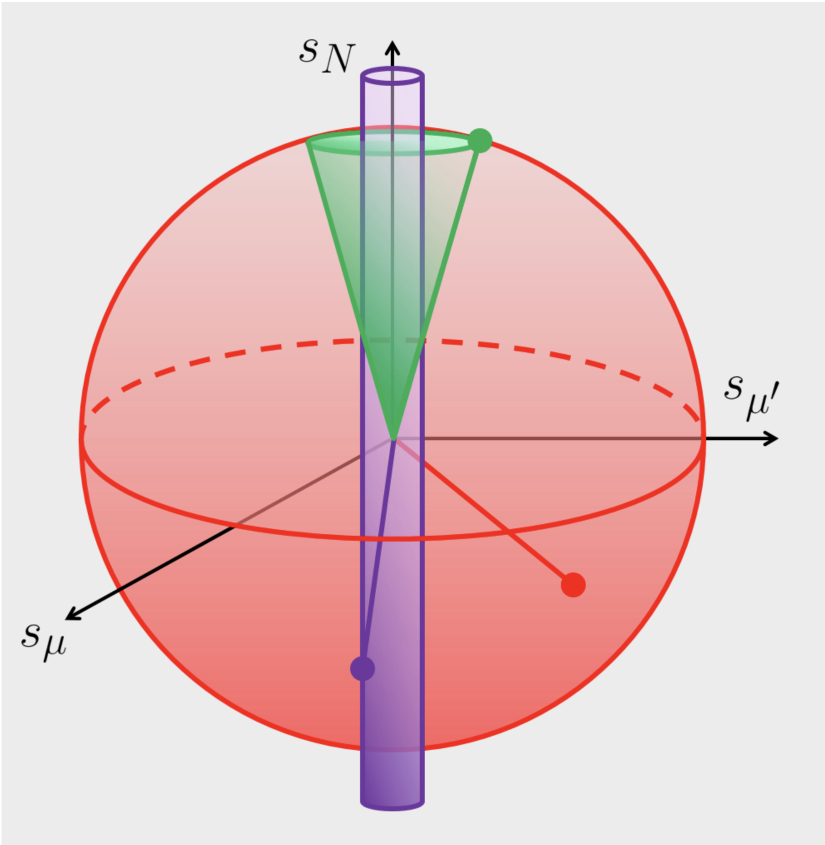
For all parameters $\lim_{t \gg t_{\rm st}} \lim_{N \gg 1} \overline{\langle s_\mu^2(t) \rangle_{i.c.}}, \overline{\langle p_\mu^2(t) \rangle_{i.c.}}$ reach constants

The two averages noted simply $\langle \ldots \rangle$ in the plot

The dynamics on the sphere

in the four Sectors of the dynamic phase diagram

Sector I & **Sector II** **Sector III** **Sector IV**



Is there a stationary asymptotic measure ?

Fourthly, establish **the GGE ensemble** and compute averages

Asymptotic measure

Is the Generalized Gibbs Ensemble the good one ?

The GGE "canonical" measure is

$$\rho_{\rm GGE}(\vec{s},\vec{p}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^{N} \gamma_\mu I_\mu(\vec{s},\vec{p})}$$

with **K. Uhlenbeck 82**

$$I_\mu = s_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(s_\mu p_\nu - s_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

(quartic & non-local) and we fix the γ_μ **on average** by imposing

$$\langle I_\mu \rangle_{\rm GGE} = \langle I_\mu \rangle_{i.c.}$$

NB in interacting quantum integrable models the charges are not known. But we do know them for this model !

The GGE

Harmonic *Ansatz*

$$\rho_{\rm GGE}(\{\vec{s},\vec{p}\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{\vec{s},\vec{p}\})}$$

Extensive expression in the exponential $\sum_{\mu=1}^N \gamma_\mu I_\mu = \mathcal{O}(N)$ if $\gamma_\mu = \mathcal{O}(1)$

GB measure recovered for $J = J_0$ with $\gamma_\mu = -\dfrac{\beta_0 \lambda_\mu}{2}$ since $\displaystyle\sum_{\mu=1}^N \frac{\lambda_\mu}{2} I_\mu = -H_J$

How to calculate $\langle s_\mu^2\rangle_{\rm GGE}$ and $\langle p_\mu^2\rangle_{\rm GGE}$? A plausible guess

$$\langle s_\mu^2\rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE}-\lambda_\mu} \qquad\qquad \langle p_\mu^2\rangle_{\rm GGE} = m\, T_\mu$$

with spherical constraint for $z_{\rm GGE}$ & the mode-temperature spectrum fixed by

$$\langle I(\lambda)\rangle_{i.c.} = \langle I(\lambda)\rangle_{\rm GGE} = \frac{2T(\lambda)}{z_{\rm GGE}-\lambda}\left[1 - \fint d\lambda' \,\frac{\rho(\lambda')T(\lambda')}{\lambda-\lambda'}\right]$$

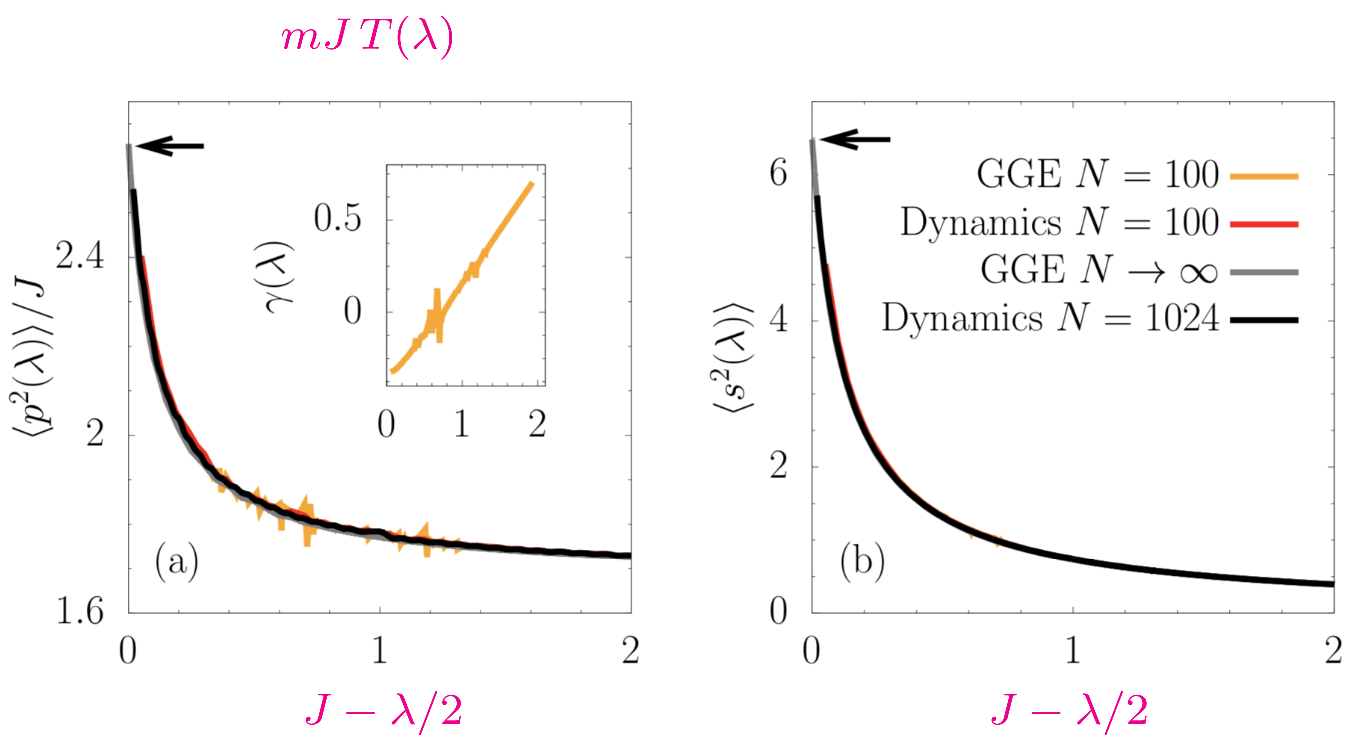
another eq. for the N-th mode for condensed $i.c.$ & eqs. for $\{\gamma_\mu\}$ **Solvable**

Dynamics vs GGE

$$\langle s_\mu^2 \rangle_{\rm GGE} = \overline{\langle s_\mu^2(t) \rangle_{i.c.}} \quad \text{and} \quad \langle p_\mu^2 \rangle_{\rm GGE} = \overline{\langle p_\mu^2(t) \rangle_{i.c.}} \quad ?$$

Dynamics vs GGE

e.g., comparison for quenches in Sector I



Similar coincidence in Sectors II, III & IV

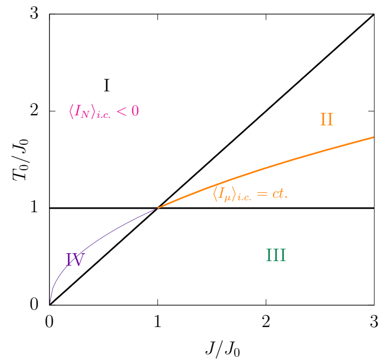
Interesting features linked to “fluctuations catastrophe” in Sector IV

Harmonic Ansatz confirmed by a saddle-point evaluation of the GGE

Dynamics vs GGE

A special case : GGE $\mapsto$ GB

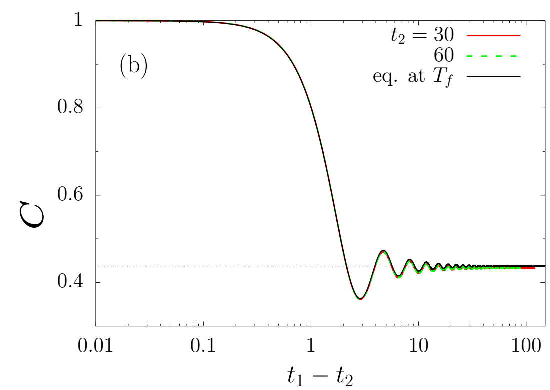
For $T_0/J_0 > 1$ and $(T_0/J_0)^2 = J/J_0$

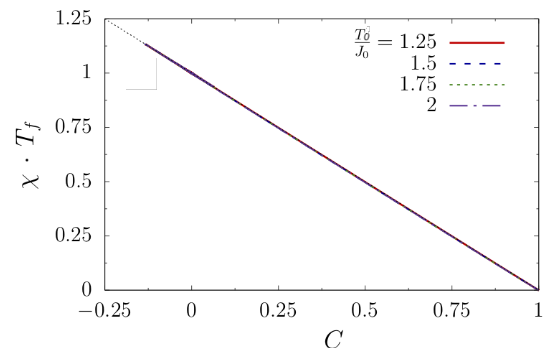


the constants of motion

are all equal

$\langle I_\mu \rangle_{i.c.} = 1$





The GGE construction yields

$$T_\mu = J \quad \text{and} \quad \gamma_\mu = -\lambda_\mu/(2J)$$

Therefore

$$-\sum_\mu \gamma_\mu I_\mu = \frac{1}{2J} \sum_\mu \lambda_\mu I_\mu = -\frac{1}{J} H$$

and the GGE reduces to the GB measure

at $T_f = J$

Stationarity & FDT OK

Fifthly, can one obtain the **mode temperatures** with a **global dynamic measurement**?

Correlation and linear response

Fluctuation-dissipation theorem in Boltzmann equilibrium

$$C(t,t') = \frac{1}{N}\sum_{\mu=1}^{N} \langle s_\mu(t) s_\mu(t')\rangle_{i.c.} \qquad \text{self correlation}$$

$$R(t,t') = \frac{1}{N}\sum_{\mu=1}^{N} \left.\frac{\delta\langle s_\mu(t)\rangle_{i.c.}}{\delta h_\mu(t')}\right|_{h=0} \qquad \text{linear response}$$

Stationary limit $C(t,t') \mapsto C_{\rm st}(t-t')$ and $R(t,t') \mapsto R_{\rm st}(t-t')$

Fourier transforms

$$\hat{C}(\omega) = \text{F.T. } C_{\rm st}(t-t')$$

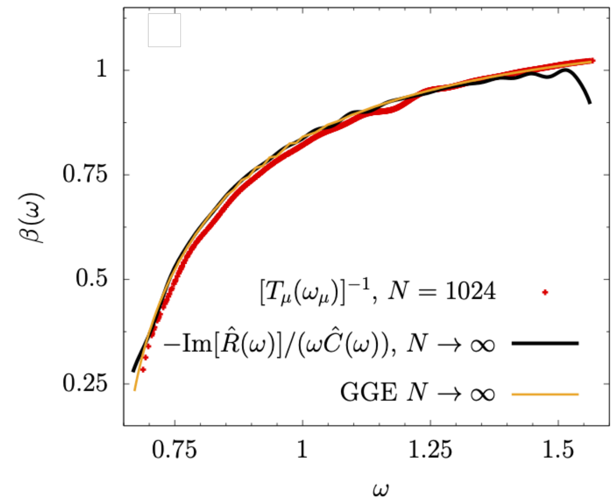
$$\hat{R}(\omega) = \text{F.T. } R_{\rm st}(t-t')$$

Fluctuation-dissipation thm

$$-\frac{\text{Im}\hat{R}(\omega)}{\omega\hat{C}(\omega)} = \beta$$

Frequency domain FDR

The T_μs from the FDR at $\omega_\mu = [(z_f - \lambda_\mu)/m]^{1/2}$ **Sector I**



A way to measure the mode temperatures with a single measurement

$$\beta_{\rm eff}(\omega_\mu) = -\mathrm{Im}\hat{R}(\omega_\mu)/(\omega_\mu \hat{C}(\omega_\mu)) = \beta_\mu$$

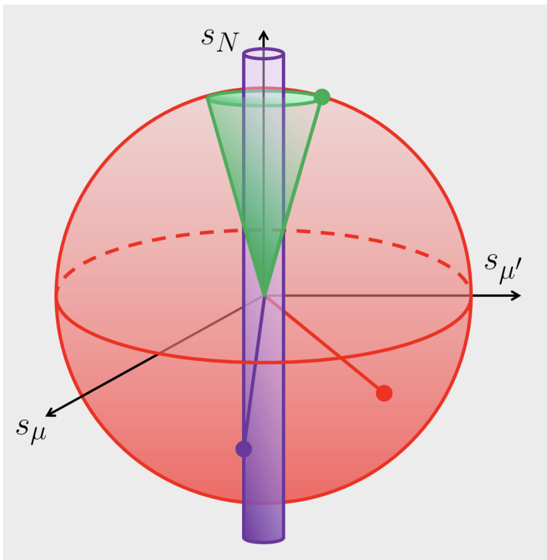
No "partial equilibration" contradiction from the effective temperature perspective. The modes are uncoupled, they do not exchange energy, and can then have different T_μs

Idea in **LFC, de Nardis, Foini, Gambassi, Konik & Panfil 17** for **quantum**

Summary

Goals achieved

In the late times limit taken after the large N limit



We solved

– the *global dynamics* with Schwinger-Dyson/ DMFT eqs.

– the *mode dynamics* with parametric oscillator techniques

of the *Soft Neumann model*

With the *GGE measure*

$$\rho_{\rm GGE}(\vec{s},\vec{p}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_\mu \gamma_\mu I_\mu(\vec{s},\vec{p}}$$

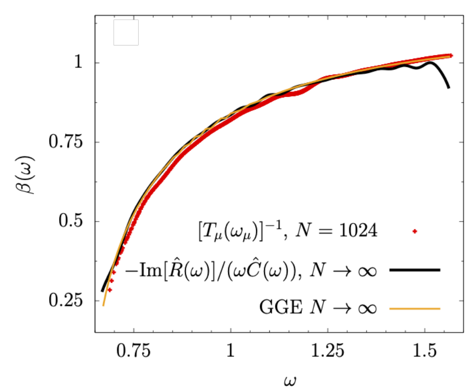
– we calculated & proved

$$\langle s_\mu^2 \rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE} - \lambda_\mu} = \overline{\langle s_\mu^2(t)\rangle_{i.c.}}$$

$$\langle p_\mu^2 \rangle_{\rm GGE} = T_\mu = \overline{\langle p_\mu^2(t)\rangle_{i.c.}}$$

obtaining also $\{T_\mu, \gamma_\mu\}$

– The $\{T_\mu\}$ are accessed by the FDR



$$\beta_{\rm eff}(\omega_\mu) = -\text{Im}\,\frac{\hat{R}(\omega_\mu)}{(\omega_\mu \hat{C}(\omega_\mu))} = \beta_\mu$$

Goals achieved

– We exhibited a *classical interacting integrable model*, the *Soft Neumann model* or *Hamiltonian spherical SK model*, the *quench dynamics* of which can be elucidated with different means.

– Rich *dynamic phase diagram*.

– We managed to calculate (mostly analytically) the GGE measure or, better said, all *GGE averaged local observables*

– We showed that *asymptotic dynamic* and *GGE averages* coincide for $N \to \infty$

– For a special set of parameters the GGE measure reduces to the GB one.

– We can also study the *fluctuations of the constraints* to prove that in I, II, III (with symmetry broken initial conditions) the **Soft Neumann Model $\equiv$ the Neumann model** with the strict spherical constraint.

Goals achieved

– We exhibited a *classical interacting integrable model*, the *Soft Neumann model* or *Hamiltonian spherical SK model*, the *quench dynamics* of which can be elucidated with different means.

– Rich *dynamic phase diagram*.

– We managed to calculate (mostly analytically) the GGE measure or, better said, all *GGE averaged local observables*

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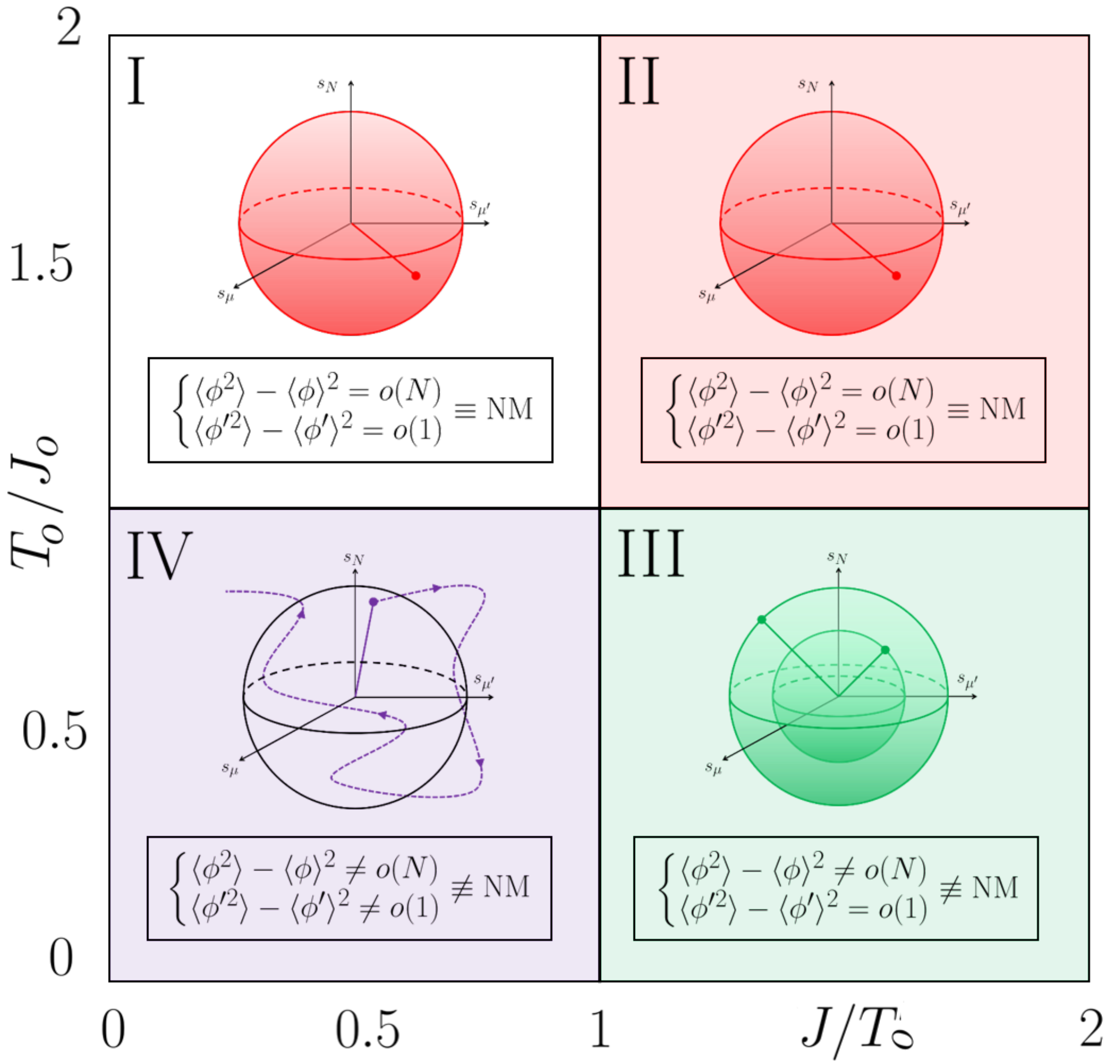
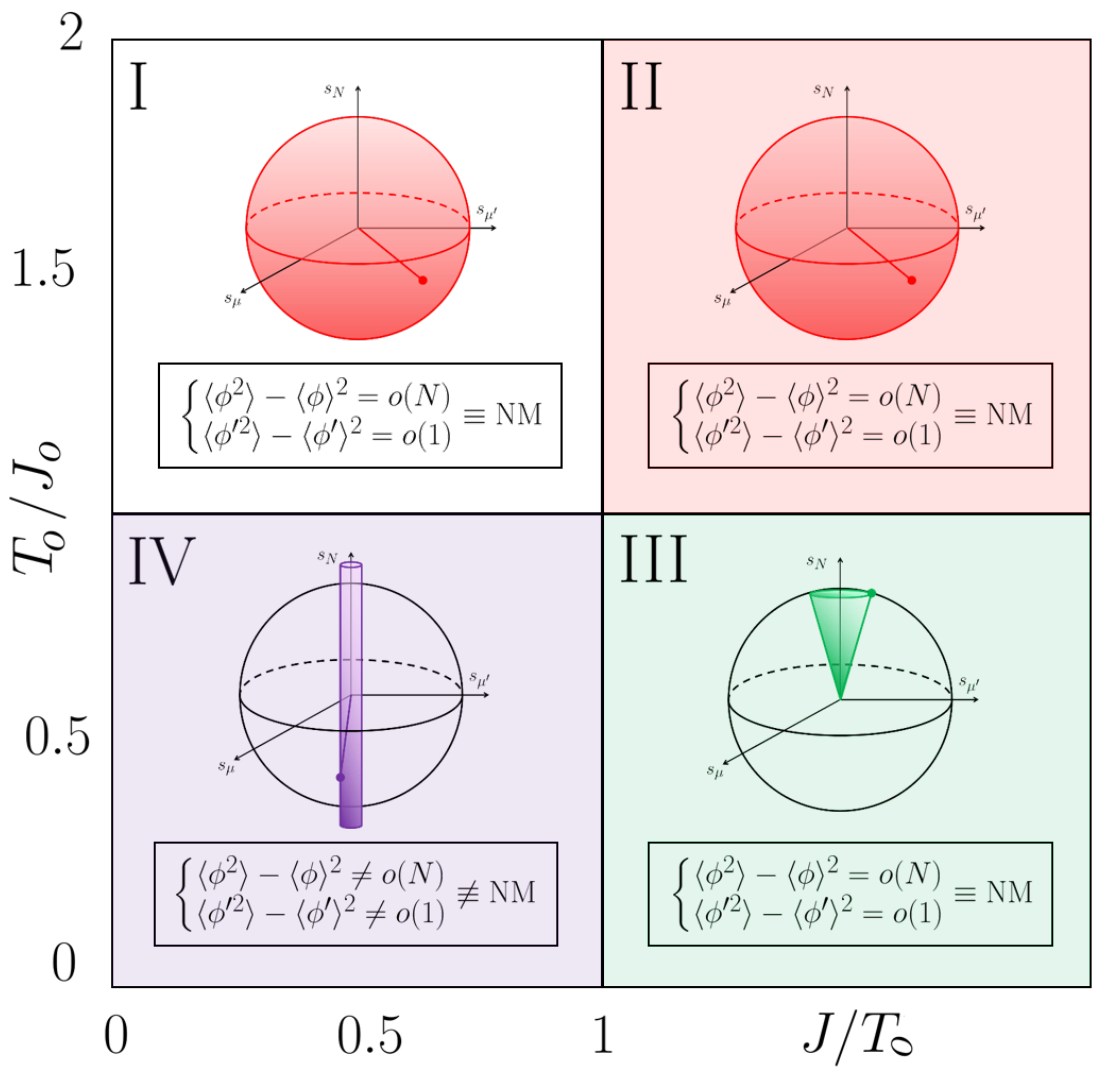
– For a special set of parameters the GGE measure reduces to the GB one.

What next ? What about the **spherical ferromagnet** ? Problems with degeneracy of eigenvalues ? Local spatial structure would be accessible.

Fluctuations



Fluctuations

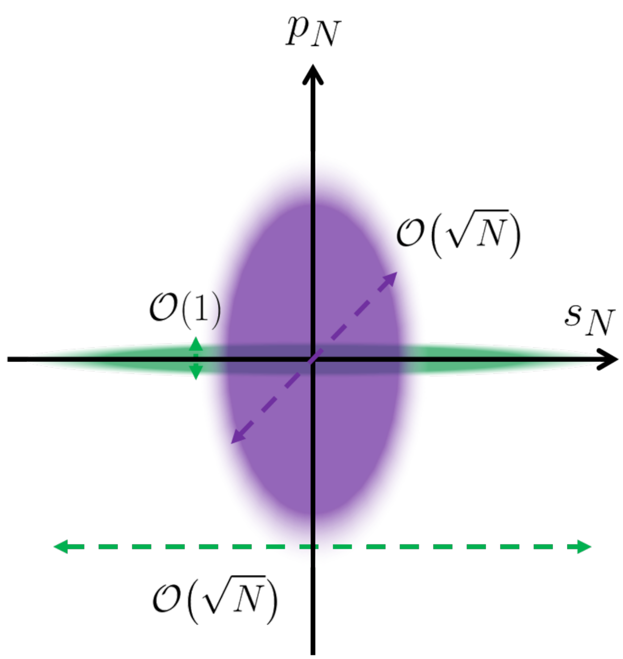


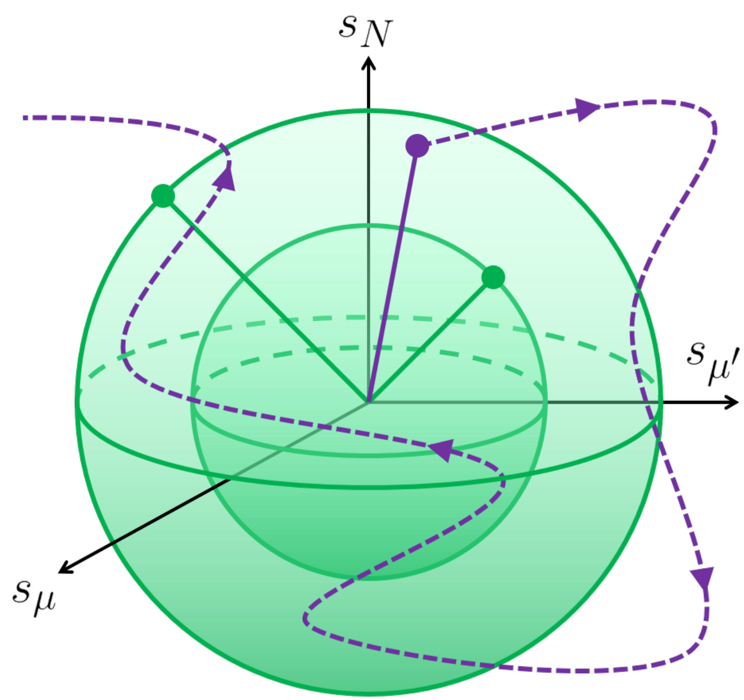
The dynamics on the sphere

and the GGE averages on the Nth mode phase space

Sector III

Sector IV





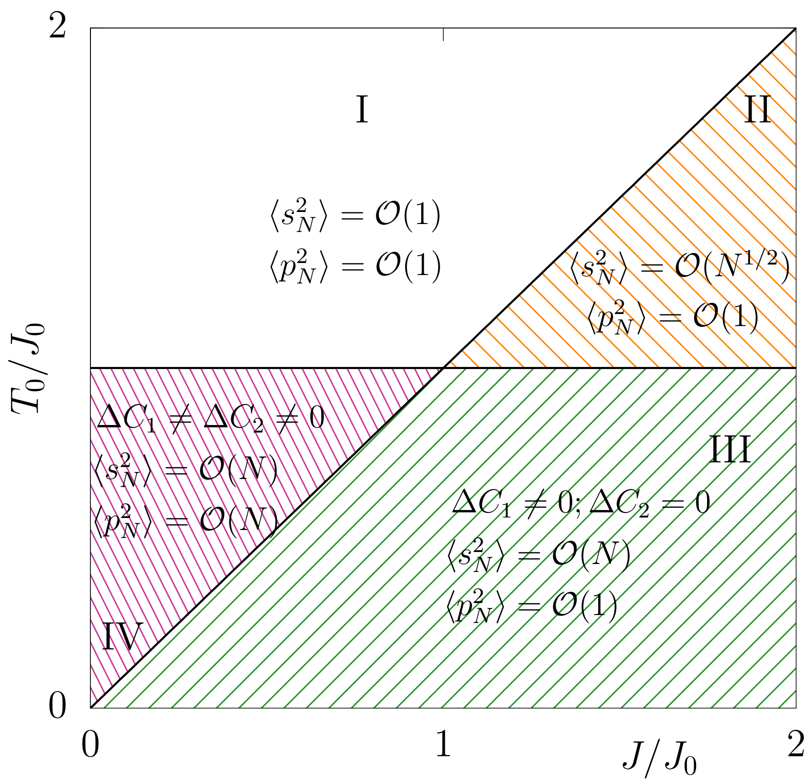
$z > \lambda_N$

$z = \lambda_N$

Fluctuations

of the primary and secondary constraints

$$\phi : \sum_{\mu=1}^{N} s_\mu^2 - N = 0 \qquad \phi' : \sum_{\mu=1}^{N} s_\mu p_\mu = 0$$



when the spherical constraint is imposed on average

Integrals of motion

From microcanonical to canonical ?

The microcanonical GGE measure is ensured **Yuzbashyan 16**

$$\rho^{\text{micro}}_{\text{GGE}}(\{\mathcal{I}_\nu\}) = c \prod_{\mu=1}^N \delta(I_\mu(\{s_\nu, p_\nu\}) - \mathcal{I}_\mu)$$

Two conditions to prove canonical from microcanonical :

(i) additivity of the energy $\to I_\mu = I_\mu^{(1)} + I_\mu^{(2)}$

(ii) extensivity of the energy $\to I_\mu = \mathcal{O}(N)$

e.g., **Campa, Dauxois, Ruffo 09** on in/equivalence of ensembles

not satisfied in our model by the I_μ's, but maybe combinations ?

Still, let's try $\rho_{\text{GGE}}(\{s_\nu, p_\nu\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{s_\nu, p_\nu\})}$

scaling with N $\boxed{\sum_{\mu=1}^N \gamma_\mu I_\mu = \mathcal{O}(N)}$ if $\gamma_\mu = \mathcal{O}(1)$

Integrals of motion

From microcanonical to canonical ?

The microcanonical GGE measure is ensured **Yuzbashyan 16**

$$\rho^{\rm micro}_{\rm GGE}(\{\mathcal{I}_\nu\}) = c\prod_{\mu=1}^N \delta(I_\mu(\{s_\nu, p_\nu\}) - \mathcal{I}_\mu)$$

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not satisfied in our model I_μ by I_μ but maybe combinations ?

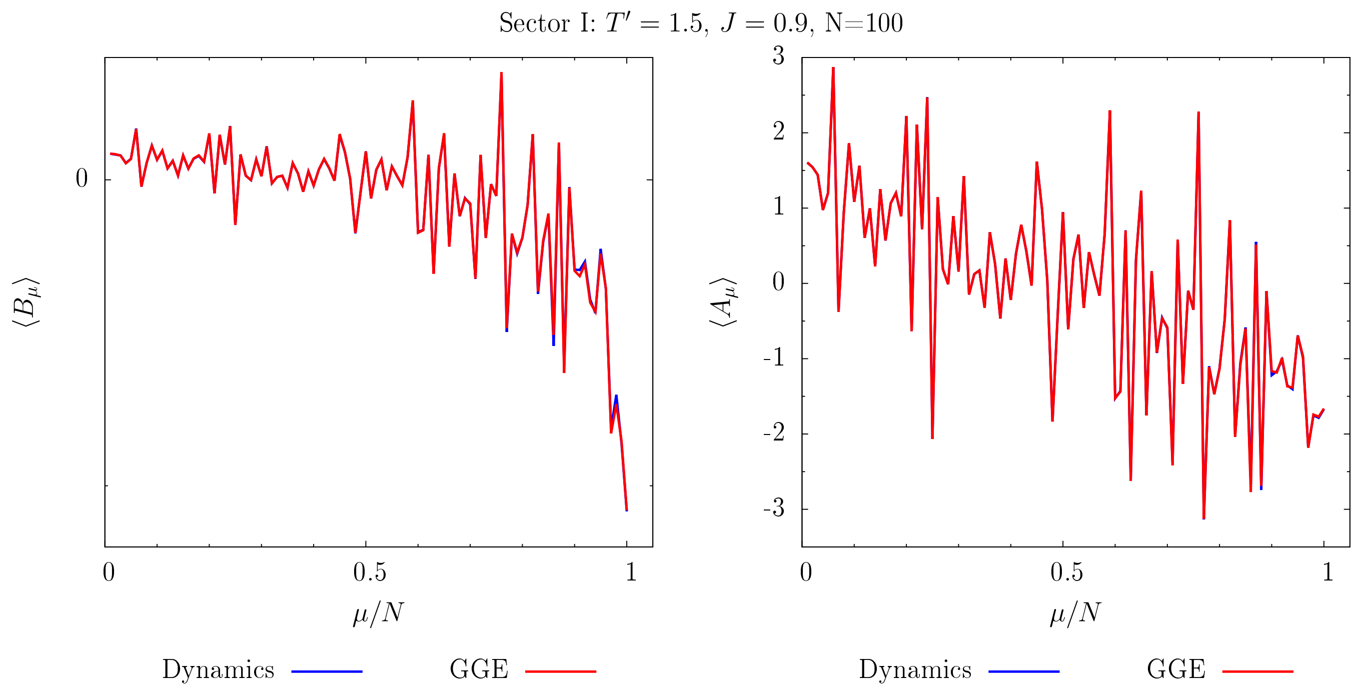
Still, let's try $\rho_{\rm GGE}(\{s_\nu, p_\nu\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{s_\nu,p_\nu\})}$

GB equilibrium for no quench $\boxed{\gamma_\mu = -\frac{\beta_0\lambda_\mu}{2}}$ since $\sum_{\mu=1}^N \frac{\lambda_\mu}{2} I_\mu = -H$

GGE calculation

The mean-fields

$\langle A_\mu^{(p)} \rangle_{\rm GGE}$ $\langle A_\mu^{(s)} \rangle_{\rm GGE}$



$\overline{A_\mu^{(p)}(t)}$ $\overline{A_\mu^{(s)}(t)}$

Classical dynamics

From spins to a particle moving on an N-dimensional sphere

Coordinate-momenta pairs $\{\vec{s}, \vec{p}\}$ and Hamiltonian (const w/Lagrange mult.)

$$H_J^{(z)} = E_{\rm kin}(\vec{p}) + V_J(\vec{s}) + \frac{z(\vec{s},\vec{p})}{2} \sum_{i=1}^N (s_i^2 - N)$$

with the kinetic energy $E_{\rm kin}(\vec{p}) = \frac{1}{2m} \sum_{i=1}^N p_i^2$

Newton-Hamilton equations

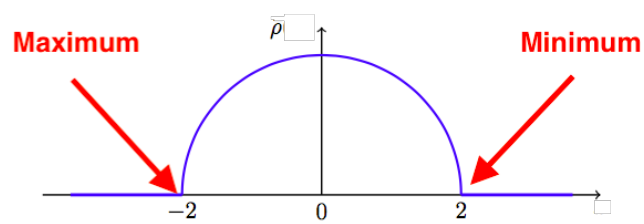
$$\dot{s}_i = p_i/m \qquad \dot{p}_i = -\delta V_J(\vec{s})/\delta s_i - z(\vec{s},\vec{p}) s_i$$

The effective potential energy landscape $2V_J^{(z)}(\vec{s}) = -\sum_{i \neq i} J_{ij} s_i s_j + z(s^2 - N)$ has

$\mu = 1, \dots, N$ saddles (including min/max)

the N eigenvectors $\vec{v}_\mu$ of the J_{ij} matrix with

$z = \lambda_\mu$ & pot. energy density $e_{\rm pot}^{(\mu)} = -\lambda_\mu/2$



$-2J$ $\qquad$ $2J = \lambda_N$

$$z(\vec{s},\vec{p}) = 2\left[e_{\rm kin}(\vec{p}) - v_J(\vec{s})\right]$$

Conservative dynamics

on average over randomness & the initial measure

In the $N \to \infty$ limit exact causal Schwinger-Dyson (DMFT) equations

$$(m\partial_t^2 - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

$$+\frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_a(t,0)C_a(t_w,0)$$

$$(m\partial_t^2 - z_t)C_a(t,0) = \int dt' \, \Sigma(t,t')C_a(t',0) + \frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_b(t,0)Q_{ab}$$

$a = 1, \ldots, n \to 0$, replica method to deal with $e^{-\beta_0 H_{J_0}^{(z)}}$ and fix Q_{ab}

Initial cond **Houghton, Jain, Young 86, Franz, Parisi 95, Barrat, Burioni, Mézard 96**

Conservative dynamics

on average over randomness and the initial measure

In the $N \to \infty$ limit exact causal Schwinger-Dyson (DMFT) equations with the post-quench self-energy and vertex

$$D(t,t_w) = J^2\, C(t,t_w)$$
$$D_a(t,0) = J^2\, C_a(t,0)$$
$$\Sigma(t,t_w) = J^2\, R(t,t_w)$$

with

$$NC(t,t') = \sum_i [\langle s_i(t) s_i(t') \rangle_{i.c}]_J$$
$$NC_a(t,0) = \sum_i [\langle s_i(t) s_i(0) \rangle_{i.c}]_J$$
$$NR(t,t') = \sum_i [\langle \frac{\delta s_i(t)}{h_i(t')}|_{\vec{h}=0} \rangle_{i.c}]_J$$

The Lagrange multiplier is fixed by $C(t,t) = 1 \Rightarrow z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)]$

Initial conditions:
- Disordered $Q_{ab} = \delta_{ab}$
- Condensed $Q_{ab} = (1-q)\delta_{ab} + q$

Solvable numerically & analytically at long times

Averaged integrals of motion

Properties, scaling and parameter dependence

'Sum rules' $\sum_\mu I_\mu = N$ and $\sum_\mu \lambda_\mu I_\mu = -2H_J$

In the $N \to \infty$ limit

$$\lim_{N\to\infty} I_\mu = I(\lambda) = s^2(\lambda) + \frac{1}{m} \int d\lambda' \, \rho(\lambda') \, \frac{[s(\lambda)p(\lambda') - s(\lambda')p(\lambda)]^2}{\lambda - \lambda'}$$

For GB equilibrium initial conditions

$$\langle I(\lambda)\rangle_{i.c.} = \langle s^2(\lambda^{(0)})\rangle_{i.c.} + \frac{1}{m} \int d\lambda' \rho(\lambda') \, \frac{\langle s^2(\lambda^{(0)})\rangle_{i.c.} \langle p^2(\lambda^{(0)\prime})\rangle_{i.c.} + \lambda^{(0)} \leftrightarrow \lambda^{(0)\prime}}{\lambda - \lambda'}$$

with $\langle s^2(\lambda^{(0)})\rangle_{i.c.} = k_B T_0/(z_0 - \lambda^{(0)})$ and $\langle p^2(\lambda^{(0)})\rangle_{i.c.} = m\, k_B T_0$

and N-th mode :

- **Extended** $\langle I_N\rangle_{i.c.} = \mathcal{O}(1)$
- **Condensed** $\langle I_N\rangle_{i.c.} = \mathcal{O}(N)$