Dynamic Mean-Field Theory

aging, weak long-term memory & time reparametrization invariance

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Introductory talk

Plan

— Many-body systems in interaction

some examples

— Collective dynamics

e.g. domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation

self-correlation and linear response

— Dynamic Mean-Field Theory

e.g., static Curie-Weiss

coarsening & glassy dynamics

separation of time scales & effective temperatures

reparametrization invariance, sigma model & fluctuations

yesterday's application : the integrable case

Many-body Systems in Interaction

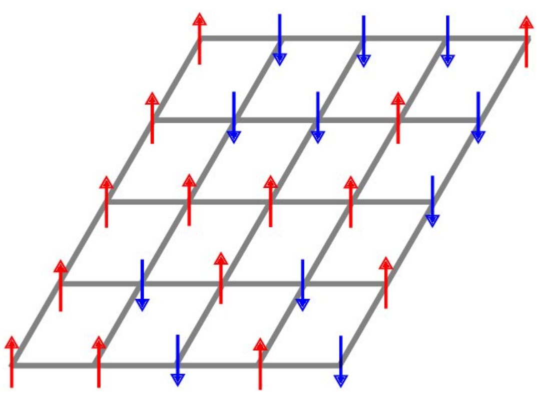
Some examples

Many-body systems

Some examples

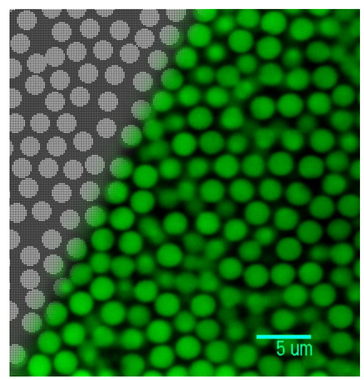
Ferromagnetic Ising Model

$$\mathcal{V} = -J \sum_{\langle ij \rangle} s_i s_j$$



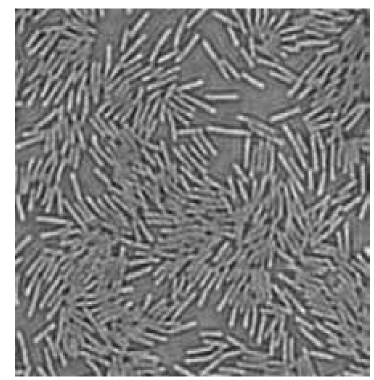
Particles in Interaction

$$\mathcal{V} = \sum_{i \neq j} V(r_{ij})$$



Active Matter

$$\vec{\mathcal{F}}_i \neq -\vec{\nabla}_i \mathcal{V}$$



In physical systems the action-reaction principle is respected, in other examples it is not

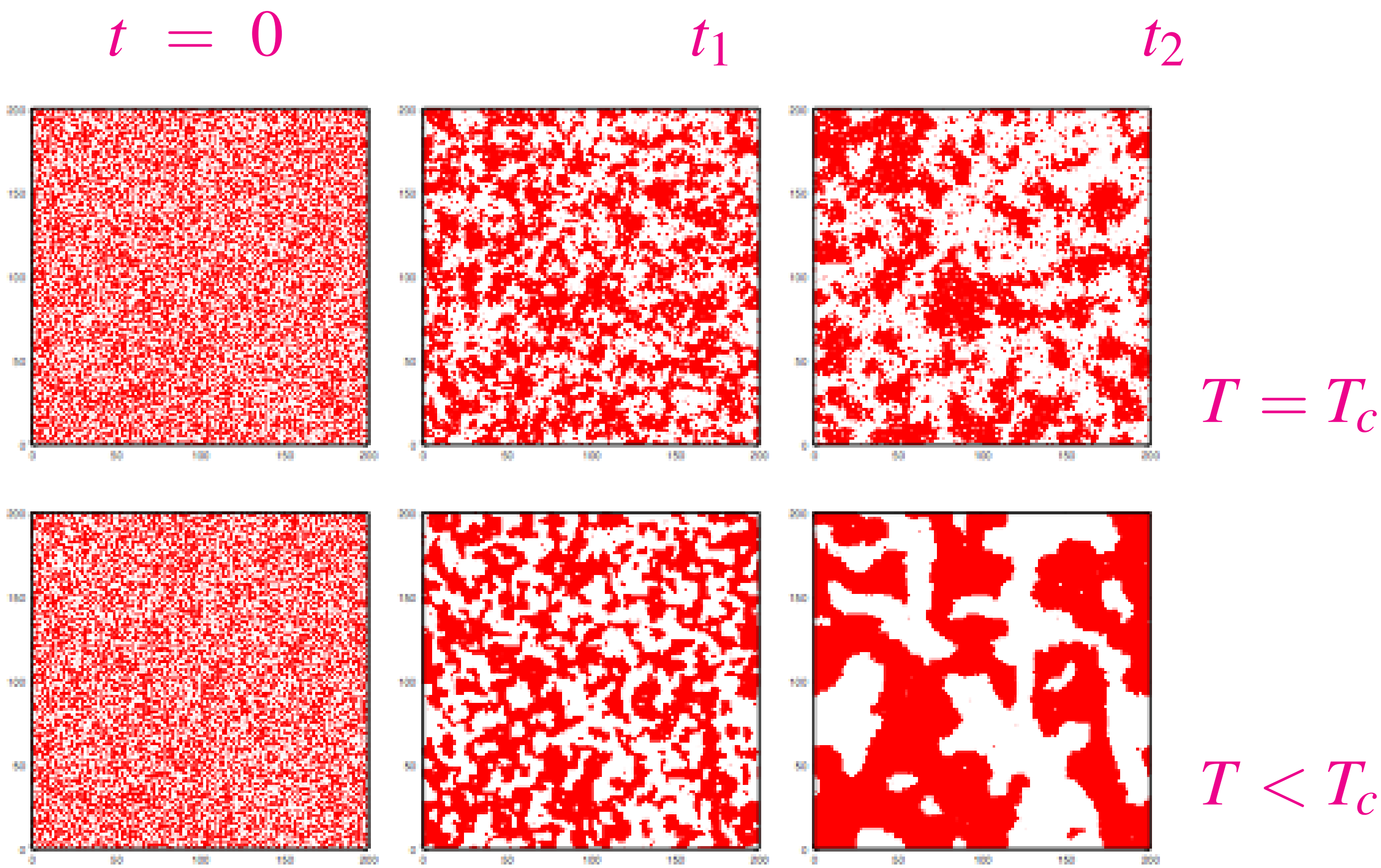
Also many examples beyond physics, like **ecosystems, markets**, etc. $\vec{\mathcal{F}}_{i \to j} \neq \vec{\mathcal{F}}_{j \to i}$

Collective dynamics

the simplest example, coarsening

$2d$ Ising model

Snapshots after an instantaneous quench from $T_0 \to \infty$ to $T \leq T_c$



At $T = T_c$ **critical dynamics** At $T < T_c$ **coarsening**

A certain number of **interfaces** or **domain walls** in the last snapshots.

Phenomenon

In both cases one sees the growth of 'red and white' patches and

interfaces surrounding such geometric domains.

Spatial regions of **local equilibrium** (with vanishing, at T_c, or non-vanishing, at $T < T_c$, order parameter) grow in time and

a single **growing length** $\mathcal{R}(t, T/J)$ can be identified and it is at the heart of *dynamic scaling*.

Global observables

Two-time correlation and linear responses

Two-time dependencies

Self-correlation and linear response

The two-time self correlation and integrated linear response

$$C(t,t_w) \equiv \frac{1}{N}\sum_i [\langle s_i(t) s_i(t_w)\rangle]$$

$$\chi(t,t_w) \equiv \frac{1}{N}\sum_i \int_{t_w}^{t} dt'\, R(t,t') = \frac{1}{N}\sum_i \int_{t_w}^{t} dt' \; [\frac{\delta\langle s_i(t)\rangle_h}{\delta h_i(t')}\Big|_{h=0}]$$

Extend the notion of **order parameter**

They are not related by FDT out of equilibrium

Magnetic notation but general

The averages are thermal (and over initial conditions) $\langle \dots \rangle$

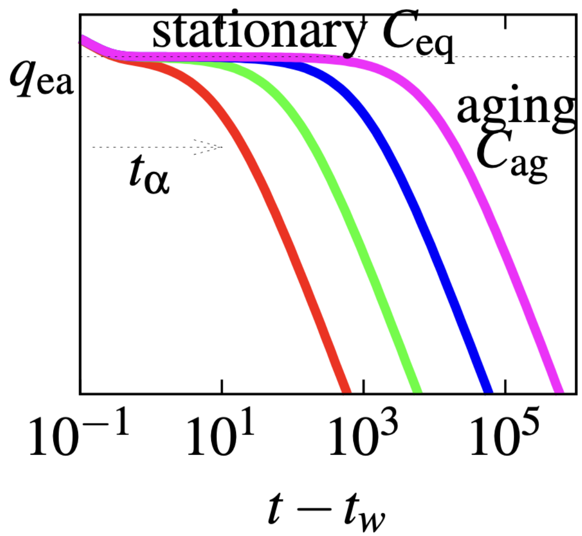
and over quenched randomness $[\dots]$ (if present)

t_w waiting-time and t measuring time

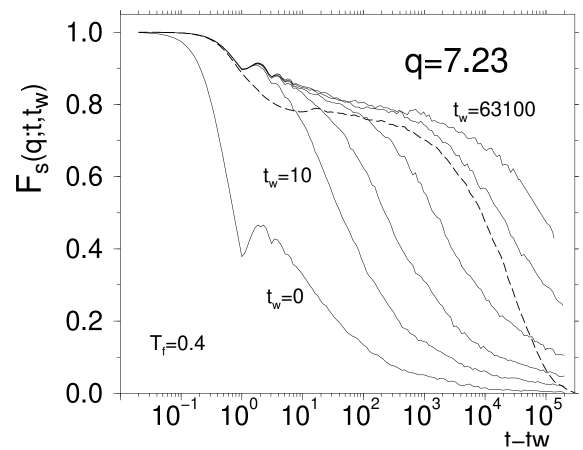
Two-time self-correlation

Also in glassy systems with no clear order growth

$T < T_c$



Lennard-Jones mixtures **Kob & Barrat 97**



Two scales $C_{eq}(t - t_w) + C_{ag}(t, t_w)$

$$C_{eq}(t - t_w) \sim f_{eq}\left(\frac{e^{-t/t_{eq}}}{e^{-t_w/t_{eq}}}\right) \qquad C_{ag}(t, t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

Also found in glassy systems for which there is no clear visualization of $\mathcal{R}$

Aging

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

Breakdown of stationarity of the self-correlation

$$C(t,t_w) \neq C(t-t_w)$$

In each regime, equilibrium and aging, scaling*

$$C(t,t_w) = C\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

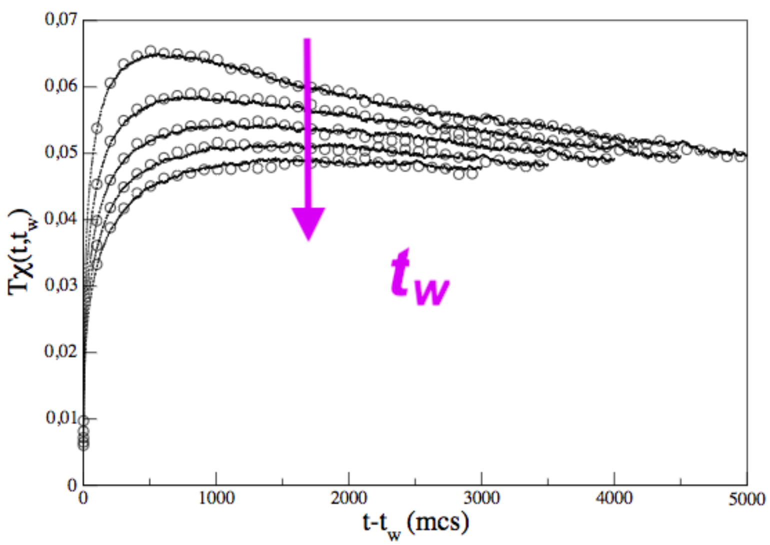
*the scaling form can be proven from general properties of temporal correlation functions

No obvious interpretation of $\mathcal{R}(t)$ in aging **glassy** systems

Two-time linear response

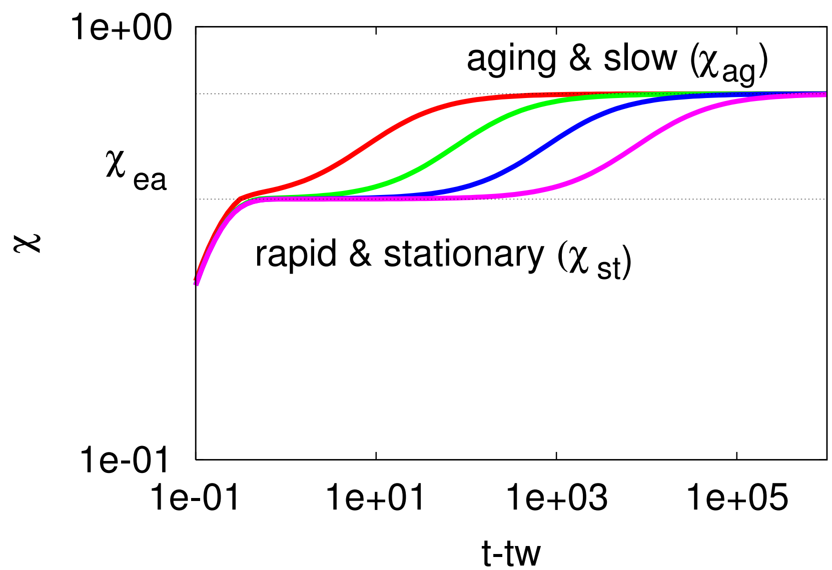
An important difference

Coarsening



Lippiello, Corberi & Zannetti 05

Glassy



Sketch **Chamon & LFC 07**

Weak long-term memory in the glassy but not in the coarsening problem.

Just the stationary part survives asymptotically, contrary to the sketch on the right valid for glasses & spin-glasses.

Memory

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

Breakdown of stationarity of the integrated linear response

$$\chi(t,t_w) \neq \chi(t-t_w)$$

In the aging regime, difference between coarsening & glassy

$$\chi(t,t_w) = t^{-a}\,\chi\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right) \quad \text{or} \quad \chi(t,t_w) = \chi\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

Coarsening — Glassy

(but no obvious interpretation of $\mathcal{R}(t)$ in aging **glassy** systems)

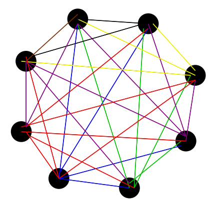
Mean-Field Modelling

Usual Curie-Weiss for PM-FM

More unusual for glasses

The Curie-Weiss model

Very well-known : for the equilibrium PM-FM phases

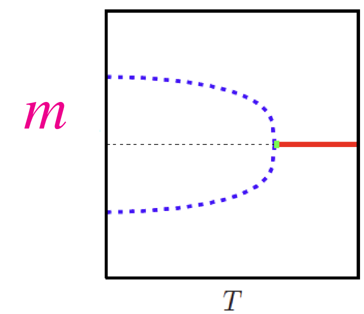


$$\mathcal{V} = -\frac{J}{N} \sum_{i \neq j} s_i s_j$$

Fully connected interactions

Ferromagnetic coupling $J > 0$

Ising spins $s_i = \pm 1$ with $i = 1, \dots, N$



The PM & FM phases are well captured <u>but not</u>

the details of the critical behavior

Similar strategy in the context of :

spin-glass models, and **interacting particle systems** (large d)

In problems beyond physics, <u>fully-connectedness</u> can be <u>the</u> precise description

Glassy mean-field models

Classical p-spin spherical

Potential energy

$$\mathcal{V} = - \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} \qquad p \text{ integer}$$

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

(over-damped) **Langevin dynamics** for continuous spins $s_i \in \mathbb{R}$

coupled to a white bath $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t-t')$

$$\gamma \frac{ds_i}{dt} = -\frac{\delta \mathcal{V}}{\delta s_i} + z_t s_i + \xi_i$$

z_t is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} s_i^2 = N$

$p = 2$ mean-field **domain growth**

$p \geq 3$ RFOT modelling of **fragile glasses**

Dynamic equations

Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\gamma\partial_t - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right] + 2\gamma k_B T R(t_w,t)$$

$$(\gamma\partial_t - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

where Σ and D are the self-energy and vertex. For the p spin models

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \qquad \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t') R(t,t')$$

The Lagrange multiplier z_t is fixed by $C(t,t) = 1$. Random initial conditions.

(Average over randomness already taken ; later, interest in noise-induced fluctuations)

See **Sompolinsky & Zippelius 82, LFC & Kurchan 93**

Dynamic equations

Generalizations - minimal changes

– Coloured baths, e.g.

$$\gamma\partial_t \to \int_0^t dt'\,\Gamma(t-t')\partial_{t'}$$

– Non-reciprocal interactions $\mathcal{F}_{i\to j} \neq \mathcal{F}_{j\to i}$: self-energy and vertex non trivially related

$$\Sigma(C,R) \neq D'(C)\,R$$

– Special initial conditions can be selected – with some added terms to the eqs.

– Closed classical problems $\gamma = 0$ and Newton dynamics

$$\gamma\partial_t - z_t \to m\partial_t^2 - z_t$$

– Quantum problems : change in differential operator, bath kernels, self-energy & vertex

(Average over randomness already taken ; later, interest in noise-induced fluctuations)

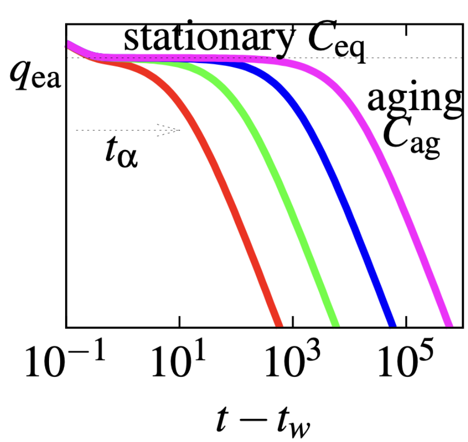
Many examples in **LFC 23**

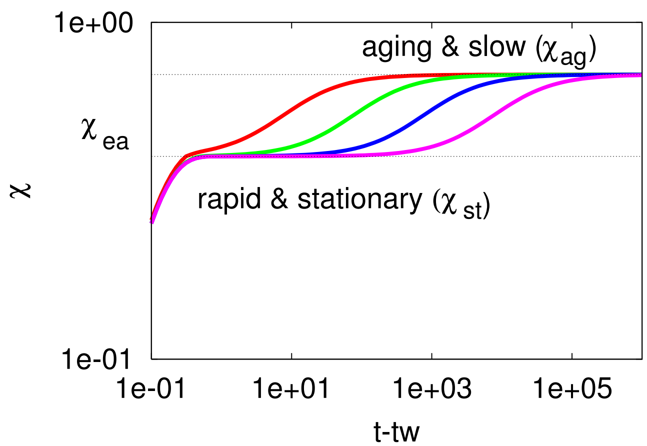
Some (surprising) Predictions

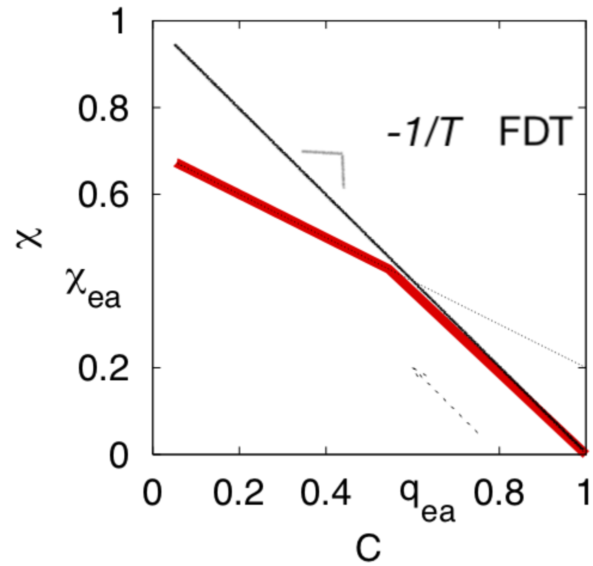
from coarsening & glassy mean-field models

Glassy Dynamics

Fluctuation-dissipation relation : parametric plot







Convergence to $\chi(C)$

two linear relations for $C \lessgtr q_{ea}$

Analytic solution to the p-spin model **LFC & J. Kurchan 93**

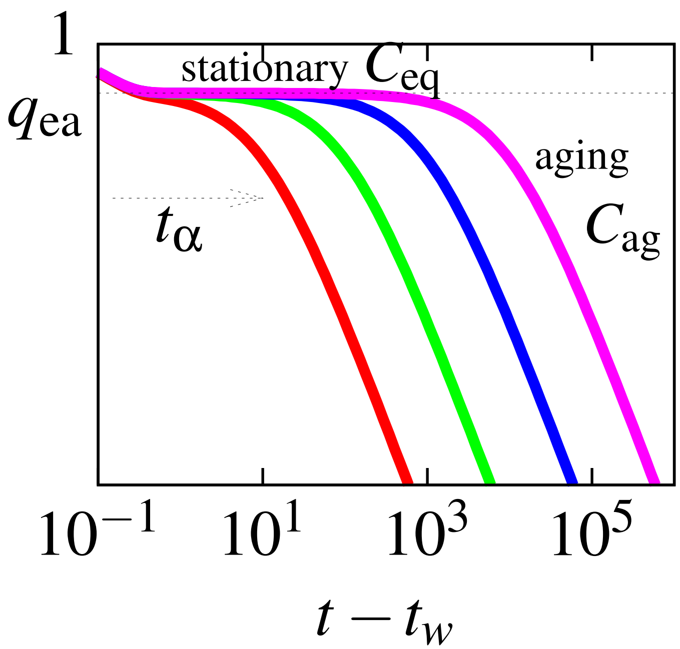
& effective temperature interpretation **LFC, Kurchan & Peliti 97**

Time reparametrization invariance

Separation of time-scales

In the long t_w limit

Fast $t - t_w \ll t_w$



The aging part is slow

Slow $\mathcal{R}(t)/\mathcal{R}(t_w) = O(1)$

$$C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t,t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t,t_w) \ll C_{ag}(t,t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{ea}$:

Approx. asymptotic time-reparametization invariance $\boxed{t \to h(t)}$

Time reparametrization

Example : the equation $(\partial_t - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w)$

- Focus on times such that $z_t \to z_\infty$, $C \sim C_{\rm ag}$ and $R \sim R_{\rm ag}$
- Separation of time-scales (drop $\partial_t R$ and approximate the integral) :

$$-z_\infty R_{\rm ag}(t,t_w) \sim \int dt' \, D'[C_{\rm ag}(t,t')] R_{\rm ag}(t,t') \, R_{\rm ag}(t',t_w) \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \qquad \left\{ \begin{array}{l} C_{\rm ag}(t,t_w) \to C_{\rm ag}(h_t,h_{t_w}) \\ R_{\rm ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} R_{\rm ag}(h_t,h_{t_w}) \end{array} \right.$$

with h_t positive and monotonic leaves eq. (1) **invariant** :

$$-z_\infty R_{\rm ag}(h_t,h_{t_w}) \sim \int dh_{t'} \, D'[C_{\rm ag}(h_t,h_{t'})] R_{\rm ag}(h_t,h_{t'}) \, R_{\rm ag}(h_{t'},h_{t_w})$$

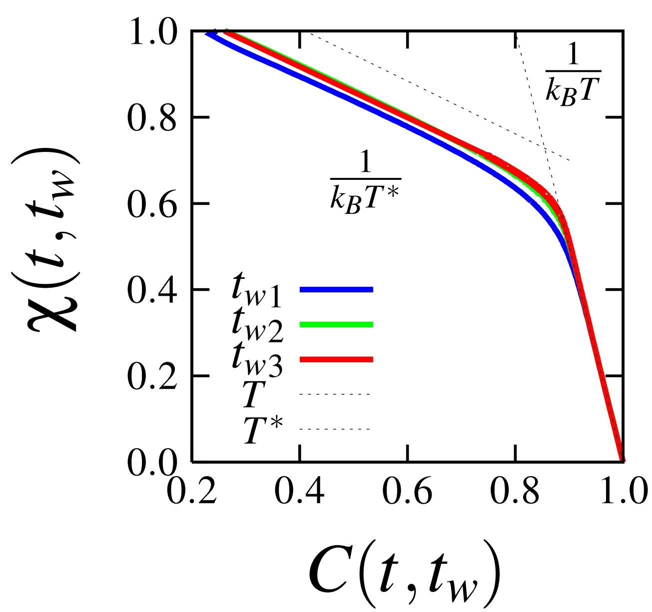
Time reparametrization

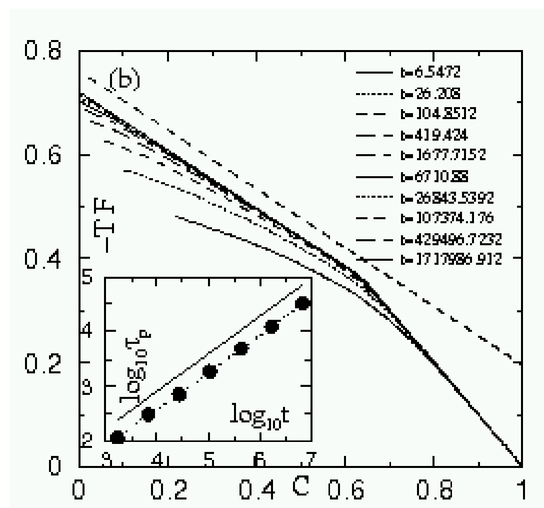
One can compute analytically f_{ag} and $\chi_{ag}(C_{ag})$

for times t and t_w such that $C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$, *e.g.*

$$\chi_{ag}(t,t_w) \sim \frac{1-q_{ea}}{T} + \frac{1}{T^*}\left[q_{ea} - C_{ag}(t,t_w)\right]$$

but not the 'clock' $\mathcal{R}(t)$





Kim & Latz 00 very precise numerical solution

Implications on Fluctuations

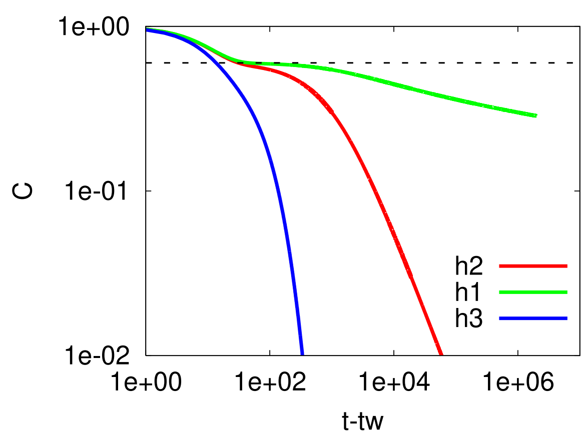
Leading fluctuations

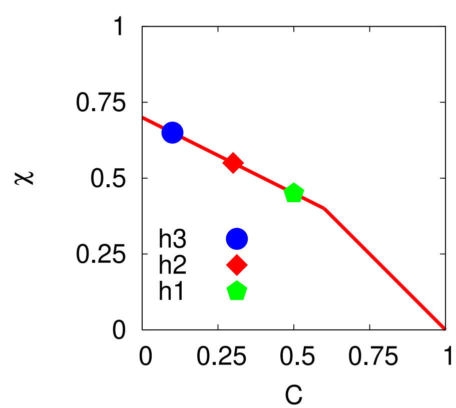
Global to local correlations & linear responses

$$C_{\rm ag}(t,t_w) \approx f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow C^{\rm ag}_{\vec r}(t,t_w) \sim f_{\rm ag}\left(\frac{h_{\vec r}(t)}{h_{\vec r}(t_w)}\right)$

Ex. $h_{\vec r_1} = \frac{t}{t_0}$, $h_{\vec r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec r_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions





Castillo, Chamon, LFC, Iguain & Kennett 02, 03

Chamon, Charbonneau, LFC, Reichman & Sellitto 04

Jaubert, Chamon, LFC & Picco 07

Conclusions on Fluctuations

Fluctuations

(Annoying) global time-reparametrization invariance $t \to h(t)$ in models in which

- $C_{ag}(t,t_w) \gg \partial_t C_{ag}(t,t_w)$ (slow dynamics)
- $\chi_{ag}(t,t_w) \gg \partial_t \chi_{ag}(t,t_w)$ (weak long-term memory)

and finite effective temperature $T_{\rm eff} < +\infty$ **Chamon, LFC & Yoshino 06**

Reason for the large dynamic fluctuations (heterogeneities) $h(\vec{r},t)$

Effective action for $\varphi(\vec{r},t)$ in $h(\vec{r},t) = e^{-\varphi(\vec{r},t)}$ **Chamon & LFC & Yoshino 07**

Quantum : the rapid equilibrium regime is modified but the slow aging one is classical controlled by a $T_{\rm eff} > 0 \Rightarrow$ the same applies

LFC & Lozano 98, 99 ; Kennett & Chamon 00, 01

Each problem

with its own peculiarities

& much more to say !

Dynamic equations

Conservative dynamics - closed classical systems

In the $\boxed{N \to \infty}$ limit exact causal Schwinger-Dyson equations

$$(m\partial_t^2 - z_t) R(t,t_w) = \int dt' \, \Sigma(t,t') R(t',t_w) + \delta(t-t_w)$$

$$(m\partial_t^2 - z_t) C(t,t_w) = \int dt' \left[\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t') \right]$$

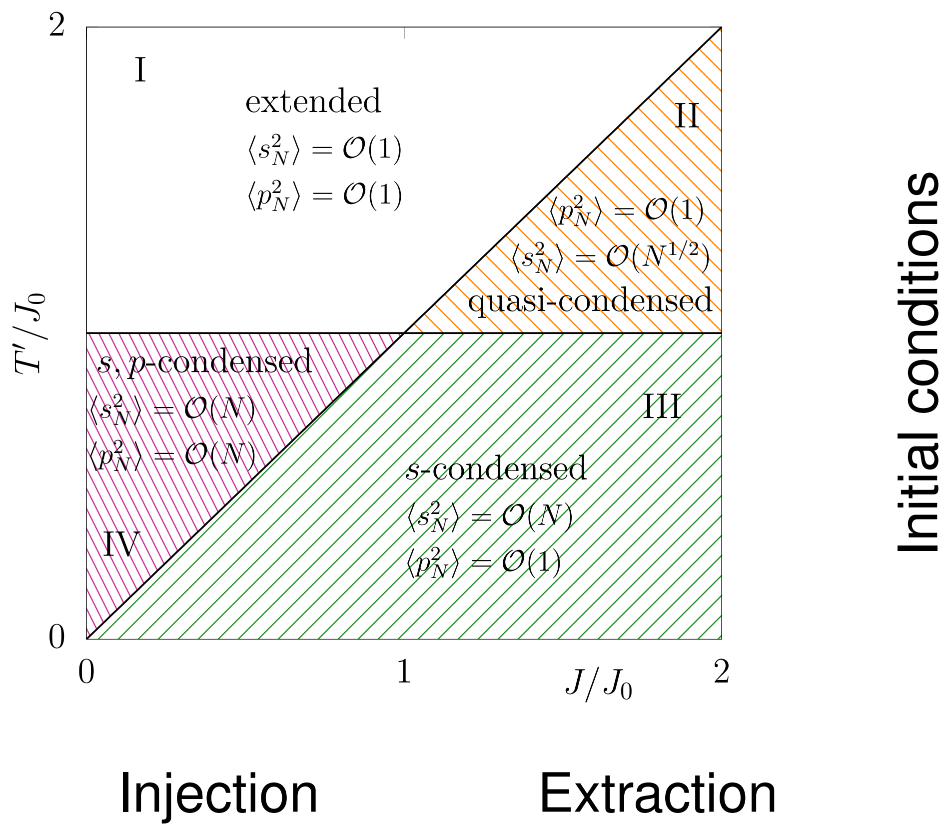
$$\boxed{+ \frac{\beta_0 J_0}{J} \sum_{a=1}^{n} D_a(t,0) C_a(t_w,0)}$$

$$\boxed{(m\partial_t^2 - z_t) C_a(t,0) = \int dt' \, \Sigma(t,t') C_a(t',0) + \frac{\beta_0 J_0}{J} \sum_{a=1}^{n} D_b(t,0) Q_{ab}}$$

$a = 1, \ldots, n \to 0$, replica method to deal with $e^{-\beta_0 \mathcal{H}_0}$ and fix Q_{ab}

The $p = 2$ integrable model

The phase diagram



For all parameters $\lim_{t \gg t_{\rm st}} \lim_{N\to\infty} \overline{\langle s_\mu^2(t) \rangle_{i.c.}} = \langle s_\mu^2 \rangle_{\rm GGE}$ etc.

Barbier, LFC, Lozano, Nessi, Picco & Tartaglia 18-22

Conclusions

Some other applications/extensions

– Large d approach to glassiness

Agoritsas, Charbonneau, Kurchan, Maimbourg, Parisi, Urbani & Zamponi, ...

– Ecological models

Altieri, Biroli, Bunin, Cammarotta & Roy, ...

– Neural networks & non-reciprocal interactions

Crisanti & Sompolinsky 80s, Brunel et al.

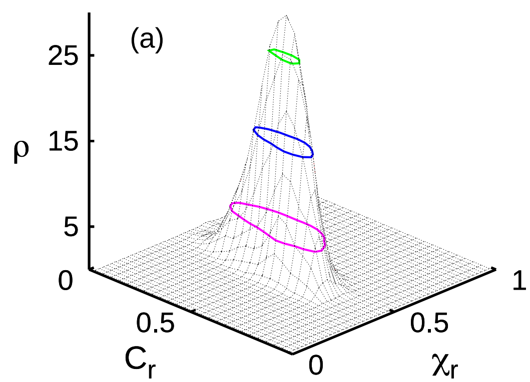
LFC, Kurchan, Le Doussal & Peliti 90s, Berthier, Barrat & Kurchan 00s

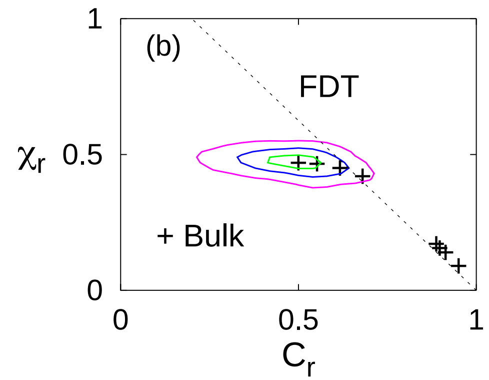
Biroli, Mignacco, Urbani, Zdeborová, ...

Local correlations & responses

$3d$ Edwards-Anderson spin-glass

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} s_i(t)s_i(t_w)\;, \quad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} \int_{t_w}^{t} dt' \; \left.\frac{\delta s_i(t)}{\delta h_i(t')}\right|_{h=0}$$





+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 $t\ (> t_w)$

ρ corresponds to the maximum t yielding the smallest C (left-most +)

Castillo, Chamon, LFC, Iguain, Kennett 02

Kinetically constrained models + Charbonneau, Reichman & Sellitto 04

Sigma Model

Conditions & expression

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad C_{ag}(\vec{r},t,t_w) = f_{ag}\left(e^{-\int_{t_w}^{t} dt'\, \partial_{t'}\varphi(\vec{r},t')}\right)$$

i. The action must be invariant under a global time reparametrization $t \to h(t)$.

ii. If our interest is in short-ranged problems, the action must be written using local terms. The action can thus contain products evaluated at a single time and point in space of terms such as $\varphi(\vec{r},t)$, $\partial_t\varphi(\vec{r},t)$, $\nabla\varphi(\vec{r},t)$, $\nabla\partial_t\varphi(\vec{r},t)$, and similar derivatives.

iii. The scaling form in eq. (29) is invariant under $\varphi(\vec{r},t) \to \varphi(\vec{r},t) + \Phi(\vec{r})$, with $\Phi(\vec{r})$ independent of time. Thus, the action must also have this symmetry.

iv. The action must be positive definite.

These requirements largely restrict the possible actions. The one with the smallest number of spatial derivatives (most relevant terms) is

$$\mathcal{S}[\varphi] = \int d^d r \int dt \left[K\, \frac{(\nabla\partial_t\varphi(\vec{r},t))^2}{\partial_t\varphi(\vec{r},t)} \right] , \qquad (30)$$

Chamon & LFC 07

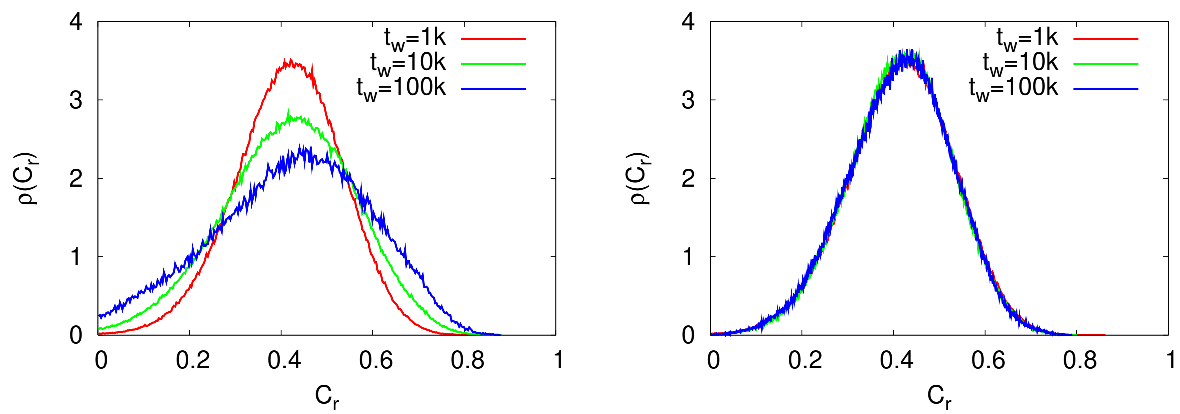
Sigma Model

Some consequences - $3d$ Edwards Anderson model

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{ag}(\vec{r},t,t_w) = f_{ag}(e^{-\int_{t_w}^{t} dt' \, \partial_{t'}\varphi(\vec{r},t')})$$

Distribution of local correlations depends on times t, t_w only through C, ξ

$$\rho(C_{\vec{r}};\, t, t_w, \ell, \xi(t,t_w)) \to \rho(C_{\vec{r}};\, C_{ag}(t,t_w), \ell/\xi(t,t_w))$$



t, t_w such that $C_{ag}(t,t_w) = C$ $\qquad$ ℓ such that $\ell/\xi = \mathrm{cst}$ $\qquad$ **Jaubert, Chamon, LFC, Picco 07**

predictions on the form of ρ derived from $S[\varphi]$ too

Tests in Lennard-Jones systems **Avila, Castillo, Mavimbela, Parsaeian 06-12**

How general is this ?

Coarsening & domain growth

e.g. the d-dimensional $O(N)$ model in the large N limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

N component field $\vec{\phi} = (\phi_1, \dots, \phi_N)$ with Langevin dynamics

$$\partial_t \phi_\alpha(\vec{r},t) = \nabla^2 \phi_\alpha(\vec{r},t) + \lambda |N^{-1}\phi^2(\vec{r},t) - 1|\, \phi_\alpha(\vec{r},t) + \xi_\alpha(\vec{r},t)$$

$\phi_\alpha(\vec{k},0)$ Gaussian distributed with variance Δ^2

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$

Same in the $p = 2$ spherical model

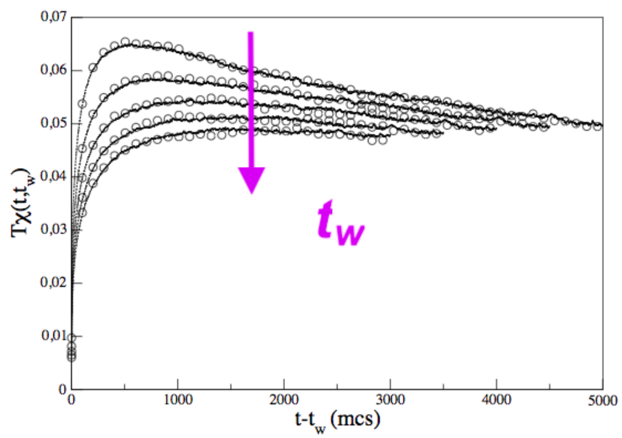
Chamon, LFC, Yoshino 06

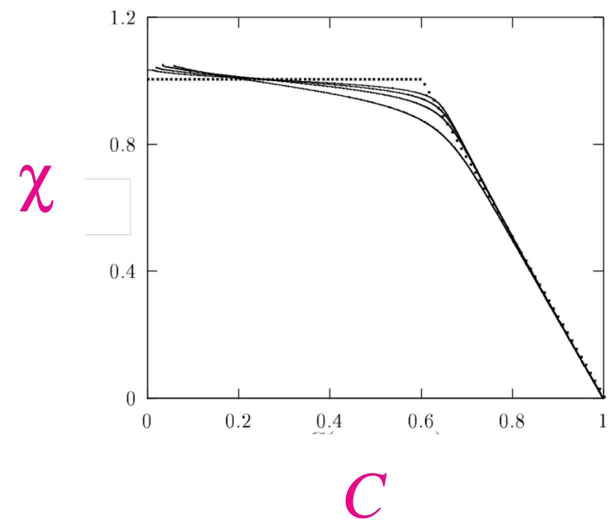
How general is this ?

Coarsening & domain growth

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$





Ising FM, $O(N)$ field theory, or $p = 2$ spherical model

Related to $T^* \to \infty$ and simplicity of free-energy landscape

Triangular relations

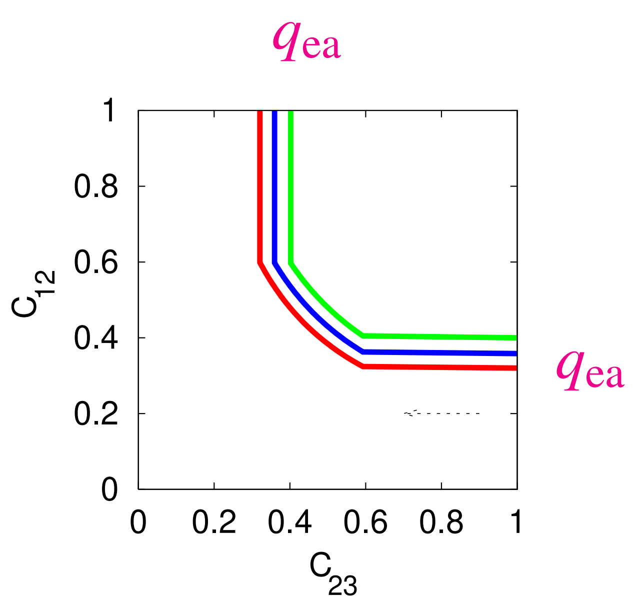
Scaling of the aging global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1,t_2),\ C(t_2,t_3),\ C(t_1,t_3)$$

If, in the aging regime $C_{\rm ag}^{ij} \equiv C_{\rm ag}(t_i,t_j) = f_{\rm ag}\left(\frac{h(t_i)}{h(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{\rm ag}^{12} = f_{\rm ag}\left(\frac{h(t_1)}{h(t_3)}\frac{h(t_3)}{h(t_2)}\right) = f_{\rm ag}\left(\frac{f_{\rm ag}^{-1}(C_{\rm ag}^{13})}{f_{\rm ag}^{-1}(C_{\rm ag}^{23})}\right)$$



choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12} = q_{\rm ea} C^{13}/C^{23}$

Triangular relations

Scaling of the slow part of the global correlation

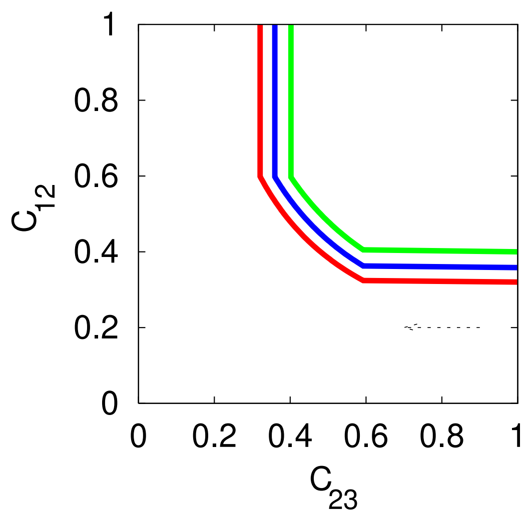
Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec{r}}(t_1,t_2),\ C_{\vec{r}}(t_2,t_3),\ C_{\vec{r}}(t_1,t_3)$$

If, in the aging regime $C_{\vec{r}}^{ij} \equiv C_{\vec{r}}(t_i,t_j) = f_{\rm ag}\left(\frac{h_{\vec{r}}(t_i)}{h_{\vec{r}}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{\vec{r}}^{12} = f_{\rm ag}\left(\frac{f_{\rm ag}^{-1}(C_{\vec{r}}^{13})}{f_{\rm ag}^{-1}(C_{\vec{r}}^{23})}\right)$$

$q_{\rm ea}$



$q_{\rm ea}$

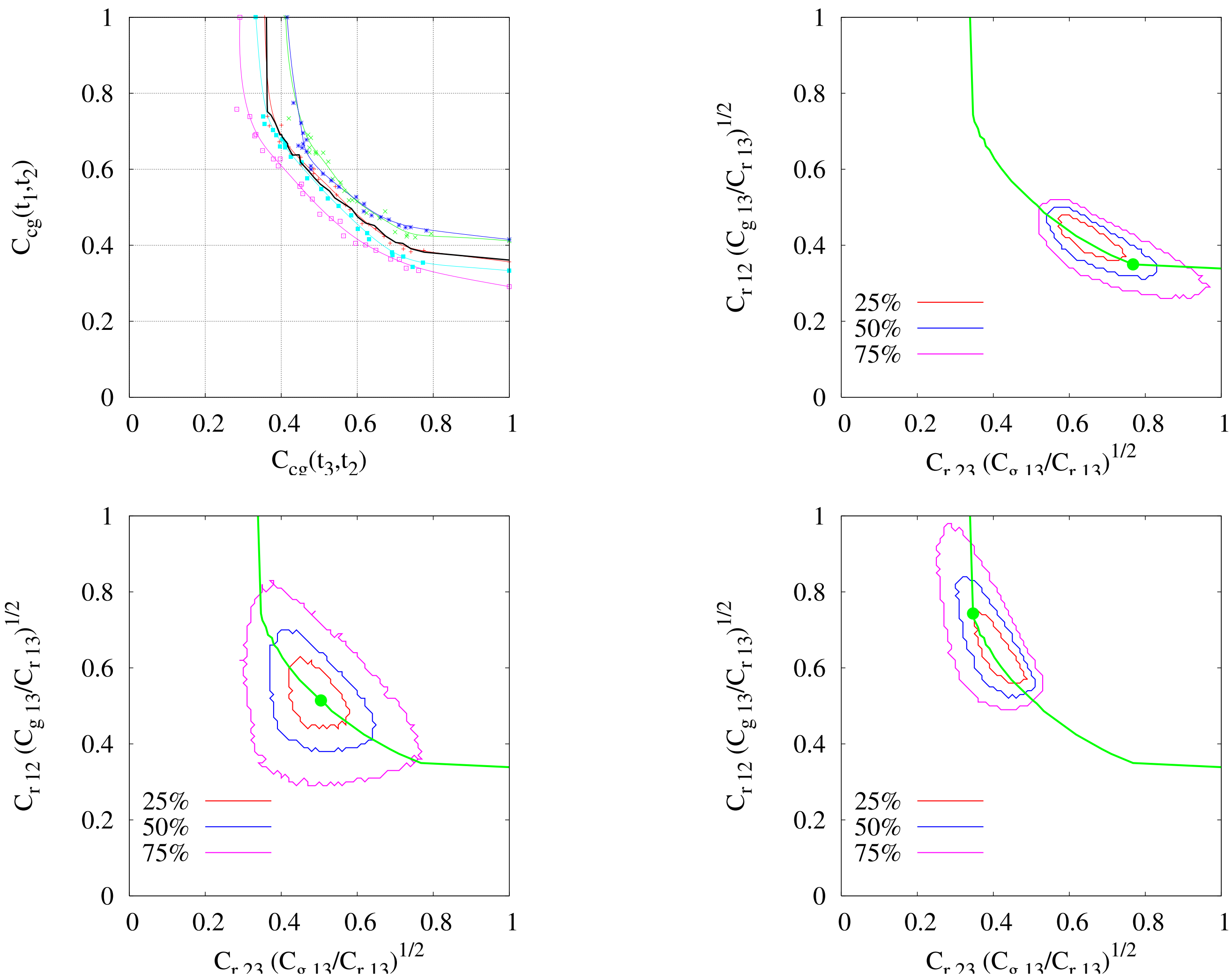
choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 'flow' from t_3 to t_1

e.g. $C_{\vec{r}}^{12} = q_{\rm ea} C_{\vec{r}}^{13}/C_{\vec{r}}^{23}$.

Triangular relations

$3d$ Edwards-Anderson model



Jaubert, Chamon, LFC & Picco 07